Kinematics of layered reinforced-concrete planar beam finite elements with embedded transversal cracking

Paulo Šćulac, Gordan Jelenić *, Leo Škec

University of Rijeka, Faculty of Civil Engineering, Radmile Matejčić 3, 51 000 Rijeka, Croatia

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In this work crack formation and development is addressed and implemented in a planar layered reinforced-concrete beam element. The crack initiation and growth is described using the strength criterion in conjunction with exact kinematics of the interlayer connection. In this way a novel embedded-discontinuity beam finite element is derived in which the tensile stresses in concrete at the crack position reaching the tensile strength will trigger a crack to open. Since the element is multi-layered, in this way the crack is allowed to propagate through the depth of the beam. The cracked layer(s) will involve discontinuity in the cross-sectional rotation equal to the crack-profile angle, as well as a discontinuity in the position vector of the layer’s reference line. A bond–slip relationship is superimposed onto this model in a kinematically consistent manner with reinforcement being treated as an additional layer of zero thickness with its own material parameters and a constitutive law implemented in the multi-layered beam element.

Emphasis in this work is placed on the definition and finite-element implementation of kinematics of such a layered beam set-up with embedded cracking, rather than on constitutional details of the concrete, steel and interface between them. Several numerical examples are presented, in which the ability of the proposed procedure to predict crack occurrence and development is investigated.

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1. Introduction

Reinforced concrete is the most widely used composite material in civil engineering. It combines the advantageous mechanical properties of concrete in compression and steel in tension so that the reinforcement bars under operating circumstances usually become stretched while the adjoining concrete experiences fully developed cracking or sizeable micro-cracked regions with limited resistance to tension (Bažant and Cedolin, 2003). A so-called tension-stiffening effect is the principal load-bearing mechanism in reinforced-concrete structures once cracking of concrete in tension and subsequent slippage of the reinforcement bars with respect to the surrounding concrete take place. In this way the tensile stresses in the reinforcement at the crack positions gradually become transferred to the concrete between the cracks.

The methods to model crack initiation and growth using the finite-element method fall into two major categories: (i) the smeared-crack methods (see e.g. Bažant, 2002; Bažant and Oh, 1983; de Borst et al., 2004; Őzbolt and Bažant, 1996; Rots et al., 2008), in which a finite band around the actual crack is considered to be either fully or partly damaged and (ii) the discrete-crack methods (see e.g. Jirásek and Zimmermann, 2001; Oliver, 1995; Oliver and Huespe, 2004; Simo et al., 1993), in which a strong discontinuity in the material takes place at the fully developed crack position. A discrete crack may be either predicted to occur at an interface element between two continuum finite elements (Alfano and Crisfield, 2001) or it may develop within an existing element (Jirásek, 2000). In all these papers, the crack formation process is utilised within the continuum based 2D or 3D elements, with the application to beam elements being less numerous. Nonetheless, to mention a few of these, Aldstedt and Bergan (1978) developed a beam element with perpendicular cracking at the element ends in which the element length was to be estimated so as to match the expected crack distance, Armero and Ehrlich (2006) studied the effects of softening in plastic hinges and proposed a beam element with an embedded developing plastic hinge as did also Jukić et al. (2013) using a stress-resultant approach applied to reinforced-concrete cross-sections, Marfia et al. (2004) considered a ‘repetitive’ element of a length equal to an assumed crack distance to study the effects of cyclic loading including the corresponding implication on the steel and the bond–slip constitutive laws, while Oliveira et al. (2008) developed a layered finite element with a length again estimated to coincide to the presumed crack distance.

* Corresponding author. Tel.: +385 (0)51 265 955.
E-mail address: gordan.jelenic@gradri.hr (G. Jelenić).

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In this work a primary fundamental ingredient of the mechanism (crack formation and development) is addressed and implemented in a reinforced-concrete beam element, which compared with the continuum-based finite elements has considerably less degrees of freedom. Here, as an underlying principle, the mechanism of crack initiation and growth is described using the laws of damage mechanics (see e.g. Bažant and Planas, 1998) in conjunction with the energy considerations originating from fracture mechanics (Hillerborg et al., 1976). In this way a novel embedded-discontinuity layered beam finite element is derived in which the tensile stresses in concrete at a pre-defined point reaching the tensile strength will trigger a crack to open. The number of layers is arbitrary and they are assembled in a beam with a rigid inter-layer connection – there is neither slip nor uplift between the layers, but the layers rotate independently. There exists another layer of zero thickness, representing the reinforcement bar. The reinforcement layer lies within a surrounding layer and may slip with respect to this layer. A transversal crack is embedded into the element, with the assumption that it propagates throughout the whole depth of the layer in which the tensile strength has been reached. Any layer which has cracked in this way will thus involve a discontinuity in the cross-sectional rotation equal to the crack-profile angle in the layer, as well as a discontinuity in the position of its reference axis.

Upon cracking, the reinforcement slips with respect to the surrounding concrete, which determines the amount of tangential stress (bond) transferred from the reinforcement to the concrete as a result of the actual shape of the bond–slip diagram as described by different authors and design codes (see e.g. Eligehausen et al., 1983; CEB-FIP Bulletin 10, 2000; CEB-FIP Model Code, 1990). A bond–slip relationship is superimposed onto this model in a kinematically consistent manner with reinforcement being treated as an additional layer with its own material parameters and a constitutive law. Further degrees of freedom are defined at the nodes to account for slippage between the reinforcement and the surrounding concrete.

The resulting multi-layer beam element is derived using the standard degrees of freedom: one rotation per layer at each of the nodes, slip of the reinforcement bar and the two displacement components at each of the nodes, and, in addition, a crack opening parameter and the crack profile angle for each layer as the internal degrees of freedom.

It should be noted that the proposed approach is capable of determining both the position of an opening crack as well as its width and depth. This puts it in stark contrast to the widely used techniques (see e.g. Figueiras, 1986; Collins and Mitchell, 1997; Stramandinoli and La Rovere, 2008; Tamai et al., 1988; Belarbi and Hsu, 1994; Wang and Hsu, 2001) whereby the tension-stiffening effect is modelled on the basis of experimentally obtained force–elongation relationships for a uniaxially loaded reinforced-concrete specimen. From these results, a mean strain of the specimen may be easily deduced and used to define the constitutive relationship needed for the numerical analyses, including those which may be performed using some of the commercial finite-element codes in which user-defined constitutive models may be integrated. Within this technique, however, the actual crack positions and other properties remain unknown.

An outline of the paper is as follows: first the multi-layer beam with a rigid interlayer connection and cracking is presented. It is then extended by the incorporation of a reinforcement layer, and followed by the definition of the total virtual work and interpolation of the test and trial functions. Finally, some numerical examples are presented in which the proposed approach is tested for linear constitutive relationship defining concrete and steel as well as the bond–slip relationship.

2. Multi-layer beam with a rigid interlayer connection and cracking

We will consider an initially straight beam of length $L$ and a cross-section composed of $M$ parts with heights $h_i$ and areas $A_i$, where $i$ is an arbitrary layer (Škéč and Jelenić, 2013). Each layer has its own material coordinate system defined by an orthonormal triad of vectors $E_{1,i}, E_{2,i}$ and $E_{3,i}$, with axes $X_{1,i}, X_{2,i}$ and $X_{3,i}$. The axes $X_{1,i}$ coincide with the reference axes of each layer which are chosen arbitrarily and are mutually parallel. The cross-sections of the layers are symmetric with respect to the vertical principal axis $X_2$ defined by a base vector $E_2 = E_{2,i}$ (a condition for a plane problem). The distance from the bottom of a layer to the layer’s reference axis is denoted as $a_i$. The reference axes of all layers in the initial undeformed state are defined by the base vector $E_0$ which closes an angle $\psi$ with respect to the axis defined by the base vector of the spatial coordinate system. The position of a material point $T(X_1,X_2)$ in the undeformed initial configuration is defined with respect to any layer by the vector

$$
\mathbf{x}_0(t_1,X_1,X_2) = \mathbf{r}_0(t_1) + X_2 t_02. \tag{1}
$$

where $\mathbf{r}_0(t_1)$ is the position of the intersection of the plane of the cross-section containing the point $T$ and the reference axis of the layer $l$ in the undeformed state. Vector $t_0$ is defined as

$$
t_0 = A_0 e_0, \quad A_0 = \begin{bmatrix} \cos \psi & - \sin \psi \\ \sin \psi & \cos \psi \end{bmatrix} = [t_{01} \quad t_{02}], \tag{2}
$$

where index $j = 1,2$ refers to the corresponding axis. The kinematics of deformation of the layer $l$ is shown in Fig. 1.

During the deformation, the plane cross-sections of the layers remain planar but not necessarily perpendicular to their deformed reference axes (Timoshenko beam theory with Bernoulli’s hypothesis). The material base vector $E_1$ remains orthogonal to the plane spanned by the spatial base vectors $e_1$ and $e_2$. Orientation of the cross-section of each layer in the deformed state is defined by the base vectors $t_{lj}$ as

$$
t_{lj} = A_l e_j, \quad A_l = \begin{bmatrix} \cos (\psi + \theta_l) & - \sin (\psi + \theta_l) \\ \sin (\psi + \theta_l) & \cos (\psi + \theta_l) \end{bmatrix}, \tag{3}
$$

which for the case of small rotations and deformations turns into

$$
A_l = \begin{bmatrix} \cos \psi & - \sin \psi \\ \sin \psi & \cos \psi \end{bmatrix} \begin{bmatrix} 1 & - \theta_l \\ \theta_l & 1 \end{bmatrix} = \begin{bmatrix} 1 & - \theta_l \\ \theta_l & 1 \end{bmatrix} \begin{bmatrix} \cos \psi & - \sin \psi \\ \sin \psi & \cos \psi \end{bmatrix}. \tag{4}
$$

![Fig. 1. Position of layer l in initial and deformed configuration.](image-url)
Rotation of the cross-section of each layer \( \theta_i \) is entirely dependent on \( X_1 \), thus \( \theta_i = \theta(X_1) \). The position of a material point \( T \) of the layer \( l \) in the deformed state can be expressed as
\[
x_l(X_1, X_{23}) = r_l(X_1) + X_{23} t_{l2}(X_1),
\]
where \( r_l(X_1) \) is the position of the intersection of the plane of the cross-section containing the point \( T \) and the reference axis of the layer \( l \) in the undeformed state. The displacement between the undeformed and deformed state is defined for each layer with respect to its reference axis, thus
\[
r_l(X_1) = r_{0l}(X_1) + u_l(X_1),
\]
where \( u_l(X_1) \) is the vector of displacement of the layer’s reference axis.

2.1. Assembly equations

2.1.1. Interlayer kinematics without cracking

Since in the present formulation neither slip nor uplift are allowed between the layers of the beam, we can express displacement of a layer \( l \) in terms of the displacement of an arbitrarily chosen main layer \( \alpha \) (denoted by \( u_\alpha \)) and the rotations \( \theta_1, \ldots, \theta_l \). The reference axis of the layer \( \alpha \) thus becomes the reference axis of the multi-layer beam. In Fig. 2, where a section of the beam with arbitrary number of layers in initial and deformed configuration is shown, we may observe the arbitrarily chosen main layer \( \alpha \), another arbitrary layer lying above layer \( \alpha \), denoted by \( \beta_+ \), and yet another arbitrary layer lying below layer \( \alpha \), denoted by \( \beta_- \). In the deformed state, the reference axes of all layers deform and the layers’ cross-sections rotate, which is defined by the unit base vectors \( t_{21}, t_{22} \) and \( t_{23} \).

