Square-Integrable Representations and the Dual Topology

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We show that the square-integrable factor representations of a connected locally compact group $G$ are precisely the normal representations whose kernels in $C^*(G)$ are open points of $\text{Pinc}_{\text{int}}(G)$ (the support of Plancherel measure). Related results hold for certain other groups. We also settle questions of Dixmier, and Duflo and Moore, by giving examples of square-integrable irreducible representations (of totally disconnected groups) which are not open in the reduced dual.

1. INTRODUCTION

In [7] Dixmier raised certain questions concerning the relation between the Fell topology and Plancherel measure on the unitary dual $\hat{G}$ of a unimodular Type I group $G$. He pointed out that points in the reduced dual $\hat{G}_r$ (the support of Plancherel measure) which are open have positive Plancherel measure, and thus define irreducible square-integrable representations of $G$; and he asked whether, conversely, square-integrable points of $\hat{G}$ are open in $\hat{G}_r$. As evidence in favor of this he proved [7, Prop. 2] that integrable points are indeed open in $\hat{G}_r$. (It was later shown by Duflo and Moore [12, Cor. 1, p. 223] that integrable points are in fact open in $\hat{G}$—this answers another question raised in [7].)

In Section 3 we construct an example which answers Dixmier's question in the negative. A related example shows that a square-integrable point of $\hat{G}$, where $G$ is now allowed to be non-unimodular, need not be locally closed (i.e. open in its closure), thus providing a negative answer to a question of Duflo and Moore [12, Problem, p. 228]. Both examples are "restricted direct product" groups, the consideration of which is suggested in part by a recent paper of Blackadar [2] where the regular representations of such groups are discussed.

On the other hand, for almost connected $G$ (i.e. $G$ such that $G/G^0$ is compact, where $G^0$ is the identity component) we are able to obtain positive results, of in fact a more general nature. Thus let $H$ be locally compact group which is

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\( \sigma \)-compact (i.e. a countable union of compact subsets) and contains a closed normal almost connected subgroup \( G \). To any unitary representation of \( G \) we may associate a certain projection-valued measure, called "Glimm measure," on the primitive ideal space \( \text{Prim} \, G \) of the group \( C^* \)-algebra \( C^*(G) \); cf. [15, Thm. 1.9]. The measure class on \( \text{Prim} \, G \) associated to the left regular representation \( \lambda \) will be called "Plancherel measure" for \( G \) (when \( G \) is Type I, this measure class is that of ordinary Plancherel measure). Plancherel measure is invariant under the action of \( H \) on \( \text{Prim} \, G \) (since \( \lambda \) is invariant under \( H \)), so its support, \( \text{Prim} \, G \), is also invariant. Our main result (proved in Section 2) is then that an \( H \)-orbit in \( \text{Prim} \, G \) has positive Plancherel measure iff it is open in \( \text{Prim} \, G \).

By combining this with results of Rosenberg [29] we deduce as a corollary that the square-integrable factor representations (in the strict sense, as defined in [29, Def. 2.2]) of a connected group \( G \) are precisely the normal (i.e. traceable factor) representations whose kernel in \( C^* (G) \) is an open point in \( \text{Prim} \, G \). In particular, a square-integrable irreducible point \( \tau \) of \( \hat{G} \) is open in \( \hat{G} \). The latter result also holds for unimodular groups \( G \) having an open normal almost connected subgroup \( N \). (For the special case that \( N \) is a compact extension of \( \mathbb{R}^n \), nilpotent Lie, or rank one semisimple, this has been proved in [30] and [21, Remark 2, p. 4161].) This implies that a representation weakly contained in \( \lambda \) and weakly containing \( \tau \) must contain a subrepresentation equivalent to \( \tau \).

As a subsidiary result used in proving the main theorem we answer a question of M. Goto by showing that every connected Lie group can be embedded as a closed normal subgroup of a "locally algebraic" Lie group (i.e. one whose Lie algebra is isomorphic to that of an algebraic group of automorphisms of some real vector space). This fact is needed in order to apply the results of [19, Section 7] (which generalize Pukanszky's theory in [26]) to Lie groups which are not necessarily simply connected. Among the other techniques used in Section 2 are those developed by Dixmier [6] and Pukanszky [25] to analyze the representations of algebraic Lie groups. The basic idea of the proof is to use the Mackey machine to reduce to the case of a reductive group with compact center, which can then be dealt with thanks to Harish-Chandra's results on the Plancherel formula of such groups.

We make free use of Mackey's theory of induced representations (for a summary of which, see [1, Chap. 1]) and of the basic facts about \( C^* \)-algebras, particularly those associated to groups (cf. [9, Chaps. 1-6 and 13-18]).

"Representation" will always mean "strongly continuous unitary representation" when referring to a (locally compact) group and "*-representation" when referring to a \( C^* \)-algebra. We will frequently use the same letter to denote a representation of a group and the associated representation of its \( C^* \)-algebra. If \( H \) is a closed subgroup of a locally compact group \( G \), and \( \pi, \eta \) representations of \( G \) and \( H \) respectively, then \( \text{Res}_H \pi \) denotes the restriction of \( \pi \) to \( H \), and \( \text{Ind}_H \eta \) the representation of \( G \) induced from \( H \). Ends of proofs are denoted by "\( \Box \)."
2. The Case of Almost Connected Groups

We begin by recalling a few facts related to the Mackey normal subgroup analysis [22], as extended by Blattner [3] and Rieffel [28]. Let $G$ be a locally compact group, $N$ a closed normal subgroup. The action of $G$ on $N$ by conjugation induces an action on $C^*(N)$, hence an action (denoted $(s, J) \mapsto sJ$ for $s \in G$, $J \in \text{Prim }N$) on Prim $N$, which turns out to be jointly continuous. A $G$-orbit $\mathcal{O}$ in Prim $N$ is said to be regular if it is locally closed and if for any $J \in \mathcal{O}$ the map

$$G|G_J \to \mathcal{O}, \quad sG_J \mapsto sJ$$

is a homeomorphism. (Here $G_J$ denotes the stabilizer in $G$ of the point $J$.) In this situation the process of inducing representations of $G_J$ up to $G$ restricts to an equivalence $\mathcal{E}_J$ between the category $\mathcal{C}_{\mathcal{O}}$ of representations $\pi$ of $G_J$ whose restrictions $\pi|_N$ are concentrated on $\{J\}$ and the category $\mathcal{C}_{\mathcal{O}}^G$ of representations of $G$ whose restrictions live on $\mathcal{O}$; furthermore this equivalence preserves weak containment of representations. If $J$ is "semi-compact"—i.e. if it is the kernel of a GCR irreducible representation $\pi_J$ of $C^*(N)$—then there is a central extension $G_J$ of $G_J/N$ by the circle group $T$ such that there is an equivalence $\mathcal{E}_J$ (again preserving weak containment) between $\mathcal{C}_{\mathcal{O}}^{G_J}$ and $\mathcal{C}_{\mathcal{O}}^{G_J}$, where $\chi$ is the representation of $T$ sending $t$ to the corresponding scalar operator on the one-dimensional Hilbert space $\mathbb{C}$. It is easy to see (for example from Mackey's description of $\mathcal{E}_J$ in [22]) that, at least when $G_J$ is second countable, $\mathcal{E}_J$ takes $\text{Ind}_{G_J}^G \pi_J$ to $\text{Ind}_{G_J}^G \chi$.

