Balanced Canonical Forms for Minimal Systems: A Normalized Coprime Factor Approach

Raimund Ober
Department of Engineering
University of Cambridge
Trumpington Street
Cambridge CB2 1PZ, England

and

Duncan McFarlane
Process Control Group
BHP Melbourne Research Laboratories
PO Box 264, Clayton 3168,
Victoria, Australia

Submitted by Jan C. Willems

ABSTRACT

Canonical forms are derived for the set of minimal systems of given order from a canonical form for a class of coinner transfer functions. One of these canonical forms is in terms of so called Riccati balanced coordinates. The application of this work to model reduction is discussed.

1. INTRODUCTION

This paper is concerned with canonical forms for linear, finite dimensional state space systems. Recently, Ober (1987a) introduced a canonical form for the special class of asymptotically stable systems in terms of balanced realizations. This canonical form gives some insight into structural properties of these systems, and is particularly useful from a model reduction point of view. A canonical form with similar properties will be derived in this paper for the set of all minimal systems of a given state space dimension.

This work was completed while the second author was a Ph.D. student at the University of Cambridge, England.

655 Avenue of the Americas, New York, NY 10010 0024-3795/89/$3.50

We define a canonical form in the following way. Let $L_n^{p,m}$ be the set of all minimal state space systems (A, B, C, D) with n dimensional state space, m dimensional input space, and p dimensional output space. Two systems (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) in $L_n^{p,m}$ are called *equivalent* if there is a nonsingular matrix T such that $A_1 = TA_2T^{-1}$, $B_1 = TB_2$, $C_1 = C_2T^{-1}$, and $D_1 = D_2$. It is well known that two minimal systems are equivalent if and only if their transfer functions are identical.

A *canonical form* is a map

$$\Gamma : L_n^{p,m} \to L_n^{p,m}$$

such that

$$\Gamma((A_1, B_1, C_1, D_1)) = \Gamma((A_2, B_2, C_2, D_2))$$

if and only if

$$(A_1, B_1, C_1, D_1) \text{ is equivalent to } (A_2, B_2, C_2, D_2).$$

Section 2 reviews the results of Ober (1987a) on balanced realizations of asymptotically stable systems. In Section 3 the concept of a normalized coprime factorization of a transfer function is introduced, and it is shown that coprime factors correspond to a particular class of asymptotically stable coinner transfer functions. We can then derive a canonical form for minimal state space systems on the basis of a canonical form for this class of coinner systems. This is done by exploiting the connection between a transfer function and its normalized coprime factors. Section 4 then gives canonical forms for minimal systems in terms of so-called "normalized left coprime factor balanced" coordinates and "Riccati balanced" coordinates. The latter is shown to be an extension of the work by Jonckheere and Silverman (1983), who gave a canonical form in terms of Riccati balanced coordinates for the special case of single input, single output systems with distinct characteristic values. The results of Sections 2–4 are then discussed in Section 5 in a model reduction framework.

2. BALANCED REALIZATIONS FOR ASYMPTOTICALLY STABLE SYSTEMS

In this section we are going to review the canonical form for asymptotically stable and minimal systems of given dimension, i.e. systems in $C_n^{p,m}$ as given in Ober (1987a). This canonical form was derived in terms of balanced realizations, which are defined as follows.

DEFINITION 2.1 (Moore, 1981). Let $(A, B, C, D) \in C_n^{p,m}$. Then (A, B, C, D) is called *balanced* if for

$$W_c = \int_0^\infty e^{tA}BB^Te^{tA^T}\,dt,$$

$$W_0 = \int_0^\infty e^{tA^T}C^TCe^{tA}\,dt$$

we have $W_c = W_0 =: \Sigma =: \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_n)$. Σ is called the *gramian* of the system (A, B, C, D). The positive numbers $\sigma_1, \sigma_2, \dots, \sigma_n$ are called the *singular values* of the system (A, B, C, D).

Alternatively, balanced realizations can be characterized as those systems whose corresponding Lyapunov equations have identical and diagonal solutions.

THEOREM 2.1 (Moore, 1981). *$(A, B, C, D) \in C_n^{p,m}$ is balanced if and only if there exists a diagonal matrix $\Sigma > 0$ such that*

$$A\Sigma + \Sigma A^T = -BB^T,$$

$$A^T\Sigma + \Sigma A = -C^TC.$$

In this case $\Sigma = W_0 = W_c$.

It was shown in Ober (1987b) that all pass systems have a particular canonical form which is a building block for the canonical form of general systems. For convenience of notation we therefore introduce the following notion, which describes the essential features of structure of the C and A matrix of an all pass system in the canonical form mentioned above.

DEFINITION 2.2. We say that

$$(C, A) \qquad A \in \mathbb{R}^{n\times n}, \quad C \in \mathbb{R}^{p\times n},$$

is in *standard all pass form* if:

(1) We have

$$C^TC = \operatorname{diag}\left(\lambda_1 I_{r(1)}, \lambda_2 I_{r(2)}, \dots, \lambda_l I_{r(l)}, 0, \dots, 0\right)$$

with $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$ and $r_0 := \sum_{i=1}^l r(i) \leqslant p$. In particular C has the following structure:

$$C = (C(1), \dots, C(i), \dots, C(l), 0, \dots, 0),$$

where for $1 \leqslant i \leqslant l$, the submatrix

$$C(i) = \left(c(i)_{st}\right)_{\substack{1\leqslant s\leqslant p\\ 1\leqslant t\leqslant r(i)}} \in \mathbb{R}^{p\times r(i)}$$

has lower triangular form specified by indices

$$1 \leqslant s(i,1) < s(i,2) < \cdots < s(i,r(i)) \leqslant p$$

in the following way:

$$\begin{aligned} c(i)_{s(i,t),t} &> 0, && 1 \leqslant t \leqslant r(i), \\ c(i)_{st} &= 0, && s < s(i,t), \quad 1 \leqslant t \leqslant r(i), \end{aligned}$$

i.e.

$$C(i) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ c(i)_{s(i,1),1} & 0 & 0 & \cdots & 0 \\ x & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ x & 0 & 0 & \cdots & 0 \\ x & c(i)_{s(i,2),2} & 0 & \cdots & 0 \\ x & x & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & c(i)_{s(i,3),3} & \cdots & 0 \\ \vdots & \vdots & x & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & 0 \\ \vdots & \vdots & x & \cdots & c(i)_{s(i,r(i)),r(i)} \\ x & x & x & \cdots & x \\ \vdots & \vdots & \vdots & & \vdots \\ x & x & x & \cdots & x \end{pmatrix}.$$

(2) For A partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad A_{11} \in \mathbb{R}^{r_0 \times r_0}$$

we have:

(i) A_{11} is skew symmetric, i.e. $A_{11}^T = -A_{11}$;
(ii) there is an integer $q \geqslant 1$ and a set of double indices

$$(g(1),h(1)),\ldots,(g(i),h(i)),\ldots,(g(q),h(q))$$

with

$$1 = g(1) < \cdots < g(i) < g(i+1) < \cdots \leqslant n - r_0,$$
$$1 = h(q) < \cdots < h(i+1) < h(i) < \cdots \leqslant r_0$$

such that for

$$A_{21} =: (a_{st})_{\substack{1 \leqslant s \leqslant n - r_0 \\ 1 \leqslant t \leqslant r_0}}$$

we have

$$\begin{aligned} a_{g(i),h(i)} &> 0, && 1 \leqslant i \leqslant q, \\ a_{g(i),t} &= 0, && t > h(i), \quad 1 \leqslant i \leqslant q, \\ a_{st} &= 0, && s > g(i), \quad t \geqslant h(i), \quad 1 \leqslant i \leqslant q, \end{aligned}$$

i.e.

$$A_{21} = \begin{pmatrix} \cdots & x & x & \cdots & x & x & x & \cdots & x & a_{g(1),h(1)} & 0 & \cdots & 0 \\ \cdots & x & x & \cdots & x & x & x & \cdots & x & 0 & 0 & \cdots & 0 \\ & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & & \\ \cdots & x & x & \cdots & x & x & x & \cdots & x & 0 & & & \\ \cdots & x & x & \cdots & x & a_{g(2),h(2)} & 0 & \cdots & 0 & 0 & & & \\ \cdots & x & x & \cdots & x & 0 & 0 & & & & & & \\ & \vdots & \vdots & & \vdots & \vdots & \vdots & & & & & & \\ \cdots & x & x & \cdots & x & 0 & 0 & & & & & & \\ \cdots & a_{g(3),h(3)} & 0 & \cdots & 0 & 0 & 0 & & & \mathbf{0} & & & \\ \cdots & 0 & 0 & & & & & & & & & & \\ & \vdots & & & & & & & & & & & \end{pmatrix};$$

(iii) $A_{12} = -A_{21}^T$;
(iv) we have

$$A_{22} = \begin{pmatrix} 0 & -\alpha_2 & & & & 0 \\ \alpha_2 & 0 & -\alpha_3 & & & \\ & \alpha_3 & 0 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & -\alpha_{n-r_0} & \\ 0 & & & a_{n-r_0} & 0 & \end{pmatrix}$$

with α_i, $2 \leqslant i \leqslant n - r_0$, given by

$$\alpha_i = \begin{cases} 0 & \text{if} \quad i = g(s) \text{ for some } 1 \leqslant s \leqslant q, \\ > 0 & \text{otherwise.} \end{cases}$$

The following result, which was proven in Ober (1987a), gives a canonical form for state space realizations of asymptotically stable and minimal systems. Conversely, it also shows that if a state space system is of this form it is automatically minimal and asymptotically stable. Let $TC_n^{p,m}$ denote the set of transfer functions of systems with state space realizations in $C_n^{p,m}$.

THEOREM 2.2. *The following two statements are equivalent*:

(1) $G(s) \in TC_n^{p,m}$.

(2) $G(s)$ *has a realization* $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ *given in the following way*: *There are positive integers*

$$n(1), \ldots, n(j), \ldots, n(k) \quad \text{such that} \quad \sum_{j=1}^{k} n(j) = n$$

and numbers

$$\sigma_1 > \cdots > \sigma_j > \cdots > \sigma_k > 0$$

such that if (A, B, C, D) *is partitioned as*

$$A =: (A(i, j))_{1 \leqslant i, j \leqslant k}, \qquad A(i, j) \in \mathbb{R}^{n(i)\times n(j)},$$

$$B =: \begin{pmatrix} B^1 \\ \vdots \\ B^j \\ \vdots \\ B^k \end{pmatrix}, \qquad B^j \in \mathbb{R}^{n(j)\times m},$$

$$C := (C^1, \ldots, C^j, \ldots, C^k), \qquad C^j \in \mathbb{R}^{p\times n(j)},$$

we have

(i) $(C^j, \tilde{A}(j,j))$ *is in standard all pass form and*

$$A(j,j) = -\frac{1}{2\sigma_j}(C^j)^T C^j + \tilde{A}(j,j)$$

with $r_0(j) := \operatorname{rank}[(C^j)^T C^j] \leqslant \min(p,m)$;

(ii) *we have*

$$B^j = \left[(C^j)^T C^j\right]^{1/2} \begin{pmatrix} U^j \\ \mathbf{0} \end{pmatrix} \quad \text{with } U^j \in \mathbb{R}^{r_0(j) \times m}, \quad U^j (U^j)^T = I_{r_0(j)};$$

(iii) *we have*

$$A(i,j) = \begin{pmatrix} \tilde{A}(i,j) & 0 \\ 0 & 0 \end{pmatrix}, \qquad 1 \leqslant i, j \leqslant k, \quad i \neq j,$$

with

$$\tilde{A}(i,j) =: \left(a(i,j)_{st}\right)_{\substack{1 \leqslant s \leqslant r_0(i) \\ 1 \leqslant t \leqslant r_0(j)}} \in \mathbb{R}^{r_0(i) \times r_0(j)}$$

such that

$$a(i,j)_{st} = \frac{1}{\sigma_i^2 - \sigma_j^2}\left[\sigma_j b(i)_s b(j)_t^T - \sigma_i c(i)_s^T c(j)_t\right]$$

$$= \frac{1}{\sigma_i^2 - \sigma_j^2}\left[\sigma_j \|c(i)_s\| \|c(j)_t\| u(i)_s u(j)_t^T - \sigma_i c(i)_s^T c(j)_t\right],$$

where $b(i)_s$ *is the* s*th row of* B^i, $u(i)_s$ *is the* s*th row of* U^i, $c(i)_s$ *is the* s*th column of* C^i, *and* $\|c(i)_s\| = \sqrt{c(i)_s^T c(i)_s}$;

(iv) $D \in \mathbb{R}^{p \times m}$.

