Theoretical Computer Science 126 (1994) 31–52
Elsevier

Decidability of reachability for disjoint union of term rewriting systems*

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Abstract

Caron, A.-C. and J.-L. Coquide, Decidability of reachability for disjoint union of term rewriting systems, Theoretical Computer Science 126 (1994) 31–52.

The reachability problem for term rewriting systems is the problem of deciding for a system S and two terms t and t', whether t can be reduced to t' with rules of S. On the one hand, we study the disjoint union of term rewriting systems the reachability problem of which is decidable and give sufficient conditions for obtaining the modularity of decidability of this problem. On the other hand, we study composition of constructor systems. This notion of composition does not imply disjointness.

1. Introduction

If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two disjoint finite alphabets, and if $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are two term rewriting systems on $\mathscr{F}_1$ and $\mathscr{F}_2$, the disjoint union $\mathscr{R}_1 \oplus \mathscr{R}_2$ of $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ is the term rewriting system (TRS) $(\mathscr{F}_1 \cup \mathscr{F}_2, \mathscr{R}_1 \cup \mathscr{R}_2)$. A property of a TRS is modular if it is preserved under disjoint union.

In programming, modularity permits a problem to be decomposed into simpler problems that are easier to solve. Several methods are known for inferring

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* This work was supported in part by PRC-GDR Mathématiques et Informatique and by ESPRIT2-working group ASMICS.

SSDI 0304-3975(93)E0122-K

properties of TRS (such as confluence or termination) but they have the greatest chance of succeeding if the systems concerned have few rewrite rules. Therefore, it is interesting to have results at our disposal which state that a TRS has a certain property P if it can be partitioned into smaller TRSs which all have the property P.

Starting with Toyama, several authors have studied modular properties. Toyama [20] showed that confluence is preserved under disjoint union (proof simplified in [8]). Toyama [19] found a counterexample for modularity of termination (also called strong normalization). Gramlich [6] studied more precisely minimal counterexamples of termination. Rusinowitch [18] and Middeldorp [13] give sufficient conditions for the termination of the disjoint union of terminating TRSs. Kurihara and Ohuchi [9] showed that termination is a modular property of TRSs whose termination can be shown by a simplification ordering. More recently, they proved that termination is a modular property of noncopying TRSs, which are restricted versions of graph rewriting systems [11]. Barendregt and Klop in [19] and Drosten [4] gave counterexamples of modularity of completeness (confluence plus termination). Toyama et al. [21] showed that the restriction to left-linear TRSs is sufficient for obtaining the modularity of completeness. Middeldorp [12] showed that the property of having unique normal forms is modular for general TRSs. He gives a survey of modular aspects of TRSs in [14] and studied more particularly conditional TRSs [15].

In this paper, we study the decidability of reachability. The reachability problem for TRSs is the problem of deciding for a system S and two terms t and t', whether t can be reduced in t' with rules of S. Reachability is generally undecidable. However, it is decidable in particular for quasi-ground rewrite systems [17] or monadic systems (see [1, 5] for a survey of monadic systems). First, we prove that decidability of reachability is not preserved under disjoint union if the two systems are not left-linear. In the second section, we study left-linear systems with or without collapsing rules: a rewrite rule $l \rightarrow r$ is collapsing if r is a variable. We prove that decidability of reachability is modular for left-linear TRSs without collapsing rules, or for linear TRSs with possibly collapsing rules but reachability becomes undecidable for left-linear TRSs with collapsing rules. In the third section, we study ground reachability, i.e. reachability restricted to ground terms and we show that decidability of ground reachability is not a modular property, even for linear TRSs without collapsing rules. Most of these results are presented in [2].

Disjoint union means union of TRSs on disjoint alphabets, and represents a strong restriction. In the last section, we study composition of constructor systems. In a constructor system, (i.e. a system which obeys the constructor discipline) all function symbols occurring at non-leftmost positions in left-hand sides of rewrite rules are constructors. Toyama and Middeldorp in [16] showed that a constructor system is complete if it can be decomposed into complete constructor systems. This notion of decomposition does not imply disjointness. Consider, for example, the constructor system given in [16]

$$\mathcal{R}=\begin{cases} 0+x & \to x,\\ S(x)+y \to S(x+y),\\ 0\times x & \to 0,\\ S(x)\times y \to x\times y+y,\\ f(0) & \to 0,\\ f(S(x)) & \to f(x)+S(x). \end{cases}$$

We can decompose $\mathcal{R}$ into

$$\mathcal{R}_1=\begin{cases} 0+x & \to x,\\ S(x)+y \to S(x+y),\\ 0\times x & \to 0,\\ S(x)\times y \to x\times y+y, \end{cases}$$

$$\mathcal{R}_2=\begin{cases} 0+x & \to x,\\ S(x)+y \to S(x+y),\\ f(0) & \to 0,\\ f(S(x)) & \to f(x)+S(x). \end{cases}$$

Both systems are complete and imply the completeness of $\mathcal{R}$. Other results concerning systems with a shared constructor can be found in [6, 10].

In the last section, we obtain the following results. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are nonterminating constructor systems the reachability problem of which is decidable, then reachability becomes undecidable for the composition $\mathcal{R}_1+\mathcal{R}_2$. If $\mathcal{R}_1$ and $\mathcal{R}_2$ are terminating, reachability remains decidable for the composition when the two systems are right-linear, and becomes undecidable otherwise.

2. Preliminaries

Most of the following definitions originate from Dershowitz and Jouannaud [3], Toyama [20] and Middeldorp [14].

First, we introduce definitions and notations concerning rewrite systems. Let $\mathcal{X}$ be a countably infinite set of *variables*. A *signature* or an *alphabet* is a set $\mathcal{F}$ of *function symbols*. Associated with every $F\in\mathcal{F}$ is a natural number denoted by its *arity*. Function symbols of arity 0 are called *constants*. The set of *terms* $\mathcal{T}_{\mathcal{F}}(\mathcal{X})$ built from a signature $\mathcal{F}$ and a countably infinite set of variables $\mathcal{X}$ with $\mathcal{F}\cap\mathcal{X}=\emptyset$ is the smallest set such that $\mathcal{X}\subset\mathcal{T}_{\mathcal{F}}(\mathcal{X})$ and if $F\in\mathcal{F}$ is an n-ary function symbol and $t_1,\ldots,t_n\in\mathcal{T}_{\mathcal{F}}(\mathcal{X})$ then $F(t_1,\ldots,t_n)\in\mathcal{T}_{\mathcal{F}}(\mathcal{X})$. The set of *ground terms* over $\mathcal{F}$ is the set of terms without variables and is denoted by $\mathcal{T}_{\mathcal{F}}$. Identity of terms is denoted by $\equiv$.

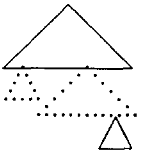


Fig. 1.

A TRS is a pair $(\mathscr{F}, \mathscr{R})$ consisting of a signature $\mathscr{F}$ and a set $\mathscr{R} \subset \mathscr{T}_{\mathscr{F}}(\mathscr{X}) \times \mathscr{T}_{\mathscr{F}}(\mathscr{X})$ of *rewrite rules*. Every rule (l, r) satisfies the following two constraints: the left-hand side l is not a variable and the variables which occur on the right-hand side r also occur in l.

A *context* $C[,\ldots,]$ is a "term" which contains at least one occurrence of a special constant $\square$. If $C[,\ldots,]$ is a context with n occurrences of $\square$ and $t_1,\ldots,t_n$ are terms then $C[t_1,\ldots,t_n]$ is the result of replacing from the left to the right the occurrences of $\square$ by $t_1,\ldots,t_n$. A context containing precisely one occurrence of $\square$ is denoted by $C[\,]$. A term s is a subterm of a term t if there exists a context $C[\,]$ such that $t \equiv C[s]$.

