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A CORRESPONDENCE BETWEEN TWO CATALAN SETS

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We set up a one-to-one correspondence between two sets whose cardinalities are both equal to the nth Catalan number, namely, the set of triangulations of a convex $(n+2)$-gon and the set of symmetric positive definite unimodular tridiagonal matrices of order n considered in [1].

1. Leighton and Newman [1] prove that if S_n is the set of positive definite unimodular matrices of the form

$$A = \mathrm{td}(a_1, \ldots, a_n) = \begin{bmatrix} a_1 & 1 & & & & \\ 1 & a_2 & 1 & & & \\ & & \ddots & & & \\ & & & 1 & a_{n-1} & 1 \\ & & & & 1 & a_n \end{bmatrix},$$

where $a_1, \ldots, a_n$ are positive integers, then the cardinality of S_n is equal to the Catalan number $C_n = \binom{2n}{n}/(n+1)$. This suggests that there might be a one-to-one correspondence between S_n and one of the other well-known sets that have cardinality C_n. It turns out that there is such a correspondence, the other set being the one that originally gave rise to the Catalan numbers, namely the set of triangulations of a convex $(n+2)$-gon by means of nonintersecting diagonals (here we use the term diagonal to denote any line joining nonadjacent vertices of the polygon).

2. The correspondence is very simple. Label the vertices of the $(n+2)$-gon successively (say clockwise) $V_0, V_1, \ldots, V_n, V_{n+1}$, and regard V_0V_{n+1} as the base (preferably drawn horizontally). Then, if we have any triangulation of the $(n+2)$-gon, a_i $(1 \leq i \leq n)$ *is the number of triangles that meet at* V_i. However, we have to prove that

(i) the matrix corresponding to a given triangulation belongs to S_n;
(ii) any matrix of S_n gives rise to a unique triangulation.

The key results are Lemmas 2 and 3 of [1], which both have analogues in the geometry.

Lemma 2 says that if $A \in S_n$, then $a_k = 1$ for at least one value of k $(1 \leq k \leq n)$. This corresponds to the fact that there is at least one vertex V_k $(1 \leq k \leq n)$ that belongs to only one triangle, that is, in the triangulation there is no diagonal through V_k. This follows geometrically from the fact that the graph formed by the diagonals in the triangulation and the vertices they join is clearly connected, and since there are $n-1$ diagonals, the number of vertices in the graph is at most n. Therefore, there are at least two vertices that have none of the diagonals through them, and these cannot be both V_0 and V_{n+1}, and so the result follows.

Lemma 3 says that if $a_k = 1$, then $A = \mathrm{td}(a_1, \ldots, a_n)$ is integrally congruent to the direct sum of the matrices [1] and $B = \mathrm{td}(a_1, \ldots, a_{k-1} - 1, a_{k+1} - 1, \ldots, a_n)$, that is, the matrix

$$\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{bmatrix}.$$

In the proof of Lemma 3, Leighton and Newman give the corresponding congruence transformations, and the result of these corresponds to deleting the vertex V_k and the edges that join it to V_{k-1} and V_{k+1}. This gives a triangulated $(n+1)$-gon, which can, of course, be relabelled in the obvious way.

We can now prove (i) and (ii).

Proof of (i). This follows easily from Lemma 3 of [1] and its geometric counterpart. By successive application of this to the triangulated $(n+2)$-gon and the triangulated polygons that arise at each stage, we see that the matrix corresponding to the given triangulation is positive definite, and it is unimodular, because all the congruence transformations in the proof of Lemma 3 correspond to matrices with determinant ± 1. □

Proof of (ii). This again follows easily from Lemma 3 of [1] and its geometric counterpart. By successive application of this to the given matrix of S_n and the matrices B that arise at each stage, we eventually arrive at the matrix [1], which corresponds to a triangle. Now, by working backwards, starting with the triangle,

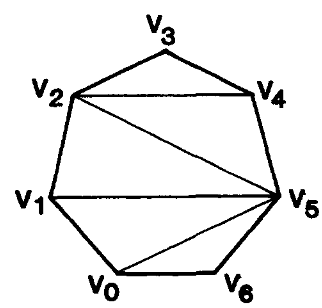

Fig. 1.

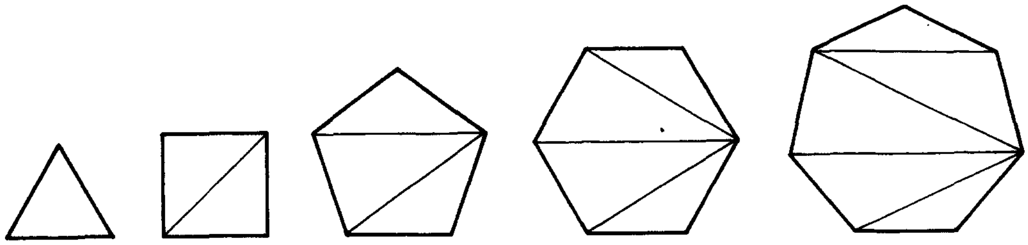

Fig. 2.

we can add one vertex at a time, and eventually arrive at the triangulated $(n+2)$-gon corresponding to the given matrix. □

The following example illustrates the method (see Fig. 1). This triangulation corresponds to the matrix td(2, 3, 1, 2, 4). Conversely, if this matrix is given, then successive applications of Lemma 3 of [1] give the matrices td(2, 2, 1, 4), td(2, 1, 3), td(1, 2), td(1). We can therefore build up the polygon as in Fig. 2.

3. In their proof that the cardinality of S_n is C_n, Leighton and Newman use the important idea of a "break-point". This is the a_r $(1 \leq r \leq n)$ such that $\mathrm{td}(a_1, \ldots, a_{r-1}) \in S_{r-1}$ and $\mathrm{td}(a_{r+1}, \ldots, a_n) \in S_{n-r}$, and they prove that it is unique. (If $r = 1$ or n, one of the conditions is redundant.) We have not used this idea, but it is interesting to note that the breakpoint a_r corresponds to the unique vertex V_r such that $V_0 V_r V_{n+1}$ is a triangle of the triangulation, that is, V_r is the other vertex of the triangle that contains the base. In the example above, this triangle is $V_0 V_5 V_6$, so the breakpoint is a_5.

Reference

[1] F.T. Leighton and M. Newman, Positive definite matrices and Catalan numbers, Proc. Amer. Math. Soc. 79 (1980) 177–181.