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The generalized uncertainty principle in (A)dS space and the modification of Hawking temperature from the minimal length

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Abstract

Recently, the Heisenberg's uncertainty principle has been extended to incorporate the existence of a large (cut-off) length scale in de Sitter or anti-de Sitter space, and the Hawking temperatures of the Schwarzshild–(anti) de Sitter black holes have been reproduced by using the extended uncertainty principle. I generalize the extended uncertainty to the case with an absolute minimum length and compute its modification to the Hawking temperature. I obtain a *general* trend that the generalized uncertainty principle due to the absolute minimum length "always" increases the Hawking temperature, implying "faster" decay, which is in conformity with the result in the asymptotically flat space. I also revisit the *black hole-string* phase transition, in the context of the generalized uncertainty principle.
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1. Introduction

The Heisenberg's uncertainty principle provides a basic limitation of measuring the classical trajectories in the atomic or sub-atomic scale. But here, there is no absolute minimum or maximum uncertainty in the position and momentum themselves, though there is "conditional" minimum in them when one of them is fixed. So, in this regards, there have been arguments that the Heisenberg's uncertainty principle needs some modifications when the gravitational interaction is considered in quantum mechanics since there is an absolute minimum uncertainty in the position of any gravitating quantum [1,2]. And also, its several interesting implications have been studied in the literatures. Especially, it has been found that the generalized uncertainty principle (GUP) increases the Hawking temperature, resulting in "faster" decay of Schwarzschild black holes in any dimension [3,4].

However, the GUP does not have any limitation on the maximum uncertainty in the position such as it cannot be naively applied to the case with the large (cut-off) length scales, like as in de Sitter or anti-de Sitter space. Actually, the Hawking temperature of black holes in (anti) de Sitter space cannot be reproduced by the Heisenberg's uncertainty principle or the GUP. Recently, an *extended* uncertainty principle (I will call this "EUP", simply) has been introduced to incorporate the existence of the large length scales and it is found that the Hawking temperatures of the Schwarzshild–(anti) de Sitter black holes have been correctly reproduced [5].

In this Letter, I generalize the EUP to the case with an absolute minimum uncertainty in the position as well and compute its modification to the Hawking temperature. I obtain a *general* trend that the generalized uncertainty principle due to the absolute minimum length always increases the Hawking temperature, implying faster decay, which is in conformity with the result of the asymptotically flat space. I also revisit the black hole-string phase transition, in the context of the generalized uncertainty principle.

2. The GUP and Hawking temperature in asymptotically flat space

In this section, I review, with some new interpretations and remarks, the GUP and the derivation of Hawking temperature

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from the uncertainty principle in the asymptotically flat space [3,4].

The GUP is given by

$$\Delta x_i \Delta p_j \geqslant \hbar\delta_{ij}\left[1+\alpha^2 l_P^2 \frac{(\Delta p_j)^2}{\hbar^2}\right], \tag{1}$$

where x_i and p_j $(i, j = 1, \ldots, d-1)$ are the spatial coordinates and momenta, respectively; $l_P = (\hbar G)^{1/(d-2)}$ is the Planck length and α is a dimensionless real constant of order one [1]. In the absence of the second term in the right-hand side, this reduces to the usual Heisenberg's uncertainty principle without any "absolute" bound of Δx_i nor Δp_j themselves. But, in the presence of the second term, there exists an absolute minimum in the position uncertainty

$$\Delta x_i \geqslant 2\alpha l_P \tag{2}$$

and the uncertainty in the momentum is given by

$$\frac{\hbar\Delta x_i}{2\alpha^2 l_P^2}\left[1-\sqrt{1-\frac{4\alpha^2 l_P^2}{(\Delta x_i)^2}}\right] \leqslant \Delta p_i \leqslant \frac{\hbar\Delta x_i}{2\alpha^2 l_P^2}\left[1+\sqrt{1-\frac{4\alpha^2 l_P^2}{(\Delta x_i)^2}}\right]. \tag{3}$$

