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The spectrum of Platonic graphs over finite fields

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Abstract

We give a decomposition theorem for Platonic graphs over finite fields and use this to determine the spectrum of these graphs. We also derive estimates for the isoperimetric numbers of the graphs.

Keywords: Cayley graph; Ramanujan graph; Isoperimetric number

1. Introduction

Let p be an odd prime, let $r \geqslant 1$, and let $\mathbb{F}_q$ be the finite field with $q = p^r$ elements. We consider a generalization of the Platonic graphs studied in [2,7–9]. In particular, using the notation from [7], we define our graphs $G^*(q)$ as follows.

Definition 1. Let the vertex set of $G^*(q)$ be

$$V(G^*(q)) = \{(\alpha \ \ \beta) | \alpha, \beta \in \mathbb{F}_q, \ (\alpha \ \ \beta) \neq (0 \ \ 0)\} / \langle \pm 1 \rangle,$$

and let $(\alpha \ \ \beta)$ be adjacent to $(\gamma \ \ \delta)$ if and only if $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm 1$.

For $q = p$, the graph $G^*(q)$ is called the qth Platonic graph. The name comes from the fact that when $p = 3$ or 5, the graphs $G^*(p)$ correspond to 1-skeletons of two of the Platonic solids, in other words, the tetrahedron and the icosahedron as in [2]. More generally, $G^*(p)$ is the 1-skeleton of a triangulation of the modular curve $X(p)$, see [7].

The spectrum Λ_G of a graph G is the set of eigenvalues of the adjacency operator, see [4]. For a finite, k-regular graph G it is known that $k \in \Lambda_G$. Further, $\lambda \in \Lambda_G$ satisfies $|\lambda| \leqslant k$. Let $\{G_m\}_{m \geqslant 1}$ be a family of finite, connected, k-regular graphs with $\lim_{m\to\infty} |V(G_m)| = \infty$. Let $\lambda_1(G_m)$ be the largest element in Λ_{G_m} distinct from k. Then it is known that $\liminf_{m\to\infty} \lambda_1(G_m) \geqslant 2\sqrt{k-1}$, see [1,5].

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A k-regular graph G is called Ramanujan if for all $\lambda \in \Lambda_G$ with $|\lambda| \neq k$ we have $|\lambda| \leqslant 2\sqrt{k-1}$. From the inequality for λ_1 above, we see that this estimate is optimal for families of regular graphs of fixed degree. It follows that Ramanujan graphs make the best expanders in the sense of [11].

In this paper, we determine the spectrum of $G^*(q)$ and thus generalize results of [7].

Theorem 1. *Let $p > 2$ be prime and $q = p^r$. The spectrum of $G^*(q)$ is q with multiplicity 1, -1 with multiplicity q, and $\pm\sqrt{q}$ each with multiplicity $(q+1)(q-3)/4$.*

It follows that the graphs $G^*(q)$ are Ramanujan.

To do this, we demonstrate a decomposition theorem for the graphs $G^*(q)$ which we also use to estimate their isoperimetric numbers. Let K_n denote the complete graph on n vertices and let K_n^m denote the complete multigraph on n vertices with m edges connecting each pair of distinct vertices. For $(0\ \ \alpha)$ in $V(G^*(q))$, let H_α be the graph defined in Section 3. Thus, H_α has $q+1$ vertices consisting of p^{r-1} disjoint p-circuits, with every vertex in a given circuit adjacent to the vertex $(0\ \ \alpha)$.

Theorem 2. *The graph $G^*(q)$ can be partitioned into $(q-1)/2$ disjoint copies of H_α with $2q$ edges joining every pair of H_α's. Alternatively, $G^*(q)$ is the complete multigraph $K_{(q-1)/2}^{2q}$ where each vertex should be viewed as a copy of H_α.*

For an arbitrary graph G and $S \subset V(G)$, the boundary of S, denoted ∂S, is the set of all edges having exactly one endpoint in S. The isoperimetric number [12] is defined to be

$$\operatorname{iso}(G) = \inf_S \frac{|\partial S|}{|S|},$$

where the infimum is over all $S \subset V(G)$ so that $|S| \leqslant |V(G)|/2$. The isoperimetric number, also called the Cheeger constant, is a measure of the connectedness of G. A set S where $\operatorname{iso}(G) = |\partial S|/|S|$ is called an isoperimetric set for G. The isoperimetric number has applications to combinatorics and computer science. For example, if a graph G is viewed as a communications network then a large $\operatorname{iso}(G)$ means that information is transmitted easily throughout the network. A small isoperimetric number means that communication can be easily disrupted.

