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Symmetric Duality with Invexity in Variational Problems

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Weak and strong duality results are established under invexity hypotheses for symmetric dual variational problems without positivity constraints. Self-dual problems and static symmetric dual programs are included as special cases. © 1990 Academic Press, Inc.

1. INTRODUCTION

Duality in nonlinear programming is usually treated using the scheme of Wolfe [9], in which the formation of a dual involves the introduction of new variables corresponding to primal constraints. Thus, except in linear programs, the primal cannot be obtained by forming the dual of the dual.

The concept of symmetric dual programs, in which the dual of the dual equals the primal, was introduced and developed in papers such as Dorn [2], and Dantzig, Eisenberg, and Cottle [1]. Mond and Hanson [5] extended symmetric duality to variational problems, giving continuous analogues of the previous results.

Assumptions common to these works are those of convexity and concavity. Since the identification of invex functions in Hanson [3], many results which formerly required convexity have been extended using invexity, including the variational problems discussed in Mond, Chandra, and Husain [4]. In this paper, we apply invexity to a symmetric dual variational problem, without the positivity constraints of Mond and Hanson [5], but with an extra condition on the invexity. The special case of self-dual variational problems, along with the reduction to static symmetric dual programs without positivity constraints when there is no time dependency, is presented.

2. NOTATION

Consider the real scalar function $f(t, x, x', y, y')$, where $t \in [t_0, t_f]$, x and y are functions of t with $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$, and x' and y' denote

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derivatives of x and y, respectively, with respect to t. Assume that f has continuous fourth-order partial derivatives with respect to x, x', y, y'.

f_x and $f_{x'}$ denote the gradient vectors of f with respect to x and x', i.e.,

$$f_x \equiv \left(\frac{\partial f}{\partial x^1}, ..., \frac{\partial f}{\partial x^n}\right)^T \quad \text{and} \quad f_{x'} \equiv \left(\frac{\partial f}{\partial x^{1'}}, ..., \frac{\partial f}{\partial x^{n'}}\right)^T$$

Similarly, f_y and $f_{y'}$ denote the gradient vectors of f with respect to y and y'.

The following observations are used for proving strong duality:

$$\frac{d}{dt} f_{y'} = f_{y't} + f_{y'y}\, y' + f_{y'y'}\, y'' + f_{y'x} x' + f_{y'x'} x''.$$

Consequently,

$$\frac{\partial}{\partial y} \frac{d}{dt} f_{y'} = \frac{d}{dt} f_{yy'}$$

$$\frac{\partial}{\partial y'} \frac{d}{dt} f_{y'} = \frac{d}{dt} f_{y'y'} + f_{y'y}$$

$$\frac{\partial}{\partial y''} \frac{d}{dt} f_{y'} = f_{y'y'}$$

$$\frac{\partial}{\partial x} \frac{d}{dt} f_{y'} = \frac{d}{dt} f_{y'x}$$

$$\frac{\partial}{\partial x'} \frac{d}{dt} f_{y'} = \frac{d}{dt} f_{y'x'} + f_{y'x}$$

$$\frac{\partial}{\partial x''} \frac{d}{dt} f_{y'} = f_{y'x'}.$$

The analogous properties of $(d/dt) f_{x'}$ could be employed for a converse duality theorem, but such a result is more efficiently established via symmetry.

Note that vector inequalities are defined component by component.

3. Symmetric Duality

We consider the problem of finding functions x: $[t_0, t_f] \to \mathbb{R}^n$ and y: $[t_0, t_f] \to \mathbb{R}^m$, with $(x'(t), y'(t))$ piecewise smooth on $[t_0, t_f]$, to solve the following pair of optimization problems.