A *position* within a term may be represented in Dewey decimal notation as a sequence of positive integers, describing the path from the outermost root symbol to the head of the subterm at that position. By t_p we denote the subterm of t rooted at position p. The *rewrite relation* $\rightarrow_{\mathscr{R}}$ is defined as follows: $s \rightarrow_{\mathscr{R}} t$ if there exists a rewrite rule $l \rightarrow r$ in $\mathscr{R}$, a substitution σ and a context $C[\,]$ such that $s \equiv C[l^\sigma]$ and $t \equiv C[r^\sigma]$. The transitive-reflexive closure of $\rightarrow_{\mathscr{R}}$ is denoted by $\xrightarrow{*}_{\mathscr{R}}$. If $s \xrightarrow{*}_{\mathscr{R}} t$ we say that *s reduces to t*.

We define now the notions of modular property and disjoint union of rewrite systems. Let $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ two TRSs such that $\mathscr{F}_1 \cap \mathscr{F}_2 = \emptyset$. The *disjoint union* $\mathscr{R}_1 \oplus \mathscr{R}_2$ of $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ is the TRS $(\mathscr{F}_1 \cup \mathscr{F}_2, \mathscr{R}_1 \cup \mathscr{R}_2)$. A property P is *modular* if ($\mathscr{R}_1 \oplus \mathscr{R}_2$ has the property $P \Leftrightarrow (\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ have the property P). Let $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ be disjoint TRSs. Every term $t \in \mathscr{T}_{\mathscr{F}_1 \cup \mathscr{F}_2}(\mathscr{X})$ can be decomposed into $\mathscr{F}_1$-parts and $\mathscr{F}_2$-parts as shown in Fig. 1.

Notation. We abbreviate $\mathscr{F}_1 \cup \mathscr{F}_2$ to $\mathscr{F}_\oplus$ and $\mathscr{T}_{\mathscr{F}_\oplus}(\mathscr{X})$ in $\mathscr{T}_\oplus$. We write $\mathscr{T}_i$ instead of $\mathscr{T}_{\mathscr{F}_i}(\mathscr{X})$ for $i = 1, 2$. The following definitions give a formalism to the structure of terms of $\mathscr{T}_\oplus$.

- The *root symbol* of a term $t \in \mathscr{T}_\oplus$, denoted root(t), is defined by

$$\mathrm{root}(t) = F \quad \text{if } t \equiv F(t_1,\ldots,t_n),$$

$$\mathrm{root}(t) = t \quad \text{if } t \in \mathscr{X}.$$

- Let $t \equiv C[t_1,\ldots,t_n]$ with $C[,\ldots,] \not\equiv \square$ and $n > 0$.
  We write $t \equiv C[\![t_1,\ldots,t_n]\!]$ if $C[,\ldots,] \in \mathscr{T}_{\mathscr{F}_a \cup \{\square\}}(\mathscr{V})$ and $\forall i \in [1, n]$ root$(t_i) \in \mathscr{F}_b$ for some $a, b \in \{1, 2\}$ with $a \neq b$. The t_i's are the *principal subterms* of t.

- The *rank* of a term $t \in \mathscr{T}_\oplus$ is defined by

$$\operatorname{rank}(t)=1 \qquad \text{if } t\in\mathscr{T}_1\cup\mathscr{T}_2,$$

$$\operatorname{rank}(t)=1+\max\{\operatorname{rank}(t_i),\ 1\leqslant i\leqslant n\} \qquad \text{if } t=C[\![t_1,\ldots,t_n]\!].$$

Remark. If $s \xrightarrow{*}_{\mathscr{R}_1\oplus\mathscr{R}_2} t$ then $\operatorname{rank}(s)\geqslant\operatorname{rank}(t)$.

- The multiset $S(t)$ of *special subterms* of a term $t\in\mathscr{T}_\oplus$ is defined as follows:

$$\begin{aligned} & S_1(t) &&= \langle t\rangle,\\ \forall n\geqslant 1\quad & S_{n+1}(t) &&= \langle\ \rangle \text{ if } \operatorname{rank}(t)=1\\ & &&= S_n(t_1)\cup\cdots\cup S_n(t_m) \text{ if } t\equiv C[\![t_1,\ldots,t_m]\!] \end{aligned}$$

$S(t)=\bigcup_{i\geqslant 1} S_i(t)$.

- The *topmost homogeneous part* of a term $t\in\mathscr{T}_\oplus$, notation $\operatorname{top}(t)$, is the result of replacing all the principal subterms of t by $\square$, i.e.

$$\begin{aligned}\operatorname{top}(t)&=t \text{ if } \operatorname{rank}(t)=1\\ &=C[,\ldots,] \text{ if } t\equiv C[\![t_1,\ldots,t_n]\!].\end{aligned}$$

- We define $\operatorname{top}'(t)$ as follows:
  (1) if $\operatorname{rank}(t)=1$ then $\operatorname{top}'(t)=t$.
  (2) else, $\operatorname{top}'(t)$ is obtained from $\operatorname{top}(t)$ by replacing the p occurrences $\square$ in $\operatorname{top}(t)$ by p distinct variables $x_1,\ldots,x_p$ such that for all $i\in[1,p]$, x_i does not occur in $\operatorname{top}(t)$. ($\operatorname{top}'(t)$ is defined only up to α-conversion).
- The multiset $P(t)$ of *parts* of a term $t\in\mathscr{T}_\oplus$ is defined by

$$P(t)=\{\operatorname{top}'(s)\mid s\in S(t)\}=\operatorname{top}'(S(t)).$$

- The multiset $E(t)$ of *erasable parts* of a term $t\in\mathscr{T}_\oplus$ is defined by

$$E(t)=P(t)-(\mathscr{T}_{\mathscr{F}_1}\cup\mathscr{T}_{\mathscr{F}_2}),$$

  i.e. $E(t)$ is the multiset of nonground trees of $P(t)$.

Example. Consider the term t shown in Fig. 2, where $\{F,G,A,B\}\subseteq\mathscr{F}_1$, $\{e,f,c\}\subseteq\mathscr{F}_2$ and $x\in\mathscr{X}$.

- The rank of t is 4.
- $t\equiv C[\![t_1,t_2,t_3]\!]$ with $C\equiv F(G(\square),F(\square,\square))$,
  $t_1\equiv e(x)$, $t_2\equiv e(G(B))$, $t_3\equiv f(e(A),e(G(c)))$.
- special subterms of t:
  $S_1(t)=\langle t\rangle$,
  $S_2(t)=\langle e(x),e(G(B)),f(e(A),e(G(c)))\rangle$,
  $S_3(t)=\langle G(B),A,G(c)\rangle$,
  $S_4(t)=\langle c\rangle$,
  $S(t)=S_1(t)\cup S_2(t)\cup S_3(t)\cup S_4(t)$.

$\{F, G, A, B\} \subseteq \mathcal{F}_1$

$\{e, f, c\} \subseteq \mathcal{F}_2$

$x \in \mathcal{X}$

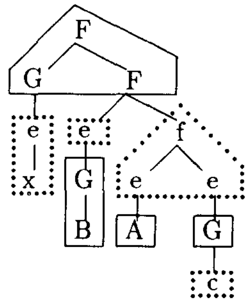


Fig. 2.

– Parts of t and erasable parts of t:
$E(t) = \langle F(G(x), F(y, z)), e(x), e(x), f(e(x), e(y)), G(x) \rangle$,
$P(t) = E(t) \cup \langle G(B), A, c \rangle$.

In the following sections, we study the modularity of decidability of the reachability problem.

