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A blow-up criterion for a 2D viscous liquid-gas two-phase flow model

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ABSTRACT

In this paper, we prove a blow-up criterion in terms of the upper bound of the liquid mass for the strong solution to the two-dimensional (2D) viscous liquid-gas two-phase flow model in a smooth bounded domain. The result also applies to three-dimensional (3D) case.

1. Introduction

In this paper, we consider the following two-dimensional (2D) viscous liquid-gas two-phase flow model

$$\begin{cases} m_t + \operatorname{div}(mu) = 0, \\ n_t + \operatorname{div}(nu) = 0, \\ (mu)_t + \operatorname{div}(mu \otimes u) + \nabla P(m,n) = \mu \Delta u + (\mu + \lambda)\nabla \operatorname{div} u \quad \text{in } \Omega \times (0,T), \end{cases} \tag{1.1}$$

with the initial and boundary conditions

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$$(m,n,u)|_{t=0}=(m_0,n_0,u_0)(x), \quad \text{in } \Omega, \tag{1.2}$$

$$u(x,t)=0, \quad \text{on } \partial\Omega\times(0,T), \tag{1.3}$$

where $\Omega\subseteq\mathbb{R}^2$ is a bounded domain with smooth boundary. Here $m=\alpha_l\rho_l$ and $n=\alpha_g\rho_g$ denote liquid mass and gas mass, respectively; μ, λ are viscosity constants, satisfying

$$\mu>0, \quad \mu+\lambda\geqslant 0. \tag{1.4}$$

The unknown variables α_l, $\alpha_g\in[0,1]$ denote liquid and gas volume fractions, satisfying the fundamental relation: $\alpha_l+\alpha_g=1$. Furthermore, the other unknown variables ρ_l and ρ_g denote liquid and gas densities, satisfying equations of state: $\rho_l=\rho_{l,0}+\frac{P-P_{l,0}}{a_l^2}$, $\rho_g=\frac{P}{a_g^2}$, where a_l, a_g are sonic speeds, respectively, in liquid and gas, and $P_{l,0}$ and $\rho_{l,0}$ are the reference pressure and density given as constants; u denotes velocities of liquid and gas; P is common pressure for both phases, which satisfies

$$P(m,n)=C^0\left(-b(m,n)+\sqrt{b(m,n)^2+c(m,n)}\right), \tag{1.5}$$

with $C^0=\frac{1}{2}a_l^2$, $k_0=\rho_{l,0}-\frac{P_{l,0}}{a_l^2}>0$, $a_0=(\frac{a_g}{a_l})^2$ and

$$b(m,n)=k_0-m-\left(\frac{a_g}{a_l}\right)^2 n=k_0-m-a_0n,$$

$$c(m,n)=4k_0\left(\frac{a_g}{a_l}\right)^2 n=4k_0a_0n.$$

For more information about the above models, we can refer to [8,10,16] and references therein.

Let us review some previous works about the viscous liquid-gas two-phase flow model. For the model (1.1) in one-dimensional (1D) case, when the liquid is incompressible and the gas is polytropic, i.e., $P(m,n)=C\rho_l^\gamma(\frac{n}{\rho_l-m})^\gamma$, Evje and Karlsen in [4] studied the existence and uniqueness of the global weak solution to the free boundary value problem with $\mu=\mu(m)=k_1\frac{m^\beta}{(\rho_l-m)^{\beta+1}}$, $\beta\in(0,\frac{1}{3})$, when the fluids connected to vacuum state discontinuously. Yao and Zhu in [14] extended the results in [4] to the case $\beta\in(0,1]$, and also obtained the asymptotic behavior and regularity of the solution. Evje, Flåtten and Friis in [2] also studied the model with $\mu=\mu(m,n)=k_2\frac{n^\beta}{(\rho_l-m)^{\beta+1}}$ ($\beta\in(0,\frac{1}{3})$) in a free boundary setting when the fluids connected to vacuum state continuously, and obtained the global existence of the weak solution. Also, for the case of connecting to vacuum state continuously, we investigated the free boundary problem to the model with constant viscosity coefficient, and obtained the global existence of the unique weak solution by the line method, where we used a new technique to get the key upper and lower bounds of gas and liquid masses n and m, cf. [15]. Specifically, when both of the two fluids are compressible, their results can consult the reference [3].