(P) Minimize

$$\int_{t_0}^{t_f} \left[f(t, x, x', y, y') - y(t)^T f_y(t, x, x', y, y') + y(t)^T \frac{d}{dt} f_{y'}(t, x, x', y, y') \right] dt$$

subject to: $x(t_0) = x_0$, $x(t_f) = x_f$, $y(t_0) = y_0$, $y(t_f) = y_f$ (1)

$$\frac{d}{dt} f_{y'}(t, x, x', y, y') \geqslant f_y(t, x, x', y, y'), \qquad t \in [t_0, t_f] \tag{2}$$

(D) Maximize

$$\int_{t_0}^{t_f} \left[f(t, x, x', y, y') - x(t)^T f_x(t, x, x', y, y') + x(t)^T \frac{d}{dt} f_{x'}(t, x, x', y, y') \right] dt$$

subject to: $x(t_0) = x_0$, $x(t_f) = x_f$, $y(t_0) = y_0$, $y(t_f) = y_f$

$$\frac{d}{dt} f_{x'}(t, x, x', y, y') \leqslant f_x(t, x, x', y, y'), \qquad t \in [t_0, t_f], \tag{3}$$

where (2) and (3) may fail to hold at corners of $(x'(t), y'(t))$, but must be satisfied for unique right- and left-hand limits.

These are Problems I and II stated in Mond and Hanson [5], with the constraint $x(t) \geqslant 0$ removed from I, and $y(t) \geqslant 0$ removed from II.

DEFINITION. The functional $\int_{t_0}^{t_f} f$ (f as given above) is invex in x and x' if for each y: $[t_0, t_f] \to \mathbb{R}^m$, with y' piecewise smooth, there exists a function η: $[t_0, t_f] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\int_{t_0}^{t_f} [f(t, x, x', y, y') - f(t, u, u', y, y')] \, dt \geqslant \int_{t_0}^{t_f} \eta(t, x, x', u, u')^T \times \left[f_x(t, u, u', y, y') - \frac{d}{dt} f_{x'}(t, u, u', y, y') \right] dt$$

for all x: $[t_0, t_f] \to \mathbb{R}^n$, u: $[t_0, t_f] \to \mathbb{R}^n$ with $(x'(t), u'(t))$ piecewise smooth on $[t_0, t_f]$.

Similarly, the functional $-\int_{t_0}^{t_f} f$ is invex in y and y' if for each $x: [t_0, t_f] \to \mathbb{R}^n$, with x' piecewise smooth, there exists a function $\xi: [t_0, t_f] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ such that

$$-\int_{t_0}^{t_f} [f(t, x, x', v, v') - f(t, x, x', y, y')]\, dt$$
$$\geqslant -\int_{t_0}^{t_f} \xi(t, v, v', y, y')^T \times \left[f_y(t, x, x', y, y') - \frac{d}{dt} f_{y'}(t, x, x', y, y') \right] dt$$

for all $v: [t_0, t_f] \to \mathbb{R}^m$, $y: [t_0, t_f] \to \mathbb{R}^m$ with $(v'(t), y'(t))$ piecewise smooth on $[t_0, t_f]$.

In the sequel, we will write $\eta(x, u)$ for $\eta(t, x, x', u, u')$, and $\xi(v, y)$ for $\xi(t, v, v', y, y')$.

As shown in Mond and Smart [7], $\int_{t_0}^{t_f} f$ is invex in x and x' iff, for each fixed y, a critical point yields a global minimum; and $-\int_{t_0}^{t_f} f$ is invex in y and y' iff, for each fixed x, a critical point yields a global maximum of $\int_{t_0}^{t_f} f$.

THEOREM 1 (Weak Duality). *If* $\int_{t_0}^{t_f} f$ *is invex in* x *and* x', *and* $-\int_{t_0}^{t_f} f$ *is invex in* y *and* y', *with* $\eta(x, u) + u(t) \geqslant 0$ *and* $\xi(v, y) + y(t) \geqslant 0$ *for all* $t \in [t_0, t_f]$ *(except perhaps at corners of* $(x'(t), y'(t))$ *or* $(u'(t), v'(t))$*) whenever* (x, y) *is feasible for* (P) *and* (u, v) *is feasible for* (D), *then* $\inf(\mathrm{P}) \geqslant \sup(\mathrm{D})$.