- Given a TRS $(\mathscr{F}, \mathscr{R})$ and two terms t_1 and t_2 in $\mathscr{T}_{\mathscr{F}}(\mathscr{X})$, the *reachability problem* for $(\mathscr{F}, \mathscr{R})$, t_1 and t_2 is to decide whether $t_1 \xrightarrow{*}_{\mathscr{R}} t_2$.
- We say that reachability is decidable for a class of TRSs if there exists an algorithm which can solve every reachability problem of this class.
- The problem of *ground reachability* is the reachability problem restricted to ground terms (i.e. without variables).

To prove undecidability of reachability for union of TRSs, we used a well-known undecidable problem: the post correspondence problem.

Definition 2.1. Let I and F be two finite alphabets. The post correspondence problem $P(\phi, \psi)$ over F is given by two morphisms ϕ and ψ from I^* to F^*. $P(\phi, \psi)$ has a solution if and only if there exists m_I in I^+ such that $\phi(m_I) = \psi(m_I)$.

3. Non-left-linear TRSs

A rewrite rule $l \to r$ is *left-linear* (resp. *right-linear*) if l (resp. r) does not contain multiple occurrences of the same variable. A TRS is left-linear (resp. right-linear) if all its rules are left-linear (resp. right-linear).

Proposition 3.1. *Decidability of reachability is not a modular property of non-left-linear TRSs.*

To prove this proposition, we build two TRSs on disjoint alphabets. Reachability is decidable for these systems and we prove that reachability on $\mathscr{R}_1 \oplus \mathscr{R}_2$ is equivalent to the post correspondence problem.

We consider the system $\mathscr{R}_1$ on the alphabet $\mathscr{F}_1 = \{f(\,,),\clubsuit\}$. f is a letter of arity 2 and $\clubsuit$ is a constant. $\mathscr{R}_1$ contains only one rule:

$$\begin{array}{ccc} & f & \\ / & & \backslash \\ x & & x \end{array} \quad \to \quad \clubsuit$$

$\mathscr{R}_1$ is not left-linear.

Lemma 3.2. *Reachability is decidable for* $(\mathscr{F}_1, \mathscr{R}_1)$.

Proof. $\mathscr{R}_1$ is terminating, so reachability is decidable for $(\mathscr{F}_1, \mathscr{R}_1)$. $\square$

We consider now a finite ranked alphabet Σ. Each letter of Σ has the arity 1. Let $\mathscr{F}_2 = \{\#(), \mathrm{phi}(), \mathrm{psi}(), \$\} \cup \Sigma$, with $\{\#(), \mathrm{phi}(), \mathrm{psi}(), \$\}$ and Σ disjoint alphabets. Let ϕ and ψ be two morphisms from Σ^* to Σ^*. $(\mathscr{F}_2, \mathscr{R}_2)$ is a left-linear TRS defined with ϕ and ψ. $\mathscr{R}_2$ is the set of rules

$$\begin{array}{c} \mathrm{phi} \\ | \\ x \\ \\ \\ \\ \\ \end{array} \to \begin{array}{c} i \\ | \\ \mathrm{phi} \\ | \\ \phi(i) \\ | \\ x \end{array} \qquad \begin{array}{c} \mathrm{psi} \\ | \\ x \\ \\ \\ \\ \\ \end{array} \to \begin{array}{c} i \\ | \\ \mathrm{psi} \\ | \\ \psi(i) \\ | \\ x \end{array}$$

$$\begin{array}{c} \mathrm{phi} \\ | \\ x \\ \\ \\ \\ \\ \end{array} \to \begin{array}{c} i \\ | \\ \# \\ | \\ \phi(i) \\ | \\ x \end{array} \qquad \begin{array}{c} \mathrm{psi} \\ | \\ x \\ \\ \\ \\ \\ \end{array} \to \begin{array}{c} i \\ | \\ \# \\ | \\ \psi(i) \\ | \\ x \end{array}$$

for all i in Σ.

Lemma 3.3. *Reachability is decidable for* $(\mathscr{F}_2, \mathscr{R}_2)$.

Proof. $\to_{\mathscr{R}_2}^{-1}$ is terminating thus reachability is decidable for $(\mathscr{F}_2, \mathscr{R}_2)$. $\square$

Lemma 3.4. *The reachability problem*

$$\begin{array}{ccc} & f & \\ / & & \backslash \\ \mathit{phi} & & \mathit{psi} \\ | & & | \\ \$ & & \$ \end{array} \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} \clubsuit$$

is equivalent to the post correspondence problem for ϕ *and* ψ.

Proof. This problem is equivalent to proving that phi($) and psi($) have a common reduct. From phi($) the system can generate all the terms of the form

$$
\begin{array}{c}
i_1\\ | \\ \vdots \\ | \\ i_n \\ | \\ \# \\ | \\ \phi(i_n) \\ | \\ \vdots \\ | \\ \phi(i_1) \\ | \\ \$
\end{array}
\quad \text{and all the terms} \quad
\begin{array}{c}
i_1\\ | \\ \vdots \\ | \\ i_n \\ | \\ \mathrm{phi} \\ | \\ \phi(i_n) \\ | \\ \vdots \\ | \\ \phi(i_1) \\ | \\ \$
\end{array}
\quad (i_1, \ldots, i_n \in \Sigma,\ n \geqslant 0)
$$

It is the same thing for psi($). Therefore, phi($) and psi($) have a common reduct if and only if there exists a solution to the post correspondence problem for ϕ and ψ, i.e. there exists a word m in Σ^* such that $\phi(m) = \psi(m)$. □

Proof of Proposition 3.1. The post correspondence problem is undecidable. Therefore, from the previous lemma, reachability is undecidable for $\mathscr{R}_1 \oplus \mathscr{R}_2$. Thus, decidability of reachability is not a modular property of non-left-linear TRSs. □

Remark. A rule is collapsing if the right-hand side is reduced to a variable. $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ are both right-linear and contain no collapsing rules. Nevertheless, reachability is not modular.

4. Left-linear TRS

4.1. *Without collapsing rules*

A rule $l \to r$ is *collapsing* if r is a variable. We want to show that decidability of reachability is a modular property of left-linear TRSs without collapsing rules. We use a result proved by Middeldorp in [14].

Definition 4.1. Let $s \to t$ by application of a rewrite rule $l \to r$. We write $s \to^{\mathrm{i}} t$ if $l \to r$ is being applied in one of the principal subterms of s and we write $s \to^{\mathrm{o}} t$ otherwise. The relation $\to^{\mathrm{i}}$ is called *inner* reduction and $\to^{\mathrm{o}}$ is called *outer* reduction.

Proposition 4.2 (Middeldorp [14]). *Let $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ be disjoint left-linear TRSs without collapsing rules. For every reduction sequence $s \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t$ there exists a term s' such that $s \xrightarrow{*}^{\mathrm{o}} s' \xrightarrow{*}^{\mathrm{i}} t$.*

Proposition 4.3. *Decidability of reachability is a modular property of left-linear TRSs without collapsing rules.*

Proof. Let $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$ be two disjoint left-linear TRSs without collapsing rules. We suppose that reachability is decidable for $(\mathscr{F}_1, \mathscr{R}_1)$ and $(\mathscr{F}_2, \mathscr{R}_2)$.

(1) Assume that $s \equiv C[\![s_1, \ldots, s_n]\!]$ and $t \equiv C'[\![t_1, \ldots, t_p]\!]$ for some n and p non-negative, C and C' contexts on the same alphabet F_a, $(a \in \{1,2\})$.

If $s \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t$ then there exists s' such that $s \xrightarrow{*}^{\mathrm{o}} s' \xrightarrow{*}^{\mathrm{i}} t$, from Proposition 4.2. Therefore, $s \equiv C[\![s_1, \ldots, s_n]\!] \xrightarrow{*}^{\mathrm{o}} s' \equiv C'[\![s_{i_1}, \ldots, s_{i_p}]\!] \xrightarrow{*}^{\mathrm{i}} t \equiv C'[\![t_1, \ldots, t_p]\!]$, $\forall j \in \{i_1, \ldots, i_p\}$, $s_j \in \{s_1, \ldots, s_n\}$.