Moreover, (A, B, C, D) *as defined in* (2) *is balanced with gramian*

$$\Sigma = \operatorname{diag}\left(\sigma_1 I_{n(1)}, \ldots, \sigma_j I_{n(j)}, \ldots, \sigma_k I_{n(k)}\right).$$

The map which assigns to each system in $C_n^{p,m}$ *the realization given in* (2) *is a canonical form.*

Proof. Follows from Theorem 6.1 and Theorem 7.1 in Ober (1987a) by taking adjoints. ■

The following corollary specializes these results to the case of single input, single output transfer functions.

COROLLARY 2.1. *The following two statements are equivalent:*

(1) $g(s) \in TC_n^{1,1}$.

(2) $g(s)$ *has a realization* $(A, b, c, d) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n} \times \mathbb{R}^{1\times 1}$, *which is given by the following parameters:*

$$\begin{array}{ll}
n(1),\ldots,n(j),\ldots,n(k), & n(j)\in\mathbb{N},\ \sum_{j=1}^{k} n(j)=n,\\
s_1,\ldots,s_j,\ldots,s_k, & s_j=\pm 1,\ 1\leqslant j\leqslant k,\\
\sigma_1>\cdots>\sigma_j>\cdots\sigma_k>0, & \sigma_j\in\mathbb{R},\ 1\leqslant j\leqslant k\\
c_1,\alpha(1)_1,\ldots,\alpha(1)_j,\ldots,\alpha(1)_{n(1)-1}, & c_1>0,\ \alpha(1)_j>0,\ 1\leqslant j\leqslant n(1)-1,\\
c_2,\alpha(2)_1,\ldots,\alpha(2)_j,\ldots,\alpha(2)_{n(2)-1}, & c_2>0,\ \alpha(2)_j>0,\ 1\leqslant j\leqslant n(2)-1,\\
\vdots & \\
c_k,\alpha(k)_1,\ldots,\alpha(k)_j,\ldots,\alpha(k)_{n(k)-1}, & c_k>0,\ \alpha(k)_j>0,\ 1\leqslant j\leqslant n(k)-1,\\
d\in\mathbb{R} &
\end{array}$$

in the following way:

(i) $c = \Big(\underbrace{c_1,0,\ldots,0}_{n(1)},\ldots,\underbrace{c_j,0,\ldots,0}_{n(j)},\ldots,\underbrace{c_k,0,\ldots,0}_{n(k)}\Big).$

(ii) $b^T = \Big(\underbrace{s_1c_1,0,\ldots,0}_{n(1)},\ldots,\underbrace{s_jc_j,0,\ldots,0}_{n(j)},\ldots,\underbrace{s_kc_k,0,\ldots,0}_{n(k)}\Big).$

(iii) For $A =: (A(i,j))_{1\leqslant i,j\leqslant k}$ *we have*

(a) *block diagonal entries* $A(j,j)$, $1\leqslant j\leqslant k$:

$$A(j,j)=\begin{pmatrix}
a(j,j) & -\alpha(j)_1 & & & & & \mathbf{0}\\
\alpha(j)_1 & 0 & -\alpha(j)_2 & & & & \\
 & \alpha(j)_2 & 0 & \cdot & & & \\
 & & \alpha(j)_3 & 0 & \cdot & & \\
 & & & \cdot & \cdot & \cdot & \\
 & & & & \cdot & \cdot & -\alpha(j)_{n(j)-1}\\
\mathbf{0} & & & & & \alpha(j)_{n(j)-1} & 0
\end{pmatrix}$$

with $a(j, j) = -(1/2\sigma_j)c_j^2$.

(b) *Off diagonal blocks* $A(i, j)$, $1 \leqslant i, j \leqslant k$, $i \neq j$:

$$A(i, j) = \begin{pmatrix} a(i, j) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

with

$$a(i, j) = -\frac{1}{\sigma_i + s_i s_j \sigma_j} c_i c_j.$$

(iv) $d \in \mathbb{R}$.

Moreover, (A, b, c, d) *as defined in* (2) *is balanced with gramian*

$$\Sigma = \operatorname{diag}\left(\sigma_1 I_{n(1)}, \dots, \sigma_j I_{n(j)}, \dots, \sigma_k I_{n(k)}\right).$$

The map which assigns to each system in $C_n^{1,1}$ *the realization given in* (2) *is a canonical form.*

It is interesting to observe that a realization (A, b, c, d) as given in part (2) of the previous corollary is in fact *sign symmetric* with respect to the sign matrix

$$S = \operatorname{diag}\left(s_1 \hat{I}_{n(1)}, \dots, s_j \hat{I}_{n(j)}, \dots, s_k \hat{I}_{n(k)}\right),$$

$$\hat{I}_{n(j)} = \operatorname{diag}(+1, -1, +1, -1, \dots) \in \mathbb{R}^{n(j) \times n(j)},$$

i.e.

$$A^T = SAS, \qquad b = Sc^T.$$

The Cauchy index of a single input, single output system, which is important in the study of the topology of transfer functions [see Brockett (1976) or Ober (1987c)] is defined as follows:

DEFINITION 2.3. Let $p(x)$ and $q(x)$ be relatively prime polynomials with real coefficients. The *Cauchy index* $C_{\text{ind}}(g(x))$ of $g(x) = p(x)/q(x)$ is defined as the number of jumps from $-\infty$ to $+\infty$ less the number of jumps from $+\infty$ to $-\infty$ of $g(x)$ when x varies from $-\infty$ to $+\infty$.

A result due to Anderson (1972) implies that if a system is sign symmetric with respect to a sign symmetry matrix S, the Cauchy index of its transfer function is given by

$$C_{\text{ind}}(g(s)) = \text{trace}(S).$$

As systems which are parametrized in the previous corollary are sign symmetric, the Cauchy index of a transfer function in $TC_n^{1,1}$ can thus be calculated on the basis of the signs $(s_i)_{1 \leqslant i \leqslant k}$ which are part of the parametrization.

3. NORMALIZED LEFT COPRIME FACTORIZATIONS

In this section we will give a canonical form for a special class of coinner systems which has a similar structure to the canonical form for $C_n^{p,m}$ in Theorem 2.2. We will also show that a *one to one correspondence* exists between this class of coinner functions and normalized coprime factor representations. Further it is shown that a function in $TL_n^{p,m}$, the set of transfer functions of systems in $L_n^{p,m}$, can be directly related to its normalized coprime factor representation. In Section 4 we are then going to give a canonical form for $L_n^{p,m}$ by exploiting these preliminary results.

Before we introduce the normalized left coprime factorization of a transfer function in $TL_n^{p,m}$, we will first discuss coinner transfer functions.

DEFINITION 3.1. A transfer function $G(s) \in TC_n^{p,m}$, $p \leqslant m$, is called *coinner* if

$$G(s)G(-s)^T = I$$

for all $s \in \mathbb{C}$.

A system theoretic criterion for a transfer function to be coinner is given in the next proposition.

PROPOSITION 3.1 (Doyle 1984). *Let* $(A, B, C, D) \in C_n^{p,m}$, $p \leqslant m$, *and let* $P = P^T > 0$ *be such that*

$$AP + PA^T = -BB^T.$$

Then $G(s) = C(sI - A)^{-1}B + D$ *is coinner if and only if*

(i) $CP + DB^T = 0$,
(ii) $DD^T = I$.

For a class of coinner functions whose realizations have a particular D-term this characterization can also be rewritten as follows.

PROPOSITION 3.2. *Let* $(A_c, B_c, C_c, D_c) \in C_n^{p,m}$, *with* $p < m$, *and partition*

$$B_c = \left[B_{c_1}, B_{c_2}\right], \qquad B_{c_2} \in \mathbb{R}^{n \times p},$$

$$D_c = \left[D_{c_1}, D_{c_2}\right], \qquad D_{c_2} \in \mathbb{R}^{p \times p}.$$

If $P = P^T > 0$ *is the solution to*

$$AP + PA^T = -BB^T$$

and $D_{c_2} = (D_{c_2})^T > 0$, *then*

$$G(s) = C_c(sI - A_c)^{-1}B_c + D_c$$

is coinner if and only if

(i) $B_{c_2} = -(PC_c^T + B_{c_1}D_{c_1}^T)D_{c_2}^{-1}$,
(ii) D_{c_1} *is such that* $I - D_{c_1}D_{c_1}^T > 0$ *and* $D_{c_2} = (I - D_{c_1}D_{c_1}^T)^{1/2}$.

Proof. Assume that $G(s)$ is coinner. Then

$$C_cP + D_cB_c^T = C_cP + D_{c_1}B_{c_1}^T + D_{c_2}B_{c_2}^T = 0$$

and hence

$$B_{c_2} = -\left(PC_c^T + B_{c_1}D_{c_1}^T\right)D_{c_2}^{-1}.$$

The fact that $D_cD_c^T = I$ together with the assumption that $D_{c_2} = D_{c_2}^T > 0$ immediately implies (ii). The converse follows similarly. ■

We denote by $TCI_n^{p,m+p}$ the class of coinner transfer functions in $TC_n^{p,m+p}$ which are such that for $G(\infty) =: [D_{c_1}, D_{c_2}]$, with $D_{c_2} \in \mathbb{R}^{p \times p}$, D_{c_2} is symmetric and positive definite and for which the largest singular value of a balanced realization of $G(s)$ is *strictly less than one*. The symbol $CI_n^{p,m+p}$ denotes the set of minimal state space realizations of transfer functions in $TCI_n^{p,m+p}$.

Before we show that the set $TCI_n^{p,m+p}$ is in fact closely linked with the set $TL_n^{p,m}$, we prove the following parametrization result for transfer functions in $TCI_n^{p,m+p}$.

THEOREM 3.1. *The following two statements are equivalent:*

(1) $G(s) \in TCI_n^{p,m+p}$.

(2) $G(s)$ *has a realization* $(A_c, B_c, C_c, D_c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times(p+m)} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times(p+m)}$ *given in the following way: There are positive integers*

$$n(1),\dots,n(j),\dots,n(k) \qquad \text{such that} \quad \sum_{j=1}^{k} n(j) = n$$

and numbers

$$1 > \sigma_1 > \cdots > \sigma_j \cdots > \sigma_k > 0$$

such that if (A_c, B_c, C_c, D_c) *is partitioned as*

$$A_c =: (A_c(i,j))_{1\leq i,j\leq k}, \qquad A_c(i,j) \in \mathbb{R}^{n(i)\times n(j)},$$

$$B_c =: \begin{pmatrix} B_{c_1} \\ \vdots \\ B_c^j \\ \vdots \\ B_c^k \end{pmatrix}, \qquad B_c^j \in \mathbb{R}^{n(j)\times(m+p)},$$

$$C_c =: \left(C_c^1,\dots,C_c^j,\dots,C_c^k\right), \qquad C_c^j \in \mathbb{R}^{p\times n(j)},$$

then:

(i) $D_c = [R^{-1/2}D_{pm}, R^{-1/2}]$ *for a matrix* $D_{pm} \in \mathbb{R}^{p\times m}$ *with* $R := I + D_{pm}D_{pm}^T$.