For an arbitrary layer \( l \) (which can lie above or below the main layer \( \alpha \)) we can express \( u_l \) in terms of displacement \( u_\alpha \) and rotations \( \theta_j \), where \( j \in [\zeta, \ldots, \zeta] \) via (see Škec and Jelenić, 2013 for the derivation of this result)
\[
u_l = u_\alpha + a_l(t_{22} - t_{02}) - a_\alpha(t_{22} - t_{02}) + \text{sgn}(l - \alpha) \sum_{j=1}^{\zeta} h_j(t_{22} - t_{02}),
\]
with
\[
\zeta = \begin{cases} \alpha, & l > \alpha \\ l, & 1 \leq \alpha < l \leq \alpha, \end{cases} \quad \zeta' = \begin{cases} l, & l > \alpha \\ \alpha, & l \leq \alpha \leq \alpha, \end{cases}
\]

2.1.2. Introduction of cracking

Concrete has limited tensile strength and once it is exceeded a crack occurs at the point where this takes place. In the present set-up it is presumed that the crack, located at \( X_{1C} \), propagates throughout the whole depth of the layer in which the tensile strength (calculated at the mid-depth) has been reached. Once cracked, the layer remains cracked for the rest of the analysis. Since the element is multi-layered, in this way the crack is allowed to propagate through the depth of the beam.

When layer \( X \) cracks, there occurs a discontinuity (Fig. 3) in its displacement field. Consequently, we then distinguish between the displacement of layer \( X, u_l(X_1) \), and the displacement of the multi-layer beam’s reference axis, denoted by \( u(X_1) \), which we define as a continuous field over \( X_1 \). The beam reference line may be imagined as a continuous conduit cast in the concrete layer \( X \). When the concrete layer cracks, it slips with respect to this conduit by \( p(X_1) \), where \( p(X_1) \) is a function accounting for the discontinuity at the crack position \( X_{1C} \). The two fields are then related via
\[
u_l(X_1) = u_l(X_1) + p(X_1) t'(X_1),
\]
which in the geometrically linear analysis turns into
\[
u_l(X_1) = u_l(X_1) + p(X_1) t_{01}, \quad \text{since} \quad t' = \frac{\partial u}{\partial X_1} \quad \text{and} \quad pt' = p[t_{01} + (t' \otimes \nabla u)]u + H.O.T. \quad = pt_{01} + H.O.T.
\]
The function \( p(X_1) \) may be approximated as
\[
p(X_1) = \delta_{\alphaC} k(X_1),
\]
where \( \delta_{\alphaC} \) is a flag denoting if layer \( \alpha \) has cracked or not:
\[
\delta_{\alphaC} = \begin{cases} 0, & \varepsilon_x < f_{ct}/E_c \\ 1, & \varepsilon_x > f_{ct}/E_c, \end{cases}
\]
with \( \varepsilon_x \) as the normal strain, \( f_{ct} \) the strength in the middle of the layer \( \alpha \) and \( E_c \) Young’s modulus of concrete. The crack opening at the middle of layer \( \alpha \) is denoted as \( w \), while \( k(X_1) \) is a step function (Fig. 4) defined as
\[
k(X_1) = \begin{cases} \frac{X_1}{X_1}, & X_1 < X_{1C} \\ \frac{1}{X_1}, & X_1 > X_{1C} \end{cases}
\]
and functionally undefined at \( X_1 = X_{1C} \).

At this point it has to be recognised that beside the discontinuity in displacements of layer \( \alpha \), there is also a rotational discontinuity in each cracked layer
\[
\theta_l = \theta_l + k(X_1) \phi_l,
\]
where \( \phi_l \) is the rotation as if there were no cracking in layer \( l \), while \( \phi_l \) is the crack profile angle of layer \( l \).

Substituting (10) into (7) we get
\[
u_l = u_l(X_1) + p(X_1) t_{01} + a_l(t_{22} - t_{02}) - a_\alpha(t_{22} - t_{02})
\]
\[
+ \text{sgn}(l - \alpha) \sum_{j=1}^{\zeta} h_j(t_{22} - t_{02}),
\]
or, in other words, the basic unknown functions of the problem are the two components of the vector \( u \) and the rotations of each layer \( \theta_l \). In addition we also have the crack opening \( w \) at the middle of layer \( \alpha \), and the crack profile angles \( \phi_l \) in each cracked layer, thus making the number of total unknown functions \( 2 + M (u, \theta_1, \ldots, \theta_M) \) and the maximum number of total unknown parameters \( 1 + M (w, \phi_1, \ldots, \phi_M) \).

Let it be mentioned that inclusion of a non-transversal shear cracking appears to be incompatible with the present idea in that the orientation of such a crack does not coincide with a layer’s cross-section. Nonetheless, such cracking might still be considered within the present concept by approximating a skew crack with an L-shaped crack with its longitudinal leg modelled by allowing the adjacent layers to slip with respect to each other and separate from
one another. This extension would require re-deriving the interlayer kinematics and is not pursued further in this paper.

2.2. Kinematic equations

Non-linear kinematic equations of the Reissner beam theory (Reissner, 1972) take the well-known Timoshenko form once the rotations of the cross-section become small:

\[ \gamma = \left\{ \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \end{array} \right\} = \Lambda^T \left( \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \end{array} \right) = \left\{ \begin{array}{c} \theta_1^r \\ \theta_2^r \\ \theta_3^r \\ \theta_4^r \\ \end{array} \right\} \] (16)

\[ \kappa = \left\{ \begin{array}{c} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \\ \end{array} \right\} = \left\{ \begin{array}{c} \kappa_1^r \\ \kappa_2^r \\ \kappa_3^r \\ \kappa_4^r \\ \end{array} \right\} \] (17)

where \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) and \( \kappa_1, \kappa_2, \kappa_3, \kappa_4 \) are the axial strain, shear strain and curvature, respectively, defined with respect to the reference axis of the layer \( l \) as functions of only \( x \).

2.3. Constitutive equations

The axial strain \( D_1 \) of a fibre at the distance \( x \) from the reference axis of the layer \( l \) is defined as

\[ D_1 = D_1(X_1, X_2, \kappa_1) = \varepsilon_1(X_1) - X_2 \kappa_1 \] (18)

and the normal stress depends on this strain, i.e.

\[ \sigma_1 = \sigma_1(D_1) \] (19)

as well as the history of straining depending on the constitutive law taken. Likewise the shear stress generally depends on a shear strain at a point which, however, in the Timoshenko beam theory is taken as constant over the height of the layer:

\[ \tau_1 = \tau_1(\gamma_1) \] (20)

The stress resultants with respect to the reference axis of layer \( l \) are obtained by integrating these stresses and the normal stress couples over the layer’s area \( A \) and read

\[ N_l = \int A_l \sigma_1 \, dA, \]

\[ T_l = A_l \tau_1, \] (21)

\[ M_l = - \int A_l \sigma_1 \, dA, \]

where \( A_l \) is the shear area. Obviously, \( N_l, T_l, M_l \) represent the normal and shear force and the bending moment in layer \( l \) in the cross-section at \( X_1 \).

2.4. Equilibrium equations

According to the principle of virtual work for a static problem, the work of internal forces in the layered beam over virtual strains (\( \mathcal{V}_{BM} \)) is equal to the work of external forces over virtual displacements (\( \mathcal{V}_{FE} \)), i.e.

\[ \mathcal{V}_{BM} = \mathcal{V}_{FE} = 0, \] (22)

where, for a multi-layer beam composed of \( M \) layers, these virtual works are defined as

\[ \mathcal{V}_{BM} = \sum_{i=1}^{M} \int_{X_i}^{X_{i+1}} \left( \gamma_i \cdot N_i + \kappa_i \cdot M_i \right) \, dX_1, \quad N_i = \left\{ \begin{array}{c} N_{i1} \\ N_{i2} \\ \end{array} \right\}, \] (23)

\[ \mathcal{V}_{FE} = \sum_{i=1}^{M} \int_{X_i}^{X_{i+1}} \left( \int_{0}^{L} \left( \mathbf{f}_i \cdot \mathbf{i} + \mathbf{t}_i \cdot \mathbf{0} \right) \, dX_1 + \mathbf{b}_{i0} \cdot \mathbf{F}_{i0} + \mathbf{b}_{i1} \cdot \mathbf{M}_{i0} + \mathbf{b}_{i2} \cdot \mathbf{F}_{i2} + \mathbf{b}_{i3} \cdot \mathbf{M}_{i2} \right), \] (24)

where \( \mathbf{F}_{i0} \) and \( \mathbf{F}_{i2} \) are boundary point forces (Fig. 5), \( \mathbf{M}_{i0} \) and \( \mathbf{M}_{i2} \) are bending moments applied at the beam ends \( 0 \) (start) or \( L \) (end). The distributed force and moment loads are denoted by \( \mathbf{f}_i \) and \( m_i \).

The virtual strains and curvature are denoted by \( \gamma_i \) and \( \kappa_i \), and the virtual displacements and rotations by \( \mathbf{u}_i \) and \( \mathbf{\theta}_i \). From (16) and (17) it follows

\[ \gamma_i = \Lambda^T \left( \begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \end{array} \right), \] (25)

\[ \kappa_i = \left( \begin{array}{c} \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \kappa_4 \\ \end{array} \right). \] (26)

Using (25) and (26) the virtual work of internal forces becomes

\[ \mathcal{V}_{BM} = \sum_{i=1}^{M} \int_{X_i}^{X_{i+1}} \left( \gamma_i \cdot N_i + \kappa_i \cdot M_i \right) \, dX_1, \] (27)
while the virtual work of external loading is
\[ V_{ex} = \sum_{l=1}^{M} \int_{0}^{L} \left( \langle \mathbf{u}^T \mathbf{b} \rangle \right) \left\{ F_{l,0} \right\} dX_l + \left( \langle \mathbf{u}^T \mathbf{b} \rangle \right) \left\{ F_{l,0} \right\}, \]
(28)
with
\[ D_l = \begin{bmatrix} \mathbf{I} & -t_{0l} \\ \mathbf{0}^T & \mathbf{0} \end{bmatrix}, \]
(29)
\[ \mathbf{L} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix}, \]
(30)
where \( \mathbf{I} \) represents the \( 2 \times 2 \) unity matrix.

From (15) we obtain (note that \( t_{0l} = \bar{b}_l \begin{bmatrix} 1 \\ -1 \end{bmatrix} \mathbf{A}_0 e_2 = -\bar{b}_l t_l \))
\[ \mathbf{\bar{u}} = \mathbf{u} + \delta k w \mathbf{t}_{0l} + \mathbf{a}_l \mathbf{t}_{0l} + \mathbf{a}_l \mathbf{t}_{0l} + \mathbf{sgn}(-\mathbf{a}_l) \sum_{l=1}^{M} b_l \mathbf{t}_{0l}, \]
(31)
which serves to define the transformation
\[ \langle \mathbf{u}^T \mathbf{b} \rangle = \langle \mathbf{u} + \delta k w \mathbf{t}_{0l} + \mathbf{a}_l \mathbf{t}_{0l} + \mathbf{a}_l \mathbf{t}_{0l} + \mathbf{sgn}(-\mathbf{a}_l) \sum_{l=1}^{M} b_l \mathbf{t}_{0l} \rangle, \]
(32)
\[ \mathbf{\bar{u}} = \mathbf{u} + \delta k w \mathbf{t}_{0l} + \mathbf{a}_l \mathbf{t}_{0l} + \mathbf{a}_l \mathbf{t}_{0l} + \mathbf{sgn}(-\mathbf{a}_l) \sum_{l=1}^{M} b_l \mathbf{t}_{0l} \] 
\[ \mathbf{B}_l = \begin{bmatrix} \mathbf{1} & 0 & \ldots & 0 \\ \mathbf{0}^T & 0 & \ldots & 0 \end{bmatrix}, \]
(33)
\[ \mathbf{\bar{B}} = \begin{bmatrix} \mathbf{1} & 0 & \ldots & 0 \\ \mathbf{0}^T & 0 & \ldots & 0 \end{bmatrix} \]
\[ \delta_{kl} = \begin{cases} 1, & l = k \\ 0, & \text{otherwise}. \end{cases} \]
(35)
\[ d_{l,s} = \mathbf{sgn}(l - \alpha)(h_s - a_s), \]
(36)