The left inequality in (3) provides some small corrections to the Heisenberg's uncertainty principle for $\Delta x_i \gg \alpha l_P$ (i.e., semi-classical regime),

$$\Delta p_i \geqslant \frac{\hbar}{\Delta x_i} + \frac{\hbar\alpha^2 l_P^2}{(\Delta x_i)^3} + \mathcal{O}\left(\frac{\hbar\alpha^4 l_P^4}{(\Delta x_i)^5}\right). \tag{4}$$

On the other hand, the right inequality implies that Δp_i cannot be arbitrarily large in order that the correction in (1) makes sense. Of course, this upper bound can be higher with the higher order terms in the right-hand side of the GUP (1), but the absolute minimum in Δx_i can be also lowered or even disappeared, depending on the parameters [6]. Another more interesting interpretation would be that the upper bound corresponds to the limit where the quantum gravity effects are very strong such as a *black hole-string* phase transition can occur [7]. Actually, the inequality can be written also as

$$\Delta p_i \leqslant \frac{\hbar\Delta x_i}{\alpha^2 l_P^2}, \tag{5}$$

which can be directly derived also from the high momentum uncertainty Δp_j limit in (1), and it is saturated by the linear relation $\Delta p_i = \hbar\Delta x_i/\alpha^2 l_P^2$, which coincides with that of strings at the high energy limit, by identifying the string scale $l_S \approx \alpha l_P$ [2,5].

Now, let me derive the Hawking temperature from the uncertainty principle and general properties of black holes. To this end, let me first consider a d-dimensional Schwarzschild black hole with a metric given by

$$ds^2 = -N^2\,dt^2 + N^{-2}\,dr^3 + r^2\,d\Omega_{d-2}^2, \tag{6}$$

where

$$N^2 = 1 - \frac{16\pi GM}{(d-2)\Omega_{d-2} r^{d-3}} \tag{7}$$

and Ω_{d-2} is the area of the unit sphere S^{n-2} [8]. By modeling a black hole as a black box with linear size r_+, the uncertainty in the position of an emitted particle by the Hawking effect is

$$\Delta x_i \approx r_+ \tag{8}$$

with the radius of the event horizon r_+. In the absence of the GUP effect, the horizon radius is given by $r_+ = [16\pi GM/(d-2)\Omega_{d-2}]^{1/(d-3)}$ from the metric (6). On the other hand, in the presence of the GUP effect, the *precise* form of the horizon radius $r_+ = r_+(M, \alpha l_P)$ is not known unless the GUP corrected metric is known, which is beyond the scope of this Letter. However, I note that the relation (8) would be generally valid even with the GUP effect, with understanding r_+ as the GUP corrected horizon already. Then, the uncertainty in the energy of the emitted particle is (by neglecting the mass of the emitted particle)[1]

$$\Delta E \approx \Delta p_i. \tag{9}$$

By assuming that ΔE, which can be identified as the characteristic temperature of the Hawking radiation, saturates the left inequality,[2] one can obtain the Hawking temperature

$$T_{\rm GUP} = \left(\frac{d-3}{4\pi}\right)\frac{\hbar r_+}{2\alpha^2 l_P^2}\left[1-\sqrt{1-\frac{4\alpha^2 l_P^2}{r_+^2}}\right]. \tag{10}$$

Here, the "calibration" factor '$(d-3)/4\pi$' has been introduced in order to have agreements with the usual Hawking temperature of the Schwarzschild black hole in the leading term, for a large black hole, i.e., $r_+ \gg \alpha l_P$ [8,9]:

$$T_{\rm GUP} = \left(\frac{d-3}{4\pi}\right)\left[\frac{\hbar}{r_+} + \frac{\hbar\alpha^2 l_P^2}{r_+^3} + \mathcal{O}\left(\frac{\hbar\alpha^4 l_P^4}{r_+^5}\right)\right]. \tag{11}$$

Before finishing this section, I remark first that the formula (8), as a result (10), is still valid even for the small black holes up to the absolute minimum, which is order of Planck length l_P, though the series formula (11) is valid only for a large r_+. The black hole evaporation stops at $r_+ = 2\alpha l_P$, where the curve ends, and this would correspond to a "melting" of the black hole which is followed by the string phase, according to the new interpretation [7]. Second, the effect of the GUP with an absolute minimum length increases the Hawking temperature always and this implies that it decays faster than the usual Schwarzschild black hole without the GUP (Fig. 1).