(ii) $\left(C_c^j, \tilde{A}_c(j,j)\right)$ *is in standard all pass form, where*

$$A_c(j,j) = -\frac{1}{2\sigma_j}\left(C_c^j\right)^T C_c^j + \tilde{A}_c(j,j)$$

with $r_0(j) := \operatorname{rank}((C_c^j)^T C_c^j) \leq \min(p,m)$.

(iii) $B_c^j =: [B_{c_1}^j, B_{c_2}^j]$, $B_{c_1}^j \in \mathbb{R}^{n(j)\times m}$, *is given by*

$$B_{c_1}^j = \tilde{B}^j S^{-1/2} - \sigma_j\left(C_c^j\right)^T R^{-1/2} D_{pm},$$

$$B_{c_2}^j = -\left[\tilde{B}^j S^{-1/2} D_{pm}^T + \sigma_j\left(C_c^j\right)^T R^{-1/2}\right],$$

where

$$\tilde{B}^j := \sqrt{1-\sigma_j^2}\left[\left(C_c^j\right)^T C_c^j\right]^{1/2}\begin{pmatrix} U^j \\ 0 \end{pmatrix},$$

with $U^j \in \mathbb{R}^{r_0(j)\times m}$, $U^j(U^j)^T = I_{r_0(j)}$, *and* $S = I + D_{pm}^T D_{pm}$.

(iv) *We have*

$$A_c(i,j) = \begin{pmatrix} \tilde{A}_c(i,j) & 0 \\ 0 & 0 \end{pmatrix}, \qquad 1 \leqslant i, j \leqslant k, \quad i \neq j,$$

with

$$\tilde{A}_c(i,j) =: \left(a_c(i,j)_{st}\right)_{\substack{1\leqslant s\leqslant r_0(j)\\ 1\leqslant t\leqslant r_0(j)}} \in \mathbb{R}^{r_0(i)\times r_0(j)}$$

such that

$$a_c(i,j)_{st} = \frac{1}{\sigma_i^2-\sigma_j^2}\left[\sigma_j \tilde{b}(i)_s \tilde{b}(j)_t^T - \sigma_i\left(1-\sigma_j^2\right) c_c(i)_s^T c_c(j)_t\right],$$

where $b(i)_s$ *is the* s*th row of* $\tilde{B}^i$ *and* $c_c(i)_s$ *is the* s*th column of* C_c^i.

Moreover, (A_c, B_c, C_c, D_c) *as defined in* (2) *is balanced with gramian*

$$\Sigma_c = \operatorname{diag}\left(\sigma_1 I_{n(1)}, \ldots, \sigma_j I_{n(j)}, \ldots, \sigma_k I_{n(k)}\right).$$

The map which assigns to each system in $CI_n^{p,m+p}$ *the realization given in* (2) *is a canonical form.*

Proof. To show that (1) implies (2) let (A_c, B_c, C_c, D_c) be a realization of $G(s)$ given in the balanced canonical form of Theorem 2.2 with gramian Σ_c.

Firstly note that as $(A_c, B_c, C_c, D_c) \in CI_n^{p,m+p}$, the singular values are such that

$$1 > \sigma_1 > \cdots > \sigma_k > 0.$$

Part (i) follows, as $G(s) \in TCI_n^{p,m+p}$ and hence, by Proposition 3.2, D_c has the form

$$D_c =: \left[D_{c_1}, D_{c_2}\right] = \left[R^{-1/2}D_{pm}, R^{-1/2}\right]$$

with $D_{pm} := D_{c_2}^{-1}D_{c_1}$ and $R := I + D_{pm}D_{pm}^T$. The inverse $D_{c_2}^{-1}$ exists because by assumption $D_{c_2} = D_{c_2}^T > 0$.

Since (A_c, B_c, C_c, D_c) is the canonical form of Theorem 2.2, $A_c(j, j)$ and C_c^j are as in (ii). To show (iii) and (iv), partition $B_c =: [B_{c_1}, B_{c_2}]$, $B_{c_1} \in \mathbb{R}^{n\times m}$, and introduce the matrix

$$\tilde{B} =: \left(B_{c_1} + \Sigma_c C_c^T R^{-1/2} D_{pm}\right) S^{1/2},$$

with $S = I + D_{pm}^T D_{pm}$. Then

$$B_{c_1} = \tilde{B}S^{-1/2} - \Sigma_c C_c^T R^{-1/2} D_{pm},$$

and with Proposition 3.2

$$\begin{aligned} B_{c_2} &= -\left(\Sigma_c C_c^T + B_{c_1} D_{pm}^T R^{-1/2}\right) R^{1/2} \\ &= -\left(\Sigma_c C_c^T R^{1/2} + \tilde{B}S^{-1/2}D_{pm}^T - \Sigma_c C_c^T R^{-1/2} D_{pm} D_{pm}^T\right) \\ &= -\left(\Sigma_c C_c^T R^{-1/2} + \tilde{B}S^{-1/2}D_{pm}^T\right). \end{aligned}$$

Thus we have that

$$\begin{aligned} B_c B_c^T &= B_{c_1}B_{c_1}^T + B_{c_2}B_{c_2}^T \\ &= \tilde{B}S^{-1}\tilde{B}^T + \tilde{B}S^{-1/2}D_{pm}^T D_{pm} S^{-1/2}\tilde{B}^T \\ &\quad + \Sigma_c C_c^T R^{-1/2} D_{pm} D_{pm}^T R^{-1/2} C_c \Sigma_c + \Sigma_c C_c^T R^{-1} C_c \Sigma_c \\ &\quad - \tilde{B}S^{-1/2}D_{pm}^T R^{-1/2} C_c \Sigma_c + \tilde{B}S^{-1/2}D_{pm}^T R^{-1/2} C_c \Sigma_c \\ &\quad - \Sigma_c C_c^T R^{-1/2} D_{pm} S^{-1/2}\tilde{B}^T + \Sigma_c C_c^T R^{-1/2} D_{pm} S^{-1/2}\tilde{B}^T \\ &= \tilde{B}\tilde{B}^T + \Sigma_c C_c^T C_c \Sigma_c. \end{aligned}$$

If we partition

$$\tilde{B} = \begin{pmatrix} \tilde{B}^1 \\ \vdots \\ \tilde{B}^j \\ \vdots \\ \tilde{B}^k \end{pmatrix}, \qquad \tilde{B}^j \in \mathbb{R}^{n(j)\times m},$$

we obtain that

$$B_c^j\left(B_c^j\right)^T = \tilde{B}\left(\tilde{B}^j\right)^T + \sigma_j^2\left(C_c^j\right)^T C_c^j,$$

and as (A, B, C, D) is balanced, we have the identity

$$B_c^j\left(B_c^j\right)^T = \left(C_c^j\right)^T C_c^j,$$

which then implies that

$$\tilde{B}^j\left(\tilde{B}^j\right)^T = \left(1-\sigma_j^2\right)\left(C_c^j\right)^T C_c^j.$$

Hence there exists a unique $U^j \in \mathbb{R}^{r_0(j)\times m}$, $U^j(U^j)^T = I_{r_0(j)}$, such that (iii) is satisfied. Moreover $r_0(j) = \operatorname{rank}\ [(C_c^j)^T C_c^j] \leqslant \min(p, m)$, which shows (iii) and completes the proof of (ii).

To complete this part of the proof it now remains to evaluate the entries $a_c(i, j)_{st}$ of $\tilde{A}_c(i, j)$, $1 \leqslant i, j \leqslant k$, $i \neq j$:

$$\begin{aligned} a_c(i, j)_{st} &= \frac{1}{\sigma_i^2 - \sigma_j^2}\left[\sigma_j b_c(i)_s b_c(j)_t^T - \sigma_i c_c(i)_s^T c_c(j)_t\right] \\ &= \frac{1}{\sigma_i^2 - \sigma_j^2}\left[\sigma_j \tilde{b}(i)_s \tilde{b}(j)_t^T - \sigma_i\left(1-\sigma_j^2\right) c_c(i)_s^T c_c(j)_t\right], \end{aligned}$$

where we have used that $B_c B_c^T = \tilde{B}\tilde{B}^T + \Sigma_c C_c^T C_c \Sigma_c$.

As a first step in proving that (2) implies (1) we show that $G(s) \in TC_n^{p,m}$ using Theorem 2.2. Analogously to the derivation above we obtain that

$$B_c B_c^T = \tilde{B}\tilde{B}^T + \Sigma_c C_c^T C_c \Sigma_c \tag{$*$}$$

and thus for $1 \leqslant j \leqslant k$,

$$\begin{aligned} B_c^j \left(B_c^j\right)^T &= \tilde{B}^j \left(\tilde{B}^j\right)^T + \sigma_j^2 \left(C_c^j\right) C_c^j \\ &= \left(1 - \sigma_j^2\right)\left(C_c^j\right)^T C_c^j + \sigma_j^2 \left(C_c^j\right)^T C_c^j \\ &= \left(C_c^j\right)^T C_c^j \end{aligned}$$

which implies that

$$B_c^j = \left[\left(C_c^j\right)^T C_c^j\right]^{1/2} \begin{pmatrix} \overline{U}^j \\ 0 \end{pmatrix}$$

for some $\overline{U}^j \in \mathbb{R}^{r_0(j) \times (m+p)}$ such that $\overline{U}^j(\overline{U}^j)^T = I_{r_0(j)}$.

Again using ($*$), it follows that

$$\tilde{b}(i)_s \tilde{b}(j)_t^T = b_c(i)_s b_c(j)_t^T - \sigma_i \sigma_j c_c(i)_s^T c_c(j)_t$$

and hence

$$\begin{aligned} a_c(i,j)_{st} &= \frac{1}{\sigma_i^2 - \sigma_j^2}\left[\sigma_j \tilde{b}(i)_s \tilde{b}(j)_t^T - \sigma_i\left(1 - \sigma_j^2\right) c_c(i)_s^T c_c(j)_t\right] \\ &= \frac{1}{\sigma_i^2 - \sigma_j^2}\left[\sigma_j b_c(i)_s b_c(j)_t^T - \sigma_i c_c(i)_s^T c_c(j)_t\right]. \end{aligned}$$

This shows that (A_c, B_c, C_c, D_c) is given in the parametrization of Theorem 2.2 and hence $G(s) \in TC_n^{p,m+p}$ with gramian Σ_c.

To complete the proof we have to show that $G(s)$ is coinner. But $D_c D_c^T = I$, so it remains to show, following Proposition 3.2, that

$$B_{c_2} = -\left(\Sigma_c C_c^T + B_{c_1} D_{pm}^T R^{-1/2}\right) R^{1/2},$$

where $B_c := [B_{c_1}, B_{c_2}]$, $B_{c_1} \in \mathbb{R}^{n\times m}$. But this is the case, since for $1 \leqslant j \leqslant k$ and $B_c^j =: [B_{c_1}^j, B_{c_2}^j]$, $B_{c_1}^j \in \mathbb{R}^{n(j)\times m}$, we have that

$$
\begin{aligned}
&-\left[\sigma_j\left(C_c^j\right)^T + B_{c_1}^j D_{pm}^T R^{-1/2}\right] R^{1/2} \\
&\quad = -\sigma_j\left(C_c^j\right)^T R^{-1/2} - \tilde{B}^j S^{-1/2} D_{pm}^T + \sigma_j\left(C_c^j\right)^T R^{-1/2} D_{pm} D_{pm}^T \\
&\quad = -\left[\sigma_j\left(C_c^j\right)^T R^{-1/2} + \tilde{B}^j S^{-1/2} D_{pm}^T\right] \\
&\quad = B_{c_2}^j.
\end{aligned}
$$

■

Again, specializing to the case of least input and output dimensions we obtain the following corollary for systems in $TCI_n^{1,2}$.

