Notes and Sources

Completing Diophantus, *De polygonis numeris*, prop. 5

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Abstract

The last proposition of Diophantus’ *De polygonis numeris*, inquiring the number of ways that a number can be polygonal and apparently aiming at “simplifying” the definitory relation established by Diophantus himself, is incomplete. Past completions of this proposition are reported in detail and discussed, and a new route to a “simplified” relation is proposed, simpler, more transparent and more “Greek looking” than the others. The issue of the application of such a simplified relation to solving the problem set out by Diophantus is also discussed in full detail.

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1. Introduction and position of the problem

The Greek mathematician Diophantus (maybe III century of our era) is well-known for his thirteen-book treatise *Arithmetica*, containing numerical problems solved by a variety of techniques. Much less known is the fact that Diophantus wrote also a short tract devoted to
polygonal numbers, that I shall designate henceforth by its traditional Latin title *De polygonis numeris*; in its actual form it contains only five propositions. The first edition of the Greek text of Diophantus’ treatises was published, along with a Latin translation and an extensive commentary, by Claude Gaspard Bachet de Méziriac in 1621; the first critical edition, and to date the only one, was achieved by Paul Tannery [1893–95].

In his *De polygonis numeris*, Diophantus succeeded in shaping a sound mathematical theory, the only one that has come to us from Greek antiquity, about objects that were part of arithmetical folklore since early times (cf. Aristotle, *Physica* III.4, 203a10–15). A treatise on polygonal numbers is ascribed by the late Byzantine lexicon *Suda* (Φ 418) to Philip of Opus, a pupil of Plato, but we know nothing of its contents. Diophantus himself states that Hypsicles (II century before our era) gave a sound mathematical definition of polygonal numbers: they can be obtained as partial sums of suitable arithmetical progressions. The II century polymath Nicomachus, *Introductio arithmetica* II.6–12, offers a long description, without a supporting demonstrative apparatus, of the construction and of the main properties of the most simple species of polygonal numbers.

Any number, if represented by a set of identical objects, can be given a “natural” geometrical shape: there are triangular, square, pentagonal, hexagonal, . . . numbers. An obvious example are the “square” numbers, whose representative sets can most naturally be arranged as squares. “Polygonal numbers” is simply the generic name for the collection of these numerical species—as a matter of fact, the set of the polygonal numbers coincides with the set of positive integers, number 2 excluded (ancient authors included the unit for reasons that are of no interest here). A sound mathematical theory of polygonal numbers can be developed once it is realized, as Hypsicles did, that all polygonal numbers of a given species can be obtained as partial sums of a suitable arithmetical progression, the parameter distinguishing among the species being the ratio of the progression. Nicomachus has it in the following terms:

(1) Each species is generated by successively adding numerical “layers”, traditionally called «gnomons», to a unit; in the case of triangular numbers the successive gnomons coincide with the integers, in the case of square numbers the gnomons are obtained from the integers by picking up every second term (these are the odd numbers), in the case of pentagonal numbers they are obtained from the integers by picking up every third term, etc. Nicomachus formulated the general rule in this way: «the difference of the gnomons of each polygonal is less by a dyad than the quantity of angles indicated in its name» (*Ar*. II.11.4).

(2) The side of any particular polygonal number of any species is greater by a unit than the number of gnomons involved in its construction (*Ar*. II.8.3).

Making a further leap into abstraction with respect to Hypsicles’ definition, the first four propositions and a subtle additional argument of Diophantus’ *De polygonis numeris* prove the validity of the following definitive relation of each species of polygonal numbers: «Every polygonal, multiplied by the octuple of the number less by a dyad than the multiplicity of the angles, and taking in addition the square on the number less by a tetrad than the multiplicity of the angles, makes a square» [Tannery, 1893–95, 450–72]. In symbolic language, a polygonal number $P$ with $v$ angles satisfies the relation

$$8P(v - 2) + (v - 4)^2 = \text{square}.$$
Prop. 4 makes it explicit what is the side of the *square*; its expression contains the side $l$ of the polygonal number $P$: the full-fledged relation is

$$8P(v - 2) + (v - 4)^2 = (2 + (v - 2)(2l - 1))^2,$$

amounting to a definition of a *specific* polygonal number, that is, to an identification of any number greater than 2 as a specific polygonal one.