Matrix \( \mathbf{B}_l \) in (33) is of the dimension \( 3 \times M \). In this way we eventually obtain the following results for the virtual work of the internal and the external forces:
\[ V_{bi} = \sum_{l=1}^{M} \int_{0}^{L} \left( \langle \mathbf{p}^T \mathbf{b} \rangle \right) \left( \mathbf{t}_{0l} \right)^T \left( \mathbf{N}_l \right) M_l \right\} dX_l, \]
(37)
\[ V_{bx} = \sum_{l=1}^{M} \left[ \int_{0}^{L} \left( \langle \mathbf{p}^T \mathbf{b} \rangle \right) \left( \mathbf{t}_{0l} \right)^T \right] \left( \mathbf{f}_l \right) M_l \right\} dX_l + \left( \langle \mathbf{p}^T \mathbf{b} \rangle \right) \left( \mathbf{f}_{0l} \right) M_{0l} \right\} \]
\[ \mathbf{L} \left( \mathbf{N}_l \right) M_l \right\} dX_l, \]
(38)
and finally (see Appendix A for the derivation)
\[ V_{bi} = \sum_{l=1}^{M} \left[ \int_{0}^{L} \left( \langle \mathbf{p}^T \mathbf{b} \rangle \right) \left( \mathbf{t}_{0l} \right)^T \right] \left( \mathbf{f}_l \right) M_l \right\} dX_l + \left( \langle \mathbf{p}^T \mathbf{b} \rangle \right) \left( \mathbf{f}_{0l} \right) M_{0l} \right\} \]
\[ \mathbf{L} \left( \mathbf{N}_l \right) M_l \right\} dX_l, \]
(39)
\[ V_{bx} = \sum_{l=1}^{M} \left[ \int_{0}^{L} \left( \langle \mathbf{p}^T \mathbf{b} \rangle \right) \left( \mathbf{t}_{0l} \right)^T \right] \left( \mathbf{f}_l \right) M_l \right\} dX_l + \left( \langle \mathbf{p}^T \mathbf{b} \rangle \right) \left( \mathbf{f}_{0l} \right) M_{0l} \right\} \]
\[ \mathbf{L} \left( \mathbf{N}_l \right) M_l \right\} dX_l, \]
(40)

The presented governing equations are highly non-linear and cannot be solved in a closed form. In order to solve them, it is necessary to choose in advance the shape of the test functions \( \langle \mathbf{u}, \mathbf{b} \rangle \), and later also the shape of the trial functions \( \langle \mathbf{u}, \mathbf{b} \rangle \).

Since there is a rotational discontinuity in each cracked layer (introduced in (14)) the vector of virtual unknown functions now becomes
\[ \mathbf{\bar{p}} = \mathbf{p} + k \left( \mathbf{0} ^T \mathbf{A}_c \mathbf{\varphi} \right), \]
(41)
with
\[ \mathbf{\bar{p}} = \langle \mathbf{u}^T, \mathbf{b}^T \rangle, \]
(42)
\[ \mathbf{A}_c = \begin{bmatrix} \delta_{lC} & 0 & \ldots & 0 \\ 0 & \delta_{lC} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \delta_{lC} \end{bmatrix}, \]
\[ \mathbf{\varphi} = \begin{bmatrix} \varphi_1 \varphi_2 \ldots \varphi_M \end{bmatrix}, \]
(43)
where \( \delta_{lC} \) is a flag denoting if layer \( l \) has cracked or not
\[ \delta_{lC} = \begin{cases} 0, & \epsilon_l < f_{lt}/E_c \\ 1, & \epsilon_l > f_{lt}/E_c \end{cases} \]
(44)
From (41) it follows
\[ \mathbf{\bar{p}} = \mathbf{p} + k \langle \mathbf{0}^T \mathbf{A}_c \mathbf{\varphi} \rangle, \]
(45)
so that the virtual work \( V_{bi} \) from (39) and \( V_{bx} \) from (40) now take the following form
\[ V_{bi} = \sum_{l=1}^{M} \int_{0}^{L} \left( \langle \mathbf{p}^T \mathbf{b} \rangle \right) \left( \mathbf{t}_{0l} \right)^T \left\{ \mathbf{N}_l \right\} M_l \right\} dX_l, \]
(46)
\[ V_{Bx} = \sum_{l=1}^{M} \left[ \int_0^L \left( p^l B^l \left\{ \frac{f_l}{m_l} \right\} + \tau^l \Delta_k k \bar{B}^l \left\{ \frac{f_l}{m_l} \right\} + \omega \delta_{xz} k \bar{t}_{xz} f_l \right) dx_1 + p^l \bar{B}^l \left\{ \frac{\bar{F}_l}{M_l} \right\} + p^l \bar{B}^l \left\{ \frac{\bar{F}_l}{M_l} \right\} \right] \]  
\( (47) \)

### 3. Incorporation of reinforcement layer

#### 3.1. Kinematics of slippage between reinforcement and concrete

In the considered multi-layer beam, the beam reference axis may be taken arbitrarily, so let us take it to be placed in the middle of the layer which surrounds the reinforcement. Let the reinforcement make an arbitrarily thin reinforcement layer of a finite cross-section, which let also be placed in the middle of the surrounding layer. The reinforcement layer, denoted as \( r \), may slip with respect to the surrounding concrete layer as shown in Fig. 6. From Fig. 6 we see that \( u_r(X_1) = u(X_1) + s(X_1)t_1(X_1) \),

\[ (48) \]

which in the geometrically linear analysis turns into \( u_r(X_1) = u(X_1) + s(X_1)t_01 \),

\[ (49) \]

since \( s_01 = s_01 + (t_1 \otimes \nabla u)_{ref} + H.O.T. \) = \( s_01 + H.O.T. \).

As shown in (10), when the concrete layer \( \alpha \) cracks it also slips with respect to the imagined continuous beam reference line. The total slip of the reinforcement with respect to the cracked concrete surrounding it is thus obtained as the magnitude of difference between the reinforcement and surrounding concrete layer \( \alpha \) displacements as

\[ f(X_1) = ||u_r(X_1) - u(X_1)|| = ||f(X_1)||, \]

\[ (50) \]

with the total slip vector \( f(X_1) \) shown in Fig. 7. From (10) and (49) it follows that in the geometrically linear case considered in this paper

\[ f(X_1) = [s(X_1) - p(X_1)]t_01, \]

\[ (51) \]

i.e. the total slip follows as

\[ f(X_1) = s(X_1) - \delta_{xz} k(X_1) w. \]

\[ (52) \]

The number of unknowns now increases by one additional function \( s(X_1) \), thus making the number of total unknown functions \( 3 + M \) plus the parameters \( w \) and crack profile angles \( \phi_1 \) in every cracked layer.

#### 3.2. Additional internal and external virtual work due to reinforcement

The internal virtual work for the multi-layer beam now needs to be extended by two additional terms. The first of these relates to the virtual work due to the axial force in the reinforcement owing to its deformability, while the second term comes as a result of slippage of the reinforcement bar with respect to the surrounding concrete and the consequent bond stresses. Additionally, since the reinforcement bar is treated as a reinforcement layer, we also assume that there exist external forces acting on that layer which are consistent with its specific properties.

##### 3.2.1. Internal virtual work due to axial force in reinforcement

Internal virtual work due to the axial force in the reinforcement is initially derived by considering reinforcement as a beam layer of finite area and zero thickness,

\[ V_{r} = \int_0^L \left( \left( \bar{u}^T - \bar{u} \right) \bar{D}^I \left\{ \begin{array}{c} \Delta \\frac{0}{0} \end{array} \right\} \right) dX_1, \]

\[ (53) \]

where

\[ -N_r = \left\{ \begin{array}{c} N_r \\frac{0}{0} \end{array} \right\}, M_r = 0, N_r = N_r(\varepsilon_r), \Delta_r = [t_01 \ t_02], \bar{D} = \left\{ \begin{array}{c} \bar{D}_{01} \ end{array} \right\}, \]

while the axial strain \( \varepsilon_r = t_01 \cdot u_r(X_1) \) follows from (49) as

\[ \varepsilon_r = \varepsilon' + t_01 \cdot u'. \]

Using (49), virtual work (53) also transforms into

\[ V_{r} = \int_0^L \left( \left( \bar{u}^T + \bar{u} \right) \bar{D}^I \left\{ \begin{array}{c} \Delta \\frac{0}{0} \end{array} \right\} \right) dX_1, \]

\[ (55) \]

or further

\[ V_{r} = \int_0^L \left( \left( \bar{u}^T + \bar{u} \right) \bar{D}^I \left\{ \begin{array}{c} \Delta \\frac{0}{0} \end{array} \right\} \right) dX_1 = \int_0^L \left( \left( \bar{u}^T + \bar{u} \right) \bar{D}^I \left\{ \begin{array}{c} \Delta \\frac{0}{0} \end{array} \right\} \right) dX_1. \]

\[ (56) \]

Since \( u^T t_01 = p^T t_01 \), we finally get

\[ V_{r} = \int_0^L \left( \bar{D}^I \left\{ \begin{array}{c} \Delta \\frac{0}{0} \end{array} \right\} \right) dX_1 = \int_0^L \left( \bar{D}^I \left\{ \begin{array}{c} \Delta \\frac{0}{0} \end{array} \right\} \right) dX_1. \]

\[ (57) \]

##### 3.2.2. Internal virtual work due to bond stresses

This virtual work comes as a result of slippage of the reinforcement bar with respect to the surrounding concrete. The adhesive shear-bond stress \( \tau \) thus produces a virtual work per unit of contact area

\[ \phi \pi \overline{f}, \]

\[ (58) \]

and the virtual work per unit of contact length

\[ \phi \pi \overline{f}. \]

\[ (59) \]

where \( \phi \) is the diameter of the reinforcement bar (or the sum of the diameters of all the reinforcement bars of the layer). Note that in the present model the reinforcement is designed to act as a layer of zero thickness and the factor \( \phi \pi \) acts as a coefficient defining the reinforcement circumference and is introduced only for the sake of convenience in modelling real problems. Along the whole length of the element we get the resulting virtual work due to the bond stresses as

\[ V_{b} = \phi \pi \int_0^L \overline{f} dX_1, \]

\[ (60) \]

and, by using (52),

\[ V_{b} = \phi \pi \int_0^L (\tau - \delta_{xz} k \bar{w}) \overline{f} dX_1, \]

\[ (61) \]
where \( \tau = \tau(f) \) is a given bond–slip relationship with the total slip defined in (52). In the present work, the actual bond–slip relationship is not the subject of any detailed investigation and will be addressed in our future work (along with the nonlinearities in concrete and reinforcement). An attempt to introduce a realistic nonlinear bond–slip relationship in the present model may require some sort of averaging if it is to be used to analyse the problems with thick reinforcement bars in which the slip at the top and the bottom of the bar may not be necessarily taken as equal.

3.2.3. External virtual work due to loading on the reinforcement

External virtual work due to the applied distributed and concentrated axial loading acting on the reinforcement may be stated as

\[
V_{r.e} = \int_0^L \left( f_0 \mathbf{u} \mathbf{t}_{01} + \mathbf{u} \mathbf{t}_{0} \mathbf{F}_{r0} \mathbf{t}_{01} + \mathbf{u} \mathbf{t}_{0} \mathbf{F}_{rl} \mathbf{t}_{01} \right) dX_1,
\]

where \( f_0 \) is a distributed axial loading along the reinforcement, while \( F_{r0} \) and \( F_{rl} \) are concentrated axial forces acting at the ends of the reinforcement. Substituting (49) into this result gives

\[
V_{r.e} = \int_0^L \left( \mathbf{u} \mathbf{t}_{01} \right)^T \left( \mathbf{f}_0 \mathbf{d} + \mathbf{u} \mathbf{e}_0 \mathbf{F}_{r0} \mathbf{t}_{01} + \mathbf{u} \mathbf{e}_0 \mathbf{F}_{rl} \mathbf{t}_{01} \right) dX_1.
\]

with \( \mathbf{u}_0, \mathbf{u}_1 \) and \( \mathbf{0}_0 \) as the boundary values (at \( X_1=0 \) and \( X_1 = L \), respectively) of the fields \( \mathbf{u} \) and \( \mathbf{u}_0 \). Since

\[
\mathbf{u}^{T} \mathbf{t}_{01} = \mathbf{p}^{T} \mathbf{t}_{01},
\]

we finally obtain

\[
V_{r.e} = \int_0^L \left( \mathbf{u}^{T} \mathbf{p} \right)^T \left[ \mathbf{f}_0 \mathbf{d} + \mathbf{e}_0 \mathbf{F}_{r0} \mathbf{t}_{01} + \mathbf{e}_0 \mathbf{F}_{rl} \mathbf{t}_{01} \right] dX_1.
\]

4. Interpolation of test functions and approximated total virtual work

Virtual work for the internal forces of the multi-layer beam \( V_{b,i} \) should be now combined with the internal virtual work in the reinforcement bar \( V_{r,i} \) and the internal virtual work of the bond stresses \( V_{b,i} \) to give the total internal virtual work

\[
V_i = V_{b,i} + V_{r,i} + V_{b,i}
\]

to be used along with the total external virtual work obtained from (47) and (65)

\[
V_e = V_{b,e} + V_{r,e},
\]

in the principle of virtual work \( V = V_i - V_e = 0 \).

