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A conjecture of Berge about linear hypergraphs and Steiner systems $S(2,4,v)$☆

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Abstract

A famous conjecture of Berge about linear hypergraphs is studied. It is proved that all nearly resolvable Steiner systems $S(2,4,v)$ and all almost nearly resolvable $S(2,4,v)$ verify this conjecture. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A hypergraph is a pair $\mathscr{H}=(X,\mathscr{E})$, where X is a finite non-empty set and $\mathscr{E}$ is a collection of subsets $E\subseteq X$, such that $E\neq\emptyset$, for every $E\in\mathscr{E}$, and $\bigcup_{E\in\mathscr{E}} E=X$. The elements of X are called the *points* (or *vertices*) of $\mathscr{H}$, the elements of $\mathscr{E}$ are called the *edges* of $\mathscr{H}$. A hypergraph $\mathscr{H}$ is called *k-uniform* if all its edges have exactly k distinct elements. For $k=2$, hypergraphs are called graphs. The degree of a point $x\in X$ is the number of edges containing x. A hypergraph is called linear if any two of its edges share at most one vertex. The *heredity* $\hat{\mathscr{H}}$ of $\mathscr{H}$ is the hypergraph having the same point-set of $\mathscr{H}$ and its edge-set is the family of all non-empty subsets of the edges of $\mathscr{H}$. A hypergraph $\mathscr{H}$ is said to be *resolvable* if there exists a partition $\Pi=\{\Pi_1,\Pi_2,\dots,\Pi_r\}$ of $\mathscr{E}$ such that, for every $i=1,2,\dots,r$, Π_i is a partition of X (therefore, if $b',\ b''\in\Pi_i,\ b'\neq b''$, then $b'\cap b''=\emptyset$). The partition Π is called a resolution of $\mathscr{H}$ and all the classes Π_i are called *parallel classes* of $\mathscr{H}$. A *complete graph*, usually indicated by K_v, has v vertices and all the pairs of its

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vertices as edges. It is resolvable iff v is even. A resolution of K_v is also called a 1-*factorization* of K_v and the parallel classes are called 1-*factors*.

A hypergraph $\mathscr{H}$ is called *nearly resolvable* if there exists a point $x \in X$ such that the hypergraph $\mathscr{H}(x) = (X - \{x\},\ \mathscr{E}(x))$, where $\mathscr{E}(x) = \{E \in \mathscr{E}\colon\ x \notin \mathscr{E}\}$, is *resolvable*.

Given a hypergraph $\mathscr{H} = (X, \mathscr{E})$, an *edge-coloring* of $\mathscr{H}$ is a mapping $K : \mathscr{E} \to C$, from the edge-set to a set C of colors, such that if $E', E'' \in \mathscr{E},\ E' \neq E'',\ E' \cap E'' \neq \emptyset$, then $K(E') \neq K(E'')$. The minimum number k, for which there exists an edge-coloring K of $\mathscr{H}$, using k colors, is denoted by $q(\mathscr{H})$ and it is called the *chromatic index* of $\mathscr{H}$. If Δ is the maximum degree of the points of $\mathscr{H}$, then it is immediate that $q(\mathscr{H}) \geqslant \Delta(\mathscr{H})$. If $q(\mathscr{H}) = \Delta(\mathscr{H})$, then we say that H has the *edge-coloring property*.