Proof. Let (x, y) be feasible for (P), and (u, v) be feasible for (D). Then

$$\int_{t_0}^{t_f} \Bigg[f(t, x, x', y, y') - y(t)^T f_y(t, x, x', y, y') + y(t)^T \frac{d}{dt} f_{y'}(t, x, x', y, y')$$
$$- \left[f(t, u, u', v, v') - u(t)^T f_x(t, u, u', v, v') + u(t)^T \frac{d}{dt} f_{x'}(t, u, u', v, v') \right] \Bigg] dt$$
$$\geqslant -\int_{t_0}^{t_f} \xi(v, y)^T \left[f_y(t, x, x', y, y') - \frac{d}{dt} f_{y'}(t, x, x', y, y') \right] dt$$
$$+ \int_{t_0}^{t_f} \eta(x, u)^T \left[f_x(t, u, u', v, v') - \frac{d}{dt} f_{x'}(t, u, u', v, v') \right] dt$$

$$+\int_{t_0}^{t_f} -y(t)^T \left[f_y(t, x, x', y, y') - \frac{d}{dt} f_{y'}(t, x, x', y, y') \right] dt$$

$$+\int_{t_0}^{t_f} u(t)^T \left[f_x(t, u, u', v, v') - \frac{d}{dt} f_{x'}(t, u, u', v, v') \right] dt$$

for some functions η and ξ by assumptions of invexity

$$= -\int_{t_0}^{t_f} (\xi(v, y) + y(t))^T$$

$$\times \left[f_y(t, x, x', y, y') - \frac{d}{dt} f_{y'}(t, x, x', y, y) \right] dt$$

$$+\int_{t_0}^{t_f} (\eta(x, u) + u(t))^T$$

$$\times \left[f_x(t, u, u', v, v') - \frac{d}{dt} f_{x'}(t, u, u', v, v') \right] dt$$

$\geqslant 0$ by (2) and (3) with $\eta(x, u) + u(t) \geqslant 0$ and $\xi(v, y) + y(t) \geqslant 0$. Hence, $\inf(\mathrm{P}) \geqslant \sup(\mathrm{D})$. ▮

Remark. If the invexity assumptions of Theorem 1 are replaced by convexity and concavity (i.e., $\int_{t_0}^{t_f} f$ is convex in x and x' for each y, and $\int_{t_0}^{t_f} f$ is concave in y and y' for each x), then the conditions $\eta(x, u) + u(t) \geqslant 0$ and $\xi(v, y) + y(t) \geqslant 0$ become $x(t) \geqslant 0$ and $v(t) \geqslant 0$. These constraints may be added to Problems (P) and (D), respectively, to obtain the dual pair of Mond and Hanson [5].

In the following theorem and proof, f^* represents $f(t, x^*, x^{*\prime}, y^*, y^{*\prime})$ and partial derivatives are similarly denoted.

THEOREM 2 (Strong Duality). *Let* (x^*, y^*) *be optimal for* (P), *and assume that the system*

$$p(t)^T \left(f^*_{yy} - \frac{d}{dt} f^*_{yy'} \right) + \frac{d}{dt} \left[p(t)^T \frac{d}{dt} f^*_{y'y'} \right] + \frac{d^2}{dt^2} \left[-p(t)^T f^*_{y'y'} \right] = 0$$

only has the solution $p(t) = 0$, $t \in [t_0, t_f]$. *Then* (x^*, y^*) *is feasible for* (D). *If, in addition, the invexity conditions of Theorem* 1 *are satisfied, then* (x^*, y^*) *is optimal for* (D), *and the extreme values of* (P) *and* (D) *are equal.*

Remark. This theorem also serves to correct the statement of the system of differential equations given in Theorem 2 of Mond and Hanson [5].

