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On the Ramsey Numbers $R(3, 8)$ and $R(3, 9)$*

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Using methods developed by Graver and Yackel, and various computer algorithms, we show that $28 \leqslant R(3, 8) \leqslant 29$, and $R(3, 9) = 36$, where $R(k, l)$ is the classical Ramsey number for 2-coloring the edges of a complete graph.

INTRODUCTION

In this paper we consider the Ramsey numbers associated with a partition of the edges of a complete graph into two sets. We consider such partitions to be represented by a graph, where one set of the partition is represented by the edges, and the other set is represented by the edges, and the other set is represented by the non-edges.

Our notation will follow that of Graver and Yackel [2], except for the definition of the Ramsey number $R(k, l)$. This difference is noted below. We now give some standard definitions, easy lemmas, and tables of numerical results.

DEFINITION 1. Given a graph G, we let $V(G)$ denote the set of vertices of G. Also, we let $|G|$ and $|V(G)|$ denote the number of edges and vertices, respectively, in G.

A set of vertices of a graph G is called an independent set if no pair of vertices in the set are adjacent, and it is called a clique if every pair of vertices in the set are adjacent. The independence number $I(G)$ and the clique number $C(G)$ are the sizes of the largest independent set and clique, respectively, in G.

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DEFINITION 2. If G is a graph with $C(G) < k$, $I(G) < l$, and G has n vertices and e edges, then G will be called a (k, l, n, e)-graph, or a (k, l, n)-graph, or a (k, l)-graph.

DEFINITION 3. The Ramsey number $R(k, l)$ is the smallest integer n such that no (k, l, n)-graph exists.

We remark that this definition is different from that in [2], but conforms with most of the literature (see [1–8]). In fact, our values of $R(k, l)$ are all one larger than those in [2]). We also remark that it is easy to see that $R(k, l) = R(l, k)$, and that $R(2, l) = l$.

DEFINITION 4. The number $e(k, l, n)$ is the minimum number of edges in any (k, l, n)-graph.

Section 1 below consists of the main lemmas needed to show that $28 \leqslant R(3, 8) \leqslant 29$, and $R(3, 9) = 36$. Section 2 contains an exposition of the computer programs which were used in the proofs of the above statements. Section 3 contains various structural results needed in the calculation of certain of the edge numbers $e(k, l, n)$. Section 4 contains some open questions in the area. Finally, the appendix contains lists of some of the more important graphs.

TABLES OF NUMERICAL RESULTS

Exact values or bounds for $R(k, l)$

$k \backslash l$	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28–29	36	39–44
4	9	18	25–28	34–44				
5	14	25–28	42–55	51–94				

Exact values or bounds for $e(3, l, n)$

l	n	$e(3, l, n)$	l	n	$e(3, l, n)$	l	n	$e(3, l, n)$
3	4	2	5	11	15	6	17	40
3	5	5	5	12	20	7	19	37
4	6	3	5	13	26	7	20	44
4	7	6	6	12	11	7	21	51
4	8	10	6	13	15	7	22	60
5	8	4	6	14	20	8	26	71–74
5	9	7	6	15	25	8	27	81–87
5	10	10	6	16	32	8	28	90–98

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One of the most fruitful ideas in the attempt to obtain good bounds for $R(k, l)$ and $e(k, l, n)$ is the idea of preferring a vertex. Given a (k, l)-graph G, and a vertex v in G, we can partition G into three subgraphs: the vertex v, the subgraph $H_1(v)$ generated by the neighbors of v, and the subgraph $H_2(v)$ generated by the remaining vertices. The vertex v is said to be the preferred vertex.

LEMMA 1. *If G is a (k, l)-graph, (and v is a vertex in G, then $H_1(v)$ is a $(k-1, l)$-graph) and $H_2(v)$ is a $(k, l-1)$-graph.*

Proof. Any clique of size x in $H_1(v)$ becomes a clique of size $x+1$ when v is added, and any independent set of size y in $H_2(v)$ becomes an independent set of size $y+1$ when v is added.

DEFINITION 5. Let G be a $(3, l)$-graph. We let $v_i = l - 1 - i$, and we define s_i to be the number of vertices of G of degree v_i.

We remark that in a $(3, l)$-graph, $H_1(v)$ is an independent set for each vertex v (this follows from Lemma 1). Therefore, the maximum possible degree in a $(3, l)$-graph is $l-1$, so in the above definition, $i \geqslant 0$. Thus, the subscript i is the difference between the degree of the vertex and the maximum possible degree in G. Note that the value of v_i depends on l as well as on i.

Given a (k, l, n)-graph, and a vertex v of degree d, we have seen that $H_2(v)$ is a $(k, l-1, n-d-1)$-graph. So, we must have $|H_2(v)| \geqslant e(k, l-1, n-d-1)$. This prompts the next definition.

DEFINITION 6. A vertex v of degree d in a (k, l, n)-graph is called full if $|H_2(v)| = e(k, l-1, n-d-1)$.

In a $(3, l)$-graph, if a vertex v is preferred, then each edge is either in $H_2(v)$ or is adjacent to exactly one vertex in $H_1(v)$, since $H_1(v)$ is an independent set.

DEFINITION 7. If a vertex v is preferred in a (k, l)-graph, we define $Z(v)$ to be the sum of the degrees of the neighbors of v. If $Z(v) = s$, we will sometimes say that v has Z-sum s.

DEFINITION 8. Given a graph G, if v is a vertex of degree d, then we say that v is a d-vertex. The subgraph of G generated by all of the d-vertices is called the d-subgraph.

The following lemma appears in [2] as Proposition 4.

LEMMA 2. *Let G be a (k, l, n, e)-graph. Let*

$$\Delta = ne - \sum_{i \geqslant 0} \{e(3, l-1, n-v_i-1) + v_i^2\}\, s_i.$$

Then, $\Delta \geqslant 0$, and there are at least $n - \Delta$ full vertices in G.

Proof. For all i and j, define

$$\begin{aligned} \beta_{ij}(v) &= (\text{the number of vertices in } H_1(v) \text{ of degree } v_j) && \text{if } \deg(v) = v_i, \\ &= 0 && \text{otherwise.} \end{aligned}$$

We note that $\sum_v \beta_{ij}(v)$ is the number of edges between vertices of degree v_i and v_j, so $\sum_v \beta_{ij}(v) = \sum_v \beta_{ji}(v)$. If v is of degree v_i, and is preferred, then

$$e = |H_2(v)| + (v_i)^2 + \sum_{j \geqslant 0} (i-j)\, \beta_{ij}(v).$$

If we sum this over all vertices of G, we have

$$ne = \sum_v |H_2(v)| + (v_i)^2\, s_i + \sum_{i \geqslant 0} \sum_{\substack{v \\ \deg v = v_i}} \sum_{j \geqslant 0} (i-j)\, \beta_{ij}(v). \tag{1}$$

In all cases, we have

$$|H_2(v)| \geqslant e(3, l-1, n-v_i-1), \tag{2}$$

and if v is a full vertex, then this is an equality.

If i and j are fixed, then the sum $\sum_v \beta_{ij}(v)$ occurs in (1) with a coefficient of $(i-j)$, and the sum $\sum_v \beta_{ji}(v)$ occurs with a coefficient of $(j-i)$. Recalling the remark at the beginning of the proof, we see that these two sums cancel in (1). Thus, using (2) in (1), we have

$$\begin{aligned} ne &\geqslant \sum_{i \geqslant 0} e(3, l-1, n-v_i-1)\, s_i + (v_i)^2\, s_i \\ &= \sum_{i \geqslant 0} \{e(3, l-1, n-v_i-1) + (v_i)^2\}\, s_i. \end{aligned}$$

This is equivalent to the statement that $\Delta \geqslant 0$, and it is easily seen that each vertex which is not full contributes at least 1 to Δ, so there must be at least $(n-\Delta)$ full vertices. This completes the proof.