The normal subgroup $N$ is said to be regularly embedded in $G$ if every $G$-orbit in Prim $N$ is regular, and if every irreducible representation of $G$ lives over some orbit in Prim $N$. (The latter condition is automatic if $N$ is second countable, or if Prim $N$ is discrete, for example.) For the purposes of our theorem we will need to know that the above equivalences are "equivariant." Thus let $H$ be a locally compact group containing $G$ as a closed normal subgroup, and $N$ a closed subgroup of $G$ which is normal in $H$. For convenience we will assume that $H$ is second countable. Let $J$ be an element of Prim $N$ such that the orbits $HJ$ and $GJ$ are both regular; then every $G$-orbit in $HJ$ is regular. Let $\mathcal{O}_{HJ}$ (resp. $\mathcal{O}_{GJ}$) be the part of Prim $G$ which lives over $HJ$ (resp. $GJ$). (We say $P \in \text{Prim }G$ lives over a Borel subset $\mathcal{O}$ of Prim $N$ if for some $\pi \in \hat{G}$ having kernel $P$, $\pi|_N$ lives on $\mathcal{O}$. The choice of $\pi$ is irrelevant when $\mathcal{O}$ is locally closed and $G$-invariant (the primary case of interest for us): By continuity of restriction [14, Thm. 2.3] the subsets $C_1$ (resp. $C_2$) of $\hat{G}$ consisting of those elements whose restrictions to $N$ live on $\mathcal{O}$ (resp. $\mathcal{O}^\circ \setminus \mathcal{O}$) are closed. Now if $\pi \in C_1 \backslash C_2$, the Glimm projection corresponding to $\mathcal{O}$ for $\pi|_N$ is non-zero and $\pi(G)$ invariant hence equal to the identity operator since $\pi$ is irreducible. It
follows that $C_1 \cap C_2 := \{ \pi \in \hat{G} : \pi \in \mathcal{N} \}$ lives on $\mathcal{O}$. Since $C_1 \cap C_2$ is locally closed and the hull-kernel topology on $\hat{G}$ does not distinguish representations having the same kernel in $\mathcal{C}(\gamma, G)$, our assertion follows. Note that this argument also shows that the part of $\text{Prim } G$ living over $\mathcal{O}$ is locally closed.) By the argument of [28, 8.1], every element of $\mathcal{C}_{HJ}$ lives over a unique $G$-orbit in $HJ$. Note that, for $s \in H$, $sJ \in GJ$ if and only if $s \in HJG$. From this and the $H$-equivariance of restriction it follows that $\mathcal{C}_{HJ}$ is stable under the $H$-action and that the map which associates to each $H$-orbit in $\mathcal{C}_{HJ}$ its intersection with $\mathcal{C}_{GJ}$ gives a bijection $\mathcal{P}^H_G$ of the orbit space $\mathcal{C}_{HJ}/H$ onto $\mathcal{C}_{GJ}/HJG$ ( = $\mathcal{C}_{GJ}/HJ$, since $G$ acts trivially on $\text{Prim } G$).

I claim that $\mathcal{P}^H_G$ is a homeomorphism: Given a net $(\mathcal{O}_a)$ in $\mathcal{C}_{HJ}/H$ converging to some element $\mathcal{O}$, we may choose (passing to a subnet if necessary) in virtue of the openness of the natural map of $\mathcal{C}_{HJ}$ onto its orbit space, representatives $P_a \in \mathcal{O}_a$ converging to $P \in \mathcal{O}$. There is a natural homeomorphism of $HJHG$ onto $HJG$ which to $sHJG$ associates the orbit $G(sJ)$; composing the inverse of this with the natural projection of $\mathcal{C}_{HJ}$ onto $HJHG$ obtained by restriction we get a continuous $H$-equivariant map of $\mathcal{O}_a$ onto $\mathcal{C}_{HJ}/HJG$. Let $s_aHJG$ denote the image of $P_a$, $s_aHJG$ that of $P$. We may choose the $s_a$ so as to converge to $s$ in $H$. Then $s_a^{-1}P_a$ converges to $s^{-1}P$, but $s_a^{-1}P_a \in \mathcal{O}_a \cap \mathcal{O}_{GJ}$ and $s^{-1}P \in \mathcal{O} \cap \mathcal{O}_{GJ}$, so we have shown that $\mathcal{P}^H_G(\mathcal{O}_a)$ converges to $\mathcal{P}^H_G(\mathcal{O})$. Hence $\mathcal{P}^H_G$ is continuous. On the other hand $(\mathcal{P}^H_G)^{-1}$ is clearly continuous, being the unique map whose composition with the projection of $\mathcal{C}_{GJ}$ onto $\mathcal{C}_{GJ}/HJ$ is equal to the continuous map obtained by composing the embedding $\mathcal{C}_{GJ} \hookrightarrow \mathcal{C}_{HJ}$ with the projection $\mathcal{C}_{HJ} \rightarrow \mathcal{C}_{HJ}/H$. So $\mathcal{P}^H_G$ is a homeomorphism, as claimed. A similar argument shows $\mathcal{P}^H_G$ preserves regularity of orbits.

Since the equivalence $\mathcal{K}_J$ defined earlier is easily seen to be $H_J$-equivariant (cf. e.g. [19, Lemma 10]) and preserves weak containment, it induces a homeomorphism of $\mathcal{O}_J/H_J$ (where $\mathcal{O}_J = \mathcal{O}_{(1)}$) onto $\mathcal{O}_{GJ}/HJ$. We have thus shown that there is a natural homeomorphism of $\mathcal{O}_{HJ}/H$ with $\mathcal{O}_{HJ}/HJ$.

Now suppose that $J$ is semi-compact. The equivalence $\mathcal{K}_J$ induces a homeomorphism between $\mathcal{O}_J$ and the part $\mathcal{E}_J$ of Prim $\hat{G}_J$ which lives over $\chi_J$; furthermore it is not hard to see that this homeomorphism is equivariant for the $H_J/T$ ($\cong H_J/N$) actions on $\mathcal{E}_J$ and $\mathcal{E}_J$. Combining this with our previous result we get a homeomorphism $\mathcal{P}^H_G$ between $\mathcal{O}_{HJ}/H$ and $\mathcal{O}_{HJ}/HJ$, which preserves regular orbits.

