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On Lin–Bose problem

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Abstract

This paper study generalized Serre problem proposed by Lin and Bose in multidimensional system theory context [Multidimens. Systems and Signal Process. 10 (1999) 379; Linear Algebra Appl. 338 (2001) 125]. This problem is stated as follows. Let $F \in A^{l\times m}$ be a full row rank matrix, and d be the greatest common divisor of all the $l \times l$ minors of F. Assume that the reduced minors of F generate the unit ideal, where $A = K[x_1, \dots, x_n]$ is the polynomial ring in n variables $x_1, \dots, x_n$ over any coefficient field K. Then there exist matrices $G \in A^{l\times l}$ and $F_1 \in A^{l\times m}$ such that $F = GF_1$ with $\det G = d$ and F_1 is a ZLP matrix. We provide an elementary proof to this problem, and treat non-full rank case.

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1. Introduction

Since the basic structure of multidimensional system theory was studied by Youla and Gnavi [13], there exist a number of research papers towards studying various prime factorizations of multivariate polynomial matrix problems [1,3–9,11,12]. Because many multidimensional systems and signal processing problems can be

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formulated as problems about multivariate polynomial matrices, it is an important topic to apply results of computational algebra into realm of multidimensional linear system theory.

In this paper, we study a problem on multivariate polynomial matrix factorizations proposed by Lin and Bose [5,7]. As pointed out in [7], there are some applications when matrices are not full rank. We study thus general cases without restriction to full rank.

The contents of this paper are as follows. In Section 2, we introduce some basic concepts, and state the Lin–Bose problem. In Section 3, we give a simple proof to Lin–Bose problem.

2. Statement of problem and result

In this section, we give some concepts which will be needed in the discussion of this paper, and state the main research problem.

Let $n \geqslant 2$ be an integer. Let K be a field, and $A = K[x_1, \ldots, x_n]$ be the polynomial ring in variables $x_1, \ldots, x_n$ over the field K. $A^{l \times m}$ the module of $l \times m$ matrices with entries in A. We also write $A^{1 \times m}$ as A^m which is a free module of rank m over A.

We first recall the following basic definitions:

Definition 2.1. Let $F \in A^{l \times m}$ be a full row rank matrix with $l \leqslant m$. Then F is said to be:

(i) zero left prime (*ZLP*) if all the $l \times l$ minors of F generate the unit ideal;
(ii) minor left prime (*MLP*) if all the $l \times l$ minors of F are relatively prime, i.e., their greatest common divisor (g.c.d.) is a non-zero constant;
(iii) factor left prime (*FLP*) if in any polynomial matrix decomposition $F = F_1 F_2$ with $F_1 \in A^{l \times l}$, F_1 is necessarily a unimodular matrix, i.e., $\det(F_1)$ is a non-zero constant in K.

Zero right prime (ZRP) and minor right prime (MRP) etc. can be similarly defined for matrices $F \in A^{m \times l}$ with $m \geqslant l$.

If K is an algebraically closed field, then ZLP $\Rightarrow$ MLP $\Rightarrow$ FLP. When $n = 2$, MLP and FLP are equivalent, and when $n \geqslant 3$, these concepts are pairwise different.

The following definition gives an invariant for multivariate polynomial matrices when the matrices are of full rank [7]:

Definition 2.2. Let $F \in A^{l \times m}$ be a matrix with rank r, and let $a_1, \ldots, a_\beta$ denote all the $r \times r$ minors of the matrix F. Let d be the greatest common divisor (g.c.d) of $a_1, \ldots, a_\beta$, and b_i such that

$$a_i = db_i, \quad i = 1, \ldots, \beta.$$

Then $b_1, \ldots, b_\beta$ are called the “reduced minors” of F.

Now we give the following problem proposed by Lin and Bose [5,7]:

Lin–Bose problem. Let $F \in A^{l\times m}$ be a full row rank matrix, d be the g.c.d of all the $l \times l$ minors of F. If the reduced minors of F generate unit ideal A, then there exists a factorization $F = GF_1$ with $G \in A^{l\times l}$, $F_1 \in A^{l\times m}$, such that $\det G = d$ and F_1 is a ZLP matrix.