4.1. Interpolation of test functions

For a finite number of nodes on the beam \( N \) it is assumed that the virtual displacements, rotations and slip are known at the nodes and interpolated between the nodes. The unknown functions \( s(X_1) \) and \( \mathbf{p}(X_1) \) are approximated as

\[
s(X_1) = \sum_{j=1}^{N} l_j(X_1) s_j,
\]

\[
\mathbf{p}(X_1) = \sum_{j=1}^{N} \mathbf{p}_j(X_1) \mathbf{p}_j,
\]

where \( l_j(X_1) \) are the standard Lagrangian interpolation functions and \( \mathbf{p}_j(X_1) \) is as yet an undefined matrix of interpolation functions, with

\[
\mathbf{p}_j = \left( \mathbf{u}_j, \mathbf{u}_{j1}, \mathbf{u}_{j2}, \ldots, \mathbf{u}_{jN} \right)^T.
\]

Substituting (68)–(70) in (46), (47), (57), (61) and (65) gives the following finite-element approximations for the virtual work of the internal and external forces

\[
V_{b,e} = \sum_{j=1}^{N} \mathbf{p}_j^{T} \mathbf{f}_0 \mathbf{d} + \sum_{j=1}^{N} \mathbf{p}_j^{T} \mathbf{e}_0 \mathbf{F}_{r0} \mathbf{t}_{01} + \sum_{j=1}^{N} \mathbf{p}_j^{T} \mathbf{e}_0 \mathbf{F}_{rl} \mathbf{t}_{01} + \sum_{j=1}^{N} \mathbf{p}_j^{T} \mathbf{F}_{r0} \mathbf{t}_{01} + \sum_{j=1}^{N} \mathbf{p}_j^{T} \mathbf{F}_{rl} \mathbf{t}_{01}.
\]

\[
V_{e} = \sum_{j=1}^{N} \mathbf{p}_j^{T} \mathbf{f}_0 \mathbf{d} + \sum_{j=1}^{N} \mathbf{p}_j^{T} \mathbf{e}_0 \mathbf{F}_{r0} \mathbf{t}_{01} + \sum_{j=1}^{N} \mathbf{p}_j^{T} \mathbf{e}_0 \mathbf{F}_{rl} \mathbf{t}_{01} + \sum_{j=1}^{N} \mathbf{p}_j^{T} \mathbf{F}_{r0} \mathbf{t}_{01} + \sum_{j=1}^{N} \mathbf{p}_j^{T} \mathbf{F}_{rl} \mathbf{t}_{01}.
\]
\[ V_{\psi} = \sum_{j=1}^{N} \left( \int_{0}^{L} I_{j} N_{X} dX + \int_{0}^{L} \psi_{j}^{T} \left[ \begin{array}{c} t_{0j} \\ 0_{M} \end{array} \right] N_{Y} dX \right). \]  

\[ V_{\phi} = \phi \pi \sum_{j=1}^{N} \left( \int_{0}^{L} I_{j} \phi^{T} dX - \phi \phi^{T} \right) \int_{0}^{L} \delta_{X} k dX. \]  

\[ V_{\chi} = \sum_{j=1}^{N} \left( \int_{0}^{L} I_{j} \chi^{T} dX + \int_{0}^{L} \chi_{j}^{T} \left[ \begin{array}{c} t_{0j} \\ 0_{M} \end{array} \right] N_{Y} dX + \int_{0}^{L} \phi_{j} \right) \int_{0}^{L} F_{\psi} dX. \]

It is instructive to consider linked interpolation, which implies higher order interpolation for the transverse displacements than the rotations, defined as \( J \) (Jelenić and Papan, 2011):

\[ u^{l} = \sum_{j=1}^{N} I_{j} u_{j} + \frac{N}{N_{1}} \prod_{j=1}^{N} \left( -\frac{1}{k} \right) \left( N - 1 \right) N_{j} \psi_{j} \beta_{x} \psi_{j} \psi_{j} \]

\[ \psi_{j} = \begin{bmatrix} l_{j} & 0 & 0 & \cdots & \frac{N}{N-1} \left( \left( N - 1 \right) \sin \psi \cdots 0 \right) \\ 0 & l_{j} & 0 & \cdots & \frac{N}{N-1} \left( \left( N - 1 \right) \cos \psi \cdots 0 \right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & l_{j} \end{bmatrix} \]

or shorter

\[ \psi_{j} = \begin{bmatrix} l_{j} & 0 & 0 & \cdots & l_{j} \end{bmatrix} \]

with \( m = \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix} \) and the unity in \( m \) at the position \( x + 2 \).

4.2. Approximation of the internal virtual work

Substituting (78) in (71) and (73) yields the discretised internal virtual work as

\[ V_{i}^{i} = V_{\phi}^{i} + V_{\psi}^{i} + V_{\chi}^{i} = \sum_{j=1}^{N} \left( \int_{0}^{L} I_{j} N_{X} dX + \int_{0}^{L} \psi_{j}^{T} \left[ \begin{array}{c} t_{0j} \\ 0_{M} \end{array} \right] N_{Y} dX \right) + \int_{0}^{L} N_{X} dX. \]  

\[ \int_{0}^{L} I_{j} \phi^{T} dX, \]  

\[ \int_{0}^{L} \psi_{j}^{T} \left[ \begin{array}{c} t_{0j} \\ 0_{M} \end{array} \right] N_{Y} dX. \]  

\[ q_{\phi}^{l} = \frac{M}{l} \int_{0}^{L} I_{j} N_{X} dX + \frac{M}{l} \int_{0}^{L} N_{X} dX. \]  

\[ q_{\psi}^{l} = \phi \pi \int_{0}^{L} I_{j} \phi^{T} dX, \]  

\[ q_{\chi}^{l} = \frac{M}{l} \int_{0}^{L} I_{j} \chi^{T} dX. \]  

4.3. Approximation of the external virtual work

Substituting (78) in (72) and (75) gives the discretised virtual work of the external forces as

\[ V_{e}^{i} = V_{\phi}^{i} + V_{\psi}^{i} \]

\[ q_{\phi}^{l} = \phi \pi \int_{0}^{L} I_{j} \phi^{T} dX + \delta_{\chi} F_{\chi} + \delta_{\phi} F_{\phi}, \]  

\[ q_{\psi}^{l} = \phi \pi \int_{0}^{L} I_{j} \phi^{T} dX + \delta_{\phi} F_{\phi} + \delta_{\chi} F_{\chi}, \]  

\[ q_{\chi}^{l} = \phi \pi \int_{0}^{L} I_{j} \phi^{T} dX + \delta_{\phi} F_{\phi} + \delta_{\chi} F_{\chi}. \]
\[ \mathbf{q}_{\text{hp}} = \sum_{i=1}^{M} \int_0^l \mathbf{B}_i^T \begin{bmatrix} \mathbf{f}_i \ \mathbf{m}_i \end{bmatrix} dX_1 + \mathbf{m} \sum_{i=1}^{M} \int_0^l \mathbf{K}_{\text{e}} \mathbf{t}_i \mathbf{f} dX_1 \\
+ \delta_1 \sum_{i=1}^{M} \mathbf{B}_i^T \begin{bmatrix} \mathbf{F}_{10} \ \mathbf{M}_{10} \end{bmatrix} + \delta_0 \sum_{i=1}^{M} \mathbf{B}_i^T \begin{bmatrix} \mathbf{F}_{1i} \ \mathbf{M}_{1i} \end{bmatrix}, \quad (94) \]

\[ \mathbf{q}_{\text{hp}}^l = \sum_{i=1}^{M} \int_0^l \begin{bmatrix} \mathbf{f}_{10} \\
\mathbf{0}_l \end{bmatrix} dX_1 + \delta_1 \begin{bmatrix} \mathbf{f}_{10} \\
\mathbf{0}_l \end{bmatrix} + \delta_0 \begin{bmatrix} \mathbf{f}_{1i} \\
\mathbf{0}_l \end{bmatrix} \quad (95) \]

\[ \mathbf{q}_{\text{hp}} = \lambda \sum_{i=1}^{M} \int_0^l k \begin{bmatrix} \mathbf{f}_i \ \mathbf{m}_i \end{bmatrix} dX_1, \quad (96) \]

\[ \mathbf{q}_{\text{hp}}^l = \delta \sum_{i=1}^{M} \int_0^l k \mathbf{t}_i \mathbf{f} dX_1. \quad (97) \]

Discretised virtual work of the external forces (92) may be expressed as

\[ V_e = \langle \mathbf{p}^T \mathbf{\varphi} \rangle \mathbf{q}_{\text{hp}}, \mathbf{q}_{\text{hp}}^l \rangle, \quad (98) \]

where

\[ \mathbf{q}_{\text{hp}} = \begin{bmatrix} \mathbf{q}_{\text{hp}} \mathbf{q}_{\text{hp}}^l \end{bmatrix}, \]

\[ \mathbf{q}_{\text{hp}}^l = \begin{bmatrix} \mathbf{q}_{\text{hp}} \mathbf{q}_{\text{hp}}^l \end{bmatrix}. \quad (99) \]

4.4. Non-linear solution procedure

From (79) and (92) we establish the principle of virtual work

\[ \mathbf{V}_e \equiv \mathbf{V}_e - \mathbf{V}_e^0 = 0 \rightarrow \mathbf{g} \equiv \begin{bmatrix} \mathbf{g}_{\text{ps}} \mathbf{q}_{\text{ps}} \mathbf{q}_{\text{ps}}^l \end{bmatrix}, \quad (100) \]

The vector of residual forces is expanded in Taylor’s series up to a linear form as

\[ \mathbf{g} + \mathbf{\Delta g} = \mathbf{0}. \quad (101) \]

Applying a linearization yields

\[ \begin{bmatrix} \mathbf{k}_{\text{pp}} & \mathbf{k}_{\text{pp}}^l \\
\mathbf{k}_{\text{pp}}^l & \mathbf{k}_{\text{pp}}^l \end{bmatrix} \begin{bmatrix} \mathbf{\Delta p}_s \\
\mathbf{\Delta p}_s^l \end{bmatrix} = - \begin{bmatrix} \mathbf{g}_{\text{ps}} \\
\mathbf{g}_{\text{ps}}^l \end{bmatrix}, \quad (102) \]

where \( \mathbf{k}_{\text{pp}}, \mathbf{k}_{\text{pp}}^l \) and \( \mathbf{k}_{\text{pp}}^l \) follow from

\[ \begin{bmatrix} \mathbf{k}_{\text{pp}} & \mathbf{k}_{\text{pp}}^l \\
\mathbf{k}_{\text{pp}}^l & \mathbf{k}_{\text{pp}}^l \end{bmatrix} \begin{bmatrix} \mathbf{\varphi} \\
\mathbf{\varphi}^l \end{bmatrix} = \begin{bmatrix} \mathbf{\varphi}_{\text{ps}} \mathbf{\varphi}_{\text{ps}}^l \end{bmatrix}, \quad (103) \]

with \( \nabla_{ps} \) and \( \nabla_{ow} \) as the vectors of partial derivatives with respect to \( \mathbf{p}_s \) and \( \mathbf{\varphi}_{ow} \). The stiffness matrix blocks \( \mathbf{k}_{\text{pp}}, \mathbf{k}_{\text{pp}}^l \) and \( \mathbf{k}_{\text{pp}}^l \) are derived in Appendix B. The solution is obtained iteratively using a Newton–Raphson procedure until a satisfying accuracy is achieved.