Suppose now that the orbit $HJ$ has positive Plancherel measure; we wish to show that, under $\mathcal{P}^H_G$, orbits in $\mathcal{O}_{HJ}$ of positive Plancherel measure correspond to orbits in $\mathcal{E}_J$ of positive Plancherel measure. Since $\mathcal{K}_J$ takes Ind$_{\mathcal{O}_J}$, $H_J$-orbits in $\mathcal{E}_J$ of positive Plancherel measure correspond to $H_J$-orbits in $\mathcal{O}_{HJ}$ having positive Glimm measure for the representation Ind$_{\mathcal{O}_J}$, these being precisely the orbits which carry the Glimm measure for some non-zero subrepresentation of Ind$_{\mathcal{O}_J}$, $\pi_J$. Now let $C \in \mathcal{O}_{HJ}/H$. If $C$ has positive Glimm measure it defines a non-zero subrepresentation of Ind$_{\mathcal{O}_J}$, $\pi_J$, which then induces to a non-zero subrepresentation of Ind$_{\mathcal{O}_J}$, Ind$_{\mathcal{O}_J}$, $\pi_J \cong$ Ind$_{\mathcal{O}_J}$, $\pi_J$ living on the $H_J$-
orbit \( \mathcal{C}_1 \) of \( \mathcal{C}_{GJ} \) which corresponds to \( \mathcal{C} \). If on the other hand the measure of \( \mathcal{C} \) is zero, then (by [13]) \( \text{Ind}_{NJ} G \pi_J \) can be written as a direct integral of “homogeneous representations” over \( \mathcal{C}_J \setminus \mathcal{C} \), and since the induction process “commutes” with direct integrals \( \text{Ind}_{N} G \pi_J \) can be written as a direct integral over \( \mathcal{C}_{GJ} \setminus \mathcal{C}_1 \). Thus \( \mathcal{C}_1 \) has positive Glimm measure for \( \text{Ind}_{N} G \pi_J \) iff \( \mathcal{C} \) does for \( \text{Ind}_{GJ} G \pi_J \). Now if \( \mathcal{C}_1 \) has positive measure, it defines a subrepresentation of \( \text{Ind}_{N} G \pi_J \) which induces to a subrepresentation of \( \text{Ind}_{H} H \pi_J \), which in turn restricts to a subrepresentation of \( \text{Res}_{C} H \text{Ind}_{N} G \pi_J \) living on the \( H \)-orbit \( \mathcal{C}_2 \) in \( \mathcal{C}_{HJ} \setminus H \) corresponding to \( \mathcal{C}_1 \); hence \( \mathcal{C}_2 \) has positive Glimm measure in \( \text{Res}_{C} H \text{Ind}_{N} H \pi_J \). If on the other hand \( \mathcal{C}_1 \) has zero measure then by considering direct integrals as before we can show that \( \mathcal{C}_2 \) does also. Observe now that \( \text{Ind}_{N} H \pi_J \) is equivalent to \( \text{Ind}_{N} H \pi_{sJ} \), for any \( s \in H \), so \( \text{Ind}_{N} H \pi_J \) is quasi-equivalent to \( \text{Ind}_{N} H \pi \) where \( \pi \) is a direct integral \( \int_{H} \pi_{sJ} ds \) (here \( ds \) is left Haar measure on \( H \)); since \( HJ \) has positive Plancherel measure, \( \pi \) is quasi-equivalent to a subrepresentation of \( \lambda_{H} \), hence \( \text{Ind}_{N} H \pi_J \) to a subrepresentation of \( \lambda_{H} \), hence \( \text{Res}_{C} H \text{Ind}_{N} H \pi_J \) to a subrepresentation of \( \lambda_{G} \) (because \( \lambda_{H} \mid G \) is quasi-equivalent to \( \lambda_{G} \)). Therefore Glimm measure in \( \text{Res}_{C} H \text{Ind}_{N} G \pi_J \) corresponds to Plancherel measure. We have proved

**Lemma 1.** Let \( H \) be a second countable locally compact group, \( G \) and \( N \) closed normal subgroups of \( H \) with \( N \subseteq G \). Suppose the \( G \)- and \( H \)-orbits of \( J \in \text{Prim} N \) are regular. Then there is a natural homeomorphism of \( \mathcal{C}_{HJ} \setminus H \) onto \( \mathcal{C}_{J} \setminus H \). If \( J \) is semi-compact there is a natural homeomorphism of \( \mathcal{C}_{H} \setminus H \) with \( \mathcal{C}_{J} \setminus H \), which, when \( HJ \) has positive Plancherel measure, preserves positivity of Plancherel measure. These homeomorphisms preserve regular orbits.

Note that by continuity of restriction \( \mathcal{C}_{HJ} \) is locally closed, and has open intersection with \( \text{Prim}_r G \) if \( HJ \) is open in \( \text{Prim}_r N \).

In the proof of the following lemma, the Lie algebra of a Lie group is denoted by the corresponding lower case letter.

**Lemma 2.** Let \( G \) be a closed connected normal subgroup of the connected Lie group \( H \). There exist connected locally algebraic Lie groups \( L, M, N \) such that \( G, H, M, \) and \( N \) are normal subgroups of \( L \) with \( G, H, \) and \( N \) closed, \( N \subseteq G \subseteq M, M/N \) is abelian, and the radical of \( N \) is locally unipotent.

**Proof.** Represent \( \mathfrak{h} \) faithfully as a Lie algebra of linear transformations on a finite dimensional real vector space \( V \), in such a way that the nilradical of \( \mathfrak{h} \) acts by nilpotent matrices [11, 2.5.5]. Let \( \mathfrak{l} \) and \( \mathfrak{m} \) be the smallest algebraic Lie algebras acting on \( V \) which contain (the images of) \( \mathfrak{h} \) and \( \mathfrak{g} \), respectively. Let \( \mathfrak{n} \) be the Lie algebra generated by the nilradical of \( \mathfrak{g} \) (which is algebraic, since contained in the nilradical of \( \mathfrak{h} \)) and \([\mathfrak{g}, \mathfrak{g}]\) (which is algebraic, by [5, Thm. 15, p. 177]); then \( \mathfrak{n} \) is algebraic by [5, Thm. 14, p. 175]. By [5, Thm. 13, p. 173] \( \mathfrak{h}, \mathfrak{m}, \mathfrak{n}, \) and \( \mathfrak{g} \) are all ideals in \( \mathfrak{l} \), and \( \mathfrak{m}/\mathfrak{n} \) is abelian.
Let $\mathcal{L}$ be a simply connected Lie group with Lie algebra $\mathfrak{l}$, $\pi_1$ the natural homomorphism of $\mathcal{L}$ into $\text{GL}(V)$. The subgroups $\tilde{H}$, $\tilde{M}$, $\tilde{N}$, and $\tilde{G}$ of $\mathcal{L}$ corresponding to $\mathfrak{h}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{g}$ are closed, normal, and simply connected.