3. The EUP and Hawking temperature in (A)dS space

The GUP cannot be naively applied to the space with the large length scales like as in (A)dS space.[3] In this section,

1 There might exist high energy modifications in the dispersion relation (9) generally [10]. But, I will not consider this possibility in this Letter.

2 This assumption would correspond to the Bekenstein bound of the entropy of an arbitrary bounded system $S(=\int T^{-1}\,dM) \leqslant S_{\rm BH}(=\int T_{\rm BH}^{-1}\,dM_{\rm BH})$ whose upper bound is saturated by that of black holes, $S_{\rm BH}$, for a given mass $M = M_{\rm BH}$ [11].

3 This has been noted earlier by Konishi et al. also [2]. See also Ref. [12] for another related work.

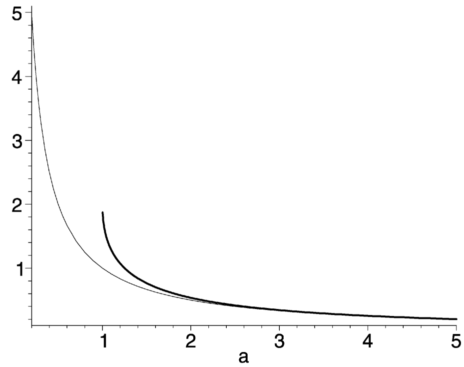

Fig. 1. Hawking temperature (divided by '$(d-3)/4\pi$') vs. the horizon radius r_+ (denoted by 'a' in the plot) in the asymptotically flat space. In the absence of the GUP, there is no absolute minimum radius for the black hole evaporation (thin line). With the GUP, the Hawking temperature becomes hotter, implying faster decay, and also there is a minimum radius $r_+ = 2\alpha l_P$ where the curve ends, implying that the black hole evaporation stops (thick line). Here, I have plotted the cases with $\hbar = l_P = 1$, $\alpha = 0.5$ and the GUP curve stops at $r_+ = 1$.

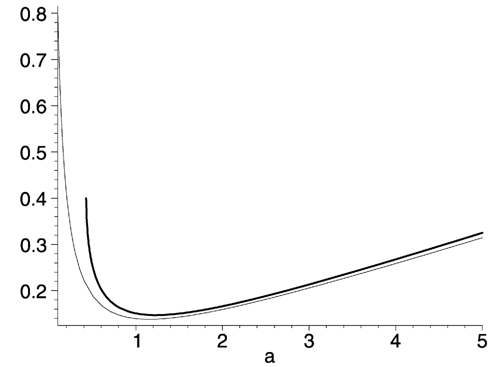

Fig. 2. Hawking temperature vs. the horizon radius r_+ in the AdS space. The EUP, but without the GUP, produces correctly the usual Hawking temperature of the Schwarzschild–AdS black holes (thin line). The existence of the absolute minimum in the temperature is a general consequence of the EUP. But, as in the case of the flat space, there is no absolute minimum radius in the absence of the GUP. With the GUP, the Hawking temperature becomes hotter also, implying faster decay, and there is a minimum radius $r_+ = 2\alpha l_P/[1 - 4\alpha^2 l_P^2 (d-1)/(d-3) l_{\mathrm{AdS}}^2]^{1/2}$ where the curve ends, implying that the black hole evaporation stops (thick line). Here, I have plotted the cases with $\hbar = l_P = 1$, $\alpha = 0.2$, $l_{\mathrm{AdS}} = 2$, $d = 4$.

I consider an extension of the uncertainty principle in order to incorporate the large-length scales and derivation of Hawking temperature from the uncertainty principle.