To complete the process it is necessary to define the constitutive laws for steel (including the information on yield stress and hardening). To account for this, the bond–slip relationship. If all these laws were linear and the tensile concrete strength high enough, Eq. (103) would result in constant stiffness matrices for a geometrically linear layered beam.

At this point a distinction should be made between two different types of elements: (i) a bar element subject to uniaxial loading and (ii) a beam element subject to uniaxial loading and bending. In case of the beam element, not all the layers in a section may crack—compression must exist at one side of the cross-section, either at the bottom or at the top. Additionally, the crack opening in the middle of layer \( z \) ceases to be independent of the crack profile angles. Instead, the relationship between these parameters may be derived from Fig. 8 as

\[ w = \sum_{l=2}^{M} \delta \mathbf{h}_l(N_{l-1}) + \delta_\sigma \mathbf{h}_s \mathbf{h}_{sw} = \mathbf{h}_{s}^T \mathbf{h}_s \mathbf{\varphi}, \quad (104) \]

where

\[ \mathbf{h}_s = \begin{bmatrix} 0 & \cdots & 0 & \mathbf{h}_s & \mathbf{h}_{sw} \end{bmatrix}^T. \quad (105) \]

Using (104), expression (79) may be rewritten so that \( w \) is no longer an unknown parameter,

\[ V_e = \sum_{j=1}^{N} \sum_{j=1}^{N} \mathbf{v}^j_{\text{ij}} (q_{\text{ij}} + q_{\text{ij}}^l) + \sum_{j=1}^{N} \mathbf{v}^j (q_{\text{ij}} + q_{\text{ij}}^l) \]

and accordingly (88) transforms into

\[ \mathbf{V}_e = \begin{bmatrix} \mathbf{q}_{\text{ps}} \\
\mathbf{q}_{\text{ps}}^l \end{bmatrix} \end{bmatrix}, \quad (107) \]

with

\[ \mathbf{q}_{\text{pp}} = \mathbf{q}_{\text{pp}}^l + \mathbf{\Delta h}_s (q_{\text{sw}}^l - q_{\text{sw}}^l). \quad (108) \]

In a similar fashion, the external virtual work (92) may be also transformed and the non-linear vector residual equation to be solved is now

\[ \mathbf{g} = \begin{bmatrix} \mathbf{g}_{\text{ps}} \\
\mathbf{g}_{\text{ps}} \end{bmatrix} - \begin{bmatrix} \mathbf{q}_{\text{ps}} \\
\mathbf{q}_{\text{ps}}^l \end{bmatrix} = \mathbf{0}, \quad (109) \]

where

\[ \mathbf{q}_{\text{pp}} = \mathbf{q}_{\text{pp}}^l + \mathbf{\Delta h}_s q_{\text{sw}}^l. \quad (110) \]

This vector of residual forces is now again expanded in Taylor’s series up to a linear form, which after the linearisation gives

\[ \begin{bmatrix} \mathbf{k}_{\text{pp}} & \mathbf{k}_{\text{pp}}^l \\
\mathbf{k}_{\text{pp}}^l & \mathbf{k}_{\text{pp}}^l \end{bmatrix} \begin{bmatrix} \mathbf{\Delta p}_s \\
\mathbf{\Delta p}_s^l \end{bmatrix} = - \begin{bmatrix} \mathbf{g}_{\text{ps}} \\
\mathbf{g}_{\text{ps}}^l \end{bmatrix}, \quad (111) \]
where \( \mathbf{k}_{pp}, \mathbf{k}_{0pp} \) and \( \mathbf{k}_{spp} \) follow from
\[
\mathbf{k}_{pp} = \mathbf{q}_p \odot \nabla \mathbf{p},
\]
\[
\mathbf{k}_{0pp} = \mathbf{q}_p \odot \nabla \mathbf{p},
\]
\[
\mathbf{k}_{spp} = \mathbf{q}_s \odot \nabla \mathbf{p},
\]
with \( \nabla \mathbf{p} \) and \( \nabla \mathbf{p} \) as the vectors of partial derivatives with respect to \( \mathbf{p} \) and \( \mathbf{q} \). The stiffness matrix blocks \( \mathbf{k}_{pp}, \mathbf{k}_{0pp} \) and \( \mathbf{k}_{spp} \) are given in Appendix C.

5. Numerical examples

In this section, a couple of examples are presented in which the novel finite element has been tested for monotonically increasing loading. In all the examples the position of the crack is assumed in the middle of an element (\( X_{IC} = L_0/2 \) where \( L_0 \) is the length of the element). In order to evaluate the integrals which contain function \( k(X_1) \), which is continuous everywhere except at the crack position, the integration interval is divided into subintervals not including the crack position: \( [0, L_0/2] \cup [L_0/2, L_0] \). Two-noded elements and Gauss quadrature of order two have been used. The Newton–Raphson tolerance for the sum of the norms is set to \( 10^{-6} \). Concrete and steel are assumed to have a linear elastic stress–strain relationship. The bond stress-slip relationship is also assumed to be linear (\( \tau = C_f \) where \( C_f \) is the bond stiffness modulus). It should be noted that all the constitutive ingredients have been chosen as linear elastic in order to concentrate on the cracking mechanism as described by the proposed multi-layered beam kinematic model.

5.1. Reinforced-concrete tie

The first example is a reinforced-concrete tie of square cross-section containing one reinforcing bar of diameter 12 mm running longitudinal through the centroid of cross-section shown in Fig. 9. The tensile axial force is applied to the ends of the reinforcing bar protruding from each end of the concrete element. The material parameters are given as: Young’s modulus of concrete \( E_s = 21000 \text{ MPa} \), Young’s modulus of steel \( E_p = 210000 \text{ MPa} \), bond stiffness modulus \( C_s = 30 \text{ 000 MPa/m} \) while concrete tensile strength is set as 2.1 MPa. Linear Lagrangian interpolation for displacements and rotations as well as for the function \( s \) has been used. Two-point Gaussian quadrature has been utilised (applied to each half of the domain for the integrals containing step function \( k(X_1) \)).

First, we will examine only the slip (without cracking) and verify it by comparing with the analytical solution (see e.g. Creazzo and Russo, 1999). The applied force equals \( F = 20 \text{ kN} \). By analysing only the right half of the tie, and placing the origin of the global coordinate system in the middle of the tie, for the slip and the displacement we get:

\[
\begin{align*}
\phi(12) &= 0.1 \text{ m} \\
L &= 2.0 \text{ m}
\end{align*}
\]

\[ s(X_1) = \frac{F}{E A_b} \sinh \frac{b X_1}{\beta} \cosh \frac{p_l}{\beta}, \]

\[ u(X_1) = -\frac{2F}{\beta^2} \sinh \frac{b X_1}{\beta} \cosh \frac{p_l}{\beta} + \frac{2F}{\beta^2} X_1, \]

with
\[
\beta = \sqrt{B_{CS}(1/E_b A_b + 1/E_p A_p)}, \quad \alpha = B_{CS}/(E_b A_b) E_p A_p, \quad B_{CS} = \phi \pi C_s,
\]

where \( l_k \) is the half-length of the tie while \( A_b \) and \( A_s \) are steel and concrete areas, respectively. In Fig. 10, the analytical solutions representing slip and displacement for the right half of the tie are depicted.

Table 1 summarises the results for the slip and displacement depending on the number of single-layer elements at the free end (\( X_1 = l_k \)), together with the analytical solution. The displacement and the slip at the left end are fixed while at the right end they are free.

A comparison of the results for slip and displacement at the free end is shown in Fig. 11. The relative errors in slip and displacement are defined as \( \epsilon_s = (s_{BCs} - s_{BCb})/s_{BCs} \) and \( \epsilon_u = u_{BCs}/u_{BCb} \), where \( s_{BCs} \) and \( u_{BCs} \) are the analytical solutions, while \( s_{BCb} \) and \( u_{BCb} \) are the proposed finite-element solutions for the slip and the displacement at the free end of the tie. We can notice that the relative error for the displacement is much smaller than for the slip for the same number of elements. As it can be seen from the graph, very good results are achieved even for a small number of elements: for just 8 elements, the relative error of the slip is less than 3.5% while for the displacement, the error is less than 0.5%, and the numerical procedure is clearly converging towards the exact solution.

In our second test the ability of the proposed procedure to predict crack occurrence and development is investigated. In order to demonstrate this process we have considered the same reinforced-concrete tie and examined different meshes of single-layer elements. The whole tie has been modelled as a simply supported beam with the left end fixed (axial displacement of the beam reference line set to zero) and the right end freely moving in the axial direction. The reinforcement has been axially loaded at both ends with the corresponding slips unrestrained. The global coordinate reference system is placed at the left end of the tie. The applied tensile force is increased monotonically, thus the cracks form one after another as soon as the tensile strength at the mid-point of an element is reached. The 1st crack opens in the middle of the tie, then by increasing the force, 2nd and 3rd crack form at the middle of the right and the left half of the tie respectively, 4th to 7th cracks form halfway between the existing cracks, and so on (Fig. 12).

Since the analytical solution for the displacement is known, the cracking forces may be easily determined – by differentiating it with respect to \( X_1 \) and multiplying by the Young’s modulus of concrete, i.e.

\[
\sigma(X_1) = \frac{2F}{\beta^2} \left( \frac{\cosh \frac{b X_1}{\beta} + 1}{\cosh \frac{p_l}{\beta}} \right) E_b,
\]

where \( l_k \) is the distance between two cracks (\( l_k = l, L/2, L/4, \ldots \)). When this stress reaches the tensile strength, a crack will open. The crack widths may be also derived from the analytical model as follows – simply by adding up the two slips (obtained for the observed cracking force) on either side of the crack. The analytical result for cracking forces and crack widths are given in Table 2.

The number of elements in meshes has been chosen in such a way so that we could track the occurrence and development of the first seven cracks. We start first with the mesh of seven elements of equal length, and then uniformly make the mesh denser always maintaining an odd number of the elements in a mesh, so that we get 15-, 31-, 63- and 127-element meshes. The expected
crack positions for these meshes together with the analytical solution from above are given in Table 3. More accurate results are expected for forces that will cause a crack to open in those meshes where the predicted crack positions are closer to the exact crack position. Obviously, in all the meshes, 4th and 5th crack will for this reason open prior to 6th and 7th crack, but the difference between the forces that cause these pairs of cracks to open will reduce with refinement of the mesh.

Table 1
Slip and displacement (mm) at the free end of the tie.

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>8</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>Analytical solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slip</td>
<td>0.0451</td>
<td>0.0797</td>
<td>0.1024</td>
<td>0.1119</td>
<td>0.1147</td>
<td>0.1154</td>
<td>0.1156</td>
<td>0.1157</td>
</tr>
<tr>
<td>Displacement</td>
<td>0.0810</td>
<td>0.0775</td>
<td>0.0752</td>
<td>0.0742</td>
<td>0.0739</td>
<td>0.0738</td>
<td>0.0738</td>
<td>0.0738</td>
</tr>
</tbody>
</table>

Fig. 10. Analytical solution for slip and displacement of the reinforced-concrete tie (right half).

Table 2
Cracking forces (kN) and crack widths (mm) – analytical model.

<table>
<thead>
<tr>
<th>Force</th>
<th>1st</th>
<th>2nd and 3rd</th>
<th>4th, 5th, 6th and 7th</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.4073</td>
<td>0.2704</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24.6691</td>
<td>0.2707</td>
<td>0.2707</td>
<td></td>
</tr>
<tr>
<td>34.1606</td>
<td>0.2849</td>
<td>0.2849</td>
<td>0.2849</td>
</tr>
</tbody>
</table>

Fig. 11. Slip and displacement at the free end: (a) analytical and numerical results, (b) relative errors of the numerical results.

Table 3
Crack positions expressed in terms of L in the existing meshes and analytical solution.