A *Steiner system* $S(h,k,v)$ [resp. a *partial Steiner system* $\mathrm{PS}(h,k,v)$] is a k-*uniform hypergraph* $\mathscr{S} = (S, \mathscr{B})$ such that every h-subsets of $\mathscr{S}$ is contained in exactly [resp. at most] one edge, also called *block*, of $\mathscr{B}$. If $h = 2$ and $k = 4$ a system $S(2,4,v)$ exists if and only if $v \equiv 1$ *or* $4 \ (\mathrm{mod.}\ 12)$. It is easy to prove that a system $S(2,4,v)$ contains exactly $|B| = [v(v-1)]/12$ blocks and that every point $x \in S$ is contained in exactly $(v-1)/3$ blocks. Two partial Steiner systems $\mathrm{PS}(h,k,n)$ $\Sigma_1 = (S, \mathscr{B}')$, $\Sigma_2 = (S, \mathscr{B}'')$ are said to be *disjoint* and *mutually balanced*, briefly DMB, if $\mathscr{B}' \cap \mathscr{B}'' = \emptyset$, and a h-subset of $\mathscr{S}$ is contained in a block of $\mathscr{B}'$ if and only if it is contained in a block of $\mathscr{B}''$. In what follows, we will consider DMB-PS$(2,4,n)$ and we will indicate them by the triple $(S; \mathscr{B}', \mathscr{B}'')$. Observe that a (partial) system $S(2,k,v)$ is a linear hypergraph.

A famous conjecture of Berge about linear hypergraphs says that: "*if* $\mathscr{H}$ *is a linear hypergraph, then* $\mathscr{H}\hat{}$ *has always the edge-coloring property*". Observe that if $\mathscr{J}$ is a regular (and linear) hypergraph then to say that "$\mathscr{J}$ *has the edge-colouring property*" is equivalent to say that "$\mathscr{J}$ *has a resolution*" [5, Lemma 2.2].

It seems that this conjecture is very difficult to prove and the known results are very few, until today. Some authors studied this conjecture for particular classes of hypergraphs. In particular, considering the importance and the interest of the design theory in combinatorics, Berge pointed out the study of the conjecture for Steiner systems $S(2,k,v)$, which are all linear hypergraphs.

In [5] it is proved that *all resolvable* $S(2,4,v)$ *verify the conjecture of Berge*. In this paper, we examine other classes of $S(2,4,v)$. We will prove that *all nearly resolvable* $S(2,4,v)$ *and all almost nearly resolvable* $S(2,4,v)$ *verify the conjecture of Berge*.

2. Nearly resolvable $S(2, 4, v)$

In what follows, we will consider nearly resolvable Steiner systems $S(2,4,v)$, which we will indicate briefly by $\mathrm{NRS}(2,4,v)$.

Theorem 1. *If* $S = (S, \mathscr{B})$ *is an* $\mathrm{NRS}(2,4,v)$, *then its hereditary closure* $S\hat{}$ *is resolvable*.

Proof. Let $\mathscr{S} = (S, \mathscr{B})$ be an $\mathrm{NRS}(2,4,v)$ and let $S = \{1, 2, \ldots, v\}$. Suppose that $x = 1$ is the point such that $(S - \{1\}, \mathscr{B}(1))$, with $\mathscr{B}(1) = \{b \in B\colon\ 1 \notin b\}$, has a resolution Σ.

Observe that

$$v \equiv 1(\mathrm{mod}.\,12),\ |B| = \frac{v\cdot(v-1)}{12},\ d(1) = (v-1)/3,\ |B(1)| = \frac{(v-1)(v-4)}{12}.$$

Further, Σ consists of $(v-4)/3$ parallel classes $\Pi_1, \Pi_2, \ldots, \Pi_{(v-4)/3}$, and each of them contains exactly $(v-1)/4$ disjoint blocks.

Let $C = \{c_1, c_2, \ldots, c_{(v-1)/3}\} = \mathscr{B} - \mathscr{B}(1)$:

$$\begin{aligned}
&c_1 = \{1,2,3,4\}\\
&c_2 = \{1,5,6,7\}\\
&c_3 = \{1,8,9,10\}\\
&\ldots\\
&\ldots\\
&c_i = \{1,3i-1,3i,3i+1\} \qquad\qquad \in C\\
&\ldots\\
&\ldots\\
&c_{(v-1)/3} = \{1, v-2, v-1, v\}.
\end{aligned}$$

It follows that $B = C \cup \Pi_1 \cup \Pi_2 \cup \cdots \cup \Pi_{(v-4)/3}$.