This relation has a natural output, namely, a polygonal number $P$, and two inputs, the number of its angles $v$ and its side $l$. Diophantus immediately applies it to set up two procedures, the one the inverse of the other. They describe how to find, once the “name” or “species” $v$ of a polygonal $P$ is assigned, the number $P$ with given side $l$ and vice versa (in the inverse procedure it is of course assumed that $P$ is polygonal). Both procedures, that are formulated in natural language and without resorting to denotative letters, always produce an output, and the output is univocally determined. These facts are warranted, and in some sense licensed, by a preliminary “analysis” formulated in the “language of the givens” [Tannery, 1893–95, 472–6].

Of course, another use of the definitory relation is possible: given a number $P$, to find the values of $v$ and $l$, if any, that satisfy the above definition—using the words of the enunciation of Diophantus’ prop. 5: «Given a number, to find in how many ways it can be polygonal». It must be stressed that this problem always has a solution: *every* number starting from 3 is polygonal with $l = 2$ and $v$ equal to the number itself. On the other hand, the problem can have non-trivial solutions: for instance, 6 is of course hexagonal with side 2, but also triangular with side 3. Now, the definitory relation $8P(v - 2) + (v - 4)^2 = (2 + (v - 2)(2l - 1))^2$ is quite unsuited to explicating $v$ once $P$ and $l$ are given. It is then reasonable to assume that prop. 5 proceeded first to “simplify” this relation, solving then the proposed problem. I say «to assume» because prop. 5 breaks off, after a good deal of manipulations that look very much like aiming at a simplification, in the middle of a passage: the text of the *De polygonis numeris* is incomplete [Tannery, 1893–95, 476–80].

In his critical edition of Diophantus’ works, Tannery deemed prop. 5 a worthless attempt of a commentator [«commentatoris vanum tentamen» at 1893–95, 477 in app.]. He thereby followed a dismissive attitude that can be traced back at least to Bachet, the first editor of the Greek Diophantus. Bachet, however, solved the problem independently by introducing a number of simplified relations that he did not try to connect with prop. 5. Let us read his 4. canon, proved in prop. 6 of the first book of the Appendix with which he supplemented his edition of the Greek text: «Sume triangulum à latere dato unitate multato, quem ducito in numerum angulorum binario multatum, producto adde datum latus, fiet quonammodo» [Bachet, 1621, 30]. This reads «Take a triangle with given side minus one, multiply it by the number of angles minus two, add the given side to the product: it becomes the sought for polygonal». The associated relation can be written $P = \frac{1}{2}l(l - 1)(v - 2) + l$; deducing it from Diophantus’ definitory relation is a matter of a few elementary steps if we are allowed to use algebraic techniques. What is more, Bachet set forth other rules equivalent to the one just given; they are formulated in the *canones* 1. to 3., proved in his props. 3 to 5, and in symbols read, respectively: $2P = ((l - 1)(v - 2) + 2)l$, $2P = l(l - 1)(v - 2) + 2l$, $2P = l^2(v - 2) - l(v - 4)$.

Bachet’s arguments in props. 1 and 2, from which the others are immediate consequences, show that *directly* proving any of these relations was within easy reach for Diophantus. In fact, the crucial steps in Bachet’s proofs—namely, (1) how to separate the contributions of the first term and of the accumulating ratios in the generic term of an arithmetic progression and (2) that each polygonal number is obtained as the sum of
a suitable arithmetic progression—are already demonstrated in *De polygonis numeris*, prop. 2 and the argument following prop. 4. One might wonder why, if Diophantus really obtained any of the simplified relations in the missing portion of his prop. 5, he did not directly prove and use it as a definitory relation. Bachet himself must have wondered why, since he claims that his *canones* are more elegant, easier, and shorter than those handed down by Diophantus [*elegantes, & quidem faciiores & compendiosiores eo quem tradit Diophantus*] at 1621, 30].