COROLLARY 3.1. *The following two statements are equivalent*:

(1) $g(s) \in TCI_n^{1,2}$.

(2) $g(s)$ *has a realization* $(A_c, b_c, c_c, d_c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 2} \times \mathbb{R}^{1\times n} \times \mathbb{R}^{1\times 2}$, *which is given by the following parameters*:

$$
\begin{array}{ll}
n(1),\dots,n(j),\dots,n(k), & n(j)\in\mathbb{N},\ \Sigma_{j=1}^k n(j) = n; \\
s_1,\dots,s_j,\dots,s_k, & s_j = \pm 1,\ 1\leqslant j\leqslant k, \\
1 > \sigma_1 > \cdots > \sigma_j > \cdots \sigma_k > 0 & \sigma_j\in\mathbb{R},\ 1\leqslant j\leqslant k \\
c_1, \alpha(1)_1,\dots,\alpha(1)_j,\dots,\alpha(1)_{n(1)-1}, & c_1 > 0,\ \alpha(1)_j > 0,\ 1\leqslant j\leqslant n(1)-1 \\
c_2, \alpha(2)_1,\dots,\alpha(2)_j,\dots,\alpha(2)_{n(2)-1}, & c_2 > 0,\ \alpha(2)_j > 0,\ 1\leqslant j\leqslant n(2)-1 \\
\vdots & \\
c_k, \alpha(k)_1,\dots,\alpha(k)_j,\dots,\alpha(k)_{n(k)-1}, & c_k > 0,\ \alpha(k)_j > 0,\ 1\leqslant j\leqslant n(k)-1 \\
d_{pm}\in\mathbb{R} &
\end{array}
$$

in the following way:

(i) $d_c = \dfrac{1}{\sqrt{1+d_{pm}^2}}[d_{pm}, 1]$.

(ii) $c_c = (\underbrace{c_1, 0, \dots, 0}_{n(1)}, \dots, \underbrace{c_j, 0, \dots, 0}_{n(j)}, \dots, \underbrace{c_k, 0, \dots, 0}_{n(k)})$.

(iii) *We have*

$$b_c = -\frac{1}{\sqrt{1+d_{pm}^2}}\left(\begin{array}{cc}
c_1\left(\sigma_1 d_{pm} - s_1\sqrt{1-\sigma_1^2}\right) & c_1\left(\sigma_1 + s_1 d_{pm}\sqrt{1-\sigma_1^2}\right)\\
0 & 0\\
\vdots & \vdots\\
0 & 0\\
\vdots & \vdots\\
c_j\left(\sigma_j d_{pm} - s_j\sqrt{1-\sigma_j^2}\right) & c_j\left(\sigma_j + s_j d_{pm}\sqrt{1-\sigma_j^2}\right)\\
0 & 0\\
\vdots & \vdots\\
0 & 0\\
\vdots & \vdots\\
c_k\left(\sigma_k d_{pm} - s_k\sqrt{1-\sigma_k^2}\right) & c_k\left(\sigma_k + s_k d_{pm}\sqrt{1-\sigma_k^2}\right)\\
0 & 0\\
\vdots & \vdots\\
0 & 0
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\} n(1)\\
\\
\left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\} n(j)\\
\\
\left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\} n(k)
\end{array}$$

(iv) *For* $A_c =: (A_c(i,j))_{1\leqslant i,j\leqslant k}$ *we have*

(a) *Block diagonal entries* $A_c(j,j)$, $1 \leqslant j \leqslant k$:

$$A_c(j,j) = \begin{pmatrix}
a_c(j,j) & -\alpha(j)_1 & & & & \mathbf{0}\\
\alpha(j)_1 & 0 & -\alpha(j)_2 & & & \\
 & \alpha(j)_2 & 0 & \ddots & & \\
 & & \alpha(j)_3 & \ddots & \ddots & \\
 & & & \ddots & \ddots & -\alpha(j)_{n(j)-1}\\
\mathbf{0} & & & & \alpha(j)_{n(j)-1} & 0
\end{pmatrix}$$

with $a_c(j,j) = -\dfrac{1}{2\sigma_j}c_j^2$.

(b) *Off diagonal blocks* $A_c(i,j)$, $1 \leqslant i, j \leqslant k$, $i \neq j$:

$$A_c(i,j) = \begin{pmatrix}
a_c(i,j) & 0 & \cdots & 0\\
0 & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots\\
0 & 0 & \cdots & 0
\end{pmatrix}$$

with

$$a_c(i,j) = -\frac{c_i c_j \sqrt{1-\sigma_j^2}}{\sigma_i \sqrt{1-\sigma_j^2} + s_i s_j \sigma_j \sqrt{1-\sigma_i^2}}.$$

Moreover, (A, b, c, d) *as defined in* (2) *is balanced with gramian*

$$\Sigma = \operatorname{diag}\left(\sigma_1 I_{n(1)}, \ldots, \sigma_j I_{n(j)}, \ldots, \sigma_k I_{n(k)}\right).$$

The map which assigns to each system in $CI_n^{1,2}$ *the realization given in* (2) *is a canonical form.*

The following definition of the normalized left coprime factorization of a transfer function shows that coinner transfer functions are naturally associated with such a factorization. For a reference on the role of coprime factorizations of transfer functions in the design of control systems see Vidyasagar (1985).

DEFINITION 3.2. Let $G(s)$ be a $p \times m$ transfer function. Then the transfer functions $\tilde{M}(s) \in TC_n^{p,p}$, $\tilde{N}(s) \in TC_n^{p,m}$ constitute a normalized left coprime factorization of $G(s)$ if:

(i) $\tilde{M}(\infty)$ is nonsingular.
(ii) $G(s) = \tilde{M}(s)^{-1}\tilde{N}(s)$.
(iii) There exist $V(s) \in TC_n^{p,p}$ and $U(s) \in TC_n^{m,p}$ such that for all $s \in \mathbb{C}$ we have

$$\tilde{M}(s)V(s) + \tilde{N}(s)U(s) = I_p.$$

(iv) $[\tilde{N}(s), \tilde{M}(s)]$ is coinner, i.e.

$$\tilde{N}(s)\tilde{N}(-s)^T + \tilde{M}(s)\tilde{M}(-s)^T = I_p$$

for all $s \in \mathbb{C}$.

The following statement formalizes the existence and uniqueness properties of these normalized left coprime factors.

PROPOSITION 3.3 (Vidyasagar, 1985). *The normalized left coprime factors* $\tilde{N}(s)$ *and* $\tilde{M}(s)$ *of a transfer function* $G(s)$ *exist and are unique to within left multiplication by a unitary matrix.*

We are now going to review a state space construction of the normalized left coprime factors of a transfer function $G(s) \in TL_n^{p,m}$. To do this we have to introduce Riccati equations for systems (A, B, C, D) in $L_n^{p,m}$.

The *generalized control algebraic Riccati equation* (GCARE) is given by

$$(A - BS^{-1}D^TC)^TX + X(A - BS^{-1}D^TC) - XBS^{-1}B^TX + C^TR^{-1}C = 0,$$

and the *generalized filtering algebraic Riccati equation* (GFARE) is given by

$$(A - BS^{-1}D^TC)Z + Z(A - BS^{-1}D^TC)^T - ZC^TR^{-1}CZ + BS^{-1}B^T = 0,$$

with $R = I + DD^T$ and $S = I + D^TD$.

These Riccati equations occur in the solution to a particular linear-quadratic-gaussian (LQG) control and filtering problem: the GCARE is the Riccati equation associated with the steady state output regulator design when input and output cost weights are chosen to be the identity. Dually, the GFARE is the Riccati equation associated with the steady state optimal filter design, where measurement and input noises have identity covariances. [More details can be found, for example, in Kwakernaak and Sivan (1972).] Note that the case $D = 0$ allows considerable simplification of these equations.

It is well known that minimality of (A, B, C, D) is sufficient to ensure that symmetric and positive definite solutions to the GCARE and the GFARE exist, are unique and are the stabilizing ones:

PROPOSITION 3.4. *If* (A, B, C, D) *is controllable (observable), then there exists a unique solution* $X = X^T > 0$ $(Z = Z^T > 0)$ *to the* GCARE (GFARE). *If the control gain* F *and the filter gain* H *are defined to be*

$$F := -S^{-1}(D^TC + B^TX),$$

$$H := -(BD^T + ZC^T)R^{-1},$$

then the eigenvalues of

$$A + BF, \qquad A + HC$$

corresponding to these solutions have strictly negative real parts.

The use of Riccati equations in this coprime factor context has its basis in the results of Nett et al. (1984), who showed that left and right coprime factors of a nominal plant can be generated from a state-feedback–observer configuration.

The following proposition shows that the LQG problem mentioned above yields the desired feedback–observer configuration for normalized coprime factors of a transfer function $G(s)$.

PROPOSITION 3.5 (Vidyasagar, 1988). *Let* $(A, B, C, D) \in L_n^{p,m}$ *with transfer function* $G(s) = C(sI - A)^{-1}B + D$, *and let* $H = -(ZC^T + BD^T)R^{-1}$ *be the filter gain corresponding to the unique positive definite solution to the GFARE. Then with* $R = I + DD^T$,

$$\tilde{N}(s) := R^{-1/2}C(sI - A - HC)^{-1}(B + HD) + R^{-1/2}D,$$

$$\tilde{M}(s) := R^{-1/2}C(sI - A - HC)^{-1}H + R^{-1/2}$$

are normalized left coprime factors of $G(s)$, *i.e.*

$$G(s) = \tilde{M}(s)^{-1}\tilde{N}(s).$$

REMARK 3.1. The previous proposition shows that each transfer function $G(s)$ has a normalized left coprime factorization $G(s) = \tilde{M}(s)^{-1}\tilde{N}(s)$ such that $\tilde{M}(\infty) = \tilde{M}(\infty)^T > 0$. It follows from Proposition 3.3 that a normalized left coprime factorization with this property is in fact unique.

Using Proposition 3.5, a realization of the transfer function $[\tilde{N}(s), \tilde{M}(s)]$ can be obtained. The next proposition shows that the positive definite solutions to the Lyapunov equations of $[\tilde{N}(s), \tilde{M}(s)]$ are closely related to the positive definitive solutions of the GCARE and GFARE of $G(s)$.

PROPOSITION 3.6 (Glover and McFarlane, 1988b). *With the notation of Proposition* 3.5, *define*

$$A_c = A + HC, \qquad B_c = [B + HD, H],$$

$$C_c = R^{-1/2}C, \qquad D_c = [R^{-1/2}D, R^{-1/2}].$$

Then (A_c, B_c, C_c, D_c) *is a minimal state space realization of* $[\tilde{N}(s), \tilde{M}(s)]$ *such that the positive definite solutions to the Lyapunov equations*

$$A_cP + PA_c^T = -B_cB_c^T,$$

$$A_c^TQ + QA_c = -C_c^TC_c$$

are given by

$$P = Z,$$

$$Q = X(I + ZX)^{-1},$$

where X, Z *are the unique positive definite solutions to the GCARE and the GFARE of* (A, B, C, D) *respectively.*

A further property of P and Q is given next.

PROPOSITION 3.7 (Meyer, 1988; Glover and McFarlane, 1988b). *With the notation above,*

$$I > PQ$$

or

$$\sigma_1 < 1,$$

where σ_1 *denotes the maximum singular value of a balanced realization of* $[\tilde{N}(s), \tilde{M}(s)]$.

REMARK 3.2. Definition 3.2 and Propositions 3.5–3.7 show immediately that the matrix $[\tilde{N}(s), \tilde{M}(s)]$ containing the normalized coprime factors of a transfer function $G(s)$ is in $TCI_n^{p,m+p}$.