Further, consider the following collections of singletons, pairs and triples of S.

1. Let $F = \{F_1, F_2, \ldots, F_v\}$ be a 1-factorization of the complete graph defined on the set $X = \{0\} \cup S$, such that $\{0, i\} \in F_i$, for every $i = 1, 2, \ldots, v$, and

   $$\{1, v\} \in F_{v-2}, \{v-2, v-1\} \in F_v.$$

2. Let $P = \{P_1, P_2, \ldots, P_v\}$ be the family obtained by the factorization F, replacing in every factor $F_i \in F$ the pair $\{0, i\}$ by the singleton $\{i\}$ (the element 0 is deleted).
3. Let $T = \{T_1, T_2, \ldots, T_v\}$ be the family of subsets of S, where T_i contains $\{i\}$ and all the triples $\{x, y, z\} \subseteq S$ such that $\{i, x, y, z\} \in \mathscr{B}$.

Now, we will construct a resolution Σ' of $\hat{S}$.

(I) For every $i = 1, 2, \ldots, (v-4)/3$, consider c_i, Π_i, and observe that in Π_i there are three disjoint blocks containing the elements $3i-1$, $3i$, $3i+1$, respectively, which belong to c_i (together with the point 1). Let

$$\left.\begin{aligned}
&d_{i,1} = \{3i-1, x_{i,1}, y_{i,1}, z_{i,1}\}\\
&d_{i,2} = \{3i, x_{i,2}, y_{i,2}, z_{i,2}\}\\
&d_{i,3} = \{3i+1, x_{i,3}, y_{i,3}, z_{i,3}\}
\end{aligned}\right. \in \Pi_i.$$

So, define the following collection of blocks:

$$\Pi'_i = \Pi_i - \{d_{i,1}, d_{i,2}, d_{i,3}\} \cup \{c_i\} \cup \{\{x_{i,1}, y_{i,1}, z_{i,1}\}, \{x_{i,2}, y_{i,2}, z_{i,2}\}, \{x_{i,3}, y_{i,3}, z_{i,3}\}\},$$

$$\forall i = 1, 2, \ldots, \frac{(v-4)}{3}.$$

If $\Pi' = \{\Pi_1, \Pi_2, \dots, \Pi'_{(v-4)/3}\}$, let $\Pi' \subseteq \Sigma'$.

(II) Now, consider the family T. Since for every $i = 1, 2, \dots, (v-4)/3$,

$$B_{3i-1} = \{\{3i-1\}, \{x_{i,1}, y_{i,1}, z_{i,1}\}\} \subseteq T_{3i-1},$$

$$B_{3i} = \{\{3i\}, \{x_{i,2}, y_{i,2}, z_{i,2}\}\} \subseteq T_{3i},$$

$$B_{3i+1} = \{\{3i+1\}, \{x_{i,3}, y_{i,3}, z_{i,3}\}\} \subseteq T_{3i+1},$$

and, among the others,

$$\{\{1\}, \{v-2, v-1, v\}\} \subseteq T_1,$$

$$\{\{v-1\}, \{1, v-2, v\}\} \subseteq T_{v-1},$$

we can define, for every $i = 1, 2, 3, \dots, (v-4)/3$

$$T'_{3i-1} = T_{3i-1} - B_{3i-1} \cup \{d_{i,1}\},$$

$$T'_{3i} = T_{3i} - B_{3i} \cup \{d_{i,2}\},$$

$$T'_{3i+1} = T_{3i+1} - B_{3i+1} \cup \{d_{i,3}\}$$

and

$$T'_1 = T_1 - \{\{1\}, \{v-2, v-1, v\}\} \cup \{c_{(v-1)/3}\},$$

$$T'_{v-2} = T_{v-2},$$

$$T'_v = T_v.$$

Let $T^* = \{T'_1, T'_2, \dots, T'_{v-2}, T'_v\} \cup \{T_{v-1}\} \subseteq \Sigma'$.