The answer is that no general definition of a polygonal number *as a numerical species* can make explicit reference to the “value” of its side, very much as happens in geometry, where a pentagon is defined as a figure with five angles (*pente gôniai* in Greek). Accordingly, Diophantus formulates his definitory relation without saying explicitly which is the side of the *square*, even if the proof of prop. 4 determines it univocally. None of the above, simplified, relations allows one to do that. This explains also the contrived deductive structure of the *De polygonis numeris*, where a “simple” relation, thought easily derivable, is obtained only as a simplification of a more complex one, taken to be primary with respect to the other. What is more, by choosing as his main definition the one that generically refers to a *square*, Diophantus exposes himself to the following objection. Take for instance $P = 2$ or 4; they respectively satisfy the relations $24P + 1 =\ square$ and $40P + 9 =\ square$, but it is obviously not the case that 2 is pentagonal or 4 is heptagonal [cf. Bachet, 1621, 21]. The condition $8P(v - 2) + (v - 4)^2 = \ square$ is sufficient but not necessary to the identification of the single species of polygonals, whereas of course the condition in which the expression of the *square* is made explicit is also necessary. In order to rescue Diophantus, it is enough to note that no ancient authour would have admitted polygonal numbers less than the number of their vertices, and all counterexamples that can be adduced are of this kind. It is easy to see that all the aberrant polygonals are such as to have either an integer, but negative, side: $l = -1$ in both cases, or a non-integer one: $l = 4/3$ or $l = 8/5$. In a modern perspective, the presence of these “spurious solutions” is quite obviously linked with the fact that the definitory relation is not the simplest possible one.

The problems I want to address in this note are the following:

1. Whether any of Bachet’s simplified relations can be obtained in a straightforward way by completing prop. 5.
2. Whether any of Bachet’s relations can be used to solve the problem of determining in how many ways a given number can be polygonal, namely, the problem posed by prop. 5.

Of course, these questions have already been given answers, in particular by Wertheim [1897] and Heath [1910, 256]: they completed or rewrote prop. 5 in order to arrive at relation $2P = l(2 + (l - 1)(v - 2))$. However, I shall explain why one should regard these answers as unsatisfactory, and shall submit ones that I think are better suited to the techniques displayed by Diophantus in his works. In addition to this, I shall add some historical flesh to my reconstruction, a move that, apparently, no interpreter regarded as worth doing.

2. Completing *De polygonis numeris* 5

That a “simplified” relation already circulated in antiquity is confirmed by a Latin agrimensural text of uncertain autorship and date (but it cannot be much later than the III century of our era), the so-called «excerpts from Epaphroditus and Vitruvius Rufus». 
It gives rules for calculating the area of a regular polygon of assigned side, from the pentagon up to the dodecagon included [Bubnov, 1899, 534–45]. These figures are in fact treated as polygonal numbers; two algorithms are described for finding the “area” of the polygon given its side and vice versa; the relation employed to this end is a simplified one: \( P = \frac{(v - 2)^2 - (v - 4)}{2} \). Let us read the procedure and the related algorithm in the case of the enneagon (ibid., 540; the aeram of the text is not a typo for aream: it designates the assigned number; the translation of the procedure is as ungrammatical as the Latin original is):


Every enneagon having equal sides, whose one side I multiply by itself and again I add seven times, I subtract five times the number itself, I take one half, I declare the enneagon.

If there is an enneagon each of whose sides is of 10 feet, I seek how many feet collects the area. Seek this way. I multiply one side by itself: it becomes 100. I add seven times: it becomes 700. Therefrom I subtract five times the number itself: it becomes 50. I take half of the remainder 650: it becomes 325. Of so many feet is the area of this enneagon.

As we shall see at the very end of this note, Bachet drew the two leitmotive of book I of his Appendix exactly from these excerpts. I now pass to the first problem stated in the introduction.

I first transcribe in symbolic notation what remains of prop. 5 (see Fig. 1), then the two reconstructions of Wertheim and Heath, explaining why I regard them as unsatisfactory, and finally propose my own reconstruction. I use the sign \( q(AB) \) for «the square on AB» and \( r(AB, B\Gamma) \) for «the rectangle contained by AB and B\Gamma», that is, the product of the numbers AB and B\Gamma; the references to the Elements are my additions.

(a) Here is the transcription of what remains of De polygonis numeris 5:

Set \( P = AB, B\Gamma = v \), and in \( B\Gamma \) set \( \Gamma\Delta = 2 \) and \( \Gamma E = 4 \). One starts with the definitory relation

\[
q(ZH) = 8r(AB, B\Delta) + q(BE),
\]

where ZH is not further specified. Set \( A\Theta = 1 \) and split up

\[
8r(AB, B\Delta) = 4r(A\Theta, B\Delta) + 4r(AB + B\Theta, B\Delta).
\]
Setting

\[ 4(AB + B\Theta) = \Delta K \]

one obtains

\[ 4r(AB + B\Theta, B\Delta) = r(K\Delta, \Delta B) \]

and

\[ 4r(A\Theta, B\Delta) = 2r(B\Delta, \Delta E) \]

for E\Delta is a dyad.