In [2], it was shown that $27 \leqslant R(3, 8) \leqslant 30$. We now develop some lemmas on the structure of (3, 8, 29)-graphs, which will allow us to show that no such graphs exist, thereby showing that $R(3, 8) \leqslant 29$.

LEMMA 3. *If G is a (3, 8, 29)-graph, then G contains a 6-vertex v with* $|H_2(v)| \leqslant 59$.

Proof. In [2], it was shown that $e(3, 8, 29) \geqslant 99$. If G contained a vertex w of degree 5 or less, then $H_2(w)$ would be a (3, 7)-graph with more than 22 vertices, which is impossible, since $R(3, 7) = 23$. Thus, every vertex is of degree 6 or 7. If edges are counted, and we recall that there are s_1 vertices of degree 6, then $|G| = 101 - \frac{1}{2}(s_1 - 1)$. This implies that $s_1 = 1$, 3, or 5.

Case 1. $s_1 = 1$.
Let v be the unique 6-vertex. Then the neighbors of v are all 7-vertices, so $Z(v) = 42$, and so $|H_2(v)| = 101 - 42 = 59$.

Case 2. $s_1 = 3$.
There must be a 6-vertex v with at most one 6-vertex as neighbor, for otherwise the three 6-vertices would form a triangle. So, $Z(v) \geqslant 41$, and so $|H_2(v)| \leqslant 100 - 41 = 59$.

Case 3. $s_1 = 5$.
If v is a 6-vertex with at least three 6-vertices as neighbors, then each of these three 6-vertices are adjacent to at most two 6-vertices, since there are no triangles. Hence, in any case, G contains a 6-vertex v with at most two 6-vertices as neighbors. So, $Z(v) \geqslant 40$, and so $|H_2(v)| \leqslant 99 - 40 = 59$.

LEMMA 4. *If G is a (3, 7, 22, e)-graph, and* $e \leqslant 62$, *then G contains no vertices of degree less than five.*

Proof. If v is a 4-vertex, then $H_2(v)$ is a (3, 6, 17)-graph. All such graphs are known (see [5]). The computer algorithms used to rule out the values of $e \leqslant 61$ are described in Section 2.

The following lemma appears in [10]. It is very similar to Lemma 2.

LEMMA 5. *In any graph G, we have* $\sum_{i=0} s_i v_i^2 = \sum_v Z(v)$.

Proof. We have

$$\begin{aligned}\sum_v Z(v) &= \sum_v \sum_{\substack{w \\ (v,w) \in G}} \deg(w) \\ &= \sum_w \sum_{\substack{v \\ (v,w) \in G}} \deg(w) \\ &= \sum_w (\deg(w))^2 \\ &= \sum_{i \geqslant 0} s_i v_i^2.\end{aligned}$$

LEMMA 6. *If G is a $(3, 7, 22)$-graph, then G has at least 60 edges.*

Proof. Lemma 4 implies that G contains only vertices of degree 5 and 6. By counting edges, we find that $s_0 = 8$ and $s_1 = 14$. We now assume that G contains no full 5 vertices. For each 5-vertex, the Z-sum is at most 26, and since $e(3, 6, 15) = 25$, we have that the Z-sum for each 6-vertex is at most 34. So,

$$\sum_v Z(v) \leqslant 14.26 + 8.34 = 636.$$

On the other hand,

$$\sum_{i \geqslant 0} s_i v_i^2 = 14.25 + 8.36 = 638.$$

This contradicts Lemma 5.

In Section 2, we show that no $(3, 7, 22, 59)$-graph can have a full 5-vertex. This establishes the lemma.

THEOREM 1. $28 \leqslant R(3, 8) \leqslant 29$.

Proof. Lemmas 3 and 6 together imply that there are no $(3, 8, 29)$-graphs. The appendix gives a $(3, 8, 27, 87)$-graph.

We now turn our attention to $R(3, 9)$. In [2], it is shown that $36 \leqslant R(3, 9) \leqslant 37$, and that if G is a $(3, 9, 36)$-graph, then it is a regular graph of degree 8. If any vertex v is preferred, then $H_2(v)$ is a $(3, 8, 27, 80)$-graph. The remainder of this section is devoted to establishing results on the structure of $(3, 8, 27, 80)$-graphs.

LEMMA 7. *If G is a $(3, 7, 19, 36)$-graph, then G contains either a full 3-vertex or a full 4-vertex.*

Proof. If v is a full 2-, 3-, or 4-vertex, then $Z(v) = 4$, 11, or 16, respectively. There are no 1-vertices, since $e(3, 6, 17) = 40$. If G contains any 2-vertices, then the 2-vertices form a component of G, since no 2-vertex v can have anything but 2-vertices as neighbors, as $Z(v) \leqslant 4$. Let the 2-subgraph be G_2. Then we have $I(G) = I(G_2) + I(G - G_2)$. But G has 19 vertices, and it is easy to check that for all possible partitions of 19 into two integers m and n, where $|V(G_2)| = m$ and $|V(G - G_2)| = n$, we must have $I(G) \geqslant 7$. So there are no 2-vertices.

We now apply Lemma 2. We have the following system:

$$\begin{aligned} \Delta &= 684 - 34s_3 - 36s_2 - 40s_1 - 47s_0 \geqslant 0, \\ 72 &= 3s_3 + 4s_2 + 5s_1 + 6s_0, \qquad (1) \\ 19 &= s_3 + s_2 + s_1 + s_0. \end{aligned}$$

We now assume that there no full 3- or 4-vertices. From Lemma 2, it follows that $19 - \Delta = s_0 + s_1$, or $\Delta \geqslant s_2 + s_3$. This relation, together with (1), implies that

$$s_0 = 2s_3 + s_2 - 23, \tag{2}$$

$$85 \geqslant 9s_3 + 4s_2. \tag{3}$$

From (2), we obtain $2s_3 + s_2 \geqslant 23$, which implies that $8s_3 + 4s_2 \geqslant 92$. This contradicts (3), which completes the proof of the lemma.

LEMMA 8. $e(3, 7, 19) = 37$.

Proof. In [2], it was shown that $36 \leqslant e(3, 7, 19) \leqslant 37$. The computer algorithms in Section 2 show that the situation described in Lemma 7 cannot occur, so $e(3, 7, 19) \geqslant 37$, which completes the proof.

LEMMA 9. *If G is a $(3, 7, 21, 50)$-graph, then G contains a full 4-vertex with two 4-vertices and two 5-vertices as neighbors.*

Proof. We recall that $e(3, 6, 17) = 40$, so if a 3-vertex v in G is preferred, then $Z(v) \leqslant 10$, which implies that v must have at least two 3-vertices, say, w_1 and w_2, as neighbors. If w_1 is preferred, then w_2 has at most two edges to other vertices in $H_2(w_1)$. But $H_2(w_1)$ is a $(3, 6, 17)$-graph, and since $R(3, 5) = 14$, $H_2(w_1)$ can have no vertices of degree 2 or less. So G contains no 3-vertices.

