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A reduced Hsieh–Clough–Tocher element with splitting based on an arbitrary interior point ☆

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Dedicated to Professor William F. Ames on his 80th birthday

Abstract

We present formulas for a reduced Hsieh–Clough–Tocher (rHCT) element with splitting based on an arbitrary interior point. These formulas use local barycentric coordinates in each of the subtriangles and are not significantly more complicated than formulas for an rHCT element with splitting based on the centroid.

Keywords: Bernstein–Bézier; Clough–Tocher element; Reduced Hsieh–Clough–Tocher element; Splitting; Subtriangle; Triangle

1. Introduction

Surfaces on triangular grids are usually represented by one linear function on each triangle. Such representations fit well with current graphics hardware and software but require many triangles to achieve mathematical and visual accuracy, which reduces computational efficiency. One

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can reduce the number of triangles and increase the computational efficiency by using flexible piecewise cubic elements rather than rigid linear elements.

The Clough–Tocher (CT) element [2,3] is a triangular C^1 piecewise cubic element that is constructed on a split of each triangle into three subtriangles. A CT element is defined by the values of the function and its first derivatives at the nodes of the triangle and the values of the derivative in the normal direction at the midpoints of the three edges of the triangle. A variant of this element, the reduced Hsieh–Clough–Tocher (rHCT) element [1,8], is fully defined by the values of the function and its first derivatives at the nodes of the triangle and does not require specification of the derivative in the normal direction at the midpoints of the three edges of the triangle. The rHCT element requires fewer parameters than the CT element and has been widely used in finite-element analysis [8] and in surface and terrain modeling [5–7,10].

In all previous work involving CT and rHCT elements except [9], the three subtriangles into which the triangle is split are created by connecting the vertices of the triangle to the centroid. Worsey and Farin [9] require a CT element with a different splitting for their generalization of CT elements to higher dimensions. The authors of the present paper are in the process of developing a tetrahedral analogue of the rHCT triangular element. To ensure tetrahedron-to-tetrahedron C^1 continuity of the tetrahedral elements that we are developing, the splittings of the rHCT elements on the faces must be based on points other than the centroids, just as was the case in [9]. The development of this new tetrahedral element has been delayed because of the absence in the literature of formulas for an rHCT element with splitting based on a point other than the centroid. The present paper fills this void.

2. rHCT element: Definition and representation

Let there be given an irregular triangulation of a domain. Denote the counterclockwise-ordered vertices of a triangle $\mathscr{T}$ in this triangulation by $V_1 = (x_1, y_1)$, $V_2 = (x_2, y_2)$ and $V_3 = (x_3, y_3)$ as indicated in Fig. 1. The values of a function $z(x, y)$ at these vertices will be denoted by z_1, z_2 and z_3, respectively. The values of the first derivatives $\partial z/\partial x$ and $\partial z/\partial y$ of this function at these vertices will be denoted by z_1^x, z_1^y, z_2^x, z_2^y, z_3^x and z_3^y (subscript = index of vertex, superscript = direction of derivative). Let $V_0 = (x_0, y_0)$ be an arbitrary point in the interior of the triangle. Connecting V_0 to each of the vertices of the triangle creates three subtriangles $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$. Edges 1, 2 and 3 denote the edges of triangle $\mathscr{T}$ that are included in subtriangles $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, respectively. We use x_{ij} and y_{ij} to denote $x_i - x_j$ and $y_i - y_j$, respectively, for $i, j = 0, 1, 2, 3$.

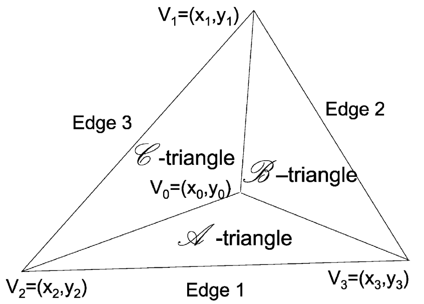

Fig. 1. Notation for triangle $\mathscr{T}$ and its splitting.

An rHCT element is a surface $z(x,y)$ on $\mathscr{T}$ defined by the following conditions:

(1) The element is a cubic function of x and y on each of the three subtriangles.
(2) At each of the three vertices of $\mathscr{T}$, the element matches the prescribed function value and the prescribed first derivatives in directions x and y.
(3) Linear Derivative Condition: On each edge of $\mathscr{T}$, the derivative of the element in the direction normal to that edge is a linear function of arclength.
(4) The element is C^1 smooth in the interior of $\mathscr{T}$.