Proof. Applying the necessary conditions of Valentine [8], if (x^*, y^*) minimizes (P), then there exists $\lambda_0 \in \mathbb{R}$ and λ: $[t_0, t_f] \to \mathbb{R}^m$ such that

$$H^* \equiv \lambda_0\left(f^* - y^{*T}f_y^* + y^{*T}\frac{d}{dt}f_{y'}^*\right) - \lambda^T\left(\frac{d}{dt}f_{y'}^* - f_y^*\right)$$

satisfies

$$H_y^* - \frac{d}{dt}H_{y'}^* + \frac{d^2}{dt^2}H_{y''}^* = 0 \tag{4}$$

$$H_x^* - \frac{d}{dt}H_{x'}^* + \frac{d^2}{dt^2}H_{x''}^* = 0 \tag{5}$$

and

$$\lambda^T\left(\frac{d}{dt}f_{y'}^* - f_y^*\right) = 0 \tag{6}$$

$$\lambda_0 \geqslant 0 \tag{7}$$

$$\lambda \geqslant 0 \tag{8}$$

throughout $[t_0, t_f]$ (except at corners of $(x^{*\prime}(t), y^{*\prime}(t))$ where (4) and (5) hold for unique right- and left-hand limits). λ_0 and $\lambda(t)$ cannot be simultaneously zero at any $t \in [t_0, t_f]$, and λ is continuous except perhaps at corners of $(x^{*\prime}(t), y^{*\prime}(t))$.

Using the observations on $(d/dt)f_{y'}$ from the notation section, (4) becomes

$$(\lambda - \lambda_0 y^*)^T\left(f_{yy}^* - \frac{d}{dt}f_{yy'}^*\right) + \frac{d}{dt}\left[(\lambda - \lambda_0 y^*)^T\frac{d}{dt}f_{y'y'}^*\right] + \frac{d^2}{dt^2}\left[-(\lambda - \lambda_0 y^*)^T f_{y'y'}^*\right] = 0 \tag{9}$$

and (5) becomes

$$\lambda_0 f_x^* + (\lambda - \lambda_0 y^*)^T\left(f_{xy}^* - \frac{d}{dt}f_{y'x}^*\right) - \lambda_0\frac{d}{dt}f_{x'}^* - \frac{d}{dt}\left[(\lambda - \lambda_0 y^*)^T\left(f_{yx'}^* - \frac{d}{dt}f_{y'x'}^* - f_{y'x}^*\right)\right] + \frac{d^2}{dt^2}\left[-(\lambda - \lambda_0 y^*)^T f_{y'x'}^*\right] = 0. \tag{10}$$

By assumption, the only solution of (9) is $\lambda - \lambda_0 y^* = 0$. This gives $\lambda_0 > 0$, since if $\lambda_0 = 0$ then $\lambda = 0$ everywhere, contradicting the necessary conditions.

Equation (10) now becomes

$$\lambda_0 f_x^* - \lambda_0 \frac{d}{dt} f_{x'}^* = 0,$$

i.e.,

$$f_x^* - \frac{d}{dt} f_{x'}^* = 0. \tag{11}$$

Equation (6) gives

$$\lambda_0 y^{*T} \left(\frac{d}{dt} f_{y'}^* - f_y^* \right) = 0,$$

i.e.,

$$- y^{*T} f_y^* + y^{*T} \frac{d}{dt} f_{y'}^* = 0. \tag{12}$$

By (11), (x^*, y^*) is feasible for (D). From (11) and (12), (P) and (D) have equal objective values at (x^*, y^*), namely $\int_{t_0}^{t_f} f(t, x^*, x^{*\prime}, y^*, y^{*\prime})\, dt$.

If the invexity conditions of Theorem 1 are satisfied, then, by weak duality, (x^*, y^*) is optimal for (D), and the extreme values of (P) and (D) are equal. ∎

A converse duality theorem may be stated; the proof would be analogous to that of Theorem 2.