We now apply Lemma 2 to obtain the following system:

$$\Delta = 1050 - 56s_0 - 50s_1 - 48s_2 \geqslant 0,$$

$$100 = 6s_0 + 5s_1 + 4s_2,$$

$$21 = s_0 + s_1 + s_2.$$

If we solve this system, we obtain

$$s_2 = 5 + s_0,$$

$$s_1 = 16 - 2s_0,$$

$$\Delta = 10 - 4s_0.$$

Since $\Delta \geqslant 0$, we must have $s_0 \leqslant 2$.

To establish the lemma, we need only show that some 4-vertex v has at most two 4-vertices as neighbors. To see why this suffices, note that since $e(3, 6, 16) = 32$, we must have $Z(v) \leqslant 18$. If v has at most two 4-vertices as neighbors, then $Z(v) \geqslant 4 + 4 + 5 + 5$, so equality must hold, and so v has exactly two 4-vertices and two 5-vertices as neighbors.

We now assume that every 4-vertex has at least three 4-vertices as neighbors. Then the 4-subgraph is a triangle-free graph with every vertex of degree at least 3. There are no such graphs with fewer than six vertices, and the only such graphs on six and seven vertices are $K_{3,3}$ and $K_{3,4}$, respectively. Since $s_2 = 5 + s_0$, we only need to consider these two cases.

In the first case, let the 4-vertices be partitioned into sets T_1 and T_2 such that $|T_1| = |T_2| = 3$, and such that every vertex in T_1 is adjacent to every vertex in T_2. There are three edges from vertices in T_1 to vertices in $G - (T_1 \cup T_2)$, so there are at least 12 vertices in $G - (T_1 \cup T_2)$ which are not adjacent to any vertex in T_1. From this set of 12 vertices, we can choose an independent set T_3 of size 4, since $R(3,4) = 9$. Then $T_1 \cup T_3$ is an independent set of size 7 in G, which is impossible.

In the second case, if the 4-subgraph is partitioned into sets T_1 and T_2, as before, with $|T_1| = 3$ and $|T_2| = 4$, then by using a similar argument, it is easy to show that T_1 is part of an independent set of size 7. This completes the proof of the lemma.

LEMMA 10. $e(3, 7, 21) = 51$.

Proof. In [2], it was shown that $50 \leqslant e(3, 7, 21) \leqslant 51$. The computer algorithms in Section 2 show that the situation described in Lemma 9 cannot occur, so $e(3, 7, 21) \geqslant 51$, which completes the proof.

LEMMA 11. *If G is a $(3, 8, 27, 80)$-graph, then G has no vertices of degree less than* 5.

Proof. In this proof, we assume that $e(3, 7, 19) = 37$, $e(3, 7, 20) = 44$, $e(3, 7, 21) = 51$, and $e(3, 7, 22) = 60$. The first, third, and fourth equalities are Lemmas 8, 10, and 4, and the third equality is given in [2].

If G contained a vertex v of degree less than 4, then $H_2(v)$ would be a $(3, 7, n)$-graph, where $n \geqslant 23$, which is impossible.

If v is a 4-vertex with two 4-vertices, say, w_1 and w_2, as neighbors, then w_2 is in $H_2(w_1)$, and w_2 has degree at most 3 in this subgraph. But $H_2(w_1)$ is a $(3, 7, 22)$-graph, and $R(3, 6) = 18$, so $H_2(w_1)$ can have no 3-vertices. Therefore, each 4-vertex in G has at most one 4-vertex as neighbor. So, if v is a 4-vertex, then $Z(v) \geqslant 19$. Therefore, $|H_2(v)| \leqslant 61$. Lemma 4 implies that $H_2(v)$ has no 4-vertices in it, so G has at most two 4-vertices, namely, v and perhaps one of its neighbors.

Using Lemma 2, we have the following system:

$$\Delta = 2160 - 86s_0 - 80s_1 - 76s_2 - 76s_3 \geqslant 0,$$

$$160 = 7s_0 + 6s_1 + 5s_2 + 4s_3,$$

$$27 = s_0 + s_1 + s_2 + s_3.$$

To solve this system, we take 2 cases:

Case 1. $s_3 = 2$.

$$\Delta = 2008 - 86s_0 - 80s_1 - 76s_2 \geqslant 0,$$
$$152 = 7s_0 + 6s_1 + 5s_2,$$
$$25 = s_0 + s_1 + s_2.$$

If s_0 and s_1 are eliminated, we get $2s_2 \leqslant -4$. This is impossible.

Case 2. $s_3 = 1$.
By solving the system we get $s_2 \leqslant 2$. Thus, the 4-vertex v has Z-sum at least $5+5+6+6=22$, so $|H_2(v)| \leqslant 58$. But $H_2(v)$ is a $(3, 7, 22)$-graph, and $e(3, 7, 22) = 60$, so we have a contradiction. This completes the proof of the lemma.

LEMMA 12. *If G is a $(3, 8, 27, 80)$-graph, then G contains either a full 6-vertex with six 6-vertices as neighbors, or a full 5-vertex with four 6-vertices and one 5-vertex as neighbors.*

Proof. We use the system of inequalities given in the proof of Lemma 11, but we set $s_3 = 0$. From this we obtain

$$s_2 = s_0 + 2,$$
$$s_1 = 25 - 2s_0,$$
$$\Delta = 8 - 2s_0 \geqslant 0.$$

Thus, $s_0 \leqslant 4$, and $2 \leqslant s_2 \leqslant 6$. We note that since $e(3, 7, 20) = 44$ and $e(3, 7, 21) = 51$, if v is a 5-vertex, then $Z(v) \leqslant 29$, and if v is a 6-vertex, then $Z(v) \leqslant 36$. In particular, the number of $(5, 6)$-edges is at least as great as the number of $(6, 7)$-edges, for otherwise some 6-vertex v would have more 7-vertices than 5-vertices as neighbors, which would force $Z(v) > 36$.

Case 1. $s_2 \leqslant 4$.
Each 5-vertex has at most four 6-vertices as neighbors, and no 6-vertex is adjacent to a 7-vertex and nor to a 5-vertex, so there are at most 16 6-vertices adjacent to either a 5-vertex or a 7-vertex. But there are at least 21 6-vertices, so there is a 6-vertex with six 6-vertices as neighbors.

Case 2. $s_2 = 5$.
If no 6-vertex exists with six 6-vertices as neighbors, then each 6-vertex is adjacent to at least one 5-vertex, and since there are 19 6-vertices, there must be at least 19 $(5, 6)$-edges. But there are only five 5-vertices, so some t-vertex v has at least four 6-vertices as neighbors. Since $Z(v) \leqslant 29$, v must have exactly four 6-vertices and one 5-vertex as neighbors.

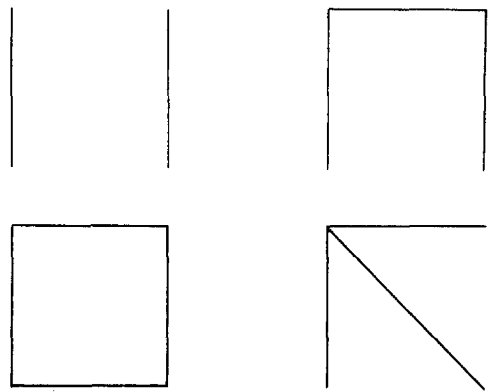


FIGURE 1

Case 3. $s_2 = 6$.

In this case, $\Delta = 0$, so Lemma 2 implies that all of the vertices are full. If v is a 7-vertex, then $Z(v) = 43$, so v must have at least one 7-vertex as neighbor. Thus, there are only four possibilities for the 7-subgraph; they are given in Fig. 1.