It is a consequence of these four conditions that an rHCT element joins C^1 smoothly with the rHCT elements in the neighboring triangles [8].

McClain and Witzgall [8] developed their rHCT element with centroid-based splitting in a framework of global barycentric coordinates in triangle $\mathscr{T}$. For representation of an rHCT element with splitting based on an arbitrary interior point, it is convenient to use local barycentric coordinates in each of the subtriangles. We denote the barycentric coordinates of subtriangles $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ by $(\alpha_1,\alpha_2,\alpha_3)$, $(\beta_1,\beta_2,\beta_3)$ and $(\gamma_1,\gamma_2,\gamma_3)$, respectively. Orientation of all barycentric coordinates is counterclockwise. Edges 1, 2 and 3 correspond to $\alpha_1=0$, $\beta_1=0$ and $\gamma_1=0$, respectively. On the subtriangles $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, cubic polynomials are represented in the Bernstein–Bézier forms

$$z^{(\mathscr{A})}(\alpha_1,\alpha_2,\alpha_3)=\sum_{i+j+k=3} a_{ijk}\frac{3!}{i!j!k!}(\alpha_1)^i(\alpha_2)^j(\alpha_3)^k, \tag{1}$$

$$z^{(\mathscr{B})}(\beta_1,\beta_2,\beta_3)=\sum_{i+j+k=3} b_{ijk}\frac{3!}{i!j!k!}(\beta_1)^i(\beta_2)^j(\beta_3)^k, \tag{2}$$

$$z^{(\mathscr{C})}(\gamma_1,\gamma_2,\gamma_3)=\sum_{i+j+k=3} c_{ijk}\frac{3!}{i!j!k!}(\gamma_1)^i(\gamma_2)^j(\gamma_3)^k, \tag{3}$$

respectively [4]. Here, the terms $(3!/(i!j!k!))(\alpha_1)^i(\alpha_2)^j(\alpha_3)^k$, $(3!/(i!j!k!))(\beta_1)^i(\beta_2)^j(\beta_3)^k$ and $(3!/(i!j!k!))(\gamma_1)^i(\gamma_2)^j(\gamma_3)^k$ are the Bernstein polynomials and the coefficients a_{ijk}, b_{ijk} and c_{ijk} are the Bézier ordinates.

3. Bézier ordinates of the rHCT element

The rHCT element is defined by the 30 Bézier ordinates in expressions (1), (2) and (3). Here we derive those ordinates omitting the detailed calculations. Equating expressions (1) and (2) to z_3 at V_3, expressions (2) and (3) to z_1 at V_1 and expressions (3) and (1) to z_2 at V_2, one obtains 6 of these ordinates:

$$a_{030}=z_2,\quad a_{003}=z_3,\qquad b_{030}=z_3,\quad b_{003}=z_1,\qquad c_{030}=z_1,\quad c_{003}=z_2. \tag{4}$$

Equating the first derivatives of the rHCT element to z_1^x, z_1^y, z_2^x, z_2^y, z_3^x and z_3^y at the vertices determines 12 more of the Bézier ordinates:

$$a_{120}=c_{102}=z_2+z_2^x\frac{x_{02}}{3}+z_2^y\frac{y_{02}}{3},$$

$$b_{120}=a_{102}=z_3+z_3^x\frac{x_{03}}{3}+z_3^y\frac{y_{03}}{3},\qquad c_{120}=b_{102}=z_1+z_1^x\frac{x_{01}}{3}+z_1^y\frac{y_{01}}{3},$$

$$a_{021}=z_2+z_2^x\frac{x_{32}}{3}+z_2^y\frac{y_{32}}{3},\qquad a_{012}=z_3+z_3^x\frac{x_{23}}{3}+z_3^y\frac{y_{23}}{3},$$

$$b_{021} = z_3 + z_3^x \frac{x_{13}}{3} + z_3^y \frac{y_{13}}{3}, \qquad b_{012} = z_1 + z_1^x \frac{x_{31}}{3} + z_1^y \frac{y_{31}}{3},$$
$$c_{021} = z_1 + z_1^x \frac{x_{21}}{3} + z_1^y \frac{y_{21}}{3}, \qquad c_{012} = z_2 + z_2^x \frac{x_{12}}{3} + z_2^y \frac{y_{12}}{3}. \tag{5}$$