As a consequence of Theorems 1 and 2, we obtain the following bounds for the isoperimetric numbers $\operatorname{iso}(G^*(q))$ of $G^*(q)$.

Corollary 1. *We have*

$$\frac{q}{2} - \frac{\sqrt{q}}{2} \leqslant \operatorname{iso}(G^*(q)) \leqslant \begin{cases} \dfrac{q(q-1)}{2(q+1)} & \text{if } q \equiv 1 \pmod 4, \\ \dfrac{q}{2} & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

In the next section, we define a Cayley graph $G(q)$ of $PSL_2(\mathbb{F}_q)$ and consider a graph $G'(q)$ obtained from the quotient of $PSL_2(\mathbb{F}_q)$ by the unipotent subgroup. We relate $G^*(q)$ to $G'(q)$ in order to prove Theorem 2. We then demonstrate bounds for the isoperimetric number of $G^*(q)$ using the methods of [8,9]. In the last section, we use the decomposition theorem to determine the spectrum of $G^*(q)$.

In [10], Li and Meemark determined the spectrum of Cayley graphs on $PGL_2(\mathbb{F}_q)$ modulo either the unipotent subgroup, the split torus, or the nonsplit torus of G. They use the Kirillov models of the representations of $PGL_2(\mathbb{F}_q)$. This is a different approach from the one used here or in [6], as [10] relies primarily on representation theory rather than an analysis of the graph structure to determine the spectrum. Note that in this paper, character sums are not needed to evaluate the eigenvalues.

2. A Cayley graph of the projective linear group

The Cayley graph of a group Γ with respect to a symmetric generating set Ω is the graph with vertex set Γ and with two vertices γ_1, γ_2 joined by an edge if we have $\gamma_1 = \alpha\gamma_2$ for some $\alpha \in \Omega$. Cayley graphs are vertex transitive and

$|\Omega|$-regular. In this section, we define a Cayley graph of $\Gamma_q = PSL_2(\mathbb{F}_q)$ for $q = p^r$ and $p > 2$ a prime. We then study a quotient graph of the Cayley graph.

For $q \equiv 1 \pmod 4$, there exists some $\omega \in \mathbb{F}_q$ so that $\omega^2 = -1$. From [3], for appropriately chosen $x \in \mathbb{F}_q$, the set

$$\Omega_q = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pm x \\ 0 & 1 \end{pmatrix} \right\}$$

is a generating set for Γ_q. For $q \equiv 3 \pmod 4$ and $q > 11$, from [3], $PSL_2(\mathbb{F}_q)$ is generated by

$$\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ -\omega^{-1} & 0 \end{pmatrix} \right\},$$

where $a^2 + b^2 = -1$, $\omega \in \mathbb{F}_q^{\times}$, and $\omega \neq \pm 1$. Since $q \equiv 3 \pmod 4$, both a and b are nonzero. Since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -a^{-1} & b \\ 0 & -a \end{pmatrix},$$

we have that $PSL_2(\mathbb{F}_q)$ is generated by

$$\Omega_q = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ -\omega^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}, \begin{pmatrix} 1 & \pm x \\ 0 & 1 \end{pmatrix} \right\}$$

for appropriately chosen $x \in \mathbb{F}_q$. We take Ω_q to be the symmetric generating set for $PSL_2(\mathbb{F}_q)$ for $q \equiv 1, 3 \pmod 4$, where $\omega^{-1} = -\omega$ in the former case. Thus, $|\Omega_q| = 4$ for $q \equiv 1 \pmod 4$ and $|\Omega_q| = 5$ for $q \equiv 3 \pmod 4$. Note that for $q \equiv 3 \pmod 4$ and $q \leqslant 11$, q is a prime. In that case, Theorems 1 and 2 follow from [7,8].

Let $G(q)$ be the Cayley graph of Γ_q with respect to Ω_q. Note that $|V(G(q))| = q(q^2 - 1)/2$. Let

$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{F}_q \right\}.$$

Thus, $|N| = q$. Let $\Gamma'_q = N \backslash \Gamma_q$ and note that

$$\Gamma'_q \cong \left\{ (\alpha \quad \beta) \mid \alpha, \beta \in \mathbb{F}_q, (\alpha \quad \beta) \neq (0 \quad 0) \right\} / \langle \pm 1 \rangle,$$

which is the vertex set of $G^*(q)$.