THEOREM 3 (Converse Duality). *Let* (x^*, y^*) *be optimal for* (D), *and assume that the system*

$$p(t)^T \left(f_{xx}^* - \frac{d}{dt} f_{xx'}^* \right) + \frac{d}{dt} \left[p(t)^T \frac{d}{dt} f_{x'x'}^* \right] + \frac{d^2}{dt^2} [-p(t)^T f_{x'x'}^*] = 0$$

only has the solution $p(t) = 0$, $t \in [t_0, t_f]$. *Then* (x^*, y^*) *is feasible for* (P). *If, in addition, the invexity conditions of Theorem* 1 *are satisfied, then* (x^*, y^*) *is optimal for* (P), *and the extreme values of* (P) *and* (D) *are equal.*

Remark. The system of differential equations in Theorem 2 may be rewritten as

$$p(t)^T \left[f_{yy}^* - \frac{d}{dt} f_{yy'}^* \right] - \frac{dp(t)^T}{dt} \left(\frac{d}{dt} f_{y'y'}^* \right) - \frac{d^2 p(t)^T}{dt^2} f_{y'y'}^* = 0.$$

If f does not explicitly depend on y', this reduces to $p(t)^T f^*_{yy} = 0$, which has only a zero solution iff f^*_{yy} is nonsingular for all $t \in [t_0, t_f]$.

4. Self-Duality

Assume that $m = n$, $f(t, x, x', y, y') = -f(t, y, y', x, x')$ (i.e., f skew-symmetric) for all $(x(t), y(t))$, $t \in [t_0, t_f]$ such that $(x'(t), y'(t))$ is piecewise smooth on $[t_0, t_f]$ and that $x_0 = y_0$, $x_f = y_f$.

It follows that (D) may be rewritten as a minimization problem:

(D′) Minimize

$$\int_{t_0}^{t_f} \left[f(t, y, y', x, x') - x(t)^T f_x(t, y, y', x, x') + x(t)^T \frac{d}{dt} f_{x'}(t, y, y', x, x') \right] dt$$

subject to: $x(t_0) = x_0,\ x(t_f) = x_f,\ y(t_0) = x_0,\ y(t_f) = x_f$

$$\frac{d}{dt} f_{x'}(t, y, y', x, x') \geqslant f_x(t, y, y', x, x').$$

(D′) is formally identical to (P); that is, the objective and constraint functions and initial conditions of (P) and (D′) are identical. This problem is said to be self-dual.

It is easily seen that whenever (x, y) is feasible for (P), then (y, x) is feasible for (D), and vice versa.

Theorem 4. *Assume* (P) *is self-dual and that the invexity conditions of Theorem* 1 *are satisfied. If* (x^*, y^*) *is optimal for* (P), *and the system given in Theorem* 2 *only has a zero solution, then* (y^*, x^*) *is optimal for both* (P) *and* (D), *and the common optimal value is* 0.

Proof. By Theorem 2, (x^*, y^*) is optimal for (D), and the extreme values of (P) and (D) are equal to $\int_{t_0}^{t_f} f(t, x^*, x^{*\prime}, y^*, y^{*\prime})\, dt$.

From self-duality, (y^*, x^*) is feasible for both (P) and (D), so Theorems 1 and 2 give optimality in both problems, and thus objective values of $\int_{t_0}^{t_f} f(t, y^*, y^{*\prime}, x^*, x^{*\prime})\, dt$.

But $\int_{t_0}^{t_f} f(t, y^*, y^{*\prime}, x^*, x^{*\prime})\, dt = -\int_{t_0}^{t_f} f(t, x^*, x^{*\prime}, y^*, y^{*\prime})\, dt$ by skew-symmetry.

Hence $\int_{t_0}^{t_f} f(t, x^*, x^{*\prime}, y^*, y^{*\prime})\, dt = -\int_{t_0}^{t_f} f(t, x^*, x^{*\prime}, y^*, y^{*\prime})\, dt = 0$. ∎

Natural boundary conditions may be dealt with as in Mond and Hanson [5], since the extra transversality conditions required for the formulation of (P) and (D) are independent of any positivity constraints on x and y.

5. Static Symmetric Dual Programs

If the time dependency of Problems (P) and (D) is removed, and f is considered to have domain $\mathbb{R}^n \times \mathbb{R}^m$, we obtain the symmetric dual pair given by

$$\text{(SP)} \quad \text{Minimize } f(x, y) - y^T f_y(x, y)$$

$$\text{subject to: } f_y(x, y) \leqslant 0$$

$$\text{(SD)} \quad \text{Maximize } f(x, y) - x^T f_x(x, y)$$

$$\text{subject to: } f_x(x, y) \geqslant 0.$$

These are the programs considered in Dantzig, Eisenberg, and Cottle [1] and Mond and Hanson [6], except that here the positivity constraints have been omitted.