Therefore,

\[ q(ZH) = r(K\Delta, \Delta B) + 2r(B\Delta, \Delta E) + q(BE). \]

But, by El. II.7,

\[ 2r(B\Delta, \Delta E) + q(BE) = q(B\Delta) + q(\Delta E). \]

Hence

\[ q(ZH) = r(K\Delta, \Delta B) + q(B\Delta) + q(\Delta E). \]

But also, by El. II.1,

\[ r(K\Delta, \Delta B) + q(B\Delta) = r(KB, B\Delta), \]

and therefore

\[ q(ZH) = r(KB, B\Delta) + q(\Delta E). \]

Now, since \( \Delta K \), that is equal to \( 4(AB + B\Theta) \), is greater than \( 4A\Theta \), that is, greater than 4, and \( \Delta \Gamma \) is a dyad, \( \Gamma K \) as a remainder is greater than a dyad \( \Gamma \Delta \): therefore, the middle point of \( \Delta K \) will fall between \( \Gamma \) and \( K \); let it be \( \Lambda \). It follows that we can apply El. II.6:

\[ r(KB, B\Delta) = q(B\Delta) - q(\Lambda \Delta), \]

or

\[ r(KB, B\Delta) + q(\Lambda \Delta) = q(\Lambda B), \]

or

\[ q(\Lambda B) - q(\Lambda \Delta) = r(KB, B\Delta). \]

Therefore

\[ q(ZH) = q(B\Delta) - q(\Lambda \Delta) + q(\Delta E). \]

Summing \( q(\Lambda \Delta) \) on both sides one gets

\[ q(ZH) + q(\Lambda \Delta) = q(B\Delta) + q(\Delta E), \]

or

\[ q(\Lambda \Delta) - q(\Delta E) = q(\Lambda B) - q(ZH). \]

Now, since E\Delta = \( \Delta \Gamma \), we can again apply El. II.6:

\[ r(EB, \Lambda \Gamma) + q(\Gamma \Delta) = q(\Lambda \Delta). \]

Therefore,

\[ q(\Lambda \Delta) - q(\Delta \Gamma) = q(\Lambda \Delta) - q(\Delta E) = r(EB, \Lambda \Gamma) = q(\Lambda B) - q(ZH). \]
Set $ZM = B\Lambda$—for $B\Lambda$ is greater than $ZH$, since it was proved that $q(ZH) + q(\Delta\Lambda) = q(B\Lambda) + q(\Delta E)$ and $q(\Delta\Lambda)$ is grater than $q(\Delta E)$ because it is also greater than $q(\Delta\Gamma)$. Therefore

$$(8) \quad q(ZM) - q(ZH) = r(E\Lambda, \Lambda\Gamma).$$

Now, since $\Delta K = 4(AB + B\Theta)$ and $\Delta K$ is bisected at $\Lambda, \Delta\Lambda = 2(AB + B\Theta)$, of which $\Delta\Gamma = 2A\Theta$; therefore, $\Gamma\Lambda = 4\Theta B$ and so, because $E\Gamma = 4 = 4A\Theta$, one has $E\Lambda = 4AB$. As a consequence,

$$r(E\Lambda, \Lambda\Gamma) = 16r(AB, B\Theta).$$

Therefore, by transitivity with (8) and El. II.4,

$$(9) \quad 16r(AB, B\Theta) = q(MZ) - q(ZH) = q(HM) + 2r(ZH, HM),$$

so that $HM$ is even. Let it be bisected at $N$. The text ends abruptly at this point.

One must stress that Diophantus, before the interruption, does not set out explicitly the expression of the side $ZH$ of the square; in particular, he does not introduce the side of the given polygon. This makes all completions of prop. 5 arbitrary to a high degree.

(b) What follows is Wertheim’s reconstruction (1897), which we find also transcribed in Heath [1910, 257–8]. It uses the same denotative letters as Diophantus’. I do not report the symbolic transcription that both Wertheim and Heath present in a facing column parallel to the text, but I shall briefly comment on it later—to give the reader an idea of the difficulties involved in this transcription, here is what is facing step (9) above:


The first step of Wertheim’s reconstruction directly links to the last one of the surviving text (see Fig. 2).