Let $G'(q)$ denote the quotient graph $N \backslash G(q)$. That is, $G'(q)$ is the multigraph whose vertices are given by the cosets Γ'_q with distinct cosets $N\gamma_1$ and $N\gamma_2$ joined by as many edges as there are edges in $G(q)$ of the form (α_1, α_2) where $\alpha_j \in N\gamma_j$. Note that Γ'_q is not a group, and thus the graphs $G'(q)$ are not themselves Cayley graphs. Also, note that the vertex set of $G'(q)$ is Γ'_q.

Lemma 1. *Let* $(\alpha \ \ \beta), (\gamma \ \ \delta) \in G'(q)$. *Then* $(\alpha \ \ \beta)$ *is adjacent to* $(\gamma \ \ \delta)$ *if and only if*

$$\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm 1, \pm \omega \quad \text{or} \quad \pm \omega^{-1}.$$

Proof. Left multiplication of $g \in \Gamma_q$ by elements of N preserves the bottom row of g. So, $g' \in G'(q)$ is adjacent to precisely those elements attained by left multiplication by $\gamma \in \Omega_q$ with $\gamma \notin N$. Left multiplication of g by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ multiplies the top row of g by -1 and switches the rows. Multiplication by $\left(\begin{smallmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{smallmatrix}\right)$ multiplies the top row of g by $-\omega$ and switches the rows, and similarly multiplication by $\left(\begin{smallmatrix} 0 & \omega \\ -\omega^{-1} & 0 \end{smallmatrix}\right)$ multiplies the top row of g by $-\omega^{-1}$ and switches the rows. So, if $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_q$ with determinant equal to 1 and $\xi g = \left(\begin{smallmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{smallmatrix}\right)$ for $\xi \in \Omega_q - N$, then $(\gamma' \ \ \delta') = -(\alpha \ \ \beta), -\omega(\alpha \ \ \beta)$, or $-\omega^{-1}(\alpha \ \ \beta)$. Therefore, we must have

$$\det \begin{pmatrix} \gamma & \delta \\ \gamma' & \delta' \end{pmatrix} = \pm 1, \pm \omega \quad \text{or} \quad \pm \omega^{-1}.$$

If $\det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \pm 1, \pm\omega$ or $\pm\omega^{-1}$ then $\begin{pmatrix}\varepsilon\alpha & \varepsilon\beta\\ \gamma & \delta\end{pmatrix} \in \Gamma_q$ for some $\varepsilon \in \{\pm 1, \pm\omega, \pm\omega^{-1}\}$. Thus, multiplication by $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $\begin{pmatrix}0 & \omega\\ -\omega^{-1} & 0\end{pmatrix}$ or $\begin{pmatrix}0 & \omega^{-1}\\ -\omega & 0\end{pmatrix}$ takes $(\gamma\ \ \delta)$ to $(\alpha\ \ \beta)$ inside Γ'_q. It follows that these vertices are adjacent in $G'(q)$. □

Note that $|V(G'(q))| = (q^2-1)/2$. Two edges incident with $g \in G(q)$ arise by multiplication by the two elements of Ω_q that are in N. The remaining $|\Omega_q| - 2$ edges come from cosets Nx, Ny with $x, y \notin N$. Further, any vertex in the graph $G'(q)$ corresponds to q vertices in $G(q)$. It follows that the regularity of $G'(q)$ is $|\Omega_q| - 2$ and

$$|E(G'(q))| = \frac{(|\Omega_q|-2)q(q^2-1)}{4}.$$

3. The decomposition theorem

Let $(\alpha\ \ \beta)$ and $(\gamma\ \ \delta)$ be adjacent vertices in $G'(q)$ and thus $\det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \pm 1, \pm\omega$ or $\pm\omega^{-1}$ from Lemma 1. If the determinant is ± 1 then call the edge incident with these vertices a 1-edge, and otherwise call the edge an ω-edge. Note that $G^*(q)$ is obtained from $G'(q)$ simply by removing all of the ω-edges. Further, it is easy to see that the number of ω-edges incident with a vertex in $G'(q)$ is $|\Omega_q| - 3$ times the number of 1-edges incident with that vertex. It follows that $G^*(q)$ is q-regular.