I claim that the center $Z(\tilde{H})$ of $\tilde{H}$ is contained in $Z(\tilde{L})$: Note first that $\pi_1(Z(\tilde{H})) \subseteq Z(\pi_1(\tilde{L}))$, since the centralizer in $\text{GL}(V)$ of $\pi_1(Z(\tilde{H}))$ is algebraic and contains $\pi_1(\tilde{H})$. So given $t \in Z(\tilde{H})$, the (continuous) map of $\tilde{H}$ into itself has image contained in $\text{Ker} \pi_1$. The latter set being discrete this image must consist of a single point, namely the identity element; so $t$ is central in $\tilde{L}$, proving the claim.

In particular the kernel $K$ of the natural homomorphism of $\tilde{H}$ onto $H$ is central in $\tilde{L}$. So we simply set $L = \tilde{L}/K$, $M = \tilde{M}K/K$, and $N = \tilde{N}K/K$, and identify $H$ with $\tilde{H}/K$. Then $G = \tilde{G}K/K$. The fact that $N$ is closed in $G$ follows from [8, Prop. 1.5]; all other assertions of the lemma are now easy to verify.

The fact that $H$ can be embedded as a closed normal subgroup of a locally algebraic groups answers a question of M. Goto, who has proved the corresponding assertion with "algebraic" replaced by "semi-algebraic" ([18, 3.4]).

The proof of our main theorem will proceed by reducing to the case of a connected reductive Lie group $G$ with compact center. Hence we must deal with that case separately. It will be convenient to work with a slightly more general class of groups, which was introduced by Wolf in [31, 0.1]: Namely, we assume that $G$ is a finite extension of a reductive Lie subgroup $G_1$ for which $G_1/Z$ (where $Z$ is the center of $G_1$) is connected, and that the adjoint action of $G$ on its complexified Lie algebra is by inner automorphisms. We will assume in addition that $Z$ is compact; this implies that Wolf's "relative discrete series" $\hat{G}_d$ consists precisely of the square-integrable irreducibles of $G$. Wolf shows (extending work of Harish-Chandra in the semi-simple case) in [31, Section 4] that to each Cartan subgroup $H$ of $G$ one can associate a series of representations of $G$ as follows: Let $P$ be a cuspidal parabolic subgroup of $G$ associated to $H$, with Langlands' decomposition $P = MAN$. Then $M$ satisfies the same assumptions as $G$, and furthermore $\tilde{M}_d$ is non-empty. The representations $\pi_{n,\phi} = \text{Ind}_P \eta \otimes \psi$, where $\eta \in \tilde{M}_d$, $\psi \in \tilde{A}$, and $\eta \otimes \psi$ is the representation of $P$ defined by $(\eta \otimes \psi)(man) = \psi(a) \eta(m)$, decompose as finite sums of irreducibles. Those irreducibles which appear (as summands) as $\eta$ and $\psi$ vary form the $H$-series of $G$ (this differs slightly from Wolf's terminology), denoted $\hat{G}_H^*$, which turns out to depend only on the conjugacy class of $H$ (and not on $P$) [31, 4.3.9]; furthermore the series associated to distinct conjugacy classes are disjoint [31, 4.4.6]. $\hat{G}_d$ is non-empty iff $G$ has a compact Cartan subgroup $B$([31, 3.5.8] and our assumption that $Z$ is compact) in which case $\hat{G}_d = \hat{G}_B$ (by Wolf's definitions—see [31, introduction to Section 4]).
Lemma 3. Let $G$ be as above. Then the $\hat{G}_H$ form a finite partition of $\hat{G}_r$ into closed (and hence also open) subsets. Furthermore $\hat{G}_d$ is discrete in the relative topology so in particular square integrable points are always open in $\hat{G}_r$.

Proof. Suppose $\hat{G}_d$ is non-empty. Then it is closed and discrete: if $G$ is connected, this follows from the proof of [21, Thm. 7.2]. Otherwise, we apply the Mackey machine to the normal subgroup $G^0$ of $G$. By our assumptions $ZG^0$ is of finite index in $G$. Since the action of $Z$ on $(G^0)^\sim$ is trivial, all $G$-orbits in $(G^0)^\sim$ are finite, from which (together with compactness of $G/G^0$) it follows easily that $\hat{G}_d$ is precisely the part of $\hat{G}$ living over $\hat{G}_d^0$. The latter set being closed, $\hat{G}_d$ is closed (by continuity of the restriction map). Since $G/G^0$ is compact the fibers of $\hat{G}_d$ over the discrete space $\hat{G}_d^0/G$ are discrete, so $\hat{G}_d$ itself is discrete.

Now let $G$ be any group satisfying our assumptions, and let $H$ be a Cartan subgroup of $G$ with corresponding cuspidal parabolic $P = MAN$. The representations of $P$ of the form $\eta \otimes \chi(\eta \in \hat{M}_d, \chi \in \hat{A})$ are precisely the irreducibles whose restrictions to $MN$ are of the form $\hat{\eta}(\eta \in \hat{M}_d)$ where $\hat{\eta}(mn) = \hat{\eta}(m)$. Since the latter representations form a closed subset of $(MN)^\sim$ the representations $\eta \otimes \chi$ form a closed subset of $\hat{P}$. Note also that the representations $ma \mapsto \eta(m)\chi(a)$ of $MA$ are obviously weakly contained in $\lambda_{MA}$, from which it follows that $\eta \otimes \chi$ is weakly contained in $\text{Ind}_N^G 1_N$ where $1_N$ is the trivial representation of $N$; since $N$ is amenable $1_N$ is weakly contained in $\lambda_N$ ([20]), and so each $\eta \otimes \chi$ is weakly contained in $\lambda_\eta$. This implies that $\pi_{n,x}$ is weakly contained in $\lambda_G$, or in other words that $\hat{G}_H \subset \hat{G}_r$.