If v is a 7-vertex which has i 7-vertices as neighbors, then v has $(i-1)$ 5-vertices as neighbors, since $Z(v) = 43$. So, in any of the four cases, there are at most four (5, 7)-edges. Since there are six 5-vertices, there must be a 5-vertex w with no 7-vertices as neighbors. Since $Z(w) = 29$, w must have four 6-vertices and one 5-vertex as neighbors. This completes the proof of the lemma.

LEMMA 13. $e(3, 8, 27) \geqslant 81$.

Proof. In [2] it is shown that $e(3, 8, 27) \geqslant 80$. In Section 2, computer algorithms are described which enable us to show that the conditions given in Lemma 12 cannot occur, so $e(3, 8, 27) \geqslant 81$.

THEOREM 2. $R(3, 9) = 36$.

Proof. By the remark made before the statement of Lemma 7, the non-existence of a (3, 8, 27, 80)-graph implies that $R(3, 9) = 36$.

2

In this section we describe the computer algorithms which are used in the proofs of Lemmas 4, 6, 8, 10, and 13.

The first algorithm generates a list of all independent sets of size j in a graph G, for any $j \leqslant I(G)$. This algorithm is a standard backtrack procedure, and is well-known.

We illustrate the other algorithms by going through a proof of Lemma 6. We assume that G is a $(3, 7, 22, 59)$-graph, and that v is a full 5-vertex. Lemma 4 states that G has no 4-vertices, and since $Z(v) = 27$, v must have three 5-vertices, say, w_1, w_2, and w_3, and two 6-vertices, say, w_4 and w_5, as neighbors. We wish to show that it is impossible to join the vertices of $H_1(v)$ to the vertices of $H_2(v)$ in such a way that the resulting graph is a $(3, 7)$-graph. We point out that given a preferred vertex v, a list of vertex degrees in $H_1(v)$, and the graph $H_2(v)$, the following algorithm will generate all possible graphs with this structure, if any exist.

In the specific case above, we note that $H_2(v)$ is a $(3, 6, 16, 32)$-graph. A complete list of these graphs is given in the appendix. We choose one of the graphs, say, H_{16a}, and proceed. In order to avoid triangles while constructing G, it is enough to join each vertex w_i in $H_1(v)$ to an independent set S_i in H_{16a}. If w_i is an m-vertex in G, then S_i is an independent set of size $(m-1)$.

Given two vertices w_i and w_j in $H_1(v)$ it is usually impossible to have $S_i = S_j$, and the condition which insures this is easy to check by hand. We proceed as follows: If $S_i = S_j$, then w_i and w_j have the same neighbors. If w_i and w_j are simultaneously preferred then G is partitioned into three graphs: $\{w_i, w_j\}$, $H_1(w_i)$ $(=H_1(w_j))$, and $H_2(w_i) - w_j$ $(=H_2(w_j) - w_i)$. This last graph we will call $H_2(w_i, w_j)$. It is easy to see that since w_1 and w_2 are not adjacent, $H_2(w_i, w_j)$ must be a $(3, 5)$-graph. Furthermore, $H_1(w_i)$ has at most six vertices, so $H_2(w_i, w_j)$ has at least $22 - 6 - 2 = 14$ vertices. But no $(3, 5, 14)$-graphs exist, since $R(3, 5) = 14$.

In order to construct G, we must choose three different 4-independent sets S_1, S_2, and S_3, and two different 5-independent sets S_4 and S_5, join w_i to each vertex in S_i, and then check to see if the resulting graph has any 7-independent sets. If there are no 7-independent sets, then G is a $(3, 7, 22, 59)$-graph.

Given a selection of sets S_i, where can the 7-independent sets be in the resulting graph G? If T is a 7-independent set in G, and T contains v, then T contains none of the vertices in $H_1(v)$. So, $T - \{v\}$ is a 6-independent set in $H_2(v)$, which is impossible. If T contains less than two vertices in $H_1(v)$, the same problem occurs. So, T must contain at least two vertices in $H_1(v)$.

Now assume that T is a 7-independent set in G, as before, and assume that T contains exactly k vertices $w_{i_1}, w_{i_2}, \ldots, w_{i_k}$ in $H_1(v)$, where $k \geqslant 2$. Let $V_2 = V(H_2(v))$. Then $T - \{w_{i_1}, \ldots, w_{i_k}\}$ is a $(7-k)$-independent set in $V_2 - (S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k})$. Conversely, if the k sets $S_{i_1}, \ldots, S_{i_k}$ have the property that when they are removed from V_2, a $(7-k)$-independent set S remains, then $S \cup \{w_{i_1}, \ldots, w_{i_k}\}$ is a 7-independent set in G. Thus, a necessary and sufficient condition that the selection $\{S_1, S_2, \ldots, S_5\}$ represents a $(3, 7)$-graph is that for each subselection $\{S_{i_1},..., S_{i_k}\}$, where $k \geqslant 2$, we have that the set $V_2 - (S_{i_1} \cup \cdots \cup S_{i_k})$ contains no $(7-k)$-independent set.

Let us say that the sets S_i and S_j form a "good" pair if $V_2 - (S_i \cup S_j)$ has no 5-independent sets. From above, we see that if the selection $\{S_1,..., S_3\}$ represents a (3, 7)-graph, then it is necessary (but not sufficient) that each pair of sets S_i, S_j be a good pair. The property of being a good pair depends only on $H_2(v)$, which in this case is H_{16a}.

We now construct three matrices M_{44}, M_{45}, and M_{55}. The first algorithm above gives us lists of 4- and 5-independent sets. Let us call them $\{S_i^4 : i \leqslant d_4\}$ and $\{S_i^5 : i \leqslant d_5\}$, where G contains d_4 4-independent sets and d_5 5-independent sets. If $i < j$, then we define the ijth entry in M_{44} to be 1 if S_i^4 and S_j^4 form a good pair, and 0 otherwise. The matrix M_{55} is defined similarly. For all i and j, we define the ijth entry in M_{45} to be 1 if S_i^4 and S_j^5 form a good pair, and 0 otherwise. An efficient way to generate these matrices will be described below.

From these matrices, we generate a list of all sets of the form $A = \{S_a^4, S_b^4, S_c^4, S_d^5, S_e^5\}$, where each pair in the set A is a good pair. To obtain all sets of this form, we pick the first row $M_{44}^{(1)}$ in the matrix M_{44}, and pick the first column j for which the ijth entry is a 1. We have now "picked" the sets S_1^4 and S_j^4. Next, the "and" operation, written $\wedge$, is applied to the first and the jth rows, and the first nonzero entry in the result, say, in the kth column, is chosen. This corresponds to picking S_k^4. Note that in the set $\{S_1^4, S_j^4, S_k^4\}$, all three pairs are good pairs. We now must pick two 5-independent sets. From the matrix M_{45}, we select the first, jth, and kth rows, and take their "and." Let us assume that in the result, the first 1 occurs in the lth column. Then the 5-independent set S_l^5 forms a good pair with each of the three 4-independent sets already chosen. Finally, the result of the last "anding" operation is "anded" with the lth row of M_{55}. If the mth column in the result is 1, then the set S_m^5 can be added to complete the set A. All 10 pairs in the set A are good pairs. We then backtrack.

If any sets of the form of A exist, then we check each set to see if for all choices of k sets from A, $k \geqslant 3$, when these k sets are removed from V_2, there are no $(7-k)$-independent sets remaining. If this is the case, then the set A represents a (3, 7, 22, 59)-graph.