Let $\alpha \in \mathbb{F}_q^\times$ and define

$$V_\alpha = \left\{ (0\quad \alpha), (\alpha^{-1}\quad \beta) \mid \beta \in \mathbb{F}_q \right\}.$$

Note that $|V_\alpha| = q+1$. If $\alpha' \in \mathbb{F}_q^\times$, $\alpha' \neq \pm\alpha$ then it is easy to see that $V_\alpha \cap V_{\alpha'} = \emptyset$. Otherwise, $V_\alpha = V_{\alpha'}$. It is also easy to see that any vertex in $G^*(q)$ lies in V_α for some $\alpha \in \mathbb{F}_q^\times$ and therefore $V(G^*(q))$ can be partitioned into disjoint V_α's. In particular, $V(G^*(q))$ can be partitioned into $|V(G^*(q))|/|V_\alpha| = (q-1)/2$ many copies of V_α. Define H_α to be the subgraph of $G^*(q)$ induced by V_α. Any $(\alpha^{-1}\ \ \beta) \in H_\alpha$ belongs to a p-circuit in H_α given by the following sequence of adjacent vertices:

$$\{(\alpha^{-1}\quad \beta), (\alpha^{-1}\quad \beta+\alpha), (\alpha^{-1}\quad \beta+2\alpha), \ldots, (\alpha^{-1}\quad \beta+(p-1)\alpha), (\alpha^{-1}\quad \beta)\}.$$

Further, any two such circuits share a common vertex $(\alpha^{-1}\ \ \beta+j\alpha) = (\alpha^{-1}\ \ \beta'+k\alpha)$ if and only if $\beta - \beta' = (k-j)\alpha$ and this occurs if and only if such circuits are identical. Thus, there are p^{r-1} disjoint p-circuits in a given H_α and each vertex in such a circuit is adjacent to $(0\ \ \alpha)$. We refer to the vertex $(0\ \ \alpha)$ as the center of H_α. As $(\alpha^{-1}\ \ \beta) \in H_\alpha$ is adjacent to $(\alpha^{-1}\ \ \beta') \in H_\alpha$ if and only if $\beta' = \pm\alpha + \beta$, then $(\alpha^{-1}\ \ \beta)$ is adjacent to only two other vertices in H_α, other than the center. This accounts for all the possible edges in H_α and so $|E(H_\alpha)| = 2q$ for any $\alpha \in \mathbb{F}_q^\times$.

Suppose $H_\alpha \neq H_{\alpha'}$ and $(\alpha^{-1}\ \ \beta) \in H_\alpha$ is adjacent to $(\alpha'^{-1}\ \ \beta) \in H_{\alpha'}$. Then

$$\det\begin{pmatrix}\alpha^{-1} & \beta\\ \alpha'^{-1} & x\end{pmatrix} = \pm 1,$$

and the number of solutions for x to this equation is independent of the choice of $\alpha' \in \mathbb{F}_q^\times$. The number of 1-edges connecting H_α to $H_{\alpha'}$ is therefore independent of α and α', where $H_{\alpha'} \neq H_\alpha$. Since H_α contains $2q$ edges and there are $(q-1)/2$ copies of H_α in $G^*(q)$, we have $\bigsqcup_{\alpha\in\mathbb{F}_q^\times} H_\alpha$ contains $q(q-1)$ edges. As $|V(G^*(q))| = (q^2-1)/2$ and $G^*(q)$ is q-regular, $|E(G^*(q))| = q(q^2-1)/4$. Thus, there are $q(q^2-1)/4 - q(q-1) = q(q-1)(q-3)/4$ edges joining different H_α's. As there are $\binom{(q-1)/2}{2}$ different ordered pairs (α, α') that give distinct H_α's and $H_{\alpha'}$'s, the number of edges joining a given H_α to a distinct $H_{\alpha'}$ is

$$\frac{q(q-1)(q-3)/4}{\binom{(q-1)/2}{2}} = 2q.$$

This gives the decomposition as stated in Theorem 2.

4. The isoperimetric number

In this section we use the decomposition theorem to estimate the isoperimetric number of $G^*(q)$ as stated in Corollary 1. In addition, we estimate the isoperimetric numbers of the related graphs $G'(q)$ and $G(q)$.