The extended uncertainty principle (EUP) is given by[4]

$$\Delta x_i \Delta p_j \geqslant \hbar \delta_{ij} \left[1 + \beta^2 \frac{(\Delta x_i)^2}{l^2} \right], \tag{12}$$

where l is the characteristic, large length scale and β is a dimensionless real constant of order one [5].[5] (For some *gedanken* experiments' derivation, even without considering black holes, see also Ref. [16].) Then, it is easy to see that there is an absolute minimum in the momentum uncertainty

$$\Delta p_i \geqslant \frac{\hbar}{\Delta x_i} + \frac{\hbar \beta^2 \Delta x_i}{l^2} \geqslant \frac{2\hbar\beta}{l}. \tag{13}$$

Here, I note that the first inequality is an "exact" relation drawn from (12), without considering any limit as in (4).

Now, using the approach in Section 2, it is straightforward to see that the Hawking temperature of the Schwarzshild–AdS black holes from the EUP (13).[6] To this end, let me first consider a d-dimensional Schwarzshild–AdS black hole with the metric function

$$N^2 = 1 + \frac{r^2}{l_{\mathrm{AdS}}^2} - \frac{16\pi G M}{(d-2)\Omega_{d-2} r^{d-3}} \tag{14}$$

in the metric (6) and a cosmological constant $\Lambda = -(d-1) \times (d-2)/2 l_{\mathrm{AdS}}^2$ [13]. Then, with the same identifications (8) and (9) for the Hawking-emitted particles, which do not depends on the large scale behaviors but only on the local structure near the horizon, one can obtain the Hawking temperature

$$T_{\mathrm{EUP}} \approx \Delta p_i,$$

$$T_{\mathrm{EUP(AdS)}} = \left(\frac{d-3}{4\pi} \right) \hbar \left[\frac{1}{r_+} + \left(\frac{d-1}{d-3} \right) \frac{r_+}{l_{\mathrm{AdS}}^2} \right], \tag{15}$$

with the same calibration factor '$(d-3)/4\pi$' as in the asymptotically flat case, implying its universality, and $\beta = \sqrt{(d-1)/(d-3)}$, $l = l_{\mathrm{AdS}}$; r_+ is the radius of the event horizon which solves $N^2(r) = 0$. Here, the existence of the absolute minimum in Δp_i and so in $T_{\mathrm{EUP(AdS)}}$ is a general consequence of the EUP of (12) (see Fig. 2 (thin line)).

So far, I have shown that the EUP in (12) applies to the AdS space. Now, the EUP for the dS space can be easily constructed by considering $l^2 \to -l^2$ in (12):

$$\Delta x_i \Delta p_j \geqslant \hbar \delta_{ij} \left[1 - \beta^2 \frac{(\Delta x_i)^2}{l^2} \right]. \tag{16}$$

Then, in contrast to (12), there is an absolute maximum in Δx_i as

$$\Delta x_i \leqslant \frac{l}{\beta} \tag{17}$$

in order that Δp_i is not negative,[7]

$$\Delta p_i \geqslant \frac{\hbar}{\Delta x_i} - \frac{\hbar \beta^2 \Delta x_i}{l^2} \geqslant 0. \tag{18}$$

Note that the absolute maximum in Δx_i does not have $\hbar$ such as this is a purely classical result.

The Hawking temperature of the Schwarzshild–dS black hole with a cosmological constant $\Lambda = +(d-1)(d-2)/2 l_{\mathrm{dS}}^2$ [14] is similarly computed as, by considering $l_{\mathrm{AdS}}^2 \to -l_{\mathrm{dS}}^2$

[4] The parameter β in Ref. [5] is related to here's by $\beta_{\mathrm{There}} = (l_P/l)\beta_{\mathrm{Here}}$.

[5] This has been considered earlier by Kempf et al. also [15], but its physical consequences have not been studied.

[6] For an alternative derivation from the laws of classical physics and Heisenberg's uncertainty principle, see Ref. [17]. But, there is no room for the GUP in that derivation.

[7] This is compared with Refs. [16,18], where the EUP (12) is considered for the particle or cosmological horizon, by giving the absolute minimum in Δp_i.