We now describe how the matrices M_{ij} are constructed. We first construct the matrix N_4 (and in a similar way, N_5). The matrix N_4 is a $d_4 \times d_5$ matrix. A 1 is placed in the (bc)th position of N_4 if and only if $S_b^4 \cap S_c^5 = \varnothing$. A 1 is placed in the (bc)th position of N_5 if and only if $S_b^5 \cap S_c^5 = \varnothing$.

Let the kth row of N_i be denoted $N_i^{(k)}$. To find the (kl)th entry of the matrix M_{44}, where $k < l$, we look at the rows $N_4^{(k)}$ and $N_4^{(l)}$. If $N_4^{(k)} \wedge N_4^{(l)}$ has any ones in it, then the (kl)th entry of M_{44} is given the value 1. To see why this works, note that there are no ones in $N_4^{(k)} \wedge N_4^{(l)}$ if and only if every 5-independent set S_j^5 intersects either $S_4^{(k)}$ or $S_4^{(l)}$. This happens if and only if the pair $(S_4^{(k)}, S_4^{(l)})$ is a good pair, which is true if and only if the (kl)th entry of M_{44} is a 1.

The above programs were run on a Honeywell Level 66 computer at Dartmouth College. This computer can perform 2×10^6 instructions per second, and we estimate the amount of time used in the search for $R(3, 9)$ to be 2.5×10^4 seconds. Thus, the number of instructions was about 5×10^{10}. The time needed could be significantly reduced in several ways. The programs would run much more quickly on a machine which has extended bit-string operations, e.g., the ability to find the "and' of two 800-bit vectors. A reduction in time would occur if the machine were able to process the programs in parallel. Finally, a fast bit-string index function, i.e., a fast method for determining the position of the first 1 after a given position in a vector, would also make the programs faster.

3

This section contains many lemmas describing the structure of $(3, l)$-graphs with the minimum possible number of edges. These lemmas are used with the algorithms described in the previous section to calculate $e(3, l, n)$ and to find all $(3, l, n)$-graph with $e(3, l, n)$ edges, for various values of l and n.

DEFINITION 9. If G is a $(3, l, n, e)$-graph with $e = e(3, l, n)$, then G is called a minimum $(3, l, n)$-graph. Let G be a $(3, l, n)$-graph with the property that if any edge is removed, then the resulting graph contains an independent set of size l. In this case, we call G a minimal $(3, l, n)$-graph.

We note that every minimum graph is minimal, but not vice versa. If the graph G is minimum and has n vertices, then we follow [2] and denote G by H_n, and we use further subscripting if there is more than 1 minimum graph. For example, there are five minimum $(3, 6, 16)$-graphs, which we denote $H_{16a}, \ldots, H_{16e}$. If G is a $(3, l, n)$-graph which is not minimum, then we usually denote it J_n, with further subscripting if necessary. No confusion arises, since the number n usually determines the values of l.

LEMMA 14. *The following parameters have unique minimum graphs associated with them. In all cases, if the graph has n vertices it is denoted* H_n.

(1) (3, 3, 5, 5)
(2) (3, 4, 8, 10)
(3) (3, 5, 9, 7)
(4) (3, 5, 10, 10)
(5) (3, 5, 11, 15)
(6) (3, 5, 12, 20)
(7) (3, 6, 13, 15)
(8) (3, 6, 15, 25)

Proof. These results were all proved in [2]. We note that the (3, 6, 13, 15)-graph is called H_{13}, even though there is a (3, 5, 13)-graph. The latter graph is not of much use in constructing larger graphs.

LEMMA 15. *If G is a* (3, 5, 11, 16)*-graph, then G contains a vertex v of degree* 2 *with* $5 \leqslant Z(v) \leqslant 6$. *Furthermore, the neighbors of v are either* 2-, 3-, *or* 4-*vertices.*

Proof. Since $R(3, 4) = 9$, every vertex in G must have degree at least 2. The average degree of the vertices in G is 32/11, so G must have a vertex of degree 2. Assume that G is the (disjoint) union of two or more connected components, say, $G = G_1 \cup G_2 \cup \cdots \cup G_k$. At least one component, say, G_1, has less than six vertices. If $|V(G_1)| < 3$, then $|V(G - G_1)| > 8$, so $I(G - G_1) \geqslant 4$, and so $I(G) \geqslant 5$, which is a contradiction. If $3 \leqslant |V(G_1)| \leqslant 5$, then $I(G_1) \geqslant 2$, and $I(G - G_1) \geqslant 3$, so $I(G) \geqslant 5$, which again is a contradiction. Therefore, G is connected.

Since G is connected, there is a 2-vertex v which is adjacent to a vertex of larger degree. So, $Z(v) \geqslant 5$. Since $e(3, 4, 8) = 10$, we must have $Z(v) \leqslant 6$. The final statement in the lemma now follows easily.

The computer algorithms in Section 2 can be applied, using the above lemma, to find all (3, 5, 11, 16)-graphs. There are six such graphs, and they are listed in the appendix. We note that three of these graphs come from H_{11} by the addition of an edge, and the other three are minimal.

LEMMA 16. *If G is a* (3, 6, 16, 32)*-graph, then either G has a full vertex or G is regular of degree* 4.

Proof. Since $R(3, 4) = 14$, G has no vertices of degree less than 2. If G has a 2-vertex v, then $H_2(v)$ is a (3, 5, 13)-graph, and there is only one such graph (see [2, 3]) and it has 26 edges. So, $Z(v) = 6$, and so there are four edges between $H_1(v)$ and $H_2(v)$. Thus, there are at least nine vertices in $H_2(v)$ which are not adjacent to any vertices in $H_1(v)$. Since $R(3, 4) = 9$, it is possible to choose an independent set S of size 4 from these nine vertices. Then $S \cup V(H_1(v))$ is an independent set of size 6, which is impossible. Hence, G has no 2-vertices.

We now apply Lemma 2, to obtain the following system:

$$\Delta = 512 - 35s_0 - 31s_1 - 29s_2 \geqslant 0,$$

$$64 = 5s_0 + 4s_1 = 3s_2, \text{ and}$$

$$16 = s_0 + s_1 + s_2.$$

If we solve this system, we obtain

$$s_1 = 16 - 2s_2,$$

$$s_0 = s_2,$$

and

$$\Delta = 16 - 2s_2.$$

If $\Delta < 16$, then G contains a full vertex. If $\Delta = 16$, then $s_2 = 0$, hence $s_0 = 0$, and so G is regular of degree 4.

Again, the computer algorithms in Section 2 can be applied, using the above lemma, to find all (3, 6, 16, 32)-graphs. Note that if G is regular, and v is any vertex in G, then $H_2(v)$ is a (3, 5, 11, 16)-graph, which is the reason that Lemma 15 is needed. There are five (3, 6, 16, 32)-graphs, two of which are regular, and they are listed in the appendix.

LEMMA 17. *If G is a* (3, 6, 15, 26)*-graph, then G contains a full vertex of degree* 2, 3, *or* 4.