Therefore,

$$4r(ZH, HN) + 4q(HN) = 16r(AB, B\Theta),$$

that is, dividing out by 4,

$$(10) \quad r(ZH, HN) + q(HN) = 4r(AB, B\Theta)$$

and hence, by El. II.1,

$$(11) \quad r(ZN, HN) = 4r(AB, B\Theta).$$

Setting $Z\Sigma = 2AB$ and $P\Sigma = HN$, which entail $H\Sigma = PN$, one obtains

$$Z\Sigma = ZP - P\Sigma = 2AB - P\Sigma,$$
$$ZN = ZP + PN = 2AB + PN,$$
$$HN = P\Sigma = 2AB - Z\Sigma,$$
and equality (11) becomes
\[(12) \quad (2AB + PN)(2AB - Z\Sigma) = 4r(AB, B\Theta),\]
or
\[(13) \quad 4q(AB) - 2r(AB, Z\Sigma - PN) - r(PN, Z\Sigma) = 4q(AB) - 4r(AB, A\Theta),\]
and therefore
\[(14) \quad 2r(AB, Z\Sigma - PN) + r(PN, Z\Sigma) = 4r(AB, A\Theta),\]
or
\[(15) \quad 2r(AB, 2A\Theta + PN - Z\Sigma) = r(PN, Z\Sigma).\]
But
\[
PN = ZN - ZP = ZM - NM - ZP = ZM - \frac{1}{2}HM - ZP
= BA - \frac{1}{2}HM - 2AB = BA + \frac{1}{2}AK - \frac{1}{2}HM - 2AB
= BA + 2AB + 2B\Theta - \frac{1}{2}HM - 2AB = BA + 2B\Theta - \frac{1}{2}HM
\]
and
\[
Z\Sigma = ZP - P\Sigma = 2AB - \frac{1}{2}HM;
\]
as a consequence
\[
PN - Z\Sigma = BA + 2B\Theta - 2AB = BA - 2A\Theta
\]
\[
PN - Z\Sigma + 2A\Theta = BA.
\]
One then gets
\[
PN = BA + 2B\Theta - \frac{1}{2}HM = BA + 2B\Theta - \frac{1}{2}BA + \frac{1}{2}ZH
= BA + 2B\Theta - \frac{1}{2}BA - \frac{1}{2}A\Theta + \frac{1}{2}ZH = \frac{1}{2}BA + 2B\Theta - \frac{1}{2}A\Theta + \frac{1}{2}ZH
= \frac{1}{2}BA + 2B\Theta - (AB + B\Theta) + \frac{1}{2}ZH = \frac{1}{2}BA + B\Theta - AB + \frac{1}{2}ZH
= \frac{1}{2}BA - A\Theta + \frac{1}{2}ZH = \frac{1}{2}(BA + ZH - 2A\Theta).
\]
But
\[
ZH = (2l - 1)BA + 2,
\]
or
\[
ZH + BA = 2/BA + 2,
\]
or
\[
ZH + BA - 2A\Theta = 2/BA,
\]
that is,
\[
PN = lBA.
\]
Equality (15) thus becomes
\[(16) \quad 2r(AB, BA) = lr(BA, Z\Sigma),\]
or, by eliminating the common height BA,
\[(17) \quad 2AB = lZ\Sigma,\]
Since AB = P and $Z \Sigma = 2 + (I - 1)(v - 2)$ one finally has

$$2P = l(2 + (I - 1)(v - 2)).$$

The drawbacks of this reconstruction are the excessive length of the intermediate transitions between equalities (15) and (16) and the ad hoc introduction of the quantity $Z \Sigma$. These difficulties are highlighted by the fact that, in the facing algebraic transcription, the almost-final equality $2P(v - 2) = l(v - 2)(2 + (I - 1)(v - 2))$ is already obtained at step 15. What follows, in fact half of the entire deduction, has the sole function of making the common height $BD = v/C_0$ appear and of “cleaning up” the deduction from the complex partitions of ZM Wertheim introduced at the beginning of it. These facts show that Wertheim’s approach is algebraically-driven, and his cumbersome restoration is an unsuccessful attempt at reshaping it in geometric terms.