So C and C' are contexts on the alphabet $\mathscr{F}_a$ for $a \in [1,2]$ and there exists n new variables $x_1, \ldots, x_n$ such that $C[x_1, \ldots, x_n] \xrightarrow{*}_{\mathscr{R}_a} C'[x_{i_1}, \ldots, x_{i_p}]$ and $s_{i_j} \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t_j$, $\forall j \in [1, p]$. C' is a context nonequal to $\square$ because $\mathscr{R}_a$ is a TRS without collapsing rules.

(2) We prove by induction on the rank of the left term that reachability is decidable.

(a) Suppose $s \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t$ with $\operatorname{rank}(s) = 1$. Reachability is decidable because $\operatorname{rank}(s) = \operatorname{rank}(t) = 1$ and reachability is decidable for $(\mathscr{F}_1, \mathscr{R}_1)$ or $(\mathscr{F}_2, \mathscr{R}_2)$.

(b) Suppose that, for some k, reachability is decidable for a left term of rank lower than k. We prove that it remains decidable for a left term of rank k. The problem is to answer the question: $s \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t$? with $s \equiv C[\![s_1, \ldots, s_n]\!]$ and $t \equiv C'[\![t_1, \ldots, t_p]\!]$.

– If C and C' are contexts on different alphabets, the answer is NO.

– Otherwise, $C, C' \in \mathscr{T}_{\mathscr{F}_a \cup \{\square\}}(\mathscr{X})$ for some $a \in \{1, 2\}$.

$s \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t \Leftrightarrow \exists s'\, s \xrightarrow{*}^{\mathrm{o}} s' \xrightarrow{*}^{\mathrm{i}} t$ with $s' \equiv C'[\![s_{i_1}, \ldots, s_{i_p}]\!]$ and $\forall j \in \{i_1, \ldots, i_p\}$, $s_j \in \{s_1, \ldots, s_n\}$. To find the term s', we choose p variables not necessarily distinct in the set $\{x_1, \ldots, x_n\}$ of distinct new variables. There is a finite number of cases. We say that $\langle x_{i_1}, \ldots, x_{i_p} \rangle$ is a multiset solution if $C[x_1, \ldots, x_n] \xrightarrow{*}_{\mathscr{R}_a} C'[x_{i_1}, \ldots, x_{i_p}]$. Finally, $s \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t$ if and only if there exists a multiset solution $\langle x_{i_1}, \ldots, x_{i_p} \rangle$ such that $s_{i_j} \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t_j$ $\forall j \in [1, p]$. Since $\operatorname{rank}(s_{i_j}) < \operatorname{rank}(s)$ for all j, the problems $s_{i_j} \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t_j$ are decidable by induction hypothesis. $\square$

We define a set of inference rules which yield "T" or "F" or a boolean combination of reachability problems on one alphabet (which can be decided), when applied on a reachability problem $s \xrightarrow{?} t$ of $\mathscr{R}_1 \oplus \mathscr{R}_2$. These inference rules give a decision algorithm for reachability on $\mathscr{R}_1 \oplus \mathscr{R}_2$ and describe the method used in the proof of Proposition 4.3.

Inference rules

Simplify: $s \xrightarrow{?} s \Rightarrow T$.

Clash: $C[\![s_1, \ldots, s_n]\!] \xrightarrow{?} C'[\![t_1, \ldots, t_p]\!] \Rightarrow F$.

If C context on $\mathscr{F}_a$ and C' context on $\mathscr{F}_b$, $\{a,b\}=\{1,2\}$.

Decompose:

$$C[\![s_1,\dots,s_n]\!]\overset{?}{\to}C'[\![t_1,\dots,t_p]\!]$$
$$\Rightarrow \bigvee_{\langle i_1,\dots,i_p\rangle} C[x_1,\dots,x_n]\overset{?}{\to}C'[x_{i_1},\dots,x_{i_p}] \wedge s_{i_1}\overset{?}{\to}t_1 \wedge \cdots \wedge s_{i_p}\overset{?}{\to}t_p.$$

If C and C' contexts on $\mathscr{F}_a$.

4.2. *With collapsing rules and linearity*

A TRS is linear if it is both left-linear and right-linear. Now, we consider disjoint linear TRSs with possibly collapsing rules.

Proposition 4.4. *Decidability of reachability is a modular property of linear TRSs with possibly collapsing rules.*

First, we introduce some notations and definitions used in the proof of the proposition.

Definition 4.5. • $\to_{u\to v,p}$ denotes the application of the rule $u\to v$ at position p.

- $\overset{=}{\to}_{u\to v}$ denotes the application of the rule $u\to v$ at most one time.
- The rewrite step $t\to_{l\to x,p.\alpha}u$ (with $\alpha\in\mathbb{N}$) is *collapsing a part of* t if $\mathrm{root}(t|_p)$ is in $\mathscr{F}_a$, $\mathrm{root}(t|_{p.\alpha})$ is in $\mathscr{F}_b$ and $\mathrm{root}(u|_{p.\alpha})$ is in $\mathscr{F}_a$ $(a\neq b)$.

To prove Proposition 4.4, we use a reduction strategy consisting in reducing the term part by part. So we can first state commuting properties which hold for linear rewrite systems.

Lemma 4.6. *Let $s\to t$, $u\to v$ be (possibly collapsing) rules; the rewrite systems are assumed to be linear. Then*

(1) *if* $t_1\to_{s\to t,p}\to_{u\to v,q}t_2$, *where p and q are incomparable positions, then* $t_1\to_{u\to v,q}\to_{s\to t,p}t_2$;

(2) *if* $t_1\to_{s\to t,q.q_1.q_2}\to_{u\to v,q}t_2$, *where q_1 is a position of a variable of u, then* $t_1\to_{u\to v,q}\overset{=}{\to}_{s\to t}t_2$;

(3) *if* $t_1\to_{u\to v,q}\to_{s\to t,q.q_1.q_2}t_2$, *where q_1 is a position of a variable of v, then* $t_1\to_{s\to t}\to_{u\to v,q}t_2$.

Proof. (1) Let $t_1\equiv C[u_1,\dots,u_n]$ such that there exist i and j $(i\neq j)$, $u_i\to_{s\to t}u_i'$ and $u_j\to_{u\to v}u_j'$. It is obvious that the two derivations are independent:

$$\text{If } t_1\equiv C[\dots,u_i,\dots,u_j,\dots]\to_{s\to t}\to_{u\to v}C[\dots,u_i',\dots,u_j',\dots]\equiv t_2$$
$$\text{then } t_1\equiv C[\dots,u_i,\dots,u_j,\dots]\to_{u\to v}\to_{s\to t}C[\dots,u_i',\dots,u_j',\dots]\equiv t_2.$$

(2) Let $t_1\equiv C[u[u_1,\dots,u_n]]$ such that there exist $i\in[1,n]$ and a substitution σ, $u_i=w[\sigma(s)]$. Let q be the position of $u[u_1,\dots,u_n]$ within t_1, $q.q_1$ the position of

u_i within t_1 and $q.q_1.q_2$ the position of $\sigma(s)$ within t_1.

$$t_1 \to_{s\to t,q.q_1.q_2} C[u[u_1,\dots,u_{i-1},w[\sigma(t)],u_{i+1},\dots,u_n]].$$

To simplify the notation, let $v_k = u_k \ \forall k \neq i$ and $v_i = w[\sigma(t)]$.
$C[u[v_1,\dots,v_n]] \to_{u\to v} C[v[v_{j_1},\dots,v_{j_m}]] \equiv t_2$, $\{j_1,\dots,j_m\} \subseteq \{1,\dots,n\}$ (if v is a variable, $t_2 \equiv C[v_{j_l}]$ for some l in $\{j_1,\dots,j_m\}$).