<table>
<thead>
<tr>
<th>Number of elements</th>
<th>6th</th>
<th>2nd</th>
<th>4th</th>
<th>1st</th>
<th>5th</th>
<th>3rd</th>
<th>7th</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0.071</td>
<td>0.214</td>
<td>0.357</td>
<td>0.500</td>
<td>0.643</td>
<td>0.786</td>
<td>0.929</td>
</tr>
<tr>
<td>15</td>
<td>0.100</td>
<td>0.233</td>
<td>0.367</td>
<td>0.500</td>
<td>0.633</td>
<td>0.767</td>
<td>0.900</td>
</tr>
<tr>
<td>31</td>
<td>0.113</td>
<td>0.242</td>
<td>0.371</td>
<td>0.500</td>
<td>0.629</td>
<td>0.758</td>
<td>0.887</td>
</tr>
<tr>
<td>63</td>
<td>0.119</td>
<td>0.246</td>
<td>0.373</td>
<td>0.500</td>
<td>0.627</td>
<td>0.754</td>
<td>0.881</td>
</tr>
<tr>
<td>127</td>
<td>0.122</td>
<td>0.248</td>
<td>0.374</td>
<td>0.500</td>
<td>0.626</td>
<td>0.752</td>
<td>0.878</td>
</tr>
</tbody>
</table>

Analytical solution 0.125 0.25 0.375 0.500 0.625 0.75 0.875

Fig. 12. Crack formation process in the reinforced-concrete tie analysed.
The forces that cause a particular crack to open for different finite-element meshes are given in Table 4 and Fig. 13. The force needed for 1st crack to open is similar in all the meshes, and the same may be said for 2nd and 3rd crack even though for the uniform meshes of an odd number of elements considered here the position of these two cracks may not be exactly predicted by the proposed uniform meshes. The following cracks (4th to 7th) should occur at the same force (see the above analytical solution), since the distances between the existing cracks are the same in the entire tie. As already explained, the reason why two by two cracks appear instead of four at once is again in the mesh; none of these pairs of cracks have the position that may be exactly predicted by the finite-element solution proposed and the two pairs of cracks form at the positions which are not equally distant from the exact positions. The more elements we use, the closer are the forces causing 4th/5th and 6th/7th crack and they would eventually merge into one for an infinite number of elements.

<table>
<thead>
<tr>
<th>Crack number</th>
<th>Number of elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>23,379</td>
<td>23,400</td>
</tr>
<tr>
<td>23,408</td>
<td>23,407</td>
</tr>
<tr>
<td>23,407</td>
<td>23,407</td>
</tr>
<tr>
<td>24,100</td>
<td>24,577</td>
</tr>
<tr>
<td>24,648</td>
<td>24,664</td>
</tr>
<tr>
<td>24,668</td>
<td>24,668</td>
</tr>
<tr>
<td>32,009</td>
<td>32,385</td>
</tr>
<tr>
<td>33,263</td>
<td>33,719</td>
</tr>
<tr>
<td>33,943</td>
<td>33,943</td>
</tr>
<tr>
<td>47,610</td>
<td>36,522</td>
</tr>
<tr>
<td>35,123</td>
<td>34,609</td>
</tr>
<tr>
<td>34,378</td>
<td>34,378</td>
</tr>
</tbody>
</table>

**Fig. 13.** Cracking forces depending on the number of elements.

Table 5
Cracking forces (kN) and crack widths (mm) for 9-element non-uniform mesh.

<table>
<thead>
<tr>
<th>Force (kN)</th>
<th>Crack number</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.386</td>
<td>0.2580</td>
</tr>
<tr>
<td>24.291</td>
<td>0.2596</td>
</tr>
<tr>
<td>35.574</td>
<td>0.2934</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Force (kN)</th>
<th>Crack number</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.405</td>
<td>0.2677</td>
</tr>
<tr>
<td>24.622</td>
<td>0.2682</td>
</tr>
<tr>
<td>34.079</td>
<td>0.2836</td>
</tr>
</tbody>
</table>

**Table 6**
Cracking forces (kN) and crack widths (mm) for 25-element non-uniform mesh.

<table>
<thead>
<tr>
<th>Force (kN)</th>
<th>Crack number</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.407</td>
<td>0.2702</td>
</tr>
<tr>
<td>24.668</td>
<td>0.2710</td>
</tr>
<tr>
<td>33.943</td>
<td>0.2844</td>
</tr>
<tr>
<td>34.378</td>
<td>0.2881</td>
</tr>
</tbody>
</table>

**Table 7**
Crack widths (mm) for 127-element uniform mesh.

<table>
<thead>
<tr>
<th>Force (kN)</th>
<th>Crack number</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.407</td>
<td>0.2881</td>
</tr>
<tr>
<td>24.668</td>
<td>0.2881</td>
</tr>
<tr>
<td>33.943</td>
<td>0.2881</td>
</tr>
</tbody>
</table>

Note that for this simple example in which it is easy to spot the exact crack positions, a much better solution may be obtained if the finite-element mesh utilised is chosen such that these positions may be exactly hit. The results for two such meshes – a 9-element mesh and a 25-element mesh with the first and the last element half the length of the other elements – are given in Table 5 and Table 6. As expected, in these meshes 4th to 7th cracks occur at a same force.

Next, we analyse the crack widths and compare it with the analytical solution given earlier (see Table 2). The results for the 25-element non-uniform mesh are given in Table 6, while the results for the 127-element uniform mesh are given in Table 7. When these results are compared to those obtained analytically (see Table 2) a very good predictive capability of the proposed formulation may be noticed.

In the third test, we analyse warping of the cross-section and the crack profile. To do so, the cross-section is divided in $n$ equal layers. Fifteen elements of equal length are used and the applied force is 23.41 kN, which causes a crack in the middle of the tie to open. In Fig. 14 warping of the cross-section at the right-hand end of the tie is shown, where the coordinate system origin is...
placed at the right-hand end of the tie in the undeformed state. In Fig. 15 the crack profile is shown with the origin of the coordinate system placed in the middle of the crack in the deformed state. In both cases it may be seen that considering the tie as a layered bar has marginal influence on the results (less than 0.6%).

5.2. Ngo and Scordelis’s beam

In our second numerical example we study a simply supported four-point bending on a beam analysed by Ngo and Scordelis (1967) shown in Fig. 16. The material parameters are given as follows: Young’s modulus of concrete \( E_c = 20\,\text{684.3 MPa} \), Poisson’s ratio 0.3, Young’s modulus of steel \( E_s = 206\,\text{843 MPa} \) and bond stiffness modulus \( C_s = 14\,\text{045 MPa/m} \). The beam is reinforced only at the bottom, with 2 bars of diameter 29 mm.

Ngo and Scordelis have performed a two-dimensional analysis using triangular plane stress finite elements for a variety of pre-defined cracking patterns, where the cracks have been modelled by separating the concrete elements on either side of the crack by using different nodal points. Four of these patterns have been chosen for this analysis (Fig. 17). Beam 0 is assumed to be uncracked, and is used as a basis for comparison of results with other beams with pre-defined cracks. Beam A has two vertical cracks in the region of constant maximum moment, symmetrically arranged against the centre-line of beam. Beam C includes two vertical cracks as model A plus two diagonal cracks that are pre-defined outside the region of maximum moment (one on each side of the midspan). In our model these diagonal cracks have been idealised by vertical cracks which reach the same depth. In Beam D four vertical cracks in the region of constant maximum moment and four diagonal cracks outside the region of constant maximum moment – again modelled with vertical cracks, are assumed to appear. The applied force \( F \) equals 44.48 kN.

Due to symmetry of geometry and boundary conditions only the right half of the beam has been modelled, with the global coordinate reference system placed in the middle of the beam. Two-noded beam elements with linked interpolation for displacements (quadratic) and rotations (linear) and linear Lagrangian interpolation for the function \( s \) are used. The number of elements equals 17, with 15 elements located between the centre-line and the support, while the other 2 elements are located at the right side of the support. The cross-section is divided in 13 equal layers and the reinforcement layer is set in layer 2. The slip at the centre-line and at the end of the beam is fixed. In those elements that have pre-defined cracks (3rd element for Beam A, 3rd and 10th element for Beam C and 2nd, 4th, 8th and 12th element for Beam D) 8 layers are taken as cracked (about 62% of the height of the beam).

The vertical midspan deflection comparison is given in Fig. 18. A very good agreement may be observed for all the cracking patterns. In Beams C and D the results differ more due to the existence of diagonal cracks – since in our model only transversal cracks are allowed, the diagonal cracks have had to be modelled as such.

The distribution of stresses in steel and bond stresses is shown in Fig. 19 and Fig. 20, respectively (due to symmetry only for the right half of the beam), together with the results obtained using a refined mesh with 85 elements (five times more elements than the original mesh) where pre-defined cracks are set in 13th element for Beam A, 13th and 48th element for Beam C and 8th, 18th, 38th and 58th element for Beam D.
Since two-noded elements are used, the stress in steel is constant over the element. The distribution of the bond stresses in Fig. 20 follow qualitatively the distribution of the bond forces at the nodal points in the reference, but there these are not given quantitatively and thus cannot be numerically related to the results obtained here.

In both figures localised effect of cracking may be noticed. At a crack location the stress in steel considerably increases, just as the bond stresses near a crack do, which is again in very good agreement with the reference results. Apart from the crack locations, the results of the 17-element mesh agree very closely with the results of the 85-element mesh and may be considered as the converged solution.

Since the primary contribution in the approach proposed in this work is prediction of occurrence and propagation of cracks, which Ngo and Scordelis have not dealt with, we will next study a situation with no pre-defined cracks but in addition we will define a value for the tensile strength of concrete, apply the monotonically
increasing force $F$ up to the value of 44.48 kN and show the results for the 17-element mesh.

A comparison of the crack positions and depths for two different tensile strengths of concrete is given in Fig. 21. When the tensile strength of concrete is set to 2.25 MPa (cca. 11% of Young’s modulus of concrete), and the force $F$ reaches 36.3 kN, the first layer of 5th element cracks. At the end of the loading process, a total of eight layers of 5th element is found to be cracked, and the midspan deflection measures 1.273 mm, which is very close to the results obtained by Ngo and Scordelis for Beam A in Fig. 18. It has to be noted, however, that the actual occurrence of the crack at the closest possible point to the applied force in the region of the constant moment (in the middle of the adjacent element to the left of the force) in the present model has a sound theoretical base. It is precisely in this element among those in the constant moment region that the reinforcement stress prior to cracking is minimal (see Fig. 18 (a)) and therefore to accommodate the constant bending moment the stress in the concrete part of the same element must crack first.

If the tensile strength of concrete is set to 1.5 MPa (cca. 7% of Young’s modulus of concrete) the first layer of 5th element cracks when the force $F$ reaches 24.2 kN. By increasing the force this crack then propagates through seven more layers and soon a new crack forms in 1st element ($F = 30.1$ kN) and later in 8th element ($F = 43.6$ kN). At the end of the loading process, 1st and 5th elements have eight layers cracked, and 8th element has seven cracked layers, while the midspan deflection equals 1.713 mm. This case may be compared with Ngo and Scordelis’s Beam D (see Fig. 18). Again, the actual occurrence of 2nd crack in the present model at the closest possible point near the centre-line has had to be expected since among the elements in the constant moment region it is this element that has the minimum reinforcement stress and consequently the maximum tensile stress in the concrete.

In Fig. 22 the distribution of stresses in steel and bond stresses for the situation with no predefined cracks but with prescribed tensile strength of concrete as 2.25 MPa is shown. The results are comparable to Ngo and Scordelis’s results for beam A (Fig. 19 (b) and Fig. 20 (b)), where a reduction in the peak reinforcement stress is now attributed not only to the mesh used, but also to the fact that the actual crack is at a more remote position measured from the centre-line.

### 6. Conclusions

In this work a novel embedded-discontinuity layered beam finite element for geometrically linear analysis of planar reinforced-concrete beams has been presented. The main characteristics of the element are:

- the number of concrete layers is arbitrary,
- the layers are assembled in a beam with a rigid interlayer connection (with neither slip nor uplift between them), but they can rotate independently of each other,
- reinforcement, treated as an additional layer of zero thickness and finite area, is placed within a surrounding concrete layer, and may slip with respect to this layer (allowance is currently made only for one reinforcement layer),
- a bond–slip relationship is superimposed onto this model,
- a transversal crack is embedded in a manner that it opens when the tensile concrete strength at a layer’s mid-depth is reached and propagates throughout the whole depth of the layer.