(III) Consider the family P. Since

$$\{\{v\}, \{v-2, v-1\}\} \subseteq P_v,$$

$$\{\{v-1\}, \{1, v-2, v\}\} \subseteq T_{v-1},$$

we can define:

$$P'_i = P_i, \text{ for every } i = 1, 2, \dots, v-3, v-1,$$

$$P'_v = P_v - \{\{v\}, \{v-2, v-1\}\} \cup \{\{v, v-2, v-1\}\},$$

$$P'_{v-2} = P_{v-2} - \{\{v-2\}, \{1, v\}\} \cup \{\{1, v-2, v\}\},$$

and if $P' = \{P'_1, P'_2, \dots, P'_v\}$, let $P' \subseteq \Sigma'$.

(IV) Finally, observe that the pairs $\{1, v\}$, $\{v-2, v-1\}$ are not contained in the families constructed so far, while the triple $\{1, v-2, v\}$ and the singleton $\{v-1\}$ are contained two times.

So, if we define

$$T'_{v-1} = T_{v-1} - \{\{v-1\}, \{1, v-2, v\}\} \cup \{\{1, v\}, \{v-2, v-1\}\},$$

and if $T' = \{T'_1, T'_2, \dots, T'_v\}$, let $T' \subseteq \Sigma'$.

The collection $\Sigma' = \Pi' \cup T' \cup P'$ contains all the quadruples of $\mathscr{B}$ and all the triples, the pairs, the singletons contained in the quadruples of $\mathscr{B}$. Σ' is a resolution of the closure $S\hat{}$ of S. This proves the theorem. □

The resolution Σ' so obtained consists of three families of subsets of S: Π', T', P'. It is clear that there exist other resolutions. In what follows, when we will use resolutions of nearly resolvable $S(2,4,v)$ constructed as Σ', we will write $\Sigma' = [\Pi', T', P']$.

3. Almost nearly resolvable and almost resolvable $S(2, 4, v)$

Let $\Sigma_1 = (S, \mathscr{B}')$, $\Sigma_2 = (S, \mathscr{B}'')$ be a pair of DMB-PS$(2,4,n)$. It is easy to see that necessarily: $|\mathscr{B}'| = |\mathscr{B}''| = m \geqslant 6$, $m \neq 7$. Further

(1) In [5] it is proved that

(I) *There are only two pairs of* DMB-PS$(2,4,n)$ *with* $m \leqslant 8$ *blocks. They are*:

$m = 6$,
$n = 11$ $(0,1,2,3,\ldots,9,A)$,

$\mathscr{B}'$:	$\mathscr{B}''$:
$\{1,3,4,5\}$	$\{1,3,6,9\}$
$\{1,6,7,8\}$	$\{1,4,7,0\}$
$\{1,9,0,A\}$	$\{1,5,8,A\}$
$\{2,3,6,9\}$	$\{2,3,4,5\}$
$\{2,4,7,0\}$	$\{2,6,7,8\}$
$\{2,5,8,A\}$	$\{2,9,0,A\}$,

$m = 8$,
$n = 14$ $(0,1,2,3,\ldots,9,A,B,C,D)$,

$\mathscr{B}'$:	$\mathscr{B}''$:
$\{1,2,3,4\}$	$\{1,2,5,8\}$
$\{1,5,6,7\}$	$\{1,3,6,B\}$
$\{1,8,9,A\}$	$\{1,4,9,C\}$
$\{1,B,C,D\}$	$\{1,7,A,D\}$
$\{0,2,5,8\}$	$\{0,2,3,4\}$
$\{0,3,6,B\}$	$\{0,5,6,7\}$
$\{0,4,9,C\}$	$\{0,8,9,A\}$
$\{0,7,A,D\}$	$\{0,B,C,D\}$.

(2) If $S_1 = (S, \mathscr{B}')$ and $S_2 = (S, \mathscr{B}'')$ are two $S(2,4,v)$, such that $\mathscr{B}' \cap \mathscr{B}'' = B$, then $\mathscr{B}' - \mathscr{B}$ and $\mathscr{B}'' - \mathscr{B}$ define a pair of DMB-PS$(2,4,n)$.