Since the system is linear, the j_l's are pairwise distinct so there exists at most one k such that $j_k = i$. If such a k does not exist, we get $t_1 \to_{u\to v,q} C[v[u_{j_1},\dots,u_{j_m}]] \equiv t_2$. Otherwise, we get $t_1 \to_{u\to v,q} C[v[u_{j_1},\dots,u_{j_k},\dots,u_{j_m}]]$ with $u_{j_k} \equiv w[\sigma(s)]$ and $C[v[\dots,u_{j_k},\dots]] \to_{s\to t} C[v[\dots,w[\sigma(t)],\dots]] \equiv C[v[v_{j_1},\dots,v_{j_k},\dots,v_{j_m}]] \equiv t_2$.

(3) Assume that $t_1 \equiv C[u[u_1,\dots,u_n]] \to_{u\to v,q} C[v[u_{j_1},\dots,u_{j_k},\dots,u_{j_m}]]$ such that $u_{j_k} = w[\sigma(s)]$. Then $C[v[u_{j_1},\dots,u_{j_k},\dots,u_{j_m}]] \to_{s\to t,q.q_1.q_2} C[v[\dots,w[\sigma(t)],\dots]] \equiv t_2$. q_1 is a position of a variable of v so there exists $i \in [1,n]$ such that $j_k = i$.

So $t_1 \to^2 t_2$ with the derivation:
$C[u[u_1,\dots,u_i,\dots,u_n]] \to_{s\to t} C[u[u_1,\dots,w[\sigma(t)],\dots,u_n]] \to_{u\to v,q} C[v[u_{j_1},\dots,u_{j_m}]]$. $\square$

Lemma 4.7. *Assume that* $s \xrightarrow{*} t \to_{l\to x,p.\alpha} u$ *collapsing a part in* $\mathcal{F}_a$ ($a \in \{1,2\}$, $\alpha \in \mathbb{N}$). *Assume that there is no rewrite step along the reduction* $s \xrightarrow{*} t$ *which is collapsing a part. Then* $s \equiv C[v[s_1,\dots,s_n]] \xrightarrow{*}_{\mathcal{R}_a} C[s_i]$ *reducing only the part* v *in* $\mathcal{F}_a$ *and* $C[s_i] \xrightarrow{*} u$.

Proof. We have to prove that every rule applied on v in the derivation $s \xrightarrow{*} t$ can be applied before every rule rewriting other parts. Assume that $t_1 \to_{r_1,p} \to_{r_2,q} t_2$, where r_2 is a rule applied on the part v and r_1 is a rule applied on another part.

If p and q are incomparable positions, from the first point of Lemma 4.6 $t_1 \to_{r_2,q} \to_{r_2,p} t_2$. If q is a prefix of p, since the two rules are not applied on the same part there exists q_1, q_2, with q_1 a position of a variable of the left-hand side of r_2 such that $p = q.q_1.q_2$. Therefore, from the second point of Lemma 4.6 $t_1 \to_{r_2} \to_{r_1} t_2$.

If p is a prefix of q, since the two rules are not applied on the same part there exists p_1, p_2, with p_1 a position of a variable of the right-hand side of r_1 such that $q = p.p_1.p_2$. Therefore, from the last point of Lemma 4.6 $t_1 \to_{r_2} \to_{r_1} t_2$. $\square$

Proof of Proposition 4.4. Let $(\mathcal{F}_1, \mathcal{R}_1)$ and $(\mathcal{F}_2, \mathcal{R}_2)$ be two disjoint linear TRSs. We suppose that reachability is decidable for $(\mathcal{F}_1, \mathcal{R}_1)$ and for $(\mathcal{F}_2, \mathcal{R}_2)$ and we prove that it remains decidable for $\mathcal{R}_1 \oplus \mathcal{R}_2$.

Let s and t be two terms. We want to solve the reachability problem $s \xrightarrow{*}_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$.

If $s \xrightarrow{*}_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ then either no erasable part has been collapsed, or this reduction has collapsed erasable parts.

Case 1: We search to solve the problem without collapsing part. From Lemma 4.6, we can reduce terms part by part. Indeed, if $t_1 \to_{r_1} \to_{r_2} t_2$ assuming that r_1 and r_2 are not applied on the same part, $t_1 \to_{r_2} \to_{r_1} t_2$. Thus, if we choose to reduce from top to bottom, $s \xrightarrow{*}_{\mathcal{R}_1 \oplus \mathcal{R}_2} t$ if and only if there exists a term s' such that $s \xrightarrow{*}^{\text{o}} s' \xrightarrow{*}^{\text{i}} t$. Therefore, we prove that reachability is decidable for $\mathcal{R}_1 \oplus \mathcal{R}_2$ with a proof similar to Proposition 4.3.

Case 2: We solve the problem $s \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t$ by collapsing at least one erasable part. If $s \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t$ collapsing a part v in $\mathscr{F}_a$, there exist s_1, s_2 such that $s \xrightarrow{*} s_1$ without collapsing part, $s_1 \xrightarrow{*} s_2$ collapsing the part v and $s_2 \xrightarrow{*} t$.

From Lemma 4.7, there exists a derivation $s \xrightarrow{*}_{\mathscr{R}_a} s'$ reducing only v and collapsing it and $s' \xrightarrow{*} s_2 \xrightarrow{*} t$.

There is a finite number of parts in s and reachability is decidable on $\mathscr{F}_a$ (for a in $\{1,2\}$). So it is decidable whether there exists a part v in s such that $s \equiv C[v[u_1, \ldots, u_n]] \xrightarrow{*} C[u_i] \equiv s'$ for i in $[1, n]$.

By induction on the number of parts of the initial term of the reachability problem, we can reduce to a problem without collapsing erasable parts. So reachability is decidable on $\mathscr{R}_1 \oplus \mathscr{R}_2$. □

We define a set of inference rules which yield T or a boolean combination of reachability problems on one alphabet (which can be decided), when applied on a reachability problem $s \xrightarrow{?} t$ of $\mathscr{R}_1 \oplus \mathscr{R}_2$. These inference rules reflect exactly the rewrite strategy which is proved to be complete by the proof of the previous proposition.

Inference rules

Simplify: $s \xrightarrow{?} s \Rightarrow T$.

Decompose 1: If C and C' are contexts on the same alphabet

$$C[\![s_1, \ldots, s_n]\!] \xrightarrow{?} C'[\![t_1, \ldots, t_m]\!]$$
$$\Rightarrow \bigvee_{\langle i_1, \ldots, i_m \rangle} (C[x_1, \ldots, x_n] \xrightarrow{?} C'[x_{i_1}, \ldots, x_{i_m}] \wedge s_{i_1} \xrightarrow{?} t_1 \wedge \cdots \wedge s_{i_m} \xrightarrow{?} t_m)$$
$$\bigvee_{\alpha \in E(C[\![s_1, \ldots, s_n]\!])} \bigvee_{i=1}^{k} (C''[u_i] \xrightarrow{?} C'[\![t_1, \ldots, t_m]\!] \wedge \alpha[x_1, \ldots, x_k] \xrightarrow{?} x_i),$$

with $C[\![s_1, \ldots, s_n]\!] \equiv C''[\alpha[u_1, \ldots, u_k]]$.