The Linear Derivative Condition (condition (3) of the rHCT element) together with Eqs. (4) and (5) determines 3 more of the Bézier ordinates:

$$a_{111} = \frac{a_{120} - a_{021} + a_{102} - a_{012}}{2} + P_{30}(z_2 - 3a_{021}) + P_{02}(z_3 - 3a_{012}),$$
$$b_{111} = \frac{b_{120} - b_{021} + b_{102} - b_{012}}{2} + P_{10}(z_3 - 3b_{021}) + P_{03}(z_1 - 3b_{012}),$$
$$c_{111} = \frac{c_{120} - c_{021} + c_{102} - c_{012}}{2} + P_{20}(z_1 - 3c_{021}) + P_{01}(z_2 - 3c_{012}), \tag{6}$$

where

$$P_{30} = \frac{x_{23}x_{30} + y_{23}y_{30}}{2({x_{23}}^2 + {y_{23}}^2)}, \qquad P_{02} = \frac{x_{02}x_{23} + y_{02}y_{23}}{2({x_{23}}^2 + {y_{23}}^2)},$$
$$P_{10} = \frac{x_{31}x_{01} + y_{31}y_{10}}{2({x_{31}}^2 + {y_{31}}^2)}, \qquad P_{03} = \frac{x_{03}x_{31} + y_{03}y_{31}}{2({x_{31}}^2 + {y_{31}}^2)},$$
$$P_{20} = \frac{x_{12}x_{20} + y_{12}y_{20}}{2({x_{12}}^2 + {y_{12}}^2)}, \qquad P_{01} = \frac{x_{01}x_{12} + y_{01}y_{12}}{2({x_{12}}^2 + {y_{12}}^2)}. \tag{7}$$

The continuity of z and of its first derivatives between subtriangles (condition (4) of the rHCT element) determines the remaining 9 Bézier ordinates:

$$a_{210} = c_{201} = \frac{c_{111} - \alpha_2^{(1)} a_{120} - \alpha_3^{(1)} a_{111}}{\alpha_1^{(1)}},$$
$$b_{210} = a_{201} = \frac{a_{111} - \beta_2^{(2)} b_{120} - \beta_3^{(2)} b_{111}}{\beta_1^{(2)}},$$
$$c_{210} = b_{201} = \frac{b_{111} - \gamma_2^{(3)} c_{120} - \gamma_3^{(3)} c_{111}}{\gamma_1^{(3)}},$$
$$a_{300} = b_{300} = c_{300} = \frac{a_{201} - \gamma_2^{(3)} b_{201} - \gamma_3^{(3)} c_{201}}{\gamma_1^{(3)}}, \tag{8}$$

where $(\alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)})$, $(\beta_1^{(2)}, \beta_2^{(2)}, \beta_3^{(2)})$ and $(\gamma_1^{(3)}, \gamma_2^{(3)}, \gamma_3^{(3)})$ are the representations of V_1, V_2 and V_3, respectively, in the barycentric coordinates of subtriangles $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$, respectively. Use of these representations of the vertices significantly simplifies formulas (8) and provides them with intuitive geometry-related internal structure.

Equations (4)–(6) and (8) define the 30 Bézier ordinates of an rHCT element with splitting based on an arbitrary interior point. The structure in the above derivation can be generalized to the derivation of the Bézier ordinates of tetrahedral analogues of rHCT elements.

4. Conclusion

In the prior literature, a CT element with splitting based on an arbitrary interior point is discussed and computational formulas for an rHCT element with splitting based on the centroid are available. However, computational formulas for CT and rHCT elements with splitting based on an arbitrary interior point are not available. The formulas given above for an rHCT element with splitting based on an arbitrary interior point are not significantly more complicated than the formulas of [8] for an rHCT element with splitting based on the centroid. The rHCT element with splitting based on an arbitrary interior point is a hitherto unused, computationally competitive, flexible modeling tool for bivariate finite element procedures and geometric modeling and for extension to tri- and multivariate finite elements and geometric modeling.

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