We are next going to establish that each transfer function in $TCI_n^{p,m+p}$ can be related to a transfer function in $TL_n^{p,m}$ in this way.

PROPOSITION 3.8. *Let* $G(s) =: [\tilde{N}(s), \tilde{M}(s)] \in TCI_n^{p,m+p}$, $\tilde{M}(s)$ *being* $p \times p$, *have a realization given by* (A_c, B_c, C_c, D_c), *where* $B_c =: [B_{c_1}, B_{c_2}]$, $D_c =: [D_{c_1}, D_{c_2}]$ *are partitioned conformally with* $[\tilde{N}(s), \tilde{M}(s)]$.

Then $\tilde{N}(s)$ *and* $\tilde{M}(s)$ *are normalized left coprime factors of the transfer function* $\tilde{G}(s) \in TL_n^{p,m}$ *defined by the state space realization* (A_0, B_0, C_0, D_0) *given by*

$$D_0 = D_{c_2}^{-1} D_{c_1},$$

$$C_0 = D_{c_2}^{-1} C_c,$$

$$B_0 = B_{c_1} - B_{c_2} D_{c_2}^{-1} D_{c_1},$$

$$A_0 = A_c - B_{c_2} D_{c_2}^{-1} C_c.$$

Proof. We first have to show that (A_0, B_0, C_0, D_0) is minimal. For this purpose we rewrite the matrices of the system (A_c, B_c, C_c, D_c) in terms of the matrices of the system (A_0, B_0, C_0, D_0). Since (A_c, B_c, C_c, D_c) is asymptotically stable, there exist $P = P^T > 0$ and $Q = Q^T > 0$ such that

$$A_cP + PA_c^T = -B_cB_c^T,$$

$$A_c^TQ + QA_c = -C_c^TC_c.$$

The fact that the system is in $CI_n^{p,m+p}$ implies by Proposition 3.2 that

$$D_{c_2} = \left(I - D_{c_1}D_{c_1}^T\right)^{1/2}$$

and

$$B_{c_2} = -\left(PC_c^T + B_{c_1}D_{c_1}^T\right)D_{c_2}^{-1}.$$

But these identities imply that

$$D_{c_1} = R^{-1/2}D_0,$$

$$D_{c_2} = R^{-1/2}$$

with $R = (I - D_{c_1}D_{c_1}^T)^{-1} = I + D_0D_0^T$ and that

$$B_{c_2} = -\left(PC_c^T + B_{c_1}D_{c_1}^T\right)R^{1/2}.$$

Using the definition $B_0 = B_{c_1} - B_{c_2}R^{1/2}D_{c_1}$ and setting $S = I + D_0^TD_0 = (I - D_{c_1}^TD_{c_1})^{-1}$, we thus have that

$$B_{c_1} = \left(B_0 - PC_0^TD_0\right)S^{-1},$$

$$B_{c_2} = -\left(B_0D_0^T + PC_0^T\right)R^{-1}$$

and hence

$$A_c = A_0 - \left(B_0D_0^T + PC_0^T\right)R^{-1}C_0$$

with

$$C_c = R^{-1/2}C_0.$$

Also note that

$$B_c B_c^T = B_0 S^{-1} B_0^T + P C_0^T R^{-1} C_0 P.$$

We are now in a position to reformulate the Lyapunov equations corresponding to (A_c, B_c, C_c, D_c). We have

$$\begin{aligned}
0 &= A_c P + P A_c^T + B_c B_c^T \\
&= \left[A_0 - \left(B_0 D_0^T + P C_0^T\right) R^{-1} C_0\right] P + P\left[A_0 - \left(B_0 D_0^T + P C_0^T\right) R^{-1} C_0\right]^T \\
&\quad + B_0 S^{-1} B_0^T + P C_0^T R^{-1} C_0 P \\
&= \left(A_0 - B_0 D_0^T R^{-1} C_0\right) P + P\left(A_0 - B_0 D_0^T R^{-1} C_0\right)^T \\
&\quad - P C_0^T R^{-1} C_0 P + B_0 S^{-1} B_0^T
\end{aligned} \tag{1}$$

and

$$\begin{aligned}
0 &= A_c^T Q + Q A_c + C_c^T C_c \\
&= \left(A_0 - B_0 D_0^T R^{-1} C_0\right)^T Q + Q\left(A_0 - B_0 D_0^T R^{-1} C_0\right) \\
&\quad + (I - QP) C_0^T R^{-1} C_0 (I - PQ) - QP C_0^T R^{-1} C_0 P Q \\
&= \left(A_0 - B_0 D_0^T R^{-1} C_0\right) Q^{-1} + Q^{-1}\left(A_0 - B_0 D_0^T R^{-1} C_0\right)^T \\
&\quad + \left(Q^{-1} - P\right) C_0^T R^{-1} C_0 \left(Q^{-1} - P\right) - P C_0^T R^{-1} C_0 P.
\end{aligned} \tag{2}$$

Subtracting Equation (1) from Equation (2), we obtain

$$\begin{aligned}
0 &= \left(A_0 - B_0 D_0^T R^{-1} C_0\right)\left(Q^{-1} - P\right) + \left(Q^{-1} - P\right)\left(A_0 - B_0 D_0^T R^{-1} C_0\right)^T \\
&\quad + \left(Q^{-1} - P\right) C_0^T R^{-1} C_0 \left(Q^{-1} - P\right) - B_0 S^{-1} B_0^T.
\end{aligned} \tag{3}$$

If (A_0, B_0, C_0, D_0) is not controllable, then there exists a vector $x_0 \neq 0$ and $\lambda \in \mathbb{C}$ such that $x_0^* A_0 = \lambda x_0^*$ and $x_0^* B_0 = 0$. Then by (3) we have

$$0 = \lambda x_0^*(Q^{-1} - P)x_0 + \bar{\lambda} x_0^*(Q^{-1} - P)x_0$$
$$+ x_0^*(Q^{-1} - P)C_0^T R^{-1} C_0 (Q^{-1} - P)x_0$$

and hence

$$\operatorname{Re}\lambda = -\frac{1}{2}\frac{x_0^*(Q^{-1} - P)C_0^T R^{-1} C_0 (Q^{-1} - P)x_0}{x_0^*(Q^{-1} - P)x_0} \leqslant 0,$$

noting that $x_0^*(Q^{-1} - P)x_0 > 0$ by Proposition 3.7.

But by (1) we have that

$$\lambda x_0^* P x_0 + \bar{\lambda} x_0^* P x_0 - x_0^* P C_0^T R^{-1} C_0 P x_0 = 0$$

and hence

$$\operatorname{Re}\lambda = \frac{1}{2}\frac{x_0^* P C_0^T R^{-1} C_0 P x_0}{x_0^* P x_0} \geqslant 0,$$

which implies that $\operatorname{Re}\lambda = 0$ and hence that

$$x_0^* P C_0^T = 0.$$

But, since $A_c = A_0 - (B_0 D_0^T + P C_0^T) R^{-1} C_0$ we now have that

$$x_0^* A_c = x_0^* \left[A_0 - \left(B_0 D_0^T + P C_0^T \right) R^{-1} C_0 \right] = x_0^* A_0 = \lambda x_0^*,$$

which implies that A_c has an eigenvalue with $\operatorname{Re}\lambda = 0$, which is a contradiction to the asymptotic stability of A_c, and hence (A_0, B_0, C_0, D_0) is controllable.

The observability of (A_0, B_0, C_0, D_0) follows by a straightforward application of the Popov-Belevitch-Hautus test.

The minimality of (A_0, B_0, C_0, D_0) now implies that $P = P^T > 0$ is in fact the unique stabilizing solution of the GFARE as given in (1).

Now let $\tilde{N}_1(s), \tilde{M}_1(s)$ be the normalized left coprime factors of $\tilde{G}(s)$ as given by the construction in Proposition 3.5. If $(\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c)$ is the corresponding realization of $[\tilde{N}_1(s), \tilde{M}_1(s)]$, then, since the filter gain of (A_0, B_0, C_0, D_0) is given by

$$\bar{H} = -\left(PC_0^T + B_0 D_0^T\right)R^{-1} = B_{c_2},$$

it follows immediately that

$$\left(\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D}_c\right) = \left(A_c, B_c, C_c, D_c\right),$$

which shows that $\tilde{N}(s) = \tilde{N}_1(s)$, $\tilde{M}(s) = \tilde{M}_1(s)$ are normalized left coprime factors of $G(s)$. ∎

4. CANONICAL FORMS FOR MINIMAL SYSTEMS

We are now going to derive a canonical form for systems in $L_n^{p,m}$ on the basis of the results in Section 3. We will exploit the relationship between the transfer functions in $TL_n^{p,m}$ and coinner functions in $TCI_n^{p,m+p}$ and make use of the canonical form for systems in $CI_n^{p,m+p}$ given in Theorem 3.1.

The relationship between $TL_n^{p,m}$ and $TCI_n^{p,m+p}$ which was partially established in the previous section is precisely formulated in the following proposition.

Proposition 4.1. *The map*

$$CF: TL_n^{p,m} \to TCI_n^{p,m+p},$$

$$G(s) \mapsto \left[\tilde{N}(s), \tilde{M}(s)\right],$$

which assigns each $G(s) \in TL_n^{p,m}$ to the coinner transfer function consisting of the normalized left coprime factors $\tilde{N}(s)$, $\tilde{M}(s)$ [i.e. $G(s) = \tilde{M}(s)^{-1}\tilde{N}(s)$], with $\tilde{M}(\infty) = \tilde{M}(\infty)^T > 0$, is a bijection.

If $(A_0, B_0, C_0, D_0) \in L_n^{p,m}$ is a realization of $G(s) \in L_n^{p,m}$, then $CF(G(s))$ has a realization (A_c, B_c, C_c, D_c) given by

$$D_c = \left[R^{-1/2}D_0, R^{-1/2}\right],$$

$$C_c = R^{-1/2}C_0,$$

$$B_c = \left[B_c + HD_0, H\right],$$

$$A_c = A_0 + HC_0$$

with $R = I + D_0 D_0^T$ and filter gain $H = -(B_0 D_0^T + ZC_0^T)R^{-1}$.

Conversely, if $(A_c, B_c, C_c, D_c) \in CI_n^{p,m+p}$ *is a realization of a transfer function* $\tilde{G}(s) \in TCI_n^{p,m+p}$, *then* $CF^{-1}(\tilde{G}(s))$ *has a realization* (A_0, B_0, C_0, D_0) *given by*

$$D_0 = D_{c_2}^{-1} D_{c_1},$$

$$C_0 = D_{c_2}^{-1} C_c,$$

$$B_0 = B_{c_1} - B_{c_2} D_{c_2}^{-1} D_{c_1},$$

$$A_0 = A_c - B_{c_2} D_{c_2}^{-1} C_c$$

with $D_c = [D_{c_1}, D_{c_2}]$, $D_{c_1} \in \mathbb{R}^{p \times m}$, $B_c = [B_{c_1}, B_{c_2}]$, $B_{c_1} \in \mathbb{R}^{n \times m}$.

Proof. It was shown in Proposition 3.8 that CF is surjective. The fact that CF is injective follows from Proposition 3.3 and Remark 3.1. The state space formulae have been established in Proposition 3.6 and Proposition 3.8. ■

The state space formulae of this proposition allow us to derive a canonical form for $L_n^{p,m}$ from the canonical form for $CI_n^{p,m+p}$.

THEOREM 4.1. *The following two statements are equivalent*:

(1) $G(s) \in TL_n^{p,m}$.