Proof. Since $e(3, 5, 13) = 26$, G contains no vertices of degree less than 2. Using Lemma 2, we obtain

$$\Delta = 390 - 32s_0 - 26s_1 - 26s_2 - 24s_3 \geqslant 0,$$

$$52 = 5s_0 + 4s_1 + 3s_2 + 2s_3,$$

and

$$15 = s_0 + s_1 + s_2 + s_3.$$

From this we obtain

$$s_1 = 23 - 2s_2 - 3s_3,$$

$$s_0 = s_2 + 2s_3 - 8,$$

and

$$\Delta = 48 - 6s_2 - 10s_3.$$

Since $s_0 \geqslant 0$, we must have $s_2 + 2s_3 \geqslant 8$, or equivalently, $5s_2 + 10s_3 \geqslant 40$. But,

$$0 \leqslant \Delta = 48 - (5s_2 + 10s_3) - s_2 \leqslant 8 - s_2 \leqslant 8,$$

so Lemma 2 implies that there are at least seven full vertices. If $s_0 \geqslant 7$, then by counting edges, it is easy to show that $s_3 \geqslant 7$, which implies that $\Delta < 0$, which is impossible. So, $s_0 < 7$, which means that there is a full vertex which is not a 5-vertex.

LEMMA 18. *There are exactly seven* (3, 6, 15, 26)*-graphs.*

Proof. Lemma 17 gives enough conditions on such graphs so that the computer algorithms in Section 2 can be used to generate a complete list of (3, 6, 15, 26)-graphs. These graphs are listed in the appendix.

LEMMA 19. *If G is a* (3, 7, 20, 44)-*graph, then G contains either a full vertex or a* 4-*vertex v with* $Z(v) = 18$.

Proof. Since $R(3, 6) = 18$, G can have no vertices of degree less than 2. If G has a vertex v of degree 2, then $Z(v) \leqslant 4$, since $e(3, 6, 17) = 40$. So the 2-subgraph of G forms a component of G. The same argument as that used at the beginning of Lemma 15 shows that it is impossible for G to be disconnected, so G has no vertices of degree 2.

Using Lemma 2, we obtain the following system:

$$\Delta = 880 - 51s_0 - 45s_1 - 41s_2 - 41s_3 \geqslant 0,$$

$$88 = 6s_0 + 5s_1 + 4s_2 + 3s_3,$$

and

$$20 = s_0 + s_1 + s_2 + s_3.$$

Solving this system, we obtain

$$\Delta = 52 - 2s_2 - 8s_3,$$

$$s_1 = 32 - 2s_2 - 3s_3,$$

and

$$s_0 = s_2 + 2s_3 - 12.$$

We now assume that G does not contain a full vertex. Thus, $\Delta \geqslant 20$. By recalling the proof of Lemma 2, we see that there can be no more than $(\Delta - 20)$ vertices v such that $Z(v)$ is 1 less than the Z-sum of a full vertex of the same degree. So, if $s_2 > (\Delta - 20)$, then some 4-vertex v has $Z(v) = 18$.

By solving the above system, we obtain nine solutions, all of which have $s_2 > \Delta - 20$, which completes the proof.

LEMMA 20. *There are exactly* 15 (3, 7, 20, 44)-*graphs.*

Proof. If G is a (3, 7, 20, 44)-graph, and v is a 4-vertex with $Z(v) = 18$, then $H_2(v)$ is a (3, 6, 15, 26)-graph. Lemma 18 gave a complete list of such graphs. The computer algorithms in Section 2 can be applied to these graphs, and to the minimum graphs H_{13}, H_{14}, H_{15}, $H_{16a}, \ldots, H_{16e}$. A complete list of (3, 7, 20, 44)-graphs, named $H_{20a}, \ldots, H_{20o}$, is given in the appendix.

LEMMA 21. *If G is a* (3, 7, 21, 51)-*graph, then G has a full vertex.*

Proof. Since $R(3, 6) = 18$, G has no vertices of degree less than 3. If v is a 3-vertex, then $Z(v) \leqslant 11$, since $e(3, 6, 17) = 40$. In this case, there are at most eight edges between $H_1(v)$ and $H_2(v)$, so there are at least nine vertices in $H_2(v)$ which are not adjacent to any vertex in $H_1(v)$. Since $R(3, 4) = 9$, there is an independent set S of size 4 in this set of nine vertices. But then $S \cup V(H_1(v))$ is an independent set of size 7 in G, which is impossible. So, $s_3 = 0$.

From Lemma 2, we obtain

$$\Delta = 1071 - 48s_2 - 50s_1 - 56s_0 \geqslant 0,$$

$$102 = 4s_2 + 5s_1 + 6s_0,$$

and

$$21 = s_2 + s_1 + s_0.$$

If we solve this system, we obtain

$$\Delta = 27 - 4s_0,$$

$$s_2 = s_0 + 3,$$

and

$$s_1 = 18 - 2s_0.$$

If $\Delta < 21$, then G has a full vertex, so we may assume that $\Delta \geqslant 21$. This implies that $s_0 = 0$ or 1.

Case 1. $s_0 = 0$.

In this case $s_2 = 3$, so at least one 4-vertex v has no more than one 4-vertex as neighbor. So, $Z(v) \geqslant 4 + 5 + 5 + 5$, so v is a full vertex.

Case 2. $s_0 = 1$.

We assume towards a contradiction that G contains no full vertices. We have $s_2 = 4$, $s_1 = 16$, and each 4-vertex has at least two 4-vertices as neighbors. Thus, the 4-subgraph must be a quadrilateral, with edges $\{(a, b), (b, c), (c, d), (d, a)\}$, say. Let v be the 6-vertex in G. Since the 4-vertices have Z-sum at most 18, they each must be adjacent to two 5-vertices. So, v is adjacent to six 5-vertices, say, $v_1, \ldots, v_6$. Since none of the 5-vertices are full, their Z-sums are at most 25, so each of the vertices $v_1, \ldots, v_6$ must be adjacent to at least one 4-vertex. Therefore, at least one of the sets $\{a, c\}$, $\{b, d\}$, has at least three neighbors in $H_1(v)$. Without loss of generality, say $\{a, c\}$ has this property. Then there is at most one 5-vertex from outside of $H_1(v)$ which is adjacent to either a or c, so there are at least nine 5-vertices which are adjacent to none of v, a, and c. Thus, an independent set S of size 4 can be found in this set of nine vertices. But then the set $S \cup \{v, a, c\}$ is an independent set of size 7, which is a contradiction. This completes the proof.

LEMMA 22. *There are exactly four* (3, 7, 21, 51)-*graphs.*

Proof. The computer algorithms in Section 2, together with Lemma 21, can be used to generate a complete list of (3, 7, 21, 51)-graphs. These graphs, denoted $H_{21a},..., H_{21d}$, are listed in the appendix.

LEMMA 23. *If G is a* (3, 7, 22, 60)-*graph, then G contains a full vertex or a* 6-*vertex v with* $Z(v) = 34$.

Proof. From Lemma 4, we know that G contains only 5-vertices and 6-vertices. By counting edges, we see that $s_0 = 12$, $s_1 = 10$. Using Lemma 2, we can show that $\Delta = 26$. So, either there is a full vertex, or else all but four of the vertices have Z-sums which are 1 less than the maximum possible Z-sum. Since there are 12 6-vertices, there must be a 6-vertex v with $Z(v) = 34$, if there are no full vertices.

LEMMA 24. *There is a unique* (3, 7, 22, 60)-*graph, and so* $e(3, 7, 22) = 60$.

Proof. Using Lemmas 18 and 23, and the computer programs in Section 2, we can show that there is exactly one (3, 7, 22, 60)-graph. This graph is denoted H_{22}, and is given in the appendix. Lemma 6 states that $e(3, 7, 22) \geqslant 60$, so the equality holds.

4

In this section, we give some unsolved problems in the area, as well as some interesting numerical phenomena which arose while this research was being done.