Occurrence and propagation of cracks predicted by this model has been demonstrated on a couple of representative examples involving linear elastic behaviour and ideally brittle concrete. The results have been found to agree well with the analytical solutions, where these exist, and the numerical solutions from literature obtained using alternative finite elements.

Emphasis has been given on verification of the developed layered-beam embedded-crack kinematics and the results presented make a sound base for future work. To complete the model, it is necessary to introduce non-linear constitutive laws for concrete and steel as well as a non-linear bond–slip relationship, which will enable testing the presented model on practical problems and comparing the results with those numerical results obtained using continuum elements and different approaches to

![Fig. 21. Crack position comparison (a) $f_t = 2.25$ MPa, (b) $f_t = 1.5$ MPa.](image1)

![Fig. 22. Crack prediction for a given concrete tensile strength of 2.25 MPa (a) stress in steel, (b) bond stress.](image2)
defining the cracking process as well as the experimental results. Additionally, the element may be recast in a full geometrically non-linear form, which may prove useful both in modelling post-critical states up to the point of collapse as well as in any further extension to problems of optimisation.

Acknowledgements

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Appendix A

In (37) the internal virtual work due to stresses in the concrete layers is given as

\[ V_{ Bj } = \sum_{ i=1 }^{ M } \int_{ 0 }^{ L } \left[ \left( \left( \frac{ d }{ d x_1 } \right)^{ T } B_i \left( \begin{array}{c} \mathbf{t}^{ 01 }_i \\ 0 \end{array} \right) \right)^{ T } \left( \begin{array}{c} \mathbf{N}_i \\ 0 \end{array} \right) \right] d x_1 \right] \tag{A.1} \]

Since

\[ \mathbf{N}_i = \left( \begin{array}{c} \mathbf{N}_i \\ \mathbf{T}_i \end{array} \right) \tag{A.2} \]

this internal work may be written as

\[ V_{ Bj } = \sum_{ i=1 }^{ M } \int_{ 0 }^{ L } \left[ \left( \frac{ d }{ d x_1 } \right)^{ T } B_i \mathbf{t}^{ 01 }_i \mathbf{N}_i \right] d x_1 \].

Further,

\[ \mathbf{L}^{ T } \mathbf{D}_i \mathbf{L}^{ T } = \left( \begin{array}{ccc} d/dx_1 & 0 & 0 \\ 0 & d/dx_1 & 0 \\ 0 & 0 & d/dx_1 \end{array} \right) + \left( \begin{array}{ccc} 0 & 0 & \sin \psi \\ 0 & 0 & -\cos \psi \\ 0 & 0 & 0 \end{array} \right) \]

and thus

\[ V_{ Bj } = \sum_{ i=1 }^{ M } \int_{ 0 }^{ L } \left[ \left( \mathbf{L}^{ T } \mathbf{D}_i \mathbf{L}^{ T } \mathbf{t}^{ 01 }_i \mathbf{N}_i \right) \right] d x_1 \] \tag{A.3}

since, according to (33), \( \mathbf{B}_i \) is constant. The above may be written as

\[ V_{ Bj } = \sum_{ i=1 }^{ M } \int_{ 0 }^{ L } \left[ \left( \mathbf{p}^{ 01 }_i B_i^{ T } L_i \mathbf{t}^{ 01 }_i + \mathbf{p}^{ 01 }_i B_i^{ T } \mathbf{N}_i - \mathbf{p}^{ 01 }_i B_i^{ T } T_i \right) 0 \right] d x_1 \] \tag{A.4}

and finally

\[ V_{ Bj } = \sum_{ i=1 }^{ M } \int_{ 0 }^{ L } \left[ \left( \mathbf{p}^{ 01 }_i B_i^{ T } L_i \mathbf{t}^{ 01 }_i + \mathbf{p}^{ 01 }_i B_i^{ T } \mathbf{N}_i - \mathbf{p}^{ 01 }_i B_i^{ T } T_i \right) 0 \right] d x_1 \] \tag{A.5}

Appendix B

To compute \( \Delta \mathbf{g} \) in (101), we consider \( \mathbf{g} \) as defined in (100) with \( \mathbf{q}^{ jp} \) and \( \mathbf{q}^{ jm} \) as given in (91) and \( \mathbf{q}^{ jn} \) and \( \mathbf{q}^{ km} \) as given in (99). It follows that we need to compute \( \Delta \mathbf{q}^{ jn} \), \( \Delta \mathbf{q}^{ jm} \), \( \Delta \mathbf{q}^{ jn} \), \( \Delta \mathbf{q}^{ nj} \), \( \Delta \mathbf{q}^{ km} \), \( \Delta \mathbf{q}^{ jm} \), \( \Delta \mathbf{q}^{ km} \), and \( \Delta \mathbf{q}^{ km} \).

B.1. Linearisation of \( \mathbf{q}^{ jn} \) (80)

Since \( N_i = N_i(\mathbf{t}_i), \Delta \mathbf{q}^{ jn} \) follows as

\[ \Delta \mathbf{q}^{ jn} = \int_{ 0 }^{ L } \frac{ d N_j }{ dx } \Delta \mathbf{t}_j d x_1. \] \tag{B.1}

From (54)

\[ \Delta \mathbf{q}^{ jn} = \frac{ \partial \mathbf{e} }{ \partial \mathbf{u} } \Delta \mathbf{u}' = \Delta \mathbf{u}' + \mathbf{t}_j \Delta \mathbf{u}' \] \tag{B.2}

and thus

\[ \Delta \mathbf{q}^{ jn} = \sum_{ k=1 }^{ N } \sum_{ k=1 }^{ N } \Delta q^{ k} \mathbf{K}_{ k k} + \sum_{ k=1 }^{ N } \Delta q^{ k} \mathbf{K}_{ k k} \Delta \mathbf{p}_k. \] \tag{B.3}

Using (68), (80) and (80) and noting that \( \mathbf{u}_k = [1, \mathbf{0}_{6x6}] \mathbf{p}_k \), we obtain

\[ \Delta \mathbf{q}^{ jn} = \mathbf{K}_{ k k} \Delta \mathbf{p}_k. \] \tag{B.4}

where

\[ \mathbf{K}_{ k k} = \int_{ 0 }^{ L } \frac{ d N_j }{ dx } \mathbf{t}_j d x_1. \]

B.2. Linearisation of \( \mathbf{q}^{ jm} \) (83)

Since \( \mathbf{q}^{ jm} = \left[ \begin{array}{c} \mathbf{t}_1 \\ 0 \end{array} \right] \) for \( \Delta \mathbf{q}^{ jm} \) we get

\[ \Delta \mathbf{q}^{ jm} = \sum_{ k=1 }^{ N } \mathbf{K}_{ k k} \Delta \mathbf{p}_k. \] \tag{B.5}

with

\[ \mathbf{K}_{ k k} = \left[ \begin{array}{c} \mathbf{t}_1 \\ 0 \end{array} \right] \mathbf{K}_{ k k}. \] \tag{B.6}

B.3. Linearisation of \( \mathbf{q}^{ km} \) (81)

Since \( \tau = \tau(\mathbf{f}) \) from (81) we get

\[ \Delta \mathbf{q}^{ km} = \phi \frac{ d }{ d \mathbf{f} } \Delta \mathbf{f} = \phi \frac{ d }{ d \mathbf{f} } \Delta \mathbf{f} = \phi \mathbf{f} \int_{ 0 }^{ L } \frac{ d x }{ d \mathbf{f} } \Delta \mathbf{f} d x_1. \] \tag{B.7}

From (52)

\[ \Delta \mathbf{f} = \frac{ d }{ d \mathbf{s} } \Delta \mathbf{s} + \frac{ d }{ d \mathbf{w} } \Delta \mathbf{w} = \Delta \mathbf{s} - \mathbf{d} \mathbf{c} \mathbf{k} \Delta \mathbf{w} \] \tag{B.8}

and

\[ \Delta \mathbf{q}^{ km} = \phi \int_{ 0 }^{ L } \mathbf{f} d x_1 + \frac{ d }{ d \mathbf{w} } \Delta \mathbf{f} d x_1. \] \tag{B.9}
After interpolating $\Delta s$ as in (68) we obtain

$$
\Delta q_{b,i}^j = \sum_{k=1}^{N} K_{b, j,k} \Delta s_k + K_{b, j,w} \Delta w,
$$

where

$$
K_{b, j,k} = \phi \pi \int_{0}^{l_k} \frac{d\tau}{df} l_1 dX_1,
$$

$$
K_{b, j,w} = -\delta_{\omega c} \phi \pi \int_{0}^{l} \frac{d\tau}{df} l_2 dX_1,
$$

B.4. Linearisation of $q_{b,i}^j$ (86)

We analogously obtain $\Delta q_{b,i}^j$ as

$$
\Delta q_{b,i}^j = \sum_{k=1}^{N} K_{b, i,k} \Delta s_k + K_{b, i,w} \Delta w,
$$

with

$$
K_{b, i,k} = \delta_{\omega c} \phi \pi \int_{0}^{l_i} \frac{d\tau}{df} l_1 dX_1,
$$

$$
K_{b, i,w} = -\delta_{\omega c} \phi \pi \int_{0}^{l} \frac{d\tau}{df} l_2 dX_1.
$$

B.5. Linearisation of $q_{i,j}^j$ (82)

In order to compute $\Delta q_{i,j}^j$, (82) may be written as

$$
q_{i,j}^j = \sum_{l=1}^{M} B_i^j L \int_{0}^{H_i} \left\{ \begin{array}{c} N_l \\ T_l \\ M_l \end{array} \right\} dX_1
+ m^i(0, 1) \sum_{l=1}^{M} K_j^l \left\{ \begin{array}{c} N_l \\ T_l \\ M_l \end{array} \right\} dX_1
$$

and then $\Delta q_{i,j}^j$ follows as

$$
\Delta q_{i,j}^j = \sum_{l=1}^{M} B_i^j L \int_{0}^{H_i} \left\{ \begin{array}{c} \Delta N_l \\ \Delta T_l \\ \Delta M_l \end{array} \right\} dX_1
+ m^i(0, 1) \sum_{l=1}^{M} K_j^l \left\{ \begin{array}{c} \Delta N_l \\ \Delta T_l \\ \Delta M_l \end{array} \right\} dX_1 = a_1 + a_2.
$$

Since $N_i = N_i(\xi_i, \kappa_i), T_i = T_i(\gamma_i), M_i = M_i(\xi_i, \kappa_i)$ we get

$$
\left\{ \begin{array}{c} \Delta N_l \\ \Delta T_l \\ \Delta M_l \end{array} \right\} = C \left\{ \begin{array}{c} \Delta \xi_i \\ \Delta \gamma_i \\ \Delta \kappa_i \end{array} \right\}.
$$

Using (16) and (17) we may write

$$
\left\{ \begin{array}{c} \xi_i \\ \gamma_i \\ \kappa_i \end{array} \right\} = \left( L^i \frac{d}{dX_i} + \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array} \right) \left\{ \begin{array}{c} u_i \\ v_i \\ \theta_i \end{array} \right\}
$$

and

$$
\left\{ \begin{array}{c} \Delta \xi_i \\ \Delta \gamma_i \\ \Delta \kappa_i \end{array} \right\} = \left( L^i \frac{d}{dX_i} + \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array} \right) \left\{ \begin{array}{c} \Delta u_i \\ \Delta v_i \\ \Delta \theta_i \end{array} \right\}.
$$

After interpolating $\Delta s$ as in (68) we obtain

$$
\Delta q_{b,i}^j = \sum_{k=1}^{N} K_{b, j,k} \Delta s_k + K_{b, j,w} \Delta w,
$$

where

$$
K_{b, j,k} = \phi \pi \int_{0}^{l_k} \frac{d\tau}{df} l_1 dX_1,
$$

$$
K_{b, j,w} = -\delta_{\omega c} \phi \pi \int_{0}^{l} \frac{d\tau}{df} l_2 dX_1.
$$