(2) $G(s)$ *has a realization* $(A_0, B_0, C_0, D_0) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}$ *given in the following way*: *There are positive integers*

$$n(1), \ldots, n(j), \ldots, n(k) \qquad \text{such that} \quad \sum_{j=1}^{k} n(j) = n$$

and numbers

$$1 > \sigma_1 > \cdots > \sigma_j > \cdots > \sigma_k > 0$$

such that if (A_0, B_0, C_0, D_0) *is partitioned as*

$$A_0 =: (A_0(i,j))_{1 \leqslant i,j \leqslant k}, \qquad A_0(i,j) \in \mathbb{R}^{n(i) \times n(j)},$$

$$B_0 =: \begin{pmatrix} B_0^1 \\ \vdots \\ B_0^j \\ \vdots \\ B_0^k \end{pmatrix}, \qquad B_0^j \in \mathbb{R}^{n(j) \times m},$$

$$C_0 =: \left(C_0^1, \dots, C_0^j, \dots, C_0^k \right), \qquad C_0^j \in \mathbb{R}^{p \times n(j)},$$

(i) $D_0 \in \mathbb{R}^{p \times m}$.

(ii) *We have*

$$B_0^j := \sqrt{1 - \sigma_j^2} \left[\left(C_0^j \right)^T R^{-1} C_0^j \right]^{1/2} \begin{pmatrix} U^j \\ 0 \end{pmatrix} S^{1/2},$$

with $U^j \in \mathbb{R}^{r_0(j) \times m}$, *such that* $U^j (U^j)^T = I_{r_0(j)}$,

$$r_0(j) := \operatorname{rank}\left[\left(C_0^j \right)^T R^{-1} C_0^j \right] \leqslant \min(p, m),$$

and $R = I + D_0 D_0^T$, $S = I + D_0^T D_0$.

(iii) $(R^{-1/2} C_0^j, \tilde{A}_0(j,j))$ *is in standard all pass form with*

$$A_0(j,j) = \frac{2\sigma_j^2 - 1}{2\sigma_j} \left(C_0^j \right)^T R^{-1} C_0^j + B_0^j D_0^T R^{-1} C_0^j + \tilde{A}_0(j,j).$$

(iv) *We have*

$$A_0(i,j) = \begin{pmatrix} \tilde{A}_0(i,j) & 0 \\ 0 & 0 \end{pmatrix}, \qquad 1 \leqslant i, j \leqslant k, \quad i \neq j,$$

with

$$\tilde{A}_0(i,j) =: \left(a_0(i,j)_{st} \right)_{\substack{1 \leqslant s \leqslant r_0(i) \\ 1 \leqslant t \leqslant r_0(j)}} \in \mathbb{R}^{r_0(i) \times r_0(j)}$$

such that

$$a_0(i,j)_{st} = \frac{1}{\sigma_i^2 - \sigma_j^2}\left[\sigma_j b_0(i)_s S^{-1} b_0(j)_t^T - \sigma_i(1-\sigma_i^2) c_0(i)_s^T R^{-1} c_0(j)_t\right]$$

$$+ b_0(i)_s D_0^T R^{-1} c_0(j)_t,$$

where $b_0(i)_s$ is the sth row of B_0^i and $c_0(i)_s$ is the sth column of C_0^i.

The map which assigns to each system in $L_n^{p,m}$ the realization given in (2) is a canonical form.

Proof. The proof is based on the canonical form for $CI_n^{p,m+p}$ and the bijection CF between $TL_n^{p,m}$ and $TCI_n^{p,m+p}$ given in Proposition 4.1.

To show (1) implies (2) let $\tilde{G}(s) := CF(G(s))$ have the realization (A_c, B_c, C_c, D_c) given in the canonical form of Theorem 3.1 with gramian Σ_c. Then by Proposition 4.1 a realization (A_0, B_0, C_0, D_0) of $G(s)$ is given by

$$D_0 = R^{1/2} D_{c_1},$$

$$C_0 = R^{1/2} C_c,$$

$$B_0 = B_{c_1} - B_{c_2} R^{1/2} D_{c_1},$$

$$A_0 = A_c - B_{c_2} R^{1/2} C_c,$$

where $R^{-1/2} = D_{c_2}$, with $D_c = [D_{c_1}, D_{c_2}]$, $D_{c_1} \in \mathbb{R}^{p\times m}$, and $B_c = [B_{c_1}, B_{c_2}]$, $B_{c_1} \in \mathbb{R}^{n\times m}$. Note that $D_0 = R^{1/2} D_{c_1}$ corresponds to the D_{pm} matrix in Theorem 3.1 and that (i) is satisfied.

If we partition the systems in the standard way according to the structural indices $n(1), n(2), \dots, n(k)$, we obtain, noting that $R = (I - D_{c_1} D_{c_1}^T)^{-1}$, $S = (I - D_{c_1}^T D_{c_1})^{-1}$,

$$\begin{aligned} B_0^j &= B_{c_1}^j - B_{c_2}^j R^{1/2} D_{c_1} \\ &= \tilde{B}^j S^{-1/2} + \tilde{B}^j S^{-1/2} D_{c_1}^T R D_{c_1} - \sigma_j (C_c^j)^T D_{c_1} + \sigma_j (C_c^j)^T D_{c_1} \\ &= \tilde{B}^j S^{1/2}, \end{aligned}$$

where $\tilde{B}$ is as in Theorem 3.1. This shows that B_0 is as postulated in (ii).

Now since

$$B_{c_2}^i R^{1/2} C_c = -\left[\sigma_i \left(C_c^i\right)^T R^{-1/2} + \tilde{B}^i S^{-1/2} D_0^T\right] C_0$$

$$= -\left[\sigma_i \left(C_0^i\right)^T + B_0^i D_0^T\right] R^{-1} C_0,$$

we have that

$$a_0(i,j)_{st} = a_c(i,j)_{st} + \sigma_i c_0(i)_s^T R^{-1} c_0(j)_t + b_0(i)_s D_0^T R^{-1} c_0(j)_t$$

$$= \frac{1}{\sigma_i^2 - \sigma_j^2}\Big[\sigma_j b_0(i)_s S^{-1} b(j)_t^T - \sigma_i\left(1-\sigma_j^2\right) c_0(i)_s^T R^{-1} c_0(j)_t$$

$$+ \sigma_i^3 c_0(i)_s^T R^{-1} c_0(j)_t - \sigma_i \sigma_j^2 c_0(i)_s^T R^{-1} c_0(j)_t\Big]$$

$$+ b_0(i)_s D_0^T R^{-1} c_0(j)_t$$

$$= \frac{1}{\sigma_i^2 - \sigma_j^2}\left[\sigma_j b_0(i)_s S^{-1} b(j)_t^T - \sigma_i\left(1-\sigma_i^2\right) c_0(i)_s^T R^{-1} c_0(j)_t\right]$$

$$+ b_0(i)_s D_0^T R^{-1} c_0(j)_t.$$

Similarly, for the block diagonal entries $A_0(j,j)$ we have

$$A_0(j,j) = A_c(j,j) - B_{c_2}^j R^{1/2} C_c^j$$

$$= -\frac{1}{2\sigma_j}\left(C_0^j\right) R^{-1} C_0^j + \tilde{A}_c(j,j) + \sigma_j \left(C_0^j\right)^T R^{-1} C_0^j + B_0^j D_0^T R^{-1} C_0^j$$

$$= \frac{2\sigma_j^2 - 1}{2\sigma_j}\left(C_0^j\right)^T R^{-1} C_0^j + B_0^j D_0^T R^{-1} C_0^j + \tilde{A}_0(j,j),$$

where we set $\tilde{A}_0(j,j) := \tilde{A}_c(j,j)$. Since (A_0, B_0, C_0, D_0) is uniquely determined by (A_c, B_c, C_c, D_c), we have thus constructed a canonical form for $TL_n^{p,m}$.

We will now show the converse, i.e. that (2) implies (1). Given a system (A_0, B_0, C_0, D_0) which is parametrized as in (2), construct the system (A_c, B_c, C_c, D_c) by

$$D_c =: \left[D_{c_1}, D_{c_2}\right] = \left[R^{-1/2}D_0, R^{-1/2}\right],$$

$$C_c = R^{-1/2}C_0,$$

$$B_c =: \left[B_{c_1}, B_{c_2}\right] = \left[B_0 + H_0 D_0, H_0\right],$$

$$A_c = A_0 + HC_0,$$

where $R = I + D_0 D_0^T$ and

$$H_0 = -\left(B_0 D_0^T + \Sigma_c C_0^T\right)R^{-1}$$

with $\Sigma_c = \operatorname{diag}(\sigma_1 I_{n(1)}, \ldots, \sigma_j I_{n(j)}, \ldots, \sigma_k I_{n(k)})$.

We have to show that (A_c, B_c, C_c, D_c) is parametrized as in the canonical form of Theorem 3.1. Partitioning in the standard way, we have

$$\begin{aligned} B_{c_1}^j &= B_0^j + H_0^j D_0 \\ &= \tilde{B}^j S^{1/2} - \tilde{B}^j S^{1/2} D_0^T R^{-1} D_0 - \sigma_j \left(C_0^j\right)^T R^{-1} D_0 \\ &= \tilde{B}^j S^{-1/2} - \sigma_j \left(C_c^j\right)^T R^{-1/2} D_0, \\ B_{c_1}^j &= H_0^j \\ &= -\left(\tilde{B}^j S^{-1/2} D_0^T + \sigma_j C_c^T R^{-1/2}\right), \end{aligned}$$

and hence, since $\tilde{B}^j := B_0^j S^{-1/2}$ is of the demanded form, B_c^j as in Theorem 3.1, where we set $D_{pm} = D_0$. It is also easily verified that A_c is parametrized as in Theorem 3.1. Thus $(A_c, B_c, C_c, D_c) \in CI_n^{p, m+p}$, and hence by Proposition 4.1 $(A_0, B_0, C_0, D_0) \in L_n^{p,m}$. ■

Specializing the statement of the previous theorem to the case of single input, single output transfer functions, we obtain the following corollary.

COROLLARY 4.1. *The following two statements are equivalent:*

(1) $g(s) \in TL_n^{1,1}$.

(2) $g(s)$ *has a realization* $(A_0, b_0, c_0, d_0) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1} \times \mathbb{R}^{1\times n} \times \mathbb{R}^{1\times 1}$, *which is given by the following parameters:*

$$\begin{array}{ll}
n(1),\dots,n(j),\dots,n(j), & n(j)\in\mathbb{N},\ \sum_{j=1}^{k} n(j) = n,\\
s_1,\dots,s_j,\dots,s_k, & s_j = \pm 1,\ 1\leqslant j\leqslant k,\\
1>\sigma_1>\cdots>\sigma_j>\cdots>\sigma_k>0, & \sigma_j\in\mathbb{R},\ 1\leqslant j\leqslant k\\
c_1,\alpha(1)_1,\dots,\alpha(1)_j,\dots,\alpha(1)_{n(1)-1}, & c_1>0,\ \alpha(1)_j>0,\ 1\leqslant j\leqslant n(1)-1,\\
c_2,\alpha(2)_1,\dots,\alpha(2)_j,\dots,\alpha(2)_{n(2)-1}, & c_2>0,\ \alpha(2)_j>0,\ 1\leqslant j\leqslant n(2)-1,\\
\vdots & \\
c_k,\alpha(k)_1,\dots,\alpha(k)_j,\dots,\alpha(k)_{n(k)-1}, & c_k>0,\ \alpha(k)_j>0,\ 1\leqslant j\leqslant n(k)-1,\\
d_0\in\mathbb{R} &
\end{array}$$

in the following way:

(i) $d_0 \in \mathbb{R}$.

(ii) $c_0 = (\underbrace{c_1,0,\dots,0}_{n(1)},\dots,\underbrace{c_j,0,\dots,0}_{n(j)},\dots,\underbrace{c_k,0,\dots,0}_{n(k)})$.