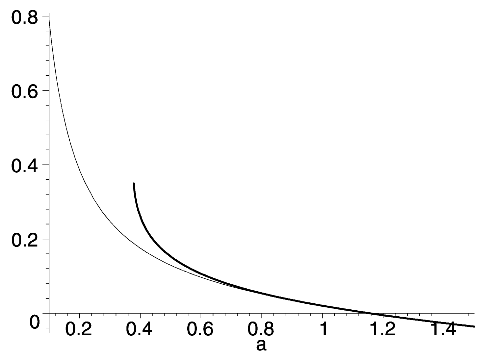

Fig. 3. Hawking temperature vs. the horizon radius r_+ in the dS space. The EUP of (16) produces correctly the usual Hawking temperature of the Schwarzshild–dS black holes (thin line), which vanishes at the Nariai bound $r_+ \leqslant \sqrt{(d-1)/(d-3)}l_{\mathrm{dS}}$; this defines the absolute maximum of the black hole horizon, but there is no absolute minimum radius. With the GUP, the Hawking temperature becomes hotter also, implying faster decay, and there is a minimum radius $r_+ = 2\alpha l_P/[1+4\alpha^2 l_P^2(d-1)/(d-3)l_{\mathrm{dS}}^2]^{1/2}$ where the curve ends, implying that the black hole evaporation stops (thick line). Here, I have plotted the cases with $\hbar = l_P = 1$, $\alpha = 0.2$, $l_{\mathrm{dS}} = 2$, $d = 4$.

in (15),

$$T_{\mathrm{EUP(dS)}} = \left(\frac{d-3}{4\pi}\right)\hbar\left[\frac{1}{r_+} - \left(\frac{d-1}{d-3}\right)\frac{r_+}{l_{\mathrm{dS}}^2}\right]. \tag{19}$$

Here, the maximum bound reads ($l = l_{\mathrm{dS}}$, $\beta = \sqrt{(d-1)/(d-3)}$)

$$r_+ \leqslant \sqrt{(d-1)/(d-3)}l_{\mathrm{dS}}, \tag{20}$$

which is the Nariai bound where the black hole horizon and the cosmological horizon meet [19]. So, the condition (17) reflects the fact that the uncertainty in the position cannot exceed the cosmological horizon, which is the size of the casually connected world in a dS space (see Fig. 3 (thin line)).

4. The generalized EUP (GEUP)

In the EUP (12), there is an absolute minimum in the uncertainty of the momentum. In this section, I generalize the EUP to have a minimum length scale as well, by combining the GUP and the EUP, and study the effect of the minimum length to the Hawking temperature from the EUP, i.e., the Hawking temperature of Schwarzshild–(A)dS black holes.

The generalized EUP (GEUP) is given by

$$\Delta x_i \Delta p_j \geqslant \hbar\delta_{ij}\left[1 + \alpha^2 l_P^2\frac{(\Delta p_j)^2}{\hbar^2} + \beta^2\frac{(\Delta x_i)^2}{l^2}\right], \tag{21}$$

where I have considered the case of the AdS space, first. Then, by inverting (21), one has the inequalities,

$$\Delta p_i^{(-)} \leqslant \Delta p_i \leqslant \Delta p_i^{(+)},$$
$$\Delta p_i^{(\pm)} = \frac{\hbar\Delta x_i}{2\alpha^2 l_P^2}\left[1 \pm \sqrt{1 - \frac{4\alpha^2 l_P^2}{(\Delta x_i)^2}\left[1 + \beta^2\frac{(\Delta x_i)^2}{l^2}\right]}\right] \tag{22}$$

and

$$\Delta x_i^{(-)} \leqslant \Delta x_i \leqslant \Delta x_i^{(+)},$$
$$\Delta x_i^{(\pm)} = \frac{l^2\Delta p_i}{2\hbar\beta^2}\left[1 \pm \sqrt{1 - \frac{4\beta^2\hbar^2}{l^2(\Delta p_i)^2}\left[1 + \frac{\alpha^2 l_P^2(\Delta p_i)^2}{\hbar^2}\right]}\right]. \tag{23}$$