According to [12], the isoperimetric number for the complete graph K_n is

$$\operatorname{iso}(K_n) = \begin{cases} n/2, & n \text{ even}, \\ (n+1)/2, & n \text{ odd}. \end{cases}$$

As a consequence of the decomposition theorem, for our set S we can take $(q-1)/4$ copies of H_α if $q \equiv 1 \pmod 4$ and $(q+1)/4$ copies of H_α if $q \equiv 3 \pmod 4$. In the former case, we have

$$|S| = (q+1)\left(\frac{q-1}{4}\right),$$

$$|\partial S| = 2q\left(\frac{q-1}{4}\right)^2,$$

and we get

$$\operatorname{iso}(G^*(q)) \leqslant \frac{|\partial S|}{|S|} = \frac{q(q-1)}{2(q+1)}.$$

The $q \equiv 3 \pmod 4$ case follows similarly and we get $\operatorname{iso}(G^*(q)) \leqslant q/2$.

We can follow the arguments from [2,9] to obtain lower bounds for $\operatorname{iso}(G^*(q))$, independent of Section 5. Thus, we do not need to determine the spectrum in order to demonstrate these graphs are expanders. As this allows us to obtain lower bounds for the isoperimetric numbers of the related graphs $G'(q)$, we include the argument. The proof for the following lemma is the same as in [2].

Lemma 2. *If* $(\alpha\ \ \beta)$, $(\alpha'\ \ \beta') \in G^*(q)$ *with* $\det\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} \neq 0$ *then there are exactly two paths of length two joining* $(\alpha\ \ \beta)$ *to* $(\alpha'\ \ \beta')$.

Partition the vertices of $G^*(q)$ into sets S_1 and S_2 with $|S_1| \leqslant |S_2|$. From Lemma 2, it follows that at least one edge from each of these paths must be cut for every pair $v_1 \in S_1$ and $v_2 \in S_2$ where v_1 and v_2 are not multiples of each other. Since any $v \in G^*(q)$ has $q-1$ nonzero multiples, the number of such pairs is at least $|S_1|(|S_2| - q + 1)$. As $G^*(q)$ is q-regular, each edge can lie in no more than $2(q-1)$ different paths of length 2. Thus, we have

$$|\partial S_1| \geqslant \frac{2|S_1|(|S_2| - q + 1)}{2(q-1)}$$

and since $|S_2| \geqslant (q^2-1)/4$, we get

$$\frac{|\partial S_1|}{|S_1|} \geqslant \frac{(|S_2| - q + 1)}{(q-1)} \geqslant \frac{q-3}{4}.$$

This gives the lower bound $(q-3)/4 \leqslant \operatorname{iso}(G^*(q))$. It follows in this elementary way that these are expander graphs in the sense of [11].

In a similar manner we can apply a version of Lemma 2 to $G'(q)$ and get the bounds

$$\frac{(q-1)(q-3)}{(2q-1)} \leqslant \operatorname{iso}(G'(q)) \leqslant \begin{cases} \dfrac{q(q-1)}{q+1} & \text{if } q \equiv 1 \pmod 4, \\[2mm] \dfrac{3q}{2} & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

By the construction of $G'(q)$, each vertex of $G'(q)$ corresponds to q vertices of $G(q)$. Further, $S' \subset V(G'(q))$ corresponds to a set $S \subset V(G(q))$ containing $q|S'|$ vertices and it is easy to see that for such sets, $|\partial S| = |\partial S'|$.

Therefore, the estimate $\mathrm{iso}(G'(q)) \leqslant |\partial S'|/|S'|$ implies the estimate $\mathrm{iso}(G(q)) \leqslant |\partial S'|/q|S'|$. Consequently, we get $\mathrm{iso}(G(q)) \leqslant (q-1)/(q+1)$ for $q \equiv 1 \pmod 4$ and $\mathrm{iso}(G(q)) \leqslant 3/2$ for $q \equiv 3 \pmod 4$. However, it does not follow that a lower bound for the isoperimetric number of a quotient graph implies a lower bound for the isoperimetric number of the larger graph. In general, we expect the isoperimetric numbers of the two graphs to be related, but an isoperimetric set for $G(q)$ may possibly be constructed that cuts through the subgraphs of $G(q)$ associated to the vertices in the quotient graph $G'(q)$. This may give a smaller isoperimetric number than $\mathrm{iso}(G'(q))/q$.

Using the results of the next section, we can get an improved lower bound for $\mathrm{iso}(G^*(q))$. For $\lambda_1 \neq k$, the second largest eigenvalue of the adjacency matrix of $G^*(q)$, we have

$$\frac{q}{2} - \frac{\lambda_1}{2} \leqslant \mathrm{iso}(G^*(q)).$$

See [1] or [5].