Decompose 2: If C and C' are not contexts on the same alphabet

$$C[\![s_1, \ldots, s_n]\!] \xrightarrow{?} C'[\![t_1, \ldots, t_m]\!]$$
$$\Rightarrow \bigvee_{\alpha \in E(C[\![s_1, \ldots, s_n]\!])} \bigvee_{i=1}^{k} (C''[u_i] \xrightarrow{?} C'[\![t_1, \ldots, t_m]\!] \wedge \alpha[x_1, \ldots, x_k] \xrightarrow{?} x_i),$$

with $C[\![s_1, \ldots, s_n]\!] \equiv C''[\alpha[u_1, \ldots, u_k]]$.

4.3. With collapsing rules and without right linearity

In [2], we conjecture that decidability of reachability was not modular for left-linear systems with collapsing rules. Here we give a proof of this conjecture.

Proposition 4.8. *Decidability of reachability is not a modular property of left-linear TRSs with collapsing rules.*

The proof is given by the two following systems.

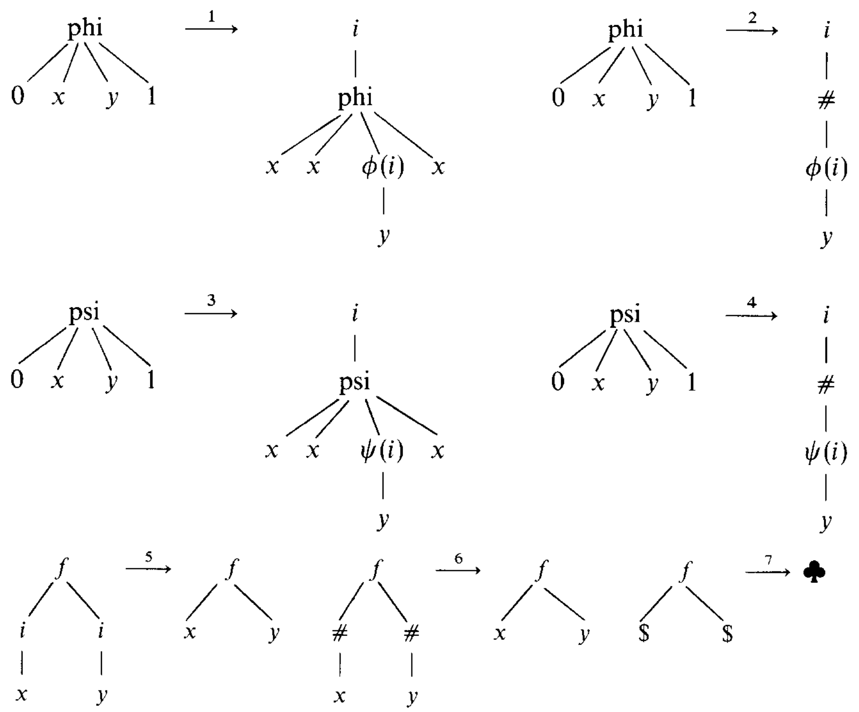
First, we consider the system $\mathscr{R}_1$ on the alphabet $\mathscr{F}_1 = \{g(\,,)\}$. It contains two rules: $g(x, y) \to x$ and $g(x, y) \to y$.

Lemma 4.9. *Reachability is decidable for* $(\mathscr{F}_1, \mathscr{R}_1)$.

Proof. $(\mathscr{F}_1, \mathscr{R}_1)$ is terminating, so reachability is decidable. $\square$

We consider now a finite ranked alphabet Σ. Each letter of Σ has the arity 1. Let $\mathscr{F}_2 = \{f(\,,), \#(), \mathrm{phi}(), \mathrm{psi}(), \$\} \cup \Sigma$, with $\{f(\,,), \#(), \mathrm{phi}(), \mathrm{psi}(), \$\}$ and Σ disjoint alphabets. Let ϕ and ψ be two morphisms from Σ^* to Σ^*. $(\mathscr{F}_2, \mathscr{R}_2)$ is a left-linear TRS defined with ϕ and ψ. $\mathscr{R}_2$ is the set of rules:

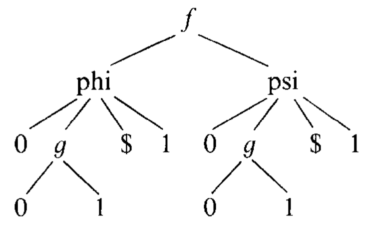
$\forall i \in \Sigma$,



Lemma 4.10. *Reachability is decidable for* $(\mathscr{F}_2, \mathscr{R}_2)$.

Proof. $(\mathscr{F}_2, \mathscr{R}_2)$ is terminating. Indeed, if $t \xrightarrow{*} t'$ then the number of symbol "phi" in t is lower than the number of phi in t'. Moreover, if $t \to t'$ by application of rule 1 on the symbol phi at position p, no rule can be applied on the corresponding phi in t', at position $p.1$. Therefore, rules 1 and 2 (resp. 3 and 4) are noetherian. It is clear that rules 5–7 are terminating. Since rules 1, 2, rules 3, 4, and rules 5–7 are independents, the system $\mathscr{R}_2$ is terminating. So reachability is decidable. $\square$

Lemma 4.11. *The reachability problem*



$\xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2}$ ♣

is equivalent to the post correspondence problem for ϕ and ψ.

Proof. Similar to the proof in Section 3. $\square$

Proof of Proposition 4.8. From the previous lemma, reachability is undecidable for $\mathscr{R}_1 \oplus \mathscr{R}_2$. Thus, decidability of reachability is not a modular property of left-linear TRSs with collapsing rules. $\square$

5. Ground reachability

Proposition 5.1. *Ground reachability is not a modular property of linear TRSs without collapsing rules.*

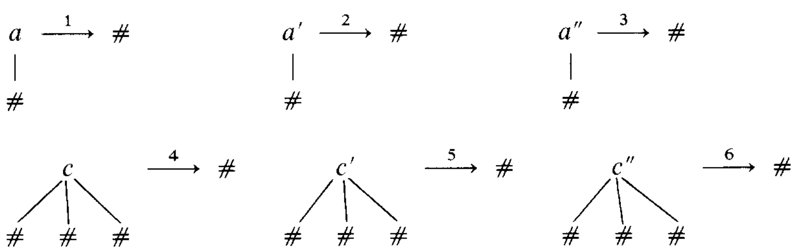
For the proof of this proposition, we build particular TRSs whose ground reachability is decidable and use the post correspondence problem to prove that ground reachability is undecidable for their disjoint union.

Let $\mathscr{F}_1 = \{\alpha(\,), 0\}$ (α is a letter of arity 1 and 0 is a constant). We consider a first TRS $(\mathscr{F}_1, \mathscr{R}_1)$. $\mathscr{R}_1$ contains no rules so it is obvious that reachability is decidable for $(\mathscr{F}_1, \mathscr{R}_1)$.

Let I be a monadic alphabet, disjoint from $\mathscr{F}_1$.

Let $\mathscr{F}_2 = \{c(,,), c'(,,), c''(,,), a(\,), a'(\,), a''(\,), \#\}$ (c, c' and c'' are letters of arity 3, a, a' and a'' are letters of arity 1 and $\#$ is a constant). $\mathscr{F}_2$ and $\mathscr{F}_1$ are disjoints. $\mathscr{G}_1$ is the set of ground terms on the alphabet $\mathscr{F}_1$ and $\mathscr{G}_2$ is the set of ground terms on the alphabet $\mathscr{F}_2$. With every pair (ϕ, ψ) of morphisms from I^* to $\{a, a'\}^*$, we associate a TRS $(\mathscr{F}_2, \mathscr{R}_2)$.

- $\mathscr{R}_2$ contains rules which permit us to reduce a ground term of $\mathscr{G}_2$ to $\#$.