Here, one finds that there are, now, both the absolute minimum in Δx_i and Δp_i

$$(\Delta x_i)^2 \geqslant \frac{4\alpha^2 l_P^2}{1 - 4\alpha^2 l_P^2\beta^2/l^2}, \tag{24}$$

$$(\Delta p_i)^2 \geqslant \frac{4\hbar^2\beta^2/l^2}{1 - 4\alpha^2 l_P^2\beta^2/l^2}, \tag{25}$$

from the *reality* of $\Delta p_i^{(\pm)}$ and $\Delta x_i^{(\pm)}$, respectively, with the condition

$$\beta^2 < \frac{l^2}{4\alpha^2 l_P^2}. \tag{26}$$

The left inequality in (22), as in (3) of the GUP, provides some small corrections to the Heisenberg's uncertainty principle, due to the minimum length and momentum, for $\alpha l_P \ll \Delta x_i \ll l/\beta$,

$$\Delta p_i \geqslant \left(1 + \frac{2\alpha^2 l_P^2\beta^2}{l^2}\right)\frac{\hbar}{\Delta x_i} + \frac{\hbar\beta^2\Delta x_i}{l^2} + \frac{\hbar\alpha^2 l_P^2}{(\Delta x_i)^3} + \mathcal{O}\left(\frac{\hbar\alpha^4 l_P^4}{(\Delta x_i)^5}, \frac{\hbar\alpha^2 l_P^2\beta^4\Delta x_i}{l^4}\right). \tag{27}$$

By repeating the same arguments as in the GUP and the EUP cases (with understanding that r_+ as the GUP corrected horizon), one can obtain the Hawking temperature $T_{\mathrm{GEUP}} \approx \Delta p_i^{(-)}$

$$T_{\mathrm{GEUP(AdS)}} = \left(\frac{d-3}{4\pi}\right)\frac{\hbar r_+}{2\alpha^2 l_P^2} \times\left[1 - \sqrt{1 - \frac{4\alpha^2 l_P^2}{r_+^2}\left[1 + \left(\frac{d-1}{d-3}\right)\frac{r_+^2}{l_{\mathrm{AdS}}^2}\right]}\right], \tag{28}$$

with the usual calibration factor '$(d-3)/4\pi$' and $\beta = \sqrt{(d-1)/(d-3)}$, $l = l_{\mathrm{AdS}}$ such as this agrees with the EUP result (15) for a semiclassical black hole with $\alpha l_P \ll r_+ \ll \sqrt{(d-3)/(d-1)}l_{\mathrm{AdS}}$,

$$T_{\mathrm{GEUP(AdS)}} \approx \left(\frac{d-3}{4\pi}\right)\hbar\left[\left\{1 + \left(\frac{d-1}{d-3}\right)\frac{2\alpha^2 l_P^2}{l_{\mathrm{AdS}}^2}\right\} \times\frac{1}{r_+} + \left(\frac{d-1}{d-3}\right)\frac{r_+}{l_{\mathrm{AdS}}^2} + \frac{\alpha^2 l_P^2}{r_+^3}\right]. \tag{29}$$

Here, the third term is purely the GUP correction and the second term in the first bracket { } is the GEUP effect, and these correction terms are all positive. This shows that the Hawking temperature of the AdS black hole is increased also by the minimum uncertainty in the position, with the GUP.

The analysis for the dS case is also straightforward. From the GEUP with $l^2 \to -l^2$, one has

$$\Delta p_i^{(-)} \leqslant \Delta p_i \leqslant \Delta p_i^{(+)},$$
$$\Delta p_i^{(\pm)} = \frac{\hbar \Delta x_i}{2\alpha^2 l_P^2}\left[1 \pm \sqrt{1 - \frac{4\alpha^2 l_P^2}{(\Delta x_i)^2}\left[1 - \beta^2 \frac{(\Delta x_i)^2}{l^2}\right]}\right], \quad (30)$$

with the minimum uncertainty in Δx_i (but none in Δp_i)

$$(\Delta x_i)^2 \geqslant \frac{4\alpha^2 l_P^2}{1 + 4\alpha^2 l_P^2 \beta^2 / l^2}. \quad (31)$$