Let us define a graph G to be bicritical if the addition of any edge of G increases $C(G)$, and the deletion of any edge increases $I(G)$, and neither G nor $\bar{G}$ is a complete graph. At the Graph Theory meeting in Kalamazoo in 1980, M. Albertson and D. Berman asked if there were any bicritical graphs other than the pentagon. Suppose that there is a unique (k, l)-graph G on n vertices. Then G must be a bicritical graph, since if any edge is added to G, the resulting graph G' i no longer a (k, l)-graph, so $C(G') \geqslant k > C(G)$, and similarly, if any edge is deleted from G, the resulting graph must have independence number at least l. There are four known examples of such graphs, namely, the graphs with parameters (3, 3, 5), (3, 5, 13), (4, 4, 17), and (3, 9, 35). We note that all of these graphs are cyclic graphs; i.e., they all have a $|V(G)|$-cycle in the automorphism group. Also, in each case, the number of vertices is one less than the corresponding Ramsey number. Are there any other cyclic bicritical graphs? Is every bicritical graph a cyclic graph? We point out that we do not know that there is only one (3, 9)-graph on 35 vertices, but it is easy to check that the only one known is bicritical.

We have shown that $e(3, 8, 27) \geqslant 81$, but the smallest known (3, 8, 27)-graph has 87 edges. In all previous cases, the lower bound for $e(3, l, n)$ given by the methods of Graves and Yackel has been within 2 of the correct value. Thus, we feel that it is likely that $e(3, 8, 27) < 87$. We can also show that $e(3, 8, 28) \geqslant 90$, but we have not found any (3, 8, 28)-graphs.

The following table gives some interesting information about the 15 (3, 7, 20, 44)-graphs. The number d_i is the number of i-independent sets. Several paterns are evident, including the fact that all four columns form arithmetic progressions. Also we note that $504 = 4 \cdot \binom{9}{4}$, $882 = 7 \cdot \binom{9}{4}$, $756 = 6 \cdot \binom{9}{4}$, and $252 = 2 \cdot \binom{9}{4}$. The numbers become even more striking when we point out that the nine graphs given in the first row of the table have five different degree sequences among them.

Graphs	d_3	d_4	d_5	d_6
$H_{20a,b,c,d,f,j,l,m,o}$	504	882	756	252
$H_{20g,h,i,k,n}$	502	871	736	240
H_{20e}	500	860	716	228

Appendix

Figure 2 gives the nine graphs listed in Lemma 14. Table 1 gives many of the graphs used in the preceding sections. We illustrate the notation with an example. The graph H_{20d} is a (3, 7, 20, 44)-graph, with the 20th vertex being preferred. The middle columns give us that the neighbors of this vertex are the vertices 19, 18, and 17, and that these vertices are also adjacent to the vertices {7, 16}, {6, 8, 9}, and {7, 13, 14, 15}, respectively. The H_2-graph for vertex 20 is given in the last column. In this case, $H_2(20) = H_{16b}$.

Table 2 gives graphs which arise from the addition of one or more edges to a graph which has already been given. For example, J_{17c} is obtained from H_{17a} by the addition of the edges (3, 7) and (4, 8).

Finally, Table 3 gives the unique (3, 7, 22, 60)-graph H_{22}, and a (3, 8, 27, 87)-graph.

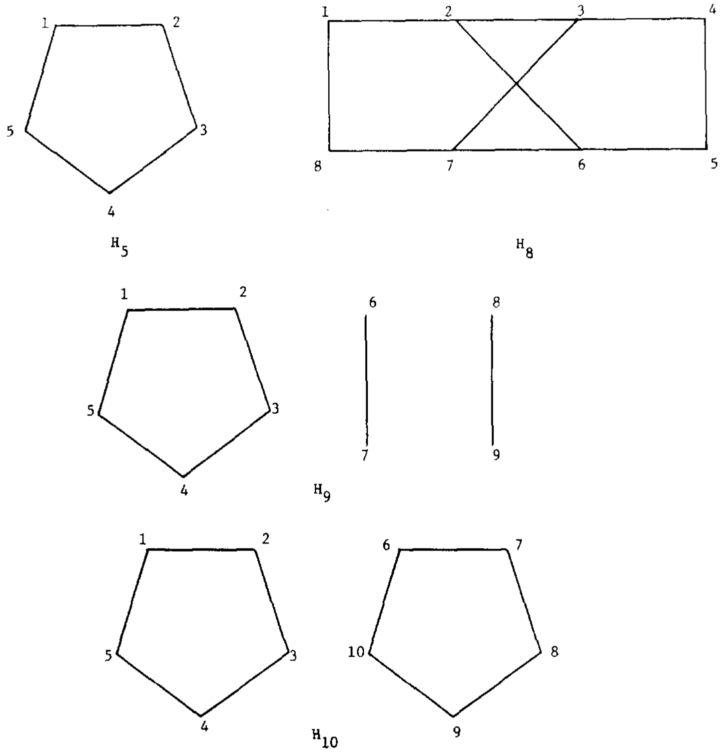


FIGURE 2a

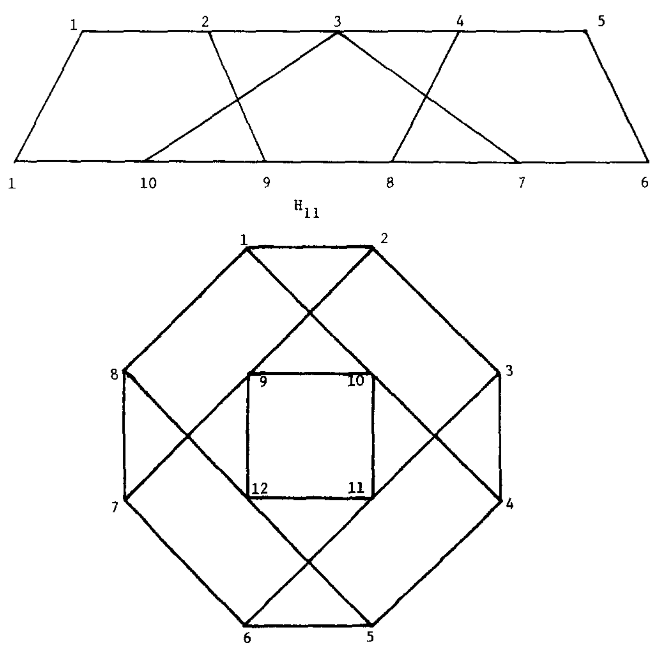
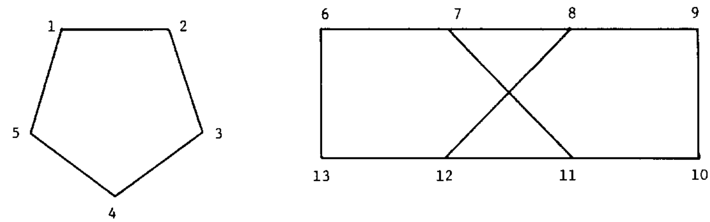
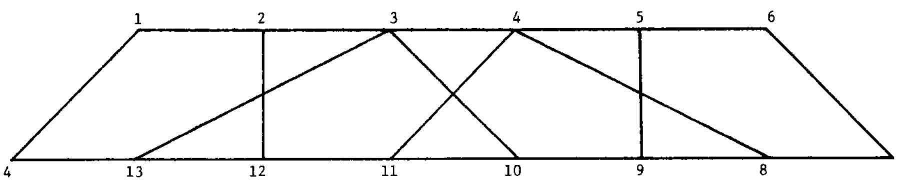