- $\mathscr{R}_2$ contains all the rules which permit us to generate every ground term of $\mathscr{G}_2$ from $\#$.

$$\# \xrightarrow{7} a(\#) \qquad \# \xrightarrow{8} a'(\#) \qquad \# \xrightarrow{9} a''(\#)$$

$$\# \xrightarrow{10} c(\#,\#,\#) \qquad \# \xrightarrow{11} c'(\#,\#,\#) \qquad \# \xrightarrow{12} c''(\#,\#,\#)$$

Therefore, we get for all ground terms t and t', $t \xrightarrow{*}_{\mathscr{R}_2} \# \xrightarrow{*}_{\mathscr{R}_2} t'$.

Thus, ground reachability is decidable for $(\mathscr{F}_2, \mathscr{R}_2)$.

- $\mathscr{R}_2$ contains rules associated with the post correspondence problem for ϕ and ψ. (adding these rules does not change the ground reducibility relation).

  for all i in I

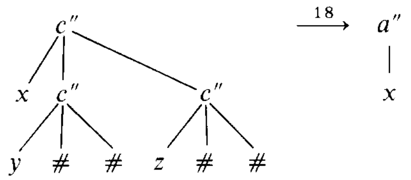
$$c(x,y,z) \xrightarrow{13} c'\big(x,\ \phi(i)(c''(y,\#,\#)),\ \psi(i)(c''(z,\#,\#))\big)$$

and

$$c'(x,y,z) \xrightarrow{14} c'\big(x,\ \phi(i)(y),\ \psi(i)(z)\big)$$

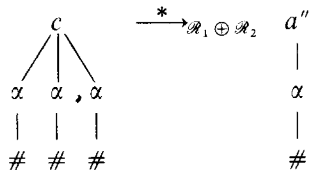
$$c'(x,y,z) \xrightarrow{15} c''(x,y,z) \qquad c''(x,a(y),a(z)) \xrightarrow{16} c''(x,y,z)$$

$$c''(x,a'(y),a'(z)) \xrightarrow{17} c''(x,y,z)$$



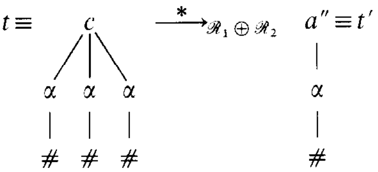
$\mathscr{R}_2$ is a linear TRS, without collapsing rules.

Lemma 5.2. *The ground reachability problem*

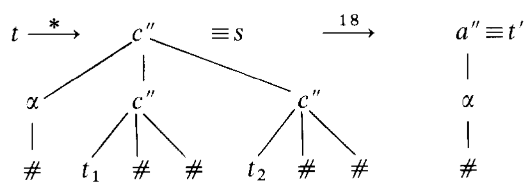


is equivalent to the post correspondence problem for ϕ and ψ.

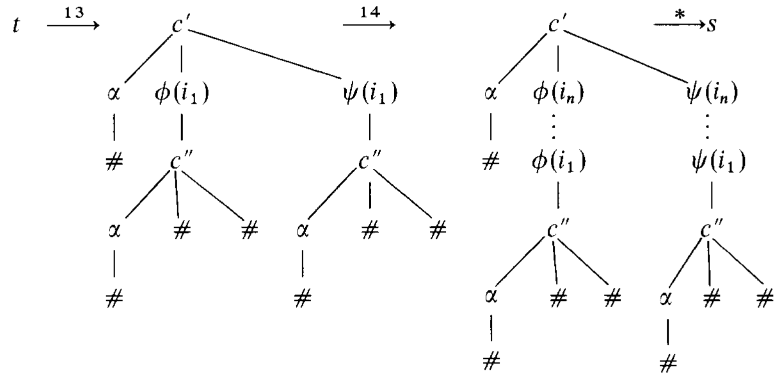
Proof. We are interested in the reachability problem:



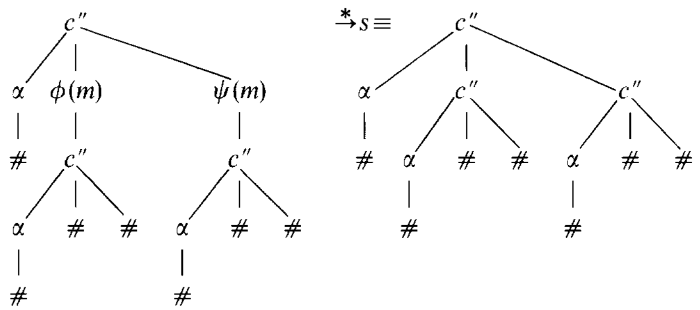
The two systems do not contain collapsing rules, so we can split the reachability problem into $t \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} s \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t'$ and say that $t \xrightarrow{*}_{\mathscr{R}_1 \oplus \mathscr{R}_2} t'$ if and only if we have applied rule 18, i.e. there exist two terms t_1 and t_2 such that



Thus, we study the new reachability problem $t \xrightarrow{*} s$. However, $t \xrightarrow{*} s$ if and only if $t_1 \equiv t_2 \equiv \alpha(\#)$ and there exist $i_1, \ldots, i_n$, $n \geqslant 1$ such that



Note that, because of the symbols α, we cannot apply rules which permit us to reduce a term of root c, c' or c'' to $\#$. s is a term of root c''. However, we rewrite a term of root c' to a term of root c'' applying rule 15. Thus, $t \xrightarrow{*} s$ if and only if there exists $m = i_n \ldots i_1$ ($n \geqslant 1$) such that



We delete the letters of $\phi(m)$ and of $\psi(m)$ with rules 16 or 17. Thus, $t \xrightarrow{*} s$ if and only if there exists $m = i_n \ldots i_1$ ($n \geqslant 1$) such that $\phi(m) = \psi(m)$. Therefore, $t \xrightarrow{*} t' \Leftrightarrow t \xrightarrow{*} s \Leftrightarrow$ the post correspondence problem for ϕ and ψ has a solution. Thus, the reachability problem $t \xrightarrow{*} t'$ is undecidable. $\square$

Proof of Proposition 5.1. From the previous lemma, ground reachability is undecidable for $\mathscr{R}_1 \oplus \mathscr{R}_2$. Therefore, decidability of ground reachability is not a modular property of linear TRSs without collapsing rules. $\square$

6. Composition of constructor systems

6.1. General case

A *constructor system* (CS for short) is a TRS $(\mathscr{F}, \mathscr{R})$ with the property that $\mathscr{F}$ can be partitioned into disjoint sets $\mathscr{D}$ and $\mathscr{C}$ such that every left-hand side $f(t_1, \ldots, t_n)$ of a rewrite rule of $\mathcal{R}$ satisfies $f \in \mathcal{D}$ and $t_1, \ldots, t_n \in \mathcal{T}_{\mathcal{C}}(\mathcal{X})$. Function symbols in $\mathcal{D}$ are called *defined* symbols and those in $\mathcal{C}$ *constructors*. To emphasize the partition of $\mathcal{F}$ into $\mathcal{D}$ and $\mathcal{C}$ we write $(\mathcal{D}, \mathcal{C}, \mathcal{R})$ instead of $(\mathcal{F}, \mathcal{R})$ and $\mathcal{T}_{\mathcal{D},\mathcal{C}}(\mathcal{X})$ instead of $\mathcal{T}_{\mathcal{F}}(\mathcal{X})$. Since the behaviour of a Turing machine can be simulated by a CS [7], constructor systems have universal computing power.

Definition 6.1 (*Middeldorp and Toyama* [16]).

(1) Let $(\mathcal{D}^{\Omega}\mathcal{C}, \mathcal{R})$ be a constructor system and suppose $\mathcal{D}' \subset \mathcal{D}$. The set $\{l \rightarrow r \in \mathcal{R} \mid \mathrm{root}(l) \in \mathcal{D}'\}$ is denoted by $\mathcal{R} \mid \mathcal{D}'$.