From the results of Section 5 we have $\lambda_1 = \sqrt{q}$ and so obtain

$$\frac{q}{2} - \frac{\sqrt{q}}{2} \leqslant \mathrm{iso}(G^*(q)),$$

which gives Corollary 1.

5. The spectrum of Platonic graphs over $\mathbb{F}_q$

Let S_q be the complex vector space of functions $s : V(G^*(q)) \to \mathbb{C}$ equipped with the canonical basis $\{e_v\}_{v \in V(G^*(q))}$ where $e_v(w) = \delta_{vw}$ and δ_{vw} is the Kronecker delta function. The adjacency operator $A_q : S_q \to S_q$ is the linear transformation defined by

$$(A_q s)(v) = \sum_{w \text{ is adjacent to } v} s(w).$$

The spectrum of $G^*(q)$, denoted Λ_q, is the set of eigenvalues of A_q. As A_q is a symmetric matrix with respect to the basis $\{e_v\}_{v \in V(G^*(q))}$ then $\Lambda_q \subset \mathbb{R}$. Further, since $G^*(q)$ is q-regular, we know that $\Lambda_q \subseteq [-q, q]$ and the eigenvalue q occurs with multiplicity 1.

Note that Γ_q acts on S_q and that this action commutes with A_q. Let v be a generator of $\mathbb{F}_q^\times$, let $\mu = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$, and let $\lambda \in \Gamma_q$ be of order $(q+1)/2$. Then, the conjugacy classes of Γ_q are those of $1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $b_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}$, μ^k for $1 \leqslant k \leqslant (q-5)/4$, and λ^ℓ for $1 \leqslant \ell \leqslant (q-1)/4$.

There are five types of representations of Γ_q, see [13] for example. There is the trivial representation **1** and the Steinberg representation ψ_{St} of degree q. For even characters $\theta : \mathbb{F}_q^\times \to \mathbb{C}^\times$, there are the principal series representations χ_θ of degree $q+1$. These occur with multiplicity 2. There are $(q-5)/4$ isomorphism classes of this representation for $q \equiv 1 \pmod 4$, and $(q-3)/4$ for $q \equiv 3 \pmod 4$. There are the $(q-1)/4$ isomorphism classes of the discrete series representations ϕ_j of degree $q-1$. Finally, for $q \equiv 1 \pmod 4$, there are the two split principal series $\xi_\pm$ of degree $(q+1)/2$. Let S_r be the irreducible submodule of S_q generated by the isomorphism classes of the representation r.

The values of the relevant irreducible characters of Γ_q on representatives of the isomorphism classes are given by the following table:

	1_2	b_1	b_2	μ^κ	λ^ℓ
1	1	1	1	1	1
ψ_{St}	q	0	0	1	-1
χ_θ	$q+1$	1	1	$\theta(v^\kappa) + \theta(v^{-\kappa})$	0
$\xi_\pm$	$(q+1)/2$	$(1 \pm \sqrt{2})/2$	$(1 \mp \sqrt{2})/2$	$(-1)^\kappa$	0

Let χ be the character of Γ_q acting on S_q. Then, trivially $\chi(1_2) = (q^2-1)/2$. Since the stabilizer in Γ_q of $(1\ 0) \in V(G^*(q))$ is isomorphic to N and any element in the stabilizer fixes only the associates of $(1\ 0)$, it follows that

$\chi(b_1) = \chi(b_2) = (q-1)/2$. From the definition of Γ_q, no other elements of Γ_q stabilize a vertex. Therefore, $\chi(c) = 0$ for c not in the classes of 1_2, b_1, and b_2. From these values of χ and the character table, we can easily determine that the representations $\mathbf{1}$, ψ_{St}, and χ_θ occur with multiplicities 1, 1, and 2, respectively. Further, for $q \equiv 1 \pmod 4$, the representations $\xi_\pm$ occur with multiplicity 1. By dimension counting, these are the only occurring representations. It follows that S_q decomposes into the direct sum of the submodules $S_{\mathbf{1}}$, $S_{\psi_{\mathrm{St}}}$, S_{χ_θ}, and $S_{\xi_\pm}$.