Moreover, in order that $\Delta p_i^{(-)}$ is not negative one obtains the same condition as (17) which is unchanged by the GUP effect (i.e., no α dependence), in contrast to the lower bound in (31). Then, one finds the Hawking temperature

$$T_{\text{GEUP(dS)}} = \left(\frac{d-3}{4\pi}\right)\frac{\hbar r_+}{2\alpha^2 l_P^2} \times \left[1 - \sqrt{1 - \frac{4\alpha^2 l_P^2}{r_+^2}\left[1 - \left(\frac{d-1}{d-3}\right)\frac{r_+^2}{l_{\text{dS}}^2}\right]}\right], \quad (32)$$

which gives

$$T_{\text{GEUP(dS)}} \approx \left(\frac{d-3}{4\pi}\right)\hbar\left[\left\{1 - \left(\frac{d-1}{d-3}\right)\frac{2\alpha^2 l_P^2}{l_{\text{dS}}^2}\right\} \times \frac{1}{r_+} - \left(\frac{d-1}{d-3}\right)\frac{r_+}{l_{\text{dS}}^2} + \frac{\alpha^2 l_P^2}{r_+^3}\right], \quad (33)$$

for semiclassical dS black holes with $\alpha l_P \ll r_+ \ll \sqrt{(d-3)/(d-1)}\, l_{\text{dS}}$. Here, note that the maximum bound of the black hole horizon (20) is not changed by the existence of the minimal length but the temperature is always increasing: The second term in the first bracket { } gives a negative correction but this is dominated by the third term, which is always positive.

Now, one finds a quite general trend that the GUP due to a minimal length increases *always* the Hawking temperature (Figs. 2, 3), regardless of being asymptotically flat or (A)dS space. This can be traced back to the *universal* appearance of the term "$+\alpha^2 l_P / r_+^3$" in the temperature formula, which makes the decay to be faster. This seems to be also true in other forms of the deformation of the uncertainty principle [20].

Finally, two remarks are in order. First, one might consider the first law of thermodynamics to compute the GUP corrected black hole entropy from the *same* ADM mass formula as that of the case without the GUP [3]. But, it still unclear how to fix *uniquely* the GUP corrected mass formula from the GUP corrected Hawking temperature, without knowing the precise form the GUP corrected gravity and its black hole solutions.

Second, I note that, in the $d = 3$ limit of the AdS black holes (i.e., the BTZ black hole limit), one has the Hawking temperature

$$T_{\text{GEUP(AdS)}}^{(d=3)} \approx \frac{\hbar}{4\pi}\left[\frac{4\alpha^2 l_P^2}{l_{\text{AdS}}^2}\frac{1}{r_+} + \frac{2r_+}{l_{\text{AdS}}^2}\right], \quad (34)$$

from the series formula (29), though it needs a scale tuning $l_{\text{AdS}} \to \infty$, $d \to 3$, with '$\sqrt{d-3}\, l_{\text{AdS}}$ = a fixed large number'. This shows also an increase of the temperature, implying faster decay from the GUP effect, compared to that of the usual BTZ black hole, $T_{\text{BTZ}} = \hbar r_+ / (2\pi l_{\text{AdS}}^2)$. But, remarkably, there is a minimum temperature at $r_+ = \sqrt{2}\alpha l_P$ and growing temperature for smaller black holes, in contrast to the monotonically decreasing temperature as r_+ becomes smaller in the BTZ black hole without the GUP. If this were true, the Hawking–Page transition [13] would occur even in three-dimensional AdS space, due to the GUP effect. But, this does not seem to occur from (24), which implies $r_+ \geqslant 2\alpha l_P$ for consistency of the exact formula (22), such as the evaporation stops before reaching the absolute minimum of the temperature at $r_+ = \sqrt{2}\alpha l_P$. This needs more rigorous analysis which can be well-defined in the three dimension, from the start [20].

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