This implies that

(II) *Two distinct Steiner systems* $S(2,4,v)$ *can have* $m = q_v - 6$ *or* $m \leqslant q_v - 8$ *blocks in common, being* $q_v = [v \cdot (v-1)]/12$.

In what follows, we will say that a Steiner system $S(2,4,v)$ is *almost nearly resolvable* [resp. *almost-resolvable*], briefly ANRS$(2,4,v)$ [resp. ARS$(2,4,v)$] if it has $q_v - 6$ or $q_v - 8$ blocks in common with a nearly resolvable [resp. resolvable] $S(2,4,v)$.

Theorem 2. *If* $S=(S,\mathscr{B})$ *is an* ANRS$(2,4,v)$, *then its hereditary closure* $S\hat{}$ *is resolvable.*

Proof. Let $\mathscr{H}=(S,C)$ be an NRS$(2,4,v)$, defined as Theorem 1, and let $S=(S,\mathscr{B})$ be an ANRS$(2,4,v)$ having q_v-6 or q_v-8 blocks in common with $\mathscr{H}$. From Theorem 1, $H\hat{}$ is resolvable having a resolution $\Sigma'=[\Pi',T',P']$.

Suppose that

(1) $|B\cap C|=q_v-6$ and let

$\mathscr{B}-C$:	$C-\mathscr{B}$:
$\{1',3',4',5'\}$	$\{1',3',6',9'\}$
$\{1',6',7',8'\}$	$\{1',4',7',0'\}$
$\{1',9',0',A'\}$	$\{1',5',8',A'\}$
$\{2',3',6',9'\}$	$\{2',3',4',5'\}$
$\{2',4',7',0'\}$	$\{2',6',7',8'\}$
$\{2',5',8',A'\}$	$\{2',9',0',A'\}$

(2) $|B\cap C|=q_v-8$,
and let

$\mathscr{B}-C$:	$C-\mathscr{B}$:
$\{1',2',3',4'\}$	$\{1',2',5',8'\}$
$\{1',5',6',7'\}$	$\{1',3',6',B'\}$
$\{1',8',9',A'\}$	$\{1',4',9',C'\}$
$\{1',B',C',D'\}$	$\{1',7',A',D'\}$
$\{0',2',5',8'\}$	$\{0',2',3',4'\}$
$\{0',3',6',B'\}$	$\{0',5',6',7'\}$
$\{0',4',9',C'\}$	$\{0',8',9',A'\}$
$\{0',7',A',D'\}$	$\{0',B',C',D'\}$.

We can observe that in both the cases (1) and (2), the quadruples of $\mathscr{B}$–C and C–$\mathscr{B}$ can be partitioned into a same set U of pairs as follows:
(1.1)

$\mathscr{B}-C$:	$C-\mathscr{B}$:
$\{1',3'-4',5'\}$	$\{1',3'-6',9'\}$
$\{1',7'-6',8'\}$	$\{1',7'-4',0'\}$
$\{1',A'-9',0'\}$	$\{1',A'-5',8'\}$
$\{2',3'-6',9'\}$	$\{2',3'-4',5'\}$
$\{2',7'-4',0'\}$	$\{2',7'-6',8'\}$
$\{2',A'-5',8'\}$	$\{2',A'-9',0'\}$,

(2.1)

$\mathscr{B}-C$:	$C-\mathscr{B}$:
$\{1',2'-3',4'\}$	$\{1',2'-5',8'\}$
$\{1',6'-5',7'\}$	$\{1',6'-3',B'\}$
$\{1',9'-8',A'\}$	$\{1',9'-4',C'\}$
$\{1',D'-B',C'\}$	$\{1',D'-7',A'\}$
$\{0',2'-5',8'\}$	$\{0',2'-3',4'\}$
$\{0',6'-3',B'\}$	$\{0'6'-5'7'\}$
$\{0',9'-4',C'\}$	$\{0',9'-8',A'\}$
$\{0',D'-7',A'\}$	$\{0'D'-B'C'\}$,

we can find another set V of pairs with the same property and without pairs in common with U:

(1.2)

$\mathscr{B}-C$:	$C-\mathscr{B}$:
$\{1',4'-3',5'\}$	$\{1',4'-7',0'\}$
$\{1',8'-6',7'\}$	$\{1',8'-A',5'\}$
$\{1',9'-0',A'\}$	$\{1',9'-3',6'\}$
$\{2',4'-7',0'\}$	$\{2',4'-3',5'\}$
$\{2',8'-A',5'\}$	$\{2',8'-6',7'\}$
$\{2',9'-3',6'\}$	$\{2',9'-0',A'\}$

(2.2)

$\mathscr{B}-C$:	$C-\mathscr{B}$:
$\{1',4'-2',3'\}$	$\{1',4'-9',C'\}$
$\{1',5'-6',7'\}$	$\{1',5'-2',8'\}$
$\{1',A'-8',9'\}$	$\{1',A'-7',D'\}$
$\{1',B'-C',D'\}$	$\{1',B'-3',6'\}$
$\{0',4'-9',C'\}$	$\{0',4'-2',3'\}$
$\{0',5'-2',8'\}$	$\{0',5'-6',7'\}$
$\{0',A'-7',D'\}$	$\{0',A'-8',9'\}$
$\{0',B'-3',6'\}$	$\{0',B'-C',D'\}$.

Therefore:

(i) if $\{1,v\},\{v-2,v-1\}\notin U$ [this implies that $\{1,v-2,v-1,v\}\in \mathrm{B}\cap C$ or, otherwise, that this quadruple is partitioned into two pairs of U different from $\{1,v\},\{v-2,v-1\}$], it is possible to consider the collection P' (defined by a factorization F, see Theorem 1) in such a way that

(1)

$\{1',3'-6',9'\}\in P_2'$
$\{1',7'-4',0'\}\in P_3'$
$\{2',3'-4',5'\}\in P_1'$
$\{2',7'-6',8'\}\in P_5'$
$\{1',A'-5',8'\}\in P_4'$
$\{2',A'-9',0'\}\in P_6'$.

(2)

$$\begin{array}{l}\{1',2'-5',8'\}\in P'_3\\ \{1',6'-3',B'\}\in P'_4\\ \{1',9'-4',C'\}\in P'_2\\ \{1',D'-7',A'\}\in P'_5\\ \{0',2'-3',4'\}\in P'_6\\ \{0',6'-5',7'\}\in P'_1\\ \{0',9'-8',A'\}\in P'_7\\ \{0',D'-B',C'\}\in P'_8.\end{array}$$

and we can define a resolution $\Sigma''=[\Pi'',T'',P'']$ of $S\hat{}$ as follows:

for every $\{x,y,z,t\}\in\mathscr{B}-C$, *if* Δ_j *is the class of* Σ' *such that* $\{x,y,z,t\}\in\Delta_j$ *and* $\{(x,y\},\{z,t\}\in U$, *then this class in* Σ'' *becomes* $\Delta'_j=\Delta_j-\{\{x,y,z,t\}\}\cup\{\{x,y\},\{z,t\}\}$;

for every $\{a,b\},\{c,d\}\in U$ *such that* $\{a,b,c,d\}\in C-\mathscr{B}$, *if* P'_i *is the class of* Σ' *such that* $\{\{a,b\},\{c,d\}\in P'_i$, *then this class in* Σ'' *becomes* $P''_i=P'_i-\{\{a,b\},\{c,d\}\}\cup\{a,b,c,d\}\}$.

(ii) *if* $\{1,v\},\{v-1,v\}\in U$ [this implies that $\{1,v-2,v-1,v\}\in\mathscr{B}-C$], then $\{1,v\},\{v-2,v-1\}\notin V$ and we can apply the same technique of (i), using the pairs of V instead of the pairs of U. □

By the same technique, used in Theorem 2, it is possible to prove the following theorem.