In [7], the authors proved equivalence of above problem and the following so called generalized Serre problem:

Generalized Serre problem. Let $F \in A^{l\times m}$ be a full row rank matrix, d the greatest common divisor of all the $l \times l$ minors of F. There exists a matrix $E \in A^{(m-l)\times m}$ such that $\det \begin{bmatrix} F \\ E \end{bmatrix} = d$.

Pommaret gave a proof to Lin–Bose problem using complicated homological algebra tools in [9]. We feel that Lin–Bose conjecture can be proved by means of an elementary argument by using Quillen–Suslin theorem.

In this paper we will study a slight general case without restriction to full rank case, i.e., we prove the following theorem:

Theorem 2.1. *Let $F \in A^{l\times m}$ be of rank r, and d be the g.c.d of all the $r \times r$ minors of F. If all the reduced minors generate unit ideal A. Then F have a factorization $F = G_1F_1$ such that $G_1 \in A^{l\times r}$ and $F_1 \in A^{r\times m}$ is a ZLP matrix.*

As an immediately corollary of Theorem 2.1, we obtain a proof for Lin–Bose conjecture:

Theorem 2.2. *Let $F \in A^{l\times m}$ be a full row rank matrix, d be the g.c.d of all the $l \times l$ minors of F. If all the reduced minors generate unit ideal A, then F have a factorization $F = GF_1$ such that* $\det G = d$ *and F_1 is a ZLP matrix.*

3. Proof of main theorem

In this section, we will give a proof of main theorem. We first recall a concept from commutative algebra [2].

Let B be an arbitrary commutative Noetherian ring, and M be a finitely generated B-module.

Let $B^l \xrightarrow{\phi} B^m \to M \to 0$ be a presentation of B-module M, where ϕ acts on the right on row vectors, i.e., $\phi(u) = uF$ for $u \in B^l$, where F is a matrix corresponding to linear mapping ϕ.

Definition 3.1. Let ϕ is defined by the matrix F for some choice of bases of B^l and B^m. For all i, we denote by $I_i(F)$ the ideal of B generated by the determinants of the $i \times i$ submatrices of F, with the convention $I_0(M) = B$. The ideals $F_j(M) = I_{m-j}(F)$ of B are called jth Fitting ideal of M.

Note that the Fitting ideals F_j are independent of the choice of the bases of B^l and B^m, and depend on the module M not on its presentation.

Now it is obvious that a full row rank matrix $F \in A^{l\times m}$ is a ZLP if and only if $F_{m-l}(M) = A$, where $M = A^m/K$, and $K = \text{rowspace}(F)$ is the submodule generated by all the row vectors of F.

From now on, we will assume that the ring B is a local integral domain, i.e., B have only a unique maximal ideal, and the product of any two non-zero elements is non-zero.

Let M be a finitely generated module over B, and have a presentation:

$$B^l \xrightarrow{\phi} B^m \to M \to 0. \tag{3.1}$$

Let $\text{Torsin}(M) = \{u \mid au = 0, 0 \neq a \in B\}$ be the torsion submodule of M. Thus if $M = B^m/K$, then $\text{Torsion}(M) = \{u + K \mid au \in K, 0 \neq a \in B\}$.

We first prove the following lemma which can be thought as a generalization of Proposition 20.8 in [2].

Lemma 3.1. *With above notation. Let B be a local integral domain, and M be a finitely generated module over B with above presentation* (3.1). *If $F_{m-r}(M)$ is a principal ideal $\langle d\rangle$ and $F_{m-r-1}(M) = 0$. Then we have* $\text{Torsion}(M) = (K' : d)/K'$, *and* $M = \text{Torsion}(M) \oplus B^{m-r}$, *where* $K' = \text{Image}(\phi)$. *In particular*, $M/\text{Torsion}(M) = B^m/(K' : d)$ *is a free module of rank* $m - r$.

Proof. Let F be the matrix corresponding to the mapping ϕ. Let all the $r \times r$ minors of F be $a_1, \dots, a_\beta$, and d_1 is the greatest common divisor of $a_1, \dots, a_\beta$. Let $a_i = d_1 b_i$ for $1 \leqslant i \leqslant \beta$. Since $a_1, \dots, a_\beta$ generate ideal $\langle d\rangle$, we have that d is a divisor of a_i, $1 \leqslant i \leqslant \beta$. Hence $d \mid d_1$. On the other hand, by $\langle a_1, \dots, a_\beta\rangle = \langle d\rangle$, we have $d = \sum_{i=1}^{\beta} a_i c_i$ for some $c_i \in B$. Thus $d = d_1 \sum_{i=1}^{\beta} b_i c_i$. Hence $d_1 \mid d$. Therefore $d_1 = d\epsilon$, ϵ is a unit in B. Thus $\langle b_1, \dots, b_\beta\rangle = B$.