(c) Heath [1910, 256] (see Fig. 3) proposed a far simpler reduction, but he did not try to complete the surviving fragment of prop. 5:

Let $FG = 2 + (2I - 1)(v - 2)$. Cut off $FR = 2$, and produce $RF$ to $S$ so that $RS = v/C_0$. We have now

$$8P.SR = q(FG) - q(SF) = q(SG - SF) - q(SF) = q(SG) - 2r(SG, SF).$$

Bisect $SG$ at $T$, and divide out by 4; therefore

$$2P.SR = q(ST) - r(ST, SF) = r(ST, ST - SF) = r(ST, FT) = r(ST, FR + RT)$$

Now, $ST = lSR$, and $FR = 2$, while $RT = (I - 1)SR = (I - 1)(v - 2)$. It follows that

$$2P = l(2 + (I - 1)(v - 2)).$$

This derivation is in fact nothing but a dressing in denotative letters of quite a straightforward algebraic reduction: it simply assumes what is set out to prove. Of course, Heath was unable to see that his derivation is in fact a circular one; the words with which he introduces it are quite typical of his way of approaching ancient texts: «The only thing, so far as I can see, tending to raise doubt as to the correctness of [Wertheim’s] restoration is the fact that [...] it can be done much more easily than it is in Diophantus’ proposition as extended by Wertheim» (ibid.).

(d) My completion of prop. 5 produces any of Bachet’s relations without resorting to algebraic manipulations, is simpler and more transparent than Wertheim’s and is, in my opinion, more “Greek looking” (see Fig. 4):

Therefore,

$$4r(ZH, HN) + 4q(HN) = 16r(AB, B \Theta),$$

or, by El. II.1,

$$4r(ZN, HN) = 16r(AB, B \Theta),$$

that is

$$r(2ZN, 2HN) = r(4AB, 4B \Theta).$$

Recalling that $ZM = B \Delta$, one has

$$ZM = B \Delta = B \Delta + \Delta \Gamma + \Gamma \Lambda.$$
Furthermore, since $\Delta \Gamma$ is a dyad,

$$ZH = \Delta \Gamma + r((B\Omega + \Phi\Omega), B\Delta) = \Delta \Gamma + 2B\Psi - B\Delta = \Delta \Gamma + 2\Delta \Psi + B\Delta,$$

where the definitory relation of a polygonal number has been used and $B\Psi = r(B\Omega, B\Delta) = B\Delta + \Delta \Psi = B\Delta + r(\Phi\Omega, B\Delta)$ has been set; $B\Omega$ is the side of the polygonal number $AB$ and a unit $B\Phi$ has been cut off from $B\Omega$ (a similar position involving the same numbers, followed by the introduction of a ‘solid’ number as I shall do below, is made in prop. 4). Therefore, recalling that also $E\Delta$ is a dyad,

$$2ZN = ZM + ZH = E\Lambda + 2B\Psi,$$

$$2HN = ZM - ZH = \Gamma\Lambda - 2\Delta \Psi,$$

and equality (11) becomes, recalling again that the definition of $\Delta K$ and its being bisected at $\Lambda$ entail $4AB = E\Lambda$ and $4B\Theta = \Gamma\Lambda$,

$$(12) \quad r((E\Lambda + 2B\Psi), (\Gamma\Lambda - 2\Delta \Psi)) = r(E\Lambda, \Gamma\Lambda).$$

This is an equality between two rectangles having the rectangular domain of sides $E\Lambda$ and $\Gamma\Lambda - 2\Delta \Psi$ in common. By subtracting it from both sides one gets

$$(13) \quad 2r(E\Lambda, \Delta \Psi) + 4r(B\Psi, \Delta \Psi) = 2r(\Gamma\Lambda, B\Psi),$$

or, setting $\Gamma\Lambda = E\Lambda - E\Gamma$ and dividing out by 2,

$$(14) \quad r(E\Lambda, \Delta \Psi) + 2r(\Delta \Psi, B\Psi) = r(E\Lambda, B\Psi) - r(E\Gamma, B\Psi).$$

Recalling that $B\Psi = B\Delta + \Delta \Psi$ and operating by partial «restoration» and «reduction»,

$$(15) \quad r(E\Lambda, \Delta \Gamma) = 2r(\Delta \Psi, B\Psi) + r(E\Gamma, B\Psi),$$

that is, since $E\Gamma$ is a tetrad and $\Delta E$ is a dyad,

$$r(E\Lambda, B\Delta) = 2r(\Delta \Psi, B\Psi) + 2r(\Delta E, B\Psi) = 2r(E\Psi, B\Psi).$$