We must show that $\hat{G}_H$ is closed. Let $I$ be the ideal of $C^*(P)$ determined by the closed set $\hat{M}_d \otimes \hat{A} \subseteq \hat{P}$—then $\pi \in C^*(P)^\sim$ has kernel containing $I$ iff the corresponding representation of $P$ lies in $\hat{M}_d \otimes \hat{A}$. We now make use of Rieffel’s approach [27] to induced representations. Thus let $X$ be the bimodule for inducing representations from $C^*(P)$ to $C^*(G)$. According to Rieffel’s version of the Mackey imprimitivity theorem, there is an action of the transformation group algebra $E = C^*(G, G/P)$ on $X$ such that the process of inducing representations of $C^*(P)$ up to $E$ gives an equivalence between the categories of representations of $E$ and of $C^*(P)$ which preserves weak containment of representations. In particular there is an ideal $J$ of $E$ such that a representation of $E$ has kernel containing $J$ iff it is induced from a representation of $C^*(P)$ whose kernel contains $I$. We now borrow an idea from [17, Section 2]: namely, that since $G/P$ is compact, there is a natural homomorphism of $C^*(G)$ onto a subalgebra of $C^*(G, G/P)$. In particular there is an ideal $K$ of $C^*(G)$ such that $C^*(G)/K$ embeds as a subalgebra of $E/J$ in such a way that if we are given a representation of $C^*(P)$ whose kernel contains $I$, the induced representation of $C^*(G)$ is obtained by first inducing to $E/J$ and then restricting to $C^*(G)/K$. Now, given $\pi \in \hat{G}_H$, it is clear that $\ker \pi$ contains $K$. I claim that conversely any $\pi \in \hat{G}$ whose kernel contains $K$ lies in $\hat{G}_H$: To see this, note that by [9, 2.10.2] we find an irreducible representation $\bar{\pi}$ of $E/J$ whose restriction to $C^*(G)/K$

contains $\pi$ as a subrepresentation. But $\tilde{\pi}$ is induced from an irreducible representation of $C^*(P)/\mathcal{I}$, which must be of the form $\eta \otimes \chi$. Therefore $\pi$ is a subrepresentation of $\pi_{\eta,\chi}$ and so lies in $\hat{G}_H$. It follows that $\hat{G}_H$ is closed, as was to be proved.

Since there are only finitely many distinct $\hat{G}_H$, their union is closed; furthermore, by Wolf's Plancherel formula [31, Section 5] this union supports Plancherel measure on $\hat{G}$, so $\bigcup_H \hat{G}_H = \hat{G}_r$ and the $\hat{G}_H$ partition $\hat{G}$, as claimed. The proof of the lemma is now complete. 

The following lemma is no doubt well-known, but we include its proof for the sake of completeness.

**Lemma 4.** Let $G$ be a (second countable) Lie group acting differentiably on the $C^\infty$-manifold $X$, and $\mathcal{O}$ an orbit in $X$ having non-zero measure (with respect to the Lebesgue measure class on $X$). Then $\mathcal{O}$ is open.

**Proof.** Let $x \in \mathcal{O}$; then the map $sG_x \mapsto sx$ gives a $C^\infty$-immersion of $G/G_x$ into $X$. By Sard's theorem this map has non-singular differential at some (hence, by $G$-transitivity, every) point of $G/G_x$. The inverse function theorem now implies that the image $\mathcal{O}$ of this map is open.

At last we are ready for the main result.

**Theorem.** Let $G$ be a closed, normal, almost connected subgroup of a $\sigma$-compact group $H$. Then an $H$-orbit in $\text{Prim} G$ has positive Plancherel measure if and only if it is open in $\text{Prim} G$, in which case it is regular.

**Proof.** The "if" implication is easy, since $\text{Prim}_G$ is the support of Plancherel measure. So suppose that the $H$-orbit $\mathcal{O}$ of $P \in \text{Prim}_G$ has positive Plancherel measure; we must show $\mathcal{O}$ is open and regular in $\text{Prim}_G$. The strategy for doing this is to use Lemma 1 to perform successive reductions on $G$ and $H$ until we are in a position to apply Lemma 3.

(a) By [16] $G$ has a maximal compact normal subgroup $K$ with $G/K$ a Lie group having finitely many components. Then $K$, being characteristic in $G$, is normal in $H$. Since $H$ is $\sigma$-compact all $H$-orbits in the discrete space $\text{Prim} K$ are countable; let $\mathcal{I}$ be the one over which $P$ lives. Then for each $Q \in \mathcal{I}$ the stabilizer $H_Q$ is of countable index in $H$, hence so is $H_1 = \bigcap_{Q \in \mathcal{I}} H_Q$. Furthermore each $G$-orbit $\mathcal{P}$ in $\mathcal{I}$ is finite (because $G$ is almost connected), so if we set $K_\mathcal{P} := \bigcap_{Q \in \mathcal{I}} \text{Ker} \pi_Q$ then $K/K_\mathcal{P}$ is a compact Lie group and $G/K_\mathcal{P}$ is a finitely connected Lie group. The automorphism group of such a group being a Lie group, it follows that $H_1/C_\mathcal{P}$ is second countable, where $C_\mathcal{P} = \{s \in H_1 : s(tK_\mathcal{P})s^{-1} = tK_\mathcal{P} \text{ for all } t \in G\}$. Hence so is $H/C$ where $C = \bigcap_{\mathcal{P} \in \mathcal{I} \subset \mathcal{I}} C_\mathcal{P}$. Note that the action of $H$ by conjugation on $G/K_i$ (where $K_i = \bigcap_{\mathcal{P} \in \mathcal{I} \subset \mathcal{I}} K_\mathcal{P}$) drops to $H/C$. Now all elements of $\mathcal{O}$ live over the trivial representation of $K_i$. 
(which is open and has positive Plancherel measure in Prim $K_1$) so without loss of generality we may replace $G$ by $G/K_1$ and $H$ by the semi-direct product $G/K_1 \rtimes H/C$ (whose orbits on Prim $G/K_1$ coincide with those of $H$) and thus assume that both $G$ and $H$ are second countable.

Now again let $K$ be the maximal compact normal subgroup of $G$. The connected component of $G/K$ acts trivially on the discrete space Prim $K$, so all stabilizers $G_0$ ($Q \in$ Prim $K$) are of finite index in $G$. Therefore, the central extensions $\tilde{G}_0$ of $G_0/K$ are finitely connected Lie groups. Since, by the discreteness of its dual, $K$ is regularly embedded in both $G$ and $H$, and every $H$-orbit in Prim $K$ is open and has positive Plancherel measure, we may use Lemma 1 to replace $G$ and $H$ by $G_0$ and $H_0$ for some $Q \in$ Prim $K$ and so assume in particular that $G$ is a finitely connected Lie group. In fact, by iterating the above construction a finite number of times if necessary we may assume that the maximal compact normal subgroup of $G$ is central in $H$ and isomorphic to $T$.

(b) Since the connected component $G_0$ of $G$ is regularly embedded in $G$ (a consequence of the fact that points in Prim $G_0$ are locally closed—cf. [23, Thm. 1]—and $G/G_0$ is finite) the $H$-orbit $\mathcal{O}$ lives over an orbit $\mathcal{O}^0$ in Prim $G_0$ which must also have positive Plancherel measure. Now $P$ lives over a (necessarily finite) $G$-orbit $\mathcal{O}_1$ in Prim $G_0$, and by [24] the part of Prim $G$ living over $\mathcal{O}_1$ is finite and $T_1$ in the relative topology. Therefore if we knew that $\mathcal{O}_0$ were regular then by Lemma 1 (with $N = G^0$) every $H$-orbit in the part $\mathcal{O}_0$ of Prim $G$ living over $\mathcal{O}_0^0$ would be regular and relatively open in $\mathcal{O}_0$; and if we knew that $\mathcal{O}_0$ were open in Prim $G_0$ then by continuity of restriction $\mathcal{O}_0$ would be open in Prim $G$. Thus it is enough to prove the result for $G_0$, so we may assume $G$ connected.