Since B is a local ring, there exists some i such that b_i is an invertible element in B. Thus by bases change of B^l and B^m, we may assume $b_1 = 1$, and we may assume that representation matrix F has the following type:

$$F = \begin{pmatrix} C_0 & C_1 \\ C_2 & C_3 \end{pmatrix},$$

where $C_0 \in B^{r\times r}$ with $\det C_0 = db_1 = d$, $C_1 \in B^{r\times(m-r)}$, $C_2 \in B^{(l-r)\times r}$, $C_3 \in B^{(l-r)\times(m-r)}$.

Now we note that $C_2C_0^{-1}$ is a matrix with entries in B, because $C_2C_0^{-1} = \frac{C_2 \operatorname{adj} C_0}{d}$, and every element of $C_2 \operatorname{adj} C_0$ is just a $r \times r$-minor (up to a sign), hence a multiple of d. Similarly $C_0^{-1}C_1$ is a matrix with entries in B.

We note the following equation:

$$\begin{pmatrix} I_r & 0 \\ -C_2C_0^{-1} & I_{l-r} \end{pmatrix} \begin{pmatrix} C_0 & C_1 \\ C_2 & C_3 \end{pmatrix} \begin{pmatrix} I_r & -C_0^{-1}C_1 \\ 0 & I_{m-r} \end{pmatrix} = \begin{pmatrix} C_0 & 0 \\ 0 & C \end{pmatrix},$$

where $C \in B^{(l-r)\times(m-r)}$ is a matrix.

Hence we may choose bases of B^l and B^m such that the representation matrix of ϕ is of type $\left(\begin{smallmatrix} C_0 & 0 \\ 0 & C \end{smallmatrix}\right)$.

Note that the representation matrix of ϕ has the rank r, thus we have $C = 0$.

Thus $K' = \text{image}(\phi) = \{(vC_0, 0) \mid v \in B^r\} = (K_1, 0)$, where $K_1 = \{vC_0 \mid v \in B^r\}$. Hence we have $M = B^m/K' \cong B^r/K_1 \oplus B^{m-r}$.

By Cramer's rule, we have $dB^r \subseteq K_1$, and no non-zero element of B^{m-r} is a torsion element, hence $(K' : d)/K' \cong (K_1 : d)/K_1 = B^r/K_1$, and $\text{Torsion}(M) = B^r/K_1 = (K' : d)/K'$. Thus $M/\text{Torsion}(M) = B^m/(K' : d) \cong B^{m-r}$ is a free module of rank $m - r$. □

Remark 3.1. Above lemma can also be obtained as a corollary of Lemma 1 in [10], which is pointed out to us by the referee of this paper.

We also need the following easy lemma:

Lemma 3.2. *Let M be a finitely generated module over A, and $T = \text{Torsion}(M)$. $\forall p \in \text{Spec}(A)$, we have $T_p = \text{Torsion}(M_p)$, where M_p denotes the localization of M at p.*

Proof. $\forall p \in \text{Spec}(A)$. Let $m = \frac{t}{s} \in T_p$, where $s \in A \setminus p$, and $t \in T$. Since $t \in T$, there exists some $0 \neq a \in A$ such that $at = 0$. Hence $am = \frac{at}{s} = 0$. Thus $m \in \text{Torsion}(M_p)$ which implies $T_p \subseteq \text{Torsion}(M_p)$.

Let $\frac{m'}{s'} \in \text{Torsion}(M_p)$, where $m' \in M$, $s' \in A \setminus p$. There exists a non-zero element $\frac{a_1}{s_1}$ such that $\frac{a_1}{s_1}\frac{m'}{s'} = 0$. Hence there exists an element $s \in A \setminus p$ such that $sa_1m' = 0$. Because of $sa_1 \neq 0$, we have $m' \in \text{Torsin}(M) = T$. Hence $\frac{m'}{s'} \in T_p$. We have $\text{Torsion}(M_p) \subseteq T_p$. Thus $T_p = \text{Torsion}(M_p)$. □

Now we can obtain the following main theorem:

Theorem 3.1. *Let $F \in A^{l\times m}$ be a matrix with rank r, d the greatest common divisor of all the $r \times r$ minors of F, and $b_1, \ldots, b_\beta$ the reduced minors of F. Let $K = \text{rowspace}(F)$, and $M = A^m/K$. Assume that $b_1, \ldots, b_\beta$ generate unit ideal A.*

Then Torsion$(M) = (K : d)/K$ *and* $M/$Torsion$(M) = A^m/(K : d)$ *is a free module of rank* $m - r$.