Bisecting $\Gamma\Lambda$ at $X$ and since $\Delta$ bisects $E\Gamma$ one has

$$2r(\Delta X, B\Delta) = 2r(E\Psi, B\Psi),$$

that is, dividing out by 2,

$$(16) \quad r(\Delta X, B\Delta) = r(E\Psi, B\Psi).$$

But if two rectangles are equal, their sides are reciprocally proportional ($El. \ VI.14$):

$$(17) \quad \Delta X : E\Psi :: B\Psi : B\Delta.$$
or, since $B\Phi$ is a unit and $\Delta E$ a dyad,

$$(18bis)\quad \Delta X = r(B\Omega, \sum\Psi) = r(B\Omega, B\Psi) - r(B\Omega, B\Delta) + r(B\Omega, \Delta E)$$

$$= s(B\Omega, B\Omega, B\Delta) - s(B\Omega, B\Phi, B\Delta) + 2r(B\Omega, B\Phi) = s(B\Omega, \Phi\Omega, B\Delta) + 2B\Omega.$$

Setting $\Delta X = 2AB = 2P$, $B\Delta = v - 2$, $\Delta E = 2$ and $BE = v - 4$, $B\Omega = l$ (and hence $\Phi\Omega = l - 1$) one can rewrite these equalities as

$$2P = l^2(v - 2) - l(v - 4),$$

or

$$2P = l(l - 1)(v - 2) + 2l,$$

from which

$$P = \frac{1}{2}l(l - 1)(v - 2) + l.$$

Of course, Diophantus would have formulated the last three lines in display in natural language. Furthermore, he would have had no qualms in proposing equalities between non-homogeneous terms, as this is already done in *De polygonis numeris*, prop. 4.

Let us now come to the second of the problems listed in the introduction. One would like to use any of the above relations to find in how many ways a number can be polygonal. This is equivalent to finding, once a polygonal $P$ is assigned, the number of its angles corresponding to a given side $l$ or vice versa. The first task was taken up by Bachet [1621, 38, and see also the conditions of possibility discussed *ibid.*, 39]:

Exponantur ab unitate omnes in infinitum ordinatim numeri, puta 1.2.3.4.5.6.7.8, & illis subiiciantur ab unitate triangulares omnes ordinate dispositi, puta 1.3.6.10.15.21.28, ut factum vides in apposita tabella, quae potest in infinitum extendi. Tum propositus numerus 120, dividatur sigillatim per numeros triangulos, & observetur, quoties residuum ex divisione aequale erit lateri proximo maioris trianguli, toties enim numerus 120, polygonus erit, cuius latus erit ipsum residuum divisionis. At quotiens ostendet differentiam progressionis huiusmodi polygonorum constitutivae, seu quod idem est, idem quotiens binario auctus, numerum angulorum indicabit.

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Set out, beginning from a unit, all numbers in order as far as infinity, say 1 2 3 4 5 6 7 8, and let all triangular numbers beginning from a unit be placed in order below them, say 1 3 6 10 15 21 28, as you can see done in the table set out below, that can be extended to infinity. Let then the assigned number 120 be divided in succession by each of the triangular numbers, and let it be remarked that the number 120 will be polygonal exactly when the remainder of the division is equal to the side of the immediately greater triangle, the side of the polygonal being the remainder itself of the division. In its turn, the quotient will show the difference of the constitutive progression of the polygonals, or, what amounts to the same, the same quotient plus two will indicate the number of angles.

In our language, dividing a polygonal $P$ by the triangular number $\frac{1}{2}l(l - 1)$ of side and order $l - 1$, one obtains the side of the triangular number of order $l$ as a remainder and a number
less by 2 than the number of angles of $P$ as a quotient: this amounts to find $v$ at fixed $P$ once $l$ is given by using relation $P = \frac{1}{2}l(l - 1)(v - 2) + l$. The procedure does not give an output in every instance, as is natural: no number is polygonal with any number of angles even though any number is polygonal with some number of angles. Bachet formulates this state of affairs with the correlative quoties... toties... by requiring that the division of the assigned number $P$ by a triangular number give as a remainder the side of the subsequent triangular number. In Greek mathematical jargon, this is a true diorismos, as it indicates in general terms the number or solutions of the problem (one or zero in this trivial instance) in function of the value of the “givens” of the problem itself. Both the “validation” of the procedure and the formulation of the diorismos can easily be formulated in Diophantine language, following the model, still contained in the De polygonis numeris as we have seen at the beginning of this note, of the procedure for finding, once the species $v$ of the polygonal is assigned, the number $P$ with given side $l$ and vice versa. Bachet’s procedure is rewritten in symbols, and adapted to his own simplified relation $2P = l(2 + (l - 1)(v - 2))$, by Wertheim [1890, 314–5], who adds 325 as an example to Bachet’s 120. The conditions of possibility are in this case that $l$ must divide $2P$ exactly and that $(2P/l - 2)/l - 1$ must be an integer. As usual, Heath [1910, 259] makes up his exposition by a verbatim plagiarism of Wertheim, without mentioning any of his predecessors.