(iii)

$$b_0^T = (\underbrace{s_1\sqrt{1-\sigma_1^2}\,c_1,0,\dots,0}_{n(1)},\dots,\underbrace{s_j\sqrt{1-\sigma_j^2}\,c_j,0,\dots,0}_{n(j)}\ \underbrace{s_k\sqrt{1-\sigma_k^2}\,c_k,0,\dots,0}_{n(k)}).$$

(iv) *For* $A_0 =: (A_0(i,j))_{1\leqslant i,j\leqslant k}$ *we have*

(a) *Block diagonal entries* $A_0(j,j)$, $1\leqslant j\leqslant k$:

$$A_0(j,j) = \begin{pmatrix}
a_0(j,j) & -\alpha(j)_1 & & & & \mathbf{0}\\
\alpha(j)_1 & 0 & -\alpha(j)_2 & & & \\
 & \alpha(j)_2 & 0 & \ddots & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & \ddots & \ddots & -\alpha(j)_{n(j)-1}\\
\mathbf{0} & & & & \alpha(j)_{n(j)-1} & 0
\end{pmatrix}$$

with

$$a_0(j,j) = -\frac{c_j^2}{1+d_0^2}\left(\frac{1-2\sigma_j^2}{2\sigma_j} - s_j d_0\sqrt{1-\sigma_j^2}\right).$$

(b) *Off diagonal blocks* $A_0(i,j)$, $1 \leqslant i, j \leqslant k$, $i \neq j$:

$$A_0(i,j) = \begin{pmatrix} a_0(i,j) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

with

$$a_0(i,j) = -\frac{c_i c_j \sqrt{1-\sigma_i^2}}{1+d_0^2}\left(\frac{\sqrt{1-\sigma_i^2}\sqrt{1-\sigma_j^2} - s_i s_j \sigma_i \sigma_j}{\sigma_i\sqrt{1-\sigma_j^2} + s_i s_j \sigma_j \sqrt{1-\sigma_i^2}} - s_i d_0\right).$$

The map which assigns to each system in $C_n^{1,1}$ *the realization given in* (2) *is a canonical form.*

REMARK 4.1. In Glover and McFarlane (1988a, 1988b) the problem of robustly stabilizing a transfer function $G(s)$ is considered. The aim is to design a feedback controller which guarantees closed loop stability for a maximum amount of uncertainty in the plant. In this case the uncertainty is modeled as additive perturbations on $\tilde{M}(s)$ and $\tilde{N}(s)$, the normalized left coprime factors of the transfer function. Thus a perturbed model is given by

$$G_\Delta(s) = \left[\tilde{M}(s) + \Delta_M(s)\right]^{-1}\left[\tilde{N}(s) + \Delta N(s)\right],$$

where $\Delta_M(s)$, $\Delta_N(s)$ are asymptotically stable, unknown transfer functions. The aim is to find a controller for which

$$\epsilon = \sup_{\omega \in \mathbb{R}} \left\| \left[\Delta_M(i\omega), \Delta_N(i\omega)\right] \right\|$$

is maximized while guaranteeing closed loop stability. It can be shown that the maximum margin for stability, $\epsilon_{\max}$, is given by

$$\epsilon_{\max} = \sqrt{1 - \sigma_1^2},$$

with σ_1 as in Theorem 4.1. ∎

REMARK 4.2. It also follows from the derivation of the canonical form of Theorem 4.1 that for a system parametrized in this canonical form, the same parameters yield a state space realization of the normalized left coprime factors of this system via the parametrization of $CI_n^{p,m+p}$ in Theorem 3.1. The canonical form of Theorem 4.1 can therefore be said to be in *normalized left coprime factor balanced* coordinates.

We are now going to use the parametrization given in the previous theorem to obtain a parametrization of so called Riccati balanced systems introduced in Jonckheere and Silverman (1983).

DEFINITION 4.1. A system (A_r, B_r, C_r, D_r) in $L_n^{p,m}$ is called *Riccati balanced* if

$$X = Z =: M =: \operatorname{diag}(\mu_1, \dots, \mu_j, \dots, \mu_n) > 0,$$

where $X = X^T > 0$ is a solution to the GCARE and $Z = Z^T > 0$ is a solution to the GFARE.

For an interpretation of Riccati balancing in the context of linear quadratic control design see Jonckheere and Silverman (1983). The following proposition states that if a system is in the canonical form of Theorem 4.1, it can be brought to an equivalent realization in Riccati balanced coordinates by a diagonal state space transformation.

PROPOSITION 4.2. *Let* $(A_0, B_0, C_0, D_0) \in L_n^{p,m}$ *be given in the canonical form of Theorem* 4.1 *with*

$$\Sigma_c = \operatorname{diag}(\sigma_1 I_{n(1)}, \dots, \sigma_j I_{n(j)}, \dots, \sigma_k I_{n(k)}).$$

Then $X = \Sigma_c(I - \Sigma_c^2)^{-1}$ *solves the GCARE, and* $Z = \Sigma_c$ *solves the GFARE. The system*

$$(A_r, B_r, C_r, D_r) := \left(TA_0T^{-1}, TB_0, C_0T^{-1}, D_0\right),$$

with $T = (I - \Sigma_c^2)^{-1/4}$, *is Riccati balanced, and the corresponding positive definite solutions to the GCARE and the GFARE of* (A_r, B_r, C_r, D_r) *are*

$$M = \Sigma_c\left(I - \Sigma_c^2\right)^{-1/2} =: \operatorname{diag}\left(\mu_1 I_{n(1)}, \dots, \mu_j I_{n(j)}, \dots, \mu_k I_{n(k)}\right).$$

Proof. The proof follows immediately from Proposition 3.6. ∎

REMARK 4.3. Using the notation of the previous proposition and noting that

$$\mu_j = \sigma_j\left(1 - \sigma_j^2\right)^{-1/2}$$

for all $1 \leqslant j \leqslant k$, it is easily verified that for $1 \leqslant j \leqslant k-1$,

$$\sigma_j > \sigma_{j+1} \quad \text{if and only if} \quad \mu_j > \mu_{j+1}.$$

Following this remark and the previous proposition, we can now write down a canonical form for $L_n^{p,m}$ in terms of Riccati balanced systems.

THEOREM 4.2. *The following two statements are equivalent*:

(1) $G(s) \in TL_n^{p,m}$.

(2) $G(s)$ *has a realization* $(A_r, B_r, C_r, D_r) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ *given in the following way*: *There are positive integers*

$$n(1), \dots, n(j), \dots, n(k) \qquad \textit{such that} \quad \sum_{j=1}^{k} n(j) = n$$

and numbers

$$\mu_1 > \cdots > \mu_j > \cdots > \mu_k > 0$$

such that if (A_r, B_r, C_r, D_r) *is partitioned as*

$$A_r =: \left(A_r(i,j)\right)_{1 \leqslant i,j \leqslant k}, \qquad A_r(i,j) \in \mathbb{R}^{n(i) \times n(j)},$$

$$B_r =: \begin{pmatrix} B_r^1 \\ \vdots \\ B_r^j \\ \vdots \\ B_r^k \end{pmatrix}, \qquad B_r^j \in \mathbb{R}^{n(j) \times m},$$

$$C_r =: \left(C_r^1, \dots, C_r^j, \dots, C_r^k\right), \qquad C_r^j \in \mathbb{R}^{p \times n(j)},$$

then:

(i) $D_r \in \mathbb{R}^{p \times m}$.

(ii) *We have*

$$B_r^j = \left[\left(C_r^j\right)^T R^{-1} C_r^j\right]^{1/2} \begin{pmatrix} U^j \\ 0 \end{pmatrix} S^{1/2},$$

with $U^j \in \mathbb{R}^{r_0(j) \times m}$, *such that* $U^j (U^j)^T = I_{r_0(j)}$, *where*

$$r_0(j) := \operatorname{rank}\left[\left(C_r^j\right)^T R^{-1} C_r^j\right] \leqslant \min(p, m)$$

and $R = I + D_r D_r^T$, $S = I + D_r^T D_r$.

(iii) $\left(R^{-1/2} C_r^j, \tilde{A}_r(j,j)\right)$ *is in standard all pass form with*

$$A_r(j,j) = \frac{\mu_i^2 - 1}{2\mu_i} \left(C_r^j\right)^T R^{-1} C_r^j + B_r^j D_r^T R^{-1} C_r^j + \tilde{A}_r(j,j).$$

(iv) *We have*

$$A_r(i,j) = \begin{pmatrix} \tilde{A}_r(i,j) & 0 \\ 0 & 0 \end{pmatrix}, \qquad 1 \leqslant i, j \leqslant k, \quad i \neq j,$$

with

$$\tilde{A}_r(i,j) =: \left(a_r(i,j)_{st}\right)_{\substack{1 \leqslant s \leqslant r_0(i) \\ 1 \leqslant t \leqslant r_0(j)}} \in \mathbb{R}^{r_0(i) \times r_0(j)}$$

such that

$$a_r(i,j)_{st} = \frac{1}{\mu_i^2 - \mu_j^2}\Big[\mu_j(1+\mu_i^2) b_r(i)_s S^{-1} b_r(j)_t^T$$

$$- \mu_i(1+\mu_j^2) c_r(i)_s^T R^{-1} c_r(j)_t\Big]$$

$$+ b_r(i)_s D_r^T R^{-1} c_r(j)_t,$$

where $b_r(i)_s$ *is the* s*th row of* B_r^i *and* $c_r(i)_s$ *is the* s*th column of* C_r^i. *Moreover,* (A_r, B_r, C_r, D_r) *is Riccati balanced such that*

$$M = \mathrm{diag}\big(\mu_1 I_{n(1)}, \ldots, \mu_j I_{n(j)}, \ldots, \mu_k I_{n(k)}\big)$$

is the unique positive definite solution to the GCARE and GFARE of (A_r, B_r, C_r, D_r). *The map which assigns to each system in* $L_n^{p,m}$ *the realization given in* (2) *is a canonical form.*

Proof. The proof follows immediately from Theorem 4.1 by performing a state space transformation with $T = (I - \Sigma_c^2)^{-1/4}$ and by setting $C_r = C_0 T^{-1}$ as well as $\mu_j = \sigma_j / \sqrt{1 - \sigma_j^2}$ for $1 \leqslant j \leqslant k$. ∎

The case of single input, single output transfer functions is considered in the following corollary.

Corollary 4.1. *The following two statements are equivalent:*

(1) $g(s) \in TL_n^{1,1}$.

(2) $g(s)$ *has a realization* $(A_r, b_r, c_r, d_r) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times 1} \times \mathbb{R}^{1 \times n} \times \mathbb{R}^{1 \times 1}$, *which is given by the following parameters:*

$$\begin{array}{ll}
n(1), \ldots, n(j), \ldots, n(k), & n(j) \in \mathbb{N}, \ \sum_{j=1}^k n(j) = n, \\
s_1, \ldots, s_j, \ldots, s_k, & s_j = \pm 1, \ 1 \leqslant j \leqslant k, \\
\mu_1 > \cdots > \mu_j > \cdots > \mu_k > 0 & \mu_j \in \mathbb{R}, \ 1 \leqslant j \leqslant k, \\
c_1, \alpha(1)_1, \ldots, \alpha(1)_j, \ldots, \alpha(1)_{n(1)-1}, & c_1 > 0, \ \alpha(1)_j > 0, \ 1 \leqslant j \leqslant n(1) - 1, \\
c_2, \alpha(2)_1, \ldots, \alpha(2)_j, \ldots, \alpha(2)_{n(2)-1}, & c_2 > 0, \ \alpha(2)_j > 0, \ 1 \leqslant j \leqslant n(2) - 1, \\
\vdots & \\
c_k, \alpha(k)_1, \ldots, \alpha(k)_j, \ldots, \alpha(k)_{n(k)-1}, & c_k > 0, \ \alpha(k)_j > 0, \ 1 \leqslant j \leqslant n(k) - 1, \\
d_r \in \mathbb{R}, &
\end{array}$$

in the following way:

(i) $d_r \in \mathbb{R}$.