Definition. The function f: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is invex in x if for each $y \in \mathbb{R}^m$, there exists a function η: $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x, y) - f(u, y) \geqslant \eta(x, u)^T f_x(u, y)$ for all x, $u \in \mathbb{R}^n$, and $-f$ is invex in y if for each $x \in \mathbb{R}^n$, there exists a function ξ: $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ such that $-f(x, v) + f(x, y) \geqslant -\xi(v, y)^T f_y(x, y)$ for all v, $y \in \mathbb{R}^m$.

The following theorems may be proved along the lines of Theorems 1, 2, and 3.

Theorem 5. *If f is invex in x, and $-f$ is invex in y, with $\eta(x, u) + u \geqslant 0$ and $\xi(u, y) + y \geqslant 0$ whenever (x, y) is feasible for* (SP) *and (u, v) is feasible for* (SD), *then* inf(SP) $\geqslant$ sup(SD).

Theorem 6. *Let (x^*, y^*) be optimal for* (SP), *and assume that f^*_{yy} is nonsingular. Then (x^*, y^*) is feasible for* (SD). *If, in addition, the invexity conditions of Theorem* 5 *are satisfied, then (x^*, y^*) is optimal for* (SD) *and the extreme values of* (SP) *and* (SD) *are equal.*

Theorem 7. *Let (x^*, y^*) be optimal for* (SD), *and assume that f^*_{xx} is nonsingular. Then (x^*, y^*) is feasible for* (SP). *If, in addition, the invexity conditions of Theorem* 5 *are satisfied, then (x^*, y^*) is optimal for* (SP), *and the extreme values of* (SP) *and* (SD) *are equal.*

The pair (SP) *and* (SD) *will be self-dual when* $m = n$ *and* f *is skew-symmetric* (*i.e.*, $f(x, y) = -f(y, x)$ *for all* $x, y \in \mathbb{R}^n$).

We state without proof a static version of Theorem 4.

THEOREM 8. *Assume* (SP) *is self-dual and that the invexity conditions of Theorem* 5 *are satisfied. If* (x^*, y^*) *is optimal for* (SP), *and* f^*_{yy} *is nonsingular* (*equivalently*, f^*_{xx} *is nonsingular*), *then* (y^*, x^*) *is optimal for both* (SP) *and* (SD), *and the common optimal value is* 0.

REFERENCES

1. G. B. DANTZIG, E. EISENBERG, AND R. W. COTTLE, Symmetric dual nonlinear programs, *Pacific J. Math.* **15** (1965), 809–812.
2. W. S. DORN, A symmetric dual theorem for quadratic programs, *J. Oper. Res. Soc. Japan* **2** (1960), 93–97.
3. M. A. HANSON, On sufficiency of the Kuhn–Tucker conditions, *J. Math. Anal. Appl.* **80** (1981), 545–550.
4. B. MOND, S. CHANDRA, AND I. HUSAIN, Duality for variational problems with invexity, *J. Math. Anal. Appl.* **134** (1988), 322–328.
5. B. MOND AND M. A. HANSON, Symmetric duality for variational problems, *J. Math. Anal. Appl.* **23** (1968), 161–172.
6. B. MOND AND M. A. HANSON, On duality with generalized convexity, *Math. Oper. Statist. Ser. Optim.* **15** (1984), 313–317.
7. B. MOND AND I. SMART, Duality and sufficiency in control problems with invexity, *J. Math. Anal. Appl.* **136** (1988), 325–333.
8. F. A. VALENTINE, The problem of Lagrange with differential inequalities as added side conditions, *in* "Contributions to the Calculus of Variations 1933–37," pp. 407–448, Univ. of Chicago Press, Chicago, 1937.
9. P. WOLFE, A duality theorem for nonlinear programming, *Quart. Appl. Math.* **19** (1961), 239–244.