In this way the problem posed in prop. 5 is solved. Still, one would like to see Diophantus setting up a direct-and-inverse procedure like the one he makes to follow to the definitory way. But let us take instead relation $2P = (v - 2)^2 - (v - 4)l$, and write it, after an operation of «restoration», $2P + (v - 4)l = (v - 2)^2$. This is an instance of the case in which «two species are left out equal to one», whose treatment Diophantus announces in the introduction of his Arithmetica [Tannery, 1893–95, 14.23–4]. Now, even if the extant Arithmetica, either in Greek or in Arabic, contains nothing systematic of this sort, in propositions such as Ar. IV.31 and 39 a procedure is explicitly described for solving this kind of equalities, and conditions are spelled out under which they can be «expressible» (that is, they have rational solutions). It is immediate to check that the above equality always has two solutions, one of which is positive. The diorismos to be required is that the expression corresponding to our discriminant be a square, and that the final division by $v - 2$ gives an integer. As is to be expected, then, also in this case the procedure may break down before producing an output. These considerations suggest that Diophantus probably derived more than one simplified relation in his prop. 5, and that the stylistic register of the De polygonis numeris, if it included techniques typical of the Arithmetica, was more mixed than we are entitled to conclude from the surviving portion.

A side problem is how to set a priori limitations on the sides or angles admissible for trial in the direct or inverse procedure solving the problem posed by prop. 5 [cf. Bachet, 1621, 39; Wertheim, 1890, 314–5; Heath, 1910, 259]. In general, it must be $2 \leq l < P$ and $3 < v \leq P$, but we can have better. Of course, as any number is polygonal with side 2 and number of angles equal to itself, the two non-strict inequalities cannot be improved. On the other hand, the side of a polygonal $P$ cannot be greater than the side of the first triangular number equal to or greater than $P$, that is, if $m$ is the side of such a triangle, $2P \leq m(m + 1)$, where $2P = m(m + 1)$ can still be solved in $m$ at given $P$ (cf. Ar. VI.6). It results that $l \leq [\sqrt{(1 + 8P)} - 1]/2$. A simpler inequality is obtained by applying the same argument to the square equal to or immediately following $P$: it results $l \leq \sqrt{2P}$. The lower bound on the number of angles cannot be improved.
Two final remarks. First, a collection of geometrical problems on polygons, published by Heiberg in his Mathematici Graeci Minores [1927, 25–65], is ascribed in the manuscript tradition to some «Diophanes», but a second hand in the principal manuscript corrected the ascription to «Diophantus» (Const. pal. vet. 1, f. 17v). The procedures solving these problems are different from those attested in the Epaphroditus and Vitruvius Rufus excerpts, yet the names of the polygons are all masculine and not neuter: they are polygonal numbers, not geometrical figures. Second, these same excerpts give, besides the procedures described above, also a simple rule for summing all polygonal numbers of species \(v\), the unit included, up to \(P\): the result is \((2P + v)(v + 1)/6\), a rule that Bachet knew and that he expressly declares to draw from a book by Epaphroditus and Vitruvius Rufus [«ex libro Apafroditi & Betrubi Rufi Architectonis» at 1621, 37–8]. Since the excerpts already transmit a valuable piece of information concerning the De polygonis numeris, I surmise that a derivation of this relation might have been contained in the Diophantine treatise, as an analogical extension of the exposition ending with prop. 5: since polygonal numbers are obtained as sums of terms having a constant first difference (= an arithmetic progression), in the same way it might be of interest to see what happens in summing terms having a constant second difference (= a progression of polygonal numbers of the same species).

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References