(c) Let $K$ be the kernel of the natural homomorphism of $H$ into the automorphism group of $G$; then $H/K$ is a Lie group. Let $H_1$ be its connected component. As $H/K$ is a second countable $H_1$ is of countable index in $H/K$ and the $H_1$-orbits in $\mathcal{O}$ form a countable partition of $\mathcal{O}$, the elements of which are permuted by $H$. Therefore the $H_1$-orbit of $P$ has positive Plancherel measure, and if it is open in Prim $G$ so is $\mathcal{O}$. $H_1$ need not contain $G$ as a subgroup but we may instead consider the semi-direct product $G \rtimes H_1$, which has the same orbits in Prim $G$ as $H_1$ does. Replacing $H$ by $G \rtimes H_1$ we may therefore assume that $H$ is a connected Lie group.

(d) Now choose locally algebraic groups $N$, $M$, and $L$ having the properties guaranteed by Lemma 2. By [8] and [25] $N$ is Type I and regularly embedded in $L$. In particular, $\mathcal{O}$ lives over some locally closed $L$-orbit $\mathcal{O}$ in Prim $N$, of positive Plancherel measure. Let $\mathcal{O}^G$ be the part of Prim $G$ which lives over $\mathcal{O}$ ($\mathcal{O}^G$ is locally closed, by continuity of restriction). By the “EH-regularity” assertion of [19, Thm. 37] (that theorem is applicable here in virtue of the argument of [19, Cor. 38], since $G$ is “wedged” in between $N$ and $M$, where $N$ is Type I and regularly embedded in $M$, and $M/N$ is abelian) every $Q \in \mathcal{O}^G$ is induced from a stability group $G_J$, where $J \in \mathcal{O}$ lies in the (locally closed)
$M$-orbit $\mathcal{Z}'$ in $\mathcal{Z}$ over which $Q$ lives. Furthermore by the argument of [19, p. 244] we have a transitive action of $(G/N)^\wedge \times M$ (where $(G/N)^\wedge$ is the dual group of $G/N$) on the part $\mathcal{R}$ of Prim $G_j$ which lives over $\mathcal{Z}'$. (The action of $\chi \in (G/N)^\wedge$ here is defined by tensoring representations of $G_j$ with $\chi |_{G_j}$.)

It is easy to see that the process of inducing primitive ideals from $G_j$ to $G$ is $(G/N)^\wedge \times M$-equivariant. From this it follows that $(G/N)^\wedge \times M$ acts transitively on the part $\mathcal{C}'$ of Prim $G$ which lives over $\mathcal{Z}'$. Since restriction from $G$ to $N$ is $L$-equivariant it is easy to see that we get a transitive action of $L' = (G/N)^\wedge \rtimes L$ on $\mathcal{Z}^G$ (where the action of $L$ on $(G/N)^\wedge$ which is used to define this semi-direct product is just the dual action to that of $L$ on $G/N$).

We want next to show that $\mathcal{Z}^G$ is homeomorphic to a coset space of $L'$. For this it is enough to show that $\{P\}$ is closed in $\mathcal{Z}^G$, since then by an easy argument $CP$ is closed for all compact subsets $C$ of $L'$, which by a standard Baire category argument (using the facts that $\mathcal{Z}^G$ is a Baire space—cf. [9, 3.4.13]— and $L'$ is $\sigma$-compact) implies that $\mathcal{Z}^G$ is homeomorphic to $L'/L'_0$. To see that $\{P\}$ is closed, note first that by [28, Prop. 7.4] $\mathcal{Z}$ is Hausdorff and homeomorphic to $L/L_0$. Therefore $\mathcal{Z}'$ is closed in $\mathcal{Z}$, so $\mathcal{C}'$ is closed in $\mathcal{Z}^G$. From [19, p. 244] it is easy to see that $\mathcal{R}$ is $T_1$, and in fact is homeomorphic to a coset space of the abelian group $M/G_j \times (G/N)^\wedge$. The EH-regularity assertion of [19, Thm. 37] together with continuity of the induction and restriction operations imply that $\mathcal{C}'$ is homeomorphic to the space of $G$-quasi-orbits of $\mathcal{R}$, which is again a coset space of $M/G_j \times (G/N)^\wedge$ and so Hausdorff. So $\{P\}$ is closed in $\mathcal{C}'$ and hence in $\mathcal{Z}^G$.

We have shown that $\mathcal{Z}^G$ is homeomorphic to a coset space of $L'$. Furthermore the restriction of $G$-Plancherel measure to $\mathcal{Z}^G$ is $L$- and $(G/N)^\wedge$-quasi-invariant (the latter because the tensor product of $\lambda_\mathcal{C}$ with a character is equivalent to $\lambda_\mathcal{C}$—cf. [14, Lemma 4.2]), hence $L'$-quasi-invariant and so lies in the Lebesgue measure class on the differentiable manifold $L'/L'_0$. Since the $H$-orbit $\mathcal{C}$ in $\mathcal{Z}^G$ has positive measure it is open in $\mathcal{Z}^G$ by Lemma 4. Therefore it is enough to prove that $\mathcal{Z}^G$ is open in Prim, $G$; this will follow if $\mathcal{Z}$ is open in Prim, $N$. In other words we may replace $G$ by $N$, $\mathcal{C}$ by $\mathcal{Z}$, and $H$ by $L$, and so assume that both $G$ and $H$ are connected locally algebraic groups. In fact, from the proof of Lemma 2 we see that we can assume that there is a faithful representation of the Lie algebra $\mathfrak{h}$ of $H$ as an algebraic Lie algebra of linear operators on a finite dimensional real vector space $V$, such that the radical $\mathfrak{n}$ of $\mathfrak{g}$ is represented by nilpotent operators.

(e) Let now $N$ be the nilradical of $G$, $Z^2 = Z^2(N)$ the “second center” of $N$ (i.e. $Z^2/Z(N)$ is the center of $N/Z(N)$). Let $A$ be the center of $Z^2$; as $N$ is a nilpotent Lie group, both $Z^2$ and $A$ are connected. Since $A \subseteq N$, $A$ is locally algebraic and so regularly embedded in $H$. Therefore $\mathcal{C}$ lives over an $H$-orbit $\mathcal{Z}$ in Prim $A$ (i.e. $A$) of positive Plancherel measure. Since Plancherel measure is the same as Haar measure on the Lie group $A$, Lemma 4 implies that $\mathcal{Z}$ is open.