(ii) $c_r = (\underbrace{c_1, 0, \ldots, 0}_{n(1)}, \ldots, \underbrace{c_j, 0, \ldots, 0}_{n(j)}, \ldots, \underbrace{c_k, 0, \ldots, 0}_{n(k)})$.

(iii) $b_r^T = (\underbrace{s_1 c_1, 0, \ldots, 0}_{n(1)}, \ldots, \underbrace{s_j c_j, 0, \ldots, 0}_{n(j)}, \ldots, \underbrace{s_k c_k, 0, \ldots, 0}_{n(k)})$.

(iv) *For* $A_r =: (A_r(i, j))_{1 \leqslant i, j \leqslant k}$ *we have*

(a) *block diagonal entries* $A_r(j, j)$, $1 \leqslant j \leqslant k$:

$$A_r(j,j) = \begin{pmatrix} a_r(j,j) & -\alpha(j)_1 & & & & & \mathbf{0} \\ \alpha(j)_1 & 0 & -\alpha(j)_2 & & & & \\ & \alpha(j)_2 & 0 & \cdot & & & \\ & & \cdot & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & -\alpha(j)_{n(j)-1} \\ \mathbf{0} & & & & & \alpha(j)_{n(j)-1} & 0 \end{pmatrix}$$

with

$$a_r(j,j) = -\frac{c_j^2}{1+d_r^2}\left(\frac{1-\mu_j^2}{2\mu_j} - s_j d_r\right);$$

(b) *off diagonal blocks* $A_r(i, j)$, $1 \leqslant i, j \leqslant k$, $i \neq j$:

$$A_r(i,j) = \begin{pmatrix} a_r(i,j) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

with

$$a_r(i,j) = -\frac{c_i c_j}{1+d_r^2}\left(\frac{1 - s_i s_j \mu_i \mu_j}{\mu_i + s_i s_j \mu_j} - s_i d_r\right).$$

Moreover, (A_r, b_r, c_r, d_r) as defined in (2) is Riccati balanced such that

$$M = \operatorname{diag}\left(\mu_1 I_{n(1)}, \dots, \mu_j I_{n(j)}, \dots, \mu_k I_{n(k)}\right)$$

is the unique positive definite solution to the GCARE and GFARE of (A_r, b_r, c_r, d_r).

The map which assigns to each system in $L_n^{1,1}$ the realization given in (2) is a canonical form.

Note that (A_r, b_r, c_r, d_r) as given in the corollary is sign symmetric with sign symmetry matrix

$$S = \operatorname{diag}\left(s_1 \hat{I}_{n(1)}, \dots, s_j \hat{I}_{n(j)}, \dots, s_k \hat{I}_{n(k)}\right)$$

i.e.

$$A^T = SAS, \qquad b = Sc^T,$$

and hence the Cauchy index of the system $g(s) = c_r(sI - A_r)^{-1} b_r + d_r$ is

$$C_{\text{ind}}(g(s)) = \operatorname{trace}(S).$$

5. MODEL REDUCTION

Balanced realizations as defined in Definition 2.1 were originally introduced to provide a simple method for model reduction (Moore, 1981). The basic idea is to consider a balanced n-dimensional system (A, B, C, D) and to partition it conformally as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \qquad \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

so that for $1 \leqslant N \leqslant n$ we have $A_{11} \in \mathbb{R}^{N \times N}$, $B_1 \in \mathbb{R}^{N \times m}$, $C_1 \in \mathbb{R}^{p \times N}$. The principal subsystem (A_{11}, B_1, C_1, D) is considered as an approximant of (A, B, C, D). It was shown in Pernebo and Silverman (1982) that this scheme has the important property that it preserves the minimality and asymptotic stability of the original system. This is restricted to the case where none of the retained singular values are identical to any of the neglected singular values.

This method of model reduction can also be applied to other types of realizations of systems. The following theorem summarizes results when this scheme is applied to systems given in one of the canonical forms of the previous sections.

THEOREM 5.1. *Let* $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$, *and let* $(\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$ *be the N-dimensional* $(1 \leqslant N \leqslant n)$ *principal subsystem of* (A, B, C, D).

(1) *If* $(A, B, C, D) \in C_n^{p,m}$ *is in the balanced canonical form of Theorem* 2.2, *then* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *is in* $C_N^{p,m}$ *and in the balanced canonical form of Theorem* 2.2. *The gramian of* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *is given by* $\Sigma_1 \in \mathbb{R}^{N\times N}$, *where* $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$ *is the gramian of* (A, B, C, D).

(2) *If* $(A, B, C, D) \in CI_n^{p,m}$ *is in the canonical form for the coinner systems in* $CI_n^{p,m}$ *of Theorem* 3.1, *then* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *is in* $CI_N^{p,m}$ *and in the canonical form of Theorem* 3.1. *The gramian of* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *is given by* $\Sigma_1 \in \mathbb{R}^{N\times N}$, *where* $\Sigma = \text{diag}(\Sigma_1, \Sigma_2)$ *is the gramian of* (A, B, C, D).

(3) *If* $(A, B, C, D) \in L_n^{p,m}$ *is in the normalized left coprime factor balanced canonical form of Theorem* 4.1, *then* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *is in* $L_N^{p,m}$ *and in the normalized left coprime factor canonical form of Theorem* 4.1.

(4) *If* $(A, B, C, D) \in L_n^{p,m}$ *is in Riccati balanced canonical form of Theorem* 4.2, *then* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *is in* $L_N^{p,m}$ *and in Riccati balanced canonical form of Theorem* 4.2. *The unique, symmetric, and positive definite solutions to the GCARE and the GFARE of* $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ *are given by* $M_1 \in \mathbb{R}^{N\times N}$, *where* $M = \text{diag}(M_1, M_2)$ *is a solution to the GCARE and GFARE of* (A, B, C, D).

Proof. The proof of the four statements is an immediate consequence of the parametrization results concerning the four different canonical forms and of the observation that the reduced order systems are parametrized in the canonical forms. ∎

REMARK 5.1. The previous theorem asserts that if a system is given in any of the canonical forms introduced in this paper, then each principal subsystem has the desired properties such as minimality and asymptotic stability in the case of balanced systems in $C_n^{p,m}$. Hence, for balanced systems which are in canonical form, the restrictive assumption that truncation has to be performed at places of nonrepeated singular values (Pernebo and Silverman, 1982) can be dropped.

REMARK 5.2. For the case of Riccati balanced systems which are not necessarily in the canonical form of Theorem 4.2, Jonckheere and Silverman (1983) proved a truncation result analogous to that of Theorem 5.1(4) for the case when the truncation is such that none of the retained "characteristic values" μ_j coincides with any of the neglected ones.

REMARK 5.3. Meyer (1988) introduced a model reduction scheme for transfer functions $G(s)$ in $TL_n^{p,m}$. The first step is to obtain a normalized left coprime factorization of $G(s) = \tilde{N}(s)\tilde{M}(s)^{-1}$. The system $[\tilde{N}(s), \tilde{M}(s)]$ is then reduced to a lower order system $[\tilde{N}(s)_r, \tilde{M}(s)_r]$. The transfer function $G(s)_r := \tilde{N}(s)_r\tilde{M}(s)_r^{-1}$ is taken to be the lower order approximant of the transfer function $G(s)$.

The results of Section 3 and 4 imply, however, that this reduction scheme in fact produces the same lower order approximants as the Riccati balancing technique. In the same way, the schemes of Theorems 5.1, part (3) and part (4), produce the same results.

REMARK 5.4. In Remark 4.1 a robust stabilization problem was discussed. The following result for a system (A, B, C, D) given in the normalized left coprime factor canonical form of Theorem 4.1 with parameters $\sigma_1 > \sigma_2 > \cdots \sigma_k$ is an immediate consequence of results by McFarlane, Glover, and Vidyasagar (1988).

Let $\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$ be the reduced order model obtained by truncating (A, B, C, D) according to Theorem 5.1(3). Let $\hat{K}(s)$ be a feedback controller designed to robustly stabilize $\hat{G}(s)$ with a corresponding maximum robustness margin $\epsilon_{\max} = \sqrt{1 - \sigma_1^2}$. Then, if

$$2 \sum_{K+1}^{k} \sigma_i < \sqrt{1 - \sigma_1^2},$$

the transfer function $G(s) = C(sI - A)^{-1}B + D$ of (A, B, C, D) is also stabilized by $\hat{K}(s)$. The robustness margin, $\epsilon'_{\max}$ of this closed loop system is now given by

$$\epsilon'_{\max} = \sqrt{1 - \sigma_1^2} - 2 \sum_{i=K+1}^{k} \sigma_i.$$

The authors would like to thank Dr. K. Glover for many helpful discussions. The first author is funded by SERC Grant GR|E|2424.8. The second *author would like to thank the Broken Hill Proprietary Co. Ltd for their support of this work.*

REFERENCES

Anderson, B. D. O. 1972. On the computation of the Cauchy index, *Quart. Appl. Math.*, pp. 577–582.

Brockett, R. W. 1976. Some geometric questions in the theory of linear systems, *IEEE Trans. Automat. Control* 21:449–454.

Doyle, J. 1984. Lecture Notes on ONR/Honeywell Workshop on Advanced in Multivariable Control, Minneapolis, 1984.

Glover, K. and McFarlane, D. 1988a. Robust stabilization of normalized coprime factors: An explicit H^{∞} solution, in *Proceedings 1988 American Control Conference*, Atlanta, Ga.

Glover, K. and McFarlane, D. 1988b. Robust Stabilization of Normalized Coprime Factor Plant Descriptions with H_{∞}-Bounded Uncertainty, to appear *IEEE Trans. Automat. Control.*

Jonckheere, E. and Silverman, L. M. 1983. A new set of invariants for linear systems —applications to reduced order compensator design, *IEEE Trans. Automat. Control* 28:953–964.

Kwakernaak, H. and Sivan, R. 1972. *Linear Optimal Control Systems*, Wiley.

McFarlane, D., Glover, K., Vidyasagar, M. 1988. Reduced Order Controller Design Using Coprime Factor Model Reduction, to appear *IEEE Trans. Automat. Control.*

Meyer, D. G. 1988. A fractional approach to model reduction, in *Proceedings 1988 American Control Conference*, Atlanta, Ga.

Moore, B. C. 1981. Principal component analysis in linear systems: Controllability, observability and model reduction, *IEEE Trans. Automat. Control* 26:17–32.

Nett, C., Jacobsen, C. A., and Balas, M. J. 1984. A connection between state space and doubly coprime fractional representations, *IEEE Trans. Automat. Control* 29:831–832.

Ober, R. 1987a. Balanced realizations: Canonical form, parametrization, model reduction, *Internat. J. Control* 46(2):643–670.

Ober, R. 1987b. Asymptotically stable all-pass transfer functions: Canonical form, parametrization and realization, in *Proceedings 1987 IFAC World Congress*, Munich.

Ober, R. 1987c. Topology of the set of asymptotically stable minimal systems, *Internat. J. Control* 46(1):263–280.

Pernebo, L. and Silverman, L. M. 1982. Model reduction via balanced state space representations, *IEEE Trans. Automat. Control* 27:282–287.

Vidyasagar, M. 1985. *Control System Synthesis: A Coprime Factorization Approach*, MIT Press.

Vidyasagar, M. 1988. Normalized coprime factorizations for non strictly proper systems, *IEEE Trans. Automat. Control* 33:300–301.

Received 29 May 1988; final manuscript accepted 30 August 1988