From (32) it further follows that

$$
\left\{ \begin{array}{c} \Delta u_i \\ \Delta \gamma_i \\ \Delta \kappa_i \end{array} \right\} = B_i \left( \Delta p_i + \delta_{\omega c} k \begin{array}{c} t_{01} \\ 0_m \end{array} \right) \Delta w = B_i \Delta p_i + \delta_{\omega c} k \begin{array}{c} t_{01} \\ 0 \end{array} \Delta w.
$$

Due to rotational discontinuity in each cracked layer $\Delta p_i = \Delta p + k \begin{array}{c} 0 \\ \Lambda_{c} \Delta \phi \end{array}$, the above expression yields

$$
\left\{ \begin{array}{c} \Delta u_i \\ \Delta \gamma_i \\ \Delta \kappa_i \end{array} \right\} = B_i \Delta p + B_i k \begin{array}{c} 0 \\ \Lambda_{c} \Delta \phi \end{array} + \delta_{\omega c} k \begin{array}{c} t_{01} \\ 0 \end{array} \Delta w.
$$

Finally,

$$
\left\{ \begin{array}{c} \Delta u_i \\ \Delta \gamma_i \\ \Delta \kappa_i \end{array} \right\} = B_i \Delta p + B_i k \Lambda_{c} \Delta \phi + \delta_{\omega c} k \begin{array}{c} t_{01} \\ 0 \end{array} \Delta w.
$$

and substituting (87),

$$
\left\{ \begin{array}{c} \Delta u_i \\ \Delta \gamma_i \\ \Delta \kappa_i \end{array} \right\} = L^i B_i \Delta p + H_i^c L^i B_i \Lambda_{c} \Delta \phi
+ \delta_{\omega c} k \begin{array}{c} t_{01} \\ 0 \end{array} \Delta w.
$$

The above expression is substituted into $\Delta q_{i,j}^j$, for simplicity divided in two terms ($a_1$ and $a_2$)

$$
a_1 = \sum_{l=1}^{M} B_i^j L \int_{0}^{H_i} H_i C \left( L^i B_i \Delta p + \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right) \left( B_i \Delta p + H_i^c L^i B_i \Lambda_{c} \Delta \phi \right) dX_1,
$$

and

$$
a_2 = m^i(0, 1) \sum_{l=1}^{M} K_j^l \left( \frac{d}{dX_1} \left( L^i B_i \Delta p + \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right) \left( B_i \Delta p + H_i^c L^i B_i \Lambda_{c} \Delta \phi \right) \right) dX_1.
$$

and after interpolation according to (69), (78), and $K_j = k(-1)^{j-1} \left( \frac{N}{j-1} \right) \Pi_{n=1}^{N} N_p.$
\[ a_1 = \sum_{l=1}^{M} B_l^2 \int_0^L H_l C \left( L^2 B \sum_{k=1}^{N} I_{l,2-M} \Delta p_k + L^2 B \sum_{k=1}^{N} K_{l,2-M} \right) \Delta p_k \]

\[ + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right] B_l \sum_{k=1}^{N} \{ t_{o_2} \} \{ t_{o_M} \} \right] \Delta w \right) dX_l. \] (B.30)

\[ a_2 = m^T (0 \ 1 \ 0) \sum_{l=1}^{M} \left[ \begin{array}{c} K_l \frac{dH_l}{dT} \end{array} \right] \left( L^2 B \sum_{k=1}^{N} I_{l,2-M} \Delta p_k + L^2 B \sum_{k=1}^{N} K_{l,2-M} \right) \Delta p_k \]

\[ + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array} \right] \left. B_l \sum_{k=1}^{N} \{ t_{o_2} \} \{ t_{o_M} \} \right] \Delta w \right) dX_l. \] (B.31)

Since

\[ g L^T \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array} \right] g = H^2 l^T, \quad H_s = \left[ \begin{array}{ccc} g^g & 0 & 0 \\ 0 & g^g & 0 \\ 0 & -g & g^g \end{array} \right] \] for \( g \)

\[ = I_l, I_k, K_k. \] (B.32)

we get

\[ a_1 = \sum_{l=1}^{M} B_l^2 L \left( \sum_{k=1}^{N} \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B \right) \]

\[ + \sum_{k=1}^{N} \left[ \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B \right] \{ t_{o_2} \} \{ t_{o_M} \} \Delta p_k \]

\[ + \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B \] (B.33)

\[ a_2 = m^T (0 \ 1 \ 0) \sum_{l=1}^{M} \left[ \begin{array}{c} K_l \frac{dH_l}{dT} \end{array} \right] \left( L^2 B \sum_{k=1}^{N} I_{l,2-M} \Delta p_k + \right) \]

\[ + \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{array} \right] \left. B_l \sum_{k=1}^{N} \{ t_{o_2} \} \{ t_{o_M} \} \right] \Delta w \] (B.34)

and finally

\[ \Delta q_{l,p,j} = \sum_{k=1}^{N} \left( K_{l,p,j,k} + K_{l,p,j,k}^{\text{link1}} + K_{l,p,j,k}^{\text{link2}} + K_{l,p,j,k}^{\text{link3}} \right) \Delta p_k \]

\[ + \left( K_{l,p,j,0} + K_{l,p,j,0}^{\text{link}} \right) \Delta \phi + K_{l,p,j,0} \Delta w, \] (B.35)

where

\[ K_{l,p,j,k} = \sum_{l=1}^{M} B_l^2 L \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B. \] (B.36)

\[ K_{l,p,j}^{\text{link1}} = \sum_{l=1}^{M} B_l^2 L \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B \left( t_{o_2} \right) \{ t_{o_M} \}. \] (B.37)

\[ K_{l,p,j}^{\text{link2}} = m^T (0 \ 1 \ 0) \sum_{l=1}^{M} \int_0^L K_l \frac{dH_l}{dT} dX_l L^2 B_t. \] (B.38)

\[ K_{l,p,j}^{\text{link3}} = m^T (0 \ 1 \ 0) \sum_{l=1}^{M} \int_0^L K_l \frac{dH_l}{dT} dX_l L^2 B_t \left( t_{o_2} \right) \{ t_{o_M} \}. \] (B.39)

\[ K_{l,p,j,0} = \sum_{l=1}^{M} B_l^2 L \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B_t. \] (B.40)

\[ K_{l,p,j,0}^{\text{link}} = m^T (0 \ 1 \ 0) \sum_{l=1}^{M} \int_0^L K_l \frac{dH_l}{dT} dX_l L^2 B_t \left( t_{o_2} \right) \{ t_{o_M} \}. \] (B.41)

\[ K_{l,p,j,0} = \sum_{l=1}^{M} B_l^2 L \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B_t. \] (B.42)

**B.6. Linearisation of \( q_{l,p} \) (84)**

From (84) \( \Delta q_{l,p} \) may be derived in the same way as \( a_1 \) in \( \Delta q_{l,p,j} \) above

\[ \Delta q_{l,p} = \sum_{k=1}^{N} \left( K_{l,p,k} + K_{l,p,k}^{\text{link1}} \right) \Delta p_k + K_{l,p,0} \Delta \phi + K_{l,p,0} \Delta w, \] (B.43)

with

\[ K_{l,p,k} = \sum_{l=1}^{M} B_l^2 L \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B_t. \] (B.44)

\[ K_{l,p,k}^{\text{link1}} = \sum_{l=1}^{M} B_l^2 L \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B_t \left( t_{o_2} \right) \{ t_{o_M} \}. \] (B.45)

\[ K_{l,p,k}^{\text{link2}} = \sum_{l=1}^{M} B_l^2 L \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B_t. \] (B.46)

\[ K_{l,p,0} = \sum_{l=1}^{M} B_l^2 L \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B_t. \] (B.47)

**B.7. Linearisation of \( q_{l,w} \) (85)**

Finally, \( \Delta q_{l,w} \) may be obtained in the same way as above. From (85)

\[ \Delta q_{l,w} = \sum_{k=1}^{N} \left( K_{l,w,k} + K_{l,w,k}^{\text{link1}} \right) \Delta p_k + K_{l,w,0} \Delta \phi + K_{l,w,0} \Delta w, \] (B.48)

where

\[ K_{l,w,k} = \sum_{l=1}^{M} B_l^2 L \int_0^L H_l C \frac{dH_l}{dT} dX_l L^2 B_t. \] (B.49)
\[ K_{bw,k}^{\text{lin}} = \delta_{ac} \sum_{i=1}^{M} \int_{0}^{l} k \left( \frac{\partial w}{\partial N_i} \right) H_{i}^T dX_i \{ t_{02} \} m. \] (B.51)

\[ K_{bw,p} = \delta_{ac} \sum_{i=1}^{M} \int_{0}^{l} k \left( \frac{\partial w}{\partial N_i} \right) H_{i}^T dX_i \{ t_{02} \} m. \] (B.52)

\[ K_{bw,w} = \delta_{ac} \sum_{i=1}^{M} \int_{0}^{l} k^2 \frac{\partial N_i}{\partial L} dX_i. \] (B.53)

### B.8. Bar element stiffness matrix

The stiffness matrix for one element with N-nodes is thus

\[
\begin{bmatrix}
\begin{array}{cccc}
K_{pp}^{11} & K_{pp}^{12} & \cdots & K_{pp}^{1N} \\
K_{pp}^{12} & K_{pp}^{22} & \cdots & K_{pp}^{2N} \\
\vdots & \vdots & \ddots & \vdots \\
K_{pp}^{1N} & K_{pp}^{2N} & \cdots & K_{pp}^{NN}
\end{array}
\end{bmatrix}
\]

\[ \text{SYMM.} \]

\[
\begin{bmatrix}
K_{pp}^{11} & K_{pp}^{21} & \cdots & K_{pp}^{N1} \\
K_{pp}^{21} & K_{pp}^{22} & \cdots & K_{pp}^{N2} \\
\vdots & \vdots & \ddots & \vdots \\
K_{pp}^{N1} & K_{pp}^{N2} & \cdots & K_{pp}^{NN}
\end{bmatrix}
\]

where

\[
K_{pp}^{ij} = K_{pp,jk} + K_{pp,jk} + K_{pp,kj} + K_{pp,kj}.
\] (B.55)

\[
K_{pp}^{ij} = K_{pp,pj} + K_{pp,pj} + K_{pp,pj} + K_{pp,pj} + K_{pp,pj} + K_{pp,pj} + K_{pp,pj}.
\] (B.55)

with

\[
K_{pp,stk} = K_{pp,stk} + K_{pp,stk} + K_{pp,stk} + K_{pp,stk}.
\]

\[
K_{pp,stk} = K_{pp,jk} + K_{pp,jk} + K_{pp,jk} + K_{pp,jk}.
\]

\[
K_{bw,stk} = K_{bw,stk} - K_{bw,stk}.
\] (B.57)

### Appendix C

In case of beam element, the stiffness matrix transforms into

\[
\begin{bmatrix}
\begin{array}{cccc}
K_{pp}^{11} & K_{pp}^{12} & \cdots & K_{pp}^{1N} \\
K_{pp}^{12} & K_{pp}^{22} & \cdots & K_{pp}^{2N} \\
\vdots & \vdots & \ddots & \vdots \\
K_{pp}^{1N} & K_{pp}^{2N} & \cdots & K_{pp}^{NN}
\end{array}
\end{bmatrix}
\]

\[ \text{SYMM.} \]

\[
\begin{bmatrix}
K_{pp}^{11} & K_{pp}^{21} & \cdots & K_{pp}^{N1} \\
K_{pp}^{21} & K_{pp}^{22} & \cdots & K_{pp}^{N2} \\
\vdots & \vdots & \ddots & \vdots \\
K_{pp}^{N1} & K_{pp}^{N2} & \cdots & K_{pp}^{NN}
\end{bmatrix}
\]

where \( K_{pp}^{ij} \) is the same as in (B.55). Since \( w \) is no longer the unknown, some of the above components rearrange and transform into

\[
\{ \begin{array}{l}
K_{pp,stk} \{ h_{i}^T \} A_c
\end{array} \}
\] (C.2)

\[
K_{bw,stk} + K_{bw,stk} + K_{bw,stk} = K_{bw,stk} + K_{bw,stk} + K_{bw,stk}.
\] (C.3)