Theorem 3. *If* $S=(S,\mathscr{B})$ *is an* ARS$(2,4,v)$, *then its heredity* $S\hat{}$ *is resolvable.*

Proof. If $S=(S,\mathscr{B})$ is an ARS$(2,4,v)$, then exists a resolvable $S(2,4,v)$ $\mathscr{H}=(S,C)$ having q_v-6 or q_v-8 blocks in common with S. The closure $\mathscr{H}\hat{}$ has a resolution, which can be obtained by

1. the resolution Π of $\mathscr{H}$,
2. the family $\{\Delta_1,\Delta_2,\dots,\Delta_v\}$, with Δ_i containing $\{i\}$ and all the triples $\{x_i,y_i,z_i\}$ such that $\{i,x_i,y_i,z_i\}\in C$,
3. a factorization $F=\{F_1,F_2,\dots,F_{v-1}\}$ of K_v.

If $\mathscr{B}-C$ and $C-\mathscr{B}$ are defined as in Theorem 2, for both cases (1) and (2), since it is possible to define F in such a way as to the conditions on P_i of Theorem 2 are verified, then we can use the same technique of case (i) of Theorem 2 and we can also prove that $S=(S,\mathscr{B})$ has the closure $S\hat{}$ with a resolution. □

In conclusion: "*All resolvable* $S(2,4,v)$, *all nearly resolvable* $S(2,4,v)$, *all almost nearly resolvable* $S(2,4,v)$ *verify the conjecture of Berge on linear hypergraphs*".

4. A nearly resolvable $S(2, 4, 25)$

The first value of v to obtain are NRS$(2,4,v)$ is $v=25$. The following is an NRS $(2,4,25)$.

C:

$\{1,2,3,4\}$	$\{1,11,12,13\}$	$\{1,20,21,22\}$
$\{1,5,6,7\}$	$\{1,14,15,16\}$	$\{1,23,24,25\}$
$\{1,8,9,10\}$	$\{1,17,18,19\}$	

$\boldsymbol{\Pi}_1$:	$\boldsymbol{\Pi}_2$:	$\boldsymbol{\Pi}_3$:
$\{2,5,8,11\}$	$\{2,7,14,19\}$	$\{2,6,21,23\}$
$\{3,6,9,12\}$	$\{3,5,15,17\}$	$\{3,7,22,24\}$
$\{4,7,10,13\}$	$\{4,6,16,18\}$	$\{4,5,20,25\}$
$\{14,17,20,23\}$	$\{8,13,22,23\}$	$\{8,12,14,18\}$
$\{15,18,21,24\}$	$\{9,11,20,24\}$	$\{9,13,15,19\}$
$\{16,19,22,25\}$	$\{10,12,21,25\}$	$\{10,11,16,17\}$

$\boldsymbol{\Pi}_4$:	$\boldsymbol{\Pi}_5$:	$\boldsymbol{\Pi}_6$:	$\boldsymbol{\Pi}_7$:
$\{2,10,15,22\}$	$\{2,9,17,25\}$	$\{2,12,16,24\}$	$\{2,13,18,20\}$
$\{3,8,16,20\}$	$\{3,10,18,23\}$	$\{3,13,14,25\}$	$\{3,11,19,21\}$
$\{4,9,14,21\}$	$\{4,8,19,24\}$	$\{4,11,15,23\}$	$\{4,12,17,22\}$
$\{5,12,19,23\}$	$\{5,13,16,21\}$	$\{5,9,18,22\}$	$\{5,10,14,24\}$
$\{6,13,17,24\}$	$\{6,11,14,22\}$	$\{6,10,19,20\}$	$\{6,8,15,25\}$
$\{7,11,18,25\}$	$\{7,12,15,20\}$	$\{7,8,17,21\}$	$\{7,9,16,23\}$.

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