(2) Two CSs $(\mathcal{D}_1, \mathcal{C}_1, \mathcal{R}_1)$ and $(\mathcal{D}_2, \mathcal{C}_2, \mathcal{R}_2)$ are composable if $\mathcal{D}_1 \cap \mathcal{C}_2 = \mathcal{D}_2 \cap \mathcal{C}_1 = \emptyset$ and $\mathcal{R}_1 \mid \mathcal{D}_2 = \mathcal{R}_2 \mid \mathcal{D}_1$. The second requirement is equivalent to the condition that both CSs contain all rewrite rules which "define" a symbol whenever that symbol is shared. The union of pairwise composable CSs $CS_1, \ldots, CS_n$ is denoted by $CS_1 + \cdots + CS_n$ and we say that $CS_1, \ldots, CS_n$ is a decomposition of $CS_1 + \cdots + CS_n$.

(3) A property P is decomposable if for all pairwise composable CSs $CS_1, \ldots, CS_n$ with the property P we have that $CS_1 + \cdots + CS_n$ has the property P.

Proposition 6.2 (Middeldorp and Toyama [16]). *Let P be a property of CSs. The following statements are equivalent:*

(1) *P is decomposable*

(2) *for all composable CSs CS_1 and CS_2 with the property P we have that $CS_1 + CS_2$ has the property P.*

In this section, we study composition of constructor systems the reachability of which is decidable. Unfortunately, in general, reachability becomes undecidable for this composition.

Proposition 6.3. *Decidability of reachability is not decomposable for linear constructor systems.*

Proof. The proof given in Section 3 can be applied for this proposition but we have to replace $\mathcal{R}_1$ by the equivalent left-linear system: $\forall i$ in Σ

$$f(i(x), i(y)) \longrightarrow f(x, y) \qquad f(\#(x), \#(y)) \longrightarrow f(x, y) \qquad f(\$, \$) \longrightarrow \clubsuit$$

Remark. In Section 3, the non-left-linearity of $\mathcal{R}_1$ was imposed by the disjointness of the two alphabets.

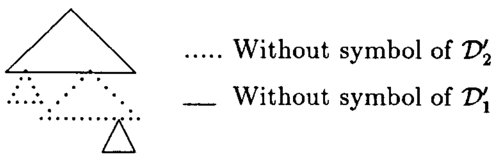


Fig. 3.

6.2. Terminating constructor systems

Since the previous example used nonterminating constructor systems, we study here the case of terminating CSs.

6.2.1. Right-linear constructor systems

Definitions. Let $(\mathscr{D}_1, \mathscr{C}_1, \mathscr{R}_1)$ and $(\mathscr{D}_2, \mathscr{C}_2, \mathscr{R}_2)$ two CSs.
Let $\mathscr{D}' = \mathscr{D}_1 \cap \mathscr{D}_2$, $\mathscr{D}_1' = \mathscr{D}_1 - \mathscr{D}'$, $\mathscr{D}_2' = \mathscr{D}_2 - \mathscr{D}'$.
We transform some of the definitions used for the disjoint union.
Here, the alphabets are not disjoint and the decomposition of a term is different from the one given in Section 2; cf. Fig. 3. Because of the properties of constructor systems, on a part which does not contain symbol of $\mathscr{D}_2'$, we can apply only rules of $\mathscr{R}_1$. (The rules of $\mathscr{R}_2$ we can apply are in $\mathscr{R}_1$ too.)

- Let $t \equiv C[t_1, \ldots, t_n]$ with $C[, \ldots,] \neq \square$. We write $t \equiv C((t_1, \ldots, t_n))$ if $C[, \ldots,]$ does not contain symbol of $\mathscr{D}_a'$ and $\mathrm{root}(t_1), \ldots, \mathrm{root}(t_n) \in \mathscr{D}_a'$ for some $a \in \{1, 2\}$. The t_i's are the *CS-principal subterms* of t.
- The *CS-rank* of a term t is defined by

$$\begin{aligned} &\text{CS-rank}(t) = 1 && \text{if } t \in \mathscr{T}_1 \cup \mathscr{T}_2, \\ &\text{CS-rank}(t) = 1 + \max\{\text{CS-rank}(t_i),\ 1 \leqslant i \leqslant n\} && \text{if } t = C((t_1, \ldots, t_n)). \end{aligned}$$

- The *CS-topmost homogeneous part* of a term t, notation CS-top(t), is the result of replacing all the principal subterms of t by $\square$, i.e.

$$\begin{aligned} \text{CS-top}(t) &= t \text{ if CS-rank}(t) = 1 \\ &= C[, \ldots,] \text{ if } t \equiv C((t_1, \ldots, t_n)). \end{aligned}$$

Decidability result

Lemma 6.4. *Composition of two terminating right-linear constructor systems is terminating.*

Proof. The systems are right-linear. Therefore, if t is reduced to s, the number of parts of s is lower than or equal to the number of parts of t. This means that, if there exists an infinite derivation starting from t, there exists a term t' such that $t \xrightarrow{*} t'$ and there is an infinite derivation starting from t' and preserving the number of parts of t'. We prove that there is no infinite derivation starting from some t and preserving the parts, by induction on the number of parts.

– Let t be a term with only one part. $(\mathscr{D}_1, \mathscr{C}_1, \mathscr{R}_1)$ and $(\mathscr{D}_2, \mathscr{C}_2, \mathscr{R}_2)$ are terminating by hypothesis so there is no infinite derivation starting from t.

– Now, we suppose Lemma 6.4 holds for terms of $n-1$ parts, for a fixed n. Suppose there exists an infinite derivation starting from t with $\mathrm{NP}(t)=n$. The systems are right-linear and no part is collapsed (we suppose the number of parts is preserved). So if $t \equiv C((t_1, \ldots, t_k)) \xrightarrow{*} s \equiv C''((s_1, \ldots, s_p))$ then $p=k$ and there exists a set $\{i_1, \ldots, i_p\} = \{1, \ldots, k\}$ such that $t_{i_1} \xrightarrow{*} s_1, \ldots, t_{i_p} \xrightarrow{*} s_p$. The number of parts of each CS-principal subterm is lower than n so by induction hypothesis there is no infinite derivation starting from it. So after a finite number of steps, we obtain a term t' such that all the rules are applied on the CS-top of t'. Suppose CS-top(t') does not contain letters of $\mathscr{D}_2$. Then, from the term CS-top(t') there exists an infinite derivation. This leads to a contradiction because $(\mathscr{D}_1, \mathscr{C}_1, \mathscr{R}_1)$ is terminating. □

As a corollary, we obtain the following proposition.

Proposition 6.5. *Decidability of reachability is decomposable for terminating right-linear constructor systems.*

Proof. The proof is obvious because termination is conserved under composition and reachability is decidable for terminating systems. □

6.2.2. Non-right-linear constructor systems

Proposition 6.6. *Decidability of reachability is not decomposable for terminating left-linear constructor systems.*

Proof. We can take the same proof as for Proposition 4.8. □

7. Conclusion

We have presented an analysis of modularity concerning the decidability of reachability. Although this property is not preserved under disjoint union in general, we have given conditions for ensuring its modularity. Moreover, we have given counterexamples based on undecidability of the post correspondence problem.

In Table 1 we summarize the results obtained in this paper.

Table 1
Decidability of reachability

	Right-linear		Non-right-linear		
	Non-left-linear	Left-linear	Non-left-linear	Left-linear	
				C.r.[a]	No c.r.
Disjoint union of TRSs	No	Yes	No	No	Yes
Composition of CSs	No	No	No	No	
Composition of terminating CSs	Yes	Yes	No	No	

[a] Collapsing rules

Acknowledgment

The authors would like to thank Aart Middeldorp, Hubert Comon and the referees for their pertinent remarks.

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