For $\varepsilon \in \mathbb{F}_q^\times$, let $s_v \in S_q$ be defined by $s_v(\varepsilon v) = q$, and $s_v(w) = -1$ for all other vertices. For v the center vertex of H_α and v' not associate to v, we can easily calculate from Theorem 2 that $(A_q s_v)(v') = 1$. Therefore, we can see that for any $v \in V(G^*(q))$,

$$(A_q s_v)(v') = \begin{cases} -q & \text{for } v' = \varepsilon v \\ 1 & \text{otherwise} \end{cases} = -s_v(v').$$

Thus, s_v is an eigenvector with eigenvalue -1. Let M be the submodule of S_q generated by all the s_v's above. We see that $s_v = s_{\varepsilon v}$ for all $v \in V(G^*(q))$ and $\varepsilon \in \mathbb{F}_q^\times$. Since $\sum_{v \in V(G^*(q))} s_v = 0$, we have $\dim M \leqslant q$. Further, since the intersection of M with the submodule generated by s_w with w adjacent to a fixed v has dimension q, it follows that $\dim M = q$. As the submodules S_r are irreducible, we must have $M = S_{\psi_{\mathrm{St}}}$. As Γ_q and A_q commute and $S_{\psi_{\mathrm{St}}}$ is irreducible, then A_q acts on $S_{\psi_{\mathrm{St}}}$ by a scalar and it follow that $S_{\psi_{\mathrm{St}}}$ is a q-dimensional eigenspace with eigenvalue -1.

Let $s^\pm_{\theta,v}(\varepsilon v') = \pm\theta(\varepsilon)\sqrt{q}$ if $v' = v$, $s^\pm_{\theta,v}(w) = \theta(\varepsilon)$ if w is adjacent to v, and 0 otherwise. From Theorem 2, we can compute

$$\begin{aligned}(A_q s^\pm_{\theta,v})(\varepsilon v) &= \sum_{w \text{ is adjacent to } \varepsilon v} s^\pm_{\theta,v}(w) \\ &= \theta(\varepsilon) q \\ &= \pm\sqrt{q}\, s^\pm_{\theta,v}(\varepsilon v)\end{aligned}$$

and for $w \sim \varepsilon v$ we have

$$\begin{aligned}(A_q s^\pm_{\theta,v})(w) &= s^\pm_{\theta,v}(v) + \sum_{\substack{w' \text{ is adjacent to } w \\ w' \neq v}} s^\pm_{\theta,v}(w') \\ &= \pm\theta(\varepsilon)\sqrt{q} + \sum_{\varepsilon' \in \mathbb{F}_q^\times} \theta(\varepsilon') \\ &= \pm\sqrt{q}\, s^\pm_{\theta,v}(w).\end{aligned}$$

Thus, $s^\pm_{\theta,v}$ is an eigenvector of A_q with eigenvalue $\pm\sqrt{q}$.

Trivially, $s^\pm_{\theta,v} \notin S_{\mathbf{1}}$ or $S_{\psi_{\mathrm{St}}}$ as they have different eigenvalues. Therefore, we must have $s^\pm_{\theta,v} \in S_{\chi_\theta}$, or $S_{\xi^\pm}$ if θ is quadratic. For any even character $\theta : \mathbb{F}_q^\times \to \mathbb{C}^\times$, the above computations show that there are eigenvectors v_1 and v_2 so that $A_q(v_i) = (-1)^i\sqrt{q}$. Further, Γ_q acts on v_i with the character θ. As S_{χ_θ} and $S_{\xi_\pm}$ are irreducible then the orbit of v_i, $\Gamma_q(v_i)$, must either be a copy of S_{χ_θ}, $S_{\xi_\pm}$, or $\{0\}$. But as $v_i \in \Gamma_q(v_i)$ then $\Gamma_q(v_i) \neq \{0\}$. As Γ_q commutes with A_q then A_q acts on $\Gamma_q(v_i)$ by a scalar. As $\Gamma_q(v_1)$and $\Gamma_q(v_2)$ are the same dimension, it follows that $\Gamma_q(v_i)$ are $(q+1)$-dimensional subspaces of S_q. That is, S_{χ_θ} and $S_{\xi_\pm}$ are eigenspaces. As $\Gamma_q(v_1) \cap \Gamma_q(v_2) = \{0\}$ since they are different eigenspaces, we have two copies of S_{χ_θ} in S_q and a copy each of $S_{\xi_\pm}$ for quadratic θ, as this applies to all the principal series representations. Counting dimensions of the relevant representations, we get all of S_q. Thus we have completely determined the spectrum of A_q and this gives Theorem 1.

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