Proof. Since $b_1, \ldots, b_\beta$ generate the unit ideal, we have $F_{m-r}(M) = \langle d \rangle$, and $F_{m-r-1}(M) = 0$.

Let $T =$ Torsion(M), then we have $(K : d)/K \subseteq T$.

$\forall p \in$ Spec(A), we have $T_p =$ Torsion(M_p) by Lemma 3.2. Now all the conditions of Theorem 3.1 are still true when we pass from A and A-module M to its localization A_p of A at p, and corresponding A_p-module M_p, i.e., from $F_{m-r}(M) = \langle d \rangle$ and $F_{m-r-1}(M) = 0$ in A we can obtain $F_{m-r}(M_p) = \langle d \rangle$ and $F_{m-r-1}(M_p) = 0$ in A_p.

Hence by Lemma 3.1 we have $(K_p : d)/K_p = T_p$, and M_p/T_p is a A_p-free module of rank $m - r$.

Since p is an arbitrary prime ideal of A, by the local-global principle for the submodule of M, we have $T = (K : d)/K$. Moreover M/T is a locally free module of constant rank $m - r$, hence a projective module over A. Thus M/T is a free module of rank of $m - r$ by Quillen–Suslin theorem. □

By Theorem 3.1 we can easily obtain the following theorem:

Theorem 3.2. *Let* $F \in A^{l\times m}$ *be a matrix with rank* r, d *the greatest common divisor of all the* $r \times r$ *minors of* F, *and the reduced minors of* F *generate unit ideal* A. *Then there exist matrices* $G_1 \in A^{l\times r}$ *and* $F_1 \in A^{r\times m}$ *such that* $F = G_1F_1$ *with* F_1 *being a ZLP matrix.*

Proof. Let $K =$ rowspace(F), and $K_1/K =$ Torsion(A^m/K). By Theorem 3.1, $A^m/K_1 = (A^m/K)/(K_1/K)$ is a free module of rank $m - r$. Hence K_1 is a free module of rank r and $K_1 \supseteq K$.

Now F defines a linear mapping ϕ from A^l to A^m such that Image$(\phi) = K$, i.e., $\phi(u) = uF$ from $u \in A^l$. Taking a system of generators of K_1 with r elements and form a $r \times m$ matrix F_1. Then F_1 also defines a linear mapping ϕ_1 from A^r to A^m such that Image$(\phi_1) = K_1$. Since $K \subseteq K_1$, ϕ can be thought as a mapping from A^l to K_1. Since A^l is a free module, there exists a linear mapping φ from A^l to A^r such that $\phi = \varphi\phi_1$. Considering the matrix corresponding to standard bases of A^l and A^m we have $F = G_1F_1$, where G_1 is the matrix corresponding to mapping φ, and F_1 is a ZLP because A^m/K_1 is a free A-module. □

As an immediately corollary of Theorem 3.2, we obtain the following result:

Theorem 3.3. *Let* $F \in A^{l\times m}$ *be a full row rank matrix*, d *the greatest common divisor of all the* $l \times l$ *minors of* F. *If the reduced minors of* F *generate the unit ideal*, *then there exist* $G \in A^{l\times l}$ *and* $F_1 \in A^{l\times m}$ *such that* $F = GF_1$, $\det G = d$ *and* F_1 *is a ZLP matrix.*

Remark 3.2. Theorem 3.3 gives an answer for Lin–Bose problem.

Remark 3.3. From proof of above theorems, we see that a factorization for a multivariate polynomial matrix $F \in A^{l\times m}$ may be obtained as follows: (i) Compute $K : d$ to obtain F_1, where K is the submodule generated by all the rows of F. (ii) Do a lifting to obtain G in the factorization. When F is full row (or column) rank, we do not need do the lifting, which have been proved in author's another paper "On multivariate polynomial matrix factorization problems".

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