We want to apply Lemma 1, but must first check that the new group we will obtain satisfies the hypotheses of the theorem—i.e. is finitely connected. Let
$J \in \mathcal{F}$. Since the real dual $\alpha'$ of $\alpha$ may be identified with $\tilde{A}_1$, where $A_1$ is the universal covering group of $A$, $J$ corresponds to some point $f$ of $\alpha'$. Let now $G^1$, $H^1$ be the algebraic groups of automorphisms of $V$ corresponding to $\mathfrak{g}$ and $\mathfrak{h}$. Their identity components $G^2$, $H^2$ are of finite index in them. Since the natural actions of $G^1$ and $H^1$ on $\alpha'$ are algebraic, the stabilizers $G^1_f$ and $H^1_f$ are algebraic, and in particular are finite extensions of their identity components. Therefore $G^2_f := (G^2 \cap G^1_f)$, $H^2_f$ are finite extensions of their identity components. If $H^1_f$ denotes the universal covering group of $H$, and $\pi_1: H_1 \to H$, $\pi_2: H_1 \to H^2$ the natural covering maps, then it is easy to see that $H_f = \pi_1(\pi_2^{-1}(H^2_f))$ and $G_f = \pi_1(\pi_2^{-1}(G^2_f)) \cap G$. Let $Z = \ker \pi_2$, which is central in $H_1$. It is clear that $H_f$ is a finite extension of $\pi_1(Z) H^1_f$ and $G_f$ is a finite extension of $Z_1 G^1_{\mathfrak{g}}$ where $Z_1 = \pi_1(Z) \cap G$. Now $Z_1$ is central in $H$, so the $H$-orbit $\mathcal{O}$ lives over a single point in $\text{Prim } Z_1$ (\textgamma $Z_1$) having positive Plancherel measure; therefore $Z_1$ must be discrete, implying that the discrete group $Z_1$ is compact and hence finite. It follows that $G_f$ is finitely connected, so the same is true of $G_f$. Thus we may use Lemma 1 to replace $G$ by $G_f$, $H$ by $H_f$ (which has the effect of replacing $A$ by $T$). Applying the reductions of (b)-(c) a finite number of times if necessary, we may assume (in addition to all previous hypotheses) that $Z(Z(N))$ is isomorphic to $T$ and is central in $H$. (Note that each reduction has the effect of “shrinking” $G$, so at most a finite number of steps are necessary.)

(f) As is well-known (cf. [6, proof of Lemme 10]) the above hypothesis on $N$ implies $N$ is either isomorphic to $T$ or is (locally isomorphic to) a “Heisenberg group.” In the second case, since $Z = Z(N)$ is central in $H$, $\mathcal{O}$ lives over a point $\chi$ in $Z$. If this point is not the trivial character, then there is a unique element $Q$ of $\text{Prim } N$ living over it, and by continuity of restriction this element is open in $\text{Prim } N$. We can then apply Lemma 1 to replace $G$ by $G_Q$ and $H$ by $H_Q$ (note that $G_Q = G$, $H_Q = H$). If $\chi$ is trivial then all points in $\mathcal{O}$ kill $Z$ and we can replace $H$ by $H/Z$ and $G$ by $G/Z$ and repeat step c). In either case, iterating this process if necessary we eventually get $G$ having radical which is isomorphic to $T$ and so compact.

(g) The connected component of the identity in the automorphism group of such a group $G$ is precisely the subgroup of inner automorphisms, so since $H$ is connected it acts trivially on $\text{Prim } G$. Then $\mathcal{O}$ consists of a single point having positive Plancherel measure, and hence corresponds to an irreducible square-integrable representation ($G$ being Type I). By an earlier argument $Z(G)$ must be compact. The desired result now follows from Lemma 3.

**Corollary 1.** Let $G$ be an almost connected group. Every square integrable factor representation of $G$ is a normal representation whose kernel in $C^\text{*}(G)$ is an open point of $\text{Prim}_r G$. If $G$ is connected there is a unique quasi-equivalence class of square-integrable factor representations corresponding to each open point $P$ in $\text{Prim}_r G$—namely, the normal representation with kernel $P$. 

Proof. The first statement follows from combining the theorem (with $H = G$) with Rosenberg's result [29, 2.13] that square-integrable factor representations of almost connected groups are normal. Since by [25, Thm. 1] there is a unique quasi-equivalence class of normal representations having kernel $P \in \text{Prim } G$ when $G$ is connected, the second statement also follows (for by [29, 2.14] an open point in $\text{Prim } G$ must be the kernel of some square-integrable factor representation).

Corollary 2. The square-integrable irreducible representations of an almost connected group $G$ are GCR and correspond to the open points of $\hat{G}$. 

Proof. This follows from Corollary 1, since a normal irreducible representation is GCR and is the unique irreducible representation (up to equivalence) having its kernel.

Corollary 3. Let $G$ be a unimodular group having an open normal almost connected subgroup $N$. Then every square-integrable irreducible representation $\pi$ of $G$ is open in $\hat{G}$. 

Proof. Since $G$ is unimodular, $\lambda_G$ is traceable (with densely defined trace) so $\pi$ must be CCR. Thus, as $C^\ast(N) \subseteq C^\ast(G)$, $\pi |_N$ (which is quasi-equivalent to a subrepresentation of $\lambda_N$) is CCR, and hence decomposes as a direct sum of square-integrable irreducible representations of $N$. It is easy to see from the irreducibility of $\pi$ that these representations form a $G$-orbit $G_\pi$ in $\hat{N}$. By Corollary 2 the orbit is discrete and open in $\hat{N}$, so in particular it is regular. We may thus apply Lemma 1 (with $H = G$; although we are not assuming $G$ is second countable, the proof of Lemma 1 goes through in this context because $G/N$ is discrete) to reduce to the group $\hat{G}_\pi$. By [30, Lemma 1.1] $\hat{G}_\pi$ has no square-integrable irreducibles unless $G_{\eta}/T$ is finite, in which case the square-integrable points are obviously open in $(\hat{G}_\pi)'$. Hence we are done. (We are using the fact that, because the various equivalences in the Mackey machine preserve irreducibility of representations, $\pi$ must come from a square-integrable representation of $\hat{G}_\pi$ which is actually irreducible.)

Corollary 4. Let $\pi$ be a representation of an almost connected group $G$ which is weakly contained in $\lambda_G$ and weakly contains a square-integrable irreducible representation $\eta$. Then $\pi$ contains a subrepresentation equivalent to $\eta$. 

Proof. The Glimm measure of $\pi$ is supported on $\text{Prim } G$ but not on $\text{Prim } G \setminus \{P\}$ where $P = \ker \eta$. Therefore $\pi$ has a non-zero subrepresentation concentrated on $\{P\}$. But any representation concentrated on $\{P\}$ is a multiple of $\eta$, since by Corollary 2 $P$ is semi-compact.