FIGURE 2b

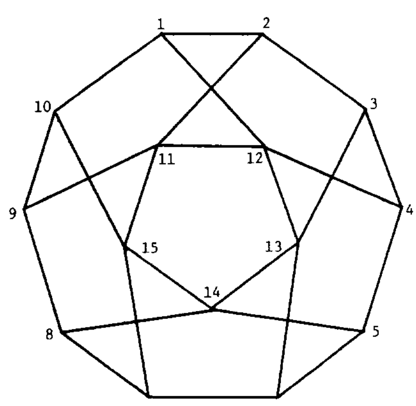


H_{13}



H_{14}

FIGURE 2c



H_{15}

FIGURE 2d

TABLE I

Graph	Parameters	Neighbors of vertices in H_1				H_2
J_{11d}	(3, 5, 11, 16)	10-2, 4	9-1, 3			H_8
J_{11e}	(3, 5, 11, 16)	10-5, 8	9-1, 4			H_8
J_{11f}	(3, 5, 11, 16)	10-3, 6	9-2, 7			H_8
J_{15e}	(3, 6, 15, 26)	14-1	13-2, 8, 10			H_{12}
J_{15f}	(3, 6, 15, 26)	14-1, 3	13-2, 8			H_{12}
J_{15g}	(3, 6, 15, 26)	14-1, 9	13-2, 10			H_{12}
H_{16a}	(3, 6, 16, 32)	15-4, 7	14-3, 5, 10	13-4, 6, 8, 9		H_{12}
H_{16b}	(3, 6, 16, 32)	15-1, 5	14-4, 6, 12	13-2, 5, 8, 10		H_{12}
H_{16c}	(3, 6, 16, 32)	15-3, 6, 9	14-2, 6, 10	13-2, 5, 11	12-1, 5, 7	J_{11a}
H_{16d}	(3, 6, 16, 32)	15-6, 8, 9	14-5, 8, 11	13-5, 7, 10	12-1, 4, 11	J_{11d}
H_{16e}	(3, 6, 16, 32)	15-5, 9, 10	14-5, 8, 11	13-4, 6, 8	12-1, 7, 11	J_{11d}
H_{17a}	(3, 6, 17, 40)	16-1, 3, 5, 9	15-1, 4, 6, 12	14-2, 5, 7, 11	13-2, 6, 8, 10	H_{12}
H_{17b}	(3, 6, 17, 40)	16-1, 3, 5	15-1, 4, 6, 12	14-2, 5, 7, 11	13-2, 6, 8, 10	J_{12}
H_{20a}	(3, 7, 20, 44)	19-3, 7	18-6, 8, 9	17-2, 4, 7, 11		H_{16b}
H_{20b}	(3, 7, 20, 44)	19-2, 16	18-1, 3, 9	17-2, 13, 14, 15		H_{16a}

TABLE I—(*Continued*)

Graph	Parameters	Neighbors of vertices in H_1				H_2
H_{20c}	(3, 7, 20, 44)	19-1, 16	18-13, 14, 15	17-2, 8, 10, 16		H_{16a}
H_{20d}	(3, 7, 20, 44)	19-7, 16	18-6, 8, 9	17-7, 13, 14, 15		H_{16b}
H_{20e}	(3, 7, 20, 44)	19-4, 7, 15	18-7, 13, 14	17-3, 5, 10, 15	16-4, 6, 8, 9	J_{15f}
H_{20f}	(3, 7, 20, 44)	19-7, 15	18-6, 8, 9	17-1, 5, 7, 16		H_{16b}
H_{20g}	(3, 7, 20, 44)	19-1, 8, 13	18-2, 8, 11	17-4, 7, 9		H_{16c}
H_{20h}	(3, 7, 20, 44)	19-8, 11	18-1, 10, 13	17-4, 7, 9, 11		H_{16c}
H_{20i}	(3, 7, 20, 44)	19-1, 13, 16	18-13, 14, 15	17-2, 8, 16		H_{16a}
H_{20j}	(3, 7, 20, 44)	19-1, 3, 9	18-2, 14, 16	17-2, 13, 15		H_{16a}
H_{20k}	(3, 7, 20, 44)	19-6, 8, 9	18-7, 14, 15	17-1, 7, 16		H_{16b}
H_{20l}	(3, 7, 20, 44)	19-6, 8, 9	18-5, 7, 16	17-7, 13, 15		H_{16b}
H_{20m}	(3, 7, 20, 44)	19-8, 9, 16	18-7, 12, 16	17-13, 14, 15		H_{16b}
H_{20n}	(3, 7, 20, 44)	19-1, 6, 15	18-2, 4, 14	17-3, 5, 14	16-1, 5, 7, 11, 15	J_{15e}
H_{20o}	(3, 7, 20, 44)	19-1, 6, 15	18-3, 10, 14	17-4, 11, 14	16-1, 5, 7, 11, 15	J_{15e}
H_{21a}	(3, 7, 21, 51)	20-13, 14, 15	19-6, 8, 9, 16	18-4, 7, 11, 16	17-3, 7, 10, 16	H_{16b}
H_{21b}	(3, 7, 21, 51)	20-6, 8, 9	19-3, 7, 14, 15	18-2, 4, 7, 11	17-1, 3, 7, 16	H_{16b}
H_{21c}	(3, 7, 21, 51)	20-6, 8, 9	19-7, 13, 14, 15	18-4, 7, 11, 16	17-3, 7, 10, 16	H_{16b}
H_{21d}	(3, 7, 21, 51)	20-6, 8, 9	19-4, 7, 11, 15	18-3, 7, 10, 15	17-1, 5, 7, 16	H_{16b}

TABLE II

Graph	Parameters	Description
J_{11a}	(3, 5, 11, 16)	$H_{11} \cup \{(1, 4)\}$
J_{11b}	(3, 5, 11, 16)	$H_{11} \cup \{(1, 8)\}$
J_{11c}	(3, 5, 11, 16)	$H_{11} \cup \{(1, 5)\}$
J_{12}	(3, 5, 12, 21)	$H_{12} \cup \{(4, 8)\}$
J_{15a}	(3, 6, 15, 26)	$H_{15} \cup \{(1, 5)\}$
J_{15b}	(3, 6, 15, 26)	$H_{15} \cup \{(1, 6)\}$
J_{15c}	(3, 6, 15, 26)	$H_{15} \cup \{(1, 14)\}$
J_{15d}	(3, 6, 15, 26)	$H_{15} \cup \{(1, 8)\}$
J_{17a}	(3, 6, 17, 41)	$H_{17a} \cup \{(3, 8)\}$
J_{17b}	(3, 6, 17, 41)	$H_{17a} \cup \{(4, 8)\}$
J_{17c}	(3, 6, 17, 42)	$H_{17a} \cup \{(3, 8), (4, 7)\}$
J_{17d}	(3, 6, 17, 42)	$H_{17a} \cup \{(3, 7), (4, 8)\}$
J_{17e}	(3, 6, 17, 41)	$H_{17b} \cup \{(3, 7)\}$

TABLE III

Graph	Parameters	Neighbors of Vertices in $H_1(v)$	$H_2(v)$
H_{22}	(3, 7, 22, 60)	21-5, 11, 13, 14 20-6, 12, 13, 14 19-1, 3, 7, 12, 15 18-2, 4, 7, 11, 15 17-3, 5, 8, 10, 15 16-4, 6, 8, 9, 15	J_{15g}
J_{27a}	(3, 8, 27, 87)	26-1, 5, 7, 11, 20 25-1, 3, 6, 9, 13, 18 24-2, 4, 6, 12, 17, 21 23-2, 5, 8, 10, 14, 19	H_{22}

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