Clearly, the above result also holds for the groups of Corollary 3.
3. The Counterexamples

We recall some facts (cf. [4, Chaps. 1-2]) about the fields $\mathbb{Q}_p$ of $p$-adic numbers, defined for each rational prime $p$. There is a multiplicative "absolute" $||\cdot||_p$ on $\mathbb{Q}_p$ which takes values in the set $\{0\} \cup \{ p^n : n \in \mathbb{Z}\}$ and induces a locally compact topology, compatible with the field operations, on $\mathbb{Q}_p$. The set $\mathbb{Z}_p$ (resp. $\mathbb{K}_p$) of elements in $\mathbb{Q}_p$ having norm $\leq 1$ (resp. $= 1$) is a compact open subgroup of the additive group $\mathbb{Q}_p^+$ (resp. multiplicative group $\mathbb{Q}_p^\times = \mathbb{Q}_p \setminus \{0\}$). We normalize Haar measure $\mu_p$ on $\mathbb{Q}_p^+$ so that $\mu_p(\mathbb{Z}_p) = 1$; then $\mu_p(\mathbb{K}_p) = 1 - 1/p$ (note that $||\cdot||_p$ is just the multiplicative "modular function" with respect to $\mu$:

$$\mu_p(rC) = ||r||_p \mu_p(C) \quad (r \in \mathbb{Q}_p)$$

for all Borel subsets $C$ of $\mathbb{Q}_p$).

Now choose a sequence $(p_i)_{i \geq 0}$ of primes such that $\Sigma_i 1/p_i < \infty$, and let $A$ be the "restricted direct product" of the $\mathbb{Q}_p^+$, with respect to the $\mathbb{Z}_p$: that is,

$$A = \left\{ (x_i)_{i} \in \prod_i \mathbb{Q}_p : x_i \in \mathbb{Z}_p \text{ for all but finitely many } i \right\}$$

equipped with the unique topology which is compatible with the (Cartesian product) group structure and for which the subgroup $\prod_i \mathbb{Z}_p$ is relatively open and carries the Cartesian product topology. Let $K = \prod_i \mathbb{K}_p$. Our first example is the semi-direct product

$$G = A \rtimes K$$

where $K$ acts on $A$ by coordinatewise multiplication. The dual group $\hat{A}$ of $A$ (which we will need to know about in order to get at $\hat{G}$) may be described as follows: for each $p$, let $\chi_p$ be a character of $\mathbb{Q}_p^+$ whose kernel is exactly $\mathbb{Z}_p$ (cf. [4, p. 309] for a proof that $\chi_p$ exists). Then the map which to $r = (r_i)_{i} \in A$ associates the character $\psi_r$ of $A$ defined by $\psi_r(s) = \prod_{i=0}^{\infty} \chi_p(r_i s_i)$ (the product on the right is actually a finite product in $T$) gives a topological isomorphism of $A$ onto $\hat{A}$. Furthermore, it is not hard to see that this isomorphism takes Haar measure on $A$ (normalized so that $\prod_i \mathbb{Z}_p$ has measure 1) to Plancherel measure on $\hat{A}$, and that it takes $K$-orbits in $A$ to $K$-orbits in $\hat{A}$. In particular, the $K$-orbit $\mathcal{O}$ of $1$ (where 1 is the element of $A$ with $1_i$ equal to the unit element of $\mathbb{Q}_p$) has Plancherel measure equal to

$$\prod_i \mu_p(\mathbb{K}_p) = \prod_i (1 - 1/p_i) > 0.$$ 

The stabilizer in $G$ of $\psi_1$ is precisely $A$, so by the Mackey machine there is a unique element $\pi = \text{Ind}_A^G \psi_1$ of $\hat{G}$ living over $\mathcal{O}$. Since $\mathcal{O}$ has positive Plancherel
measure it defines a non-zero subrepresentation \( \eta \) of \( \lambda_A \), and the induced representation \( \text{Ind}_A^G \eta \) must then be a subrepresentation of \( \lambda_G \) which is concentrated on \( \{\pi\} \)—i.e. it decomposes as a direct sum of copies of \( \pi \). So \( \pi \) is square integrable.

We may choose a sequence \((r^{(j)})_{j=0,1,2,\ldots} \) in \( A \) with the property that

\[
\begin{align*}
    r_i^{(j)} &= 1_i & i \leq j \\
    \|r_i^{(j)}\|_{p_i} &= 1/p_i & i > j.
\end{align*}
\]

Then \((r^{(j)})\) converges to 1 in \( A \). Furthermore, the stabilizer of \( \psi_{r^{(j)}} \) in \( G \) is precisely \( A \), so \( \text{Ind}_A^G \psi_{r^{(j)}} \) is an irreducible representation \( \pi_j \) of \( G \) which is not equal to \( \pi \) since the \( G \)-orbits of \( \psi_{r^{(j)}} \) and \( \psi_1 \) are disjoint. By continuity of induction [14] \( \pi_j \) converges to \( \pi \), and as \( G \) is amenable [20] all \( \pi_j \) are in the reduced dual \( \hat{G} \); it follows that \( \pi \) is not open in \( \hat{G} \). (Note that \( G \) is CCR by [10, Prop. 8], and is unimodular, so it satisfies all the hypotheses of Dixmier's problem [7, Prob. c, p. 96].)

Our second example is similar; take

\[ G = A \times L \]

where \( L \) is now the restricted product of the \( Q_{p_i}^* \) with respect to the open subgroups \( K_{p_i} \). Again \( \text{Ind}_A^G \psi_1 \) is an irreducible square integrable representation, denoted \( \pi_1 \) of \( G \), to which the irreducible representations \( \pi_j = \text{Ind}_A^G \psi_{r^{(j)}} \) (which are distinct from \( \pi \)) converge. But note now that all \( r^{(j)} \) are in the same \( G \)-orbit, so the \( \pi_j \) are all equivalent and define an element of \( \hat{G} \) (\( = \hat{G}_r \)) whose closure contains \( \pi \). But \( \pi_0 \) is also in the closure of \( \pi \), since the elements \( s^{(j)} \) of \( A \) defined by

\[
    s_i^{(j)} = r_i^{(0)} \quad i \leq j
\]

\[
    = 1_i \quad i > j
\]

are in the \( G \)-orbit of 1, and converge to \( r^{(0)} \). It follows that \( \pi \) and \( \pi_0 \) have the same closure in \( \hat{G} \), so in particular \( \{\pi\} \) is not locally closed in \( \hat{G} \). This answers [12, Problem, p. 228] in the negative.

We note one further curious fact about the above representation \( \pi \). It is not hard to see that the \( G \)-orbit of 1 carries the entire Haar measure of \( G \), from which it follows that \( \lambda_G \) is a multiple of \( \pi \). Now \( \lambda_G \) is integrable since it includes continuous functions of compact support among its coefficients; but \( \pi \) cannot be integrable, since it is not locally closed. (Cf. [12. Cor. 1, p. 223].) Thus we see that integrability is not necessarily preserved under quasi-equivalence of representations (unlike square integrability, which is).
SQUARE-INTEGRABLE REPRESENTATIONS

REFERENCES