But there are few results about the multidimensional model of this kind. Recently, Yao, Zhang and Zhu in [16] obtained the existence of the global weak solution to the 2D model when the initial energy is small. And this can be viewed to be a generalization of the results in [3] from one-dimensional to two-dimensional. In this paper, we prove a blow-up criterion in terms of the upper bound of the liquid mass for the strong solution to the 2D viscous liquid-gas two-phase flow model in a smooth bounded domain.

Before giving the main result, we state the following local existence of the unique strong solution without initial vacuum, the proof of which is similar to that in [1]. In fact, Cho, Choe and Kim in [1] deal with the local existence of the unique strong solution with initial vacuum for the single-phase Navier–Stokes equation, where the initial data must satisfy a natural compatibility condition. The initial assumptions and the properties of pressure in the present paper satisfy the assumptions in [1], and there is no initial vacuum, so the proof of the local existence of the unique strong solution for the viscous liquid-gas two-phase flow model is simpler, compared to [1]. We omit the details of the proof here.

Theorem 1.1 *(Local existence). Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and $q > 2$. Assume that there exist constants $\underline{m}_1$, $\bar{m}_1$, $\underline{n}_1$ and $\bar{n}_1$ with $0 < \underline{m}_1 \leqslant \bar{m}_1 < \infty$, $0 < \underline{n}_1 \leqslant \bar{n}_1 < \infty$, such that the initial data m_0, n_0, u_0 satisfy*

$$\begin{gathered} 0 < \underline{m}_1 \leqslant \inf_x m_0 \leqslant \sup_x m_0 \leqslant \bar{m}_1 < \infty, \qquad 0 < \underline{n}_1 \leqslant \inf_x n_0 \leqslant \sup_x n_0 \leqslant \bar{n}_1 < \infty, \\ m_0, n_0 \in W^{1,q}(\Omega), \qquad u_0 \in H_0^1(\Omega) \cap H^2(\Omega). \end{gathered} \tag{1.6}$$

Then, there exist a $T_1 > 0$ and a unique strong solution $(m, n, u)(x, t)$ to the problem (1.1)–(1.3)*, such that*

$$\begin{gathered} m, n > 0, \quad m, n \in C\big([0, T_1], W^{1,q}(\Omega)\big), \qquad m_t, n_t \in C\big([0, T_1], L^q(\Omega)\big), \\ u \in C\big([0, T_1], H_0^1(\Omega) \cap H^2(\Omega)\big) \cap L^2\big(0, T_1; W^{2,q}(\Omega)\big), \\ u_t \in L^\infty\big(0, T_1; L^2(\Omega)\big) \cap L^2\big(0, T_1; H_0^1(\Omega)\big). \end{gathered} \tag{1.7}$$

The following is then the main result of this paper.

Theorem 1.2. *Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and $q \in (2, \infty)$. Assume that the initial data m_0, n_0, u_0 satisfy* (1.6)*. If $T^* < \infty$ is the maximal existence time for strong solution $(m, n, u)(x, t)$ to the problem* (1.1)–(1.3) *stated in Theorem* 1.1*, then*

$$\limsup_{T \to T^*} \|m\|_{L^\infty(0,T;L^\infty(\Omega))} = \infty. \tag{1.8}$$

Remark 1.1. It is easy to verify

$$\begin{cases} P_m = \dfrac{\partial P}{\partial m} = C^0 \left\{ 1 - \dfrac{b}{\sqrt{b^2 + c}} \right\} > 0, \\ P_n = \dfrac{\partial P}{\partial n} = C^0 \left\{ a_0 + \dfrac{a_0}{\sqrt{b^2 + c}} (m + a_0 n + k_0) \right\} > 0, \quad m, n > 0. \end{cases} \tag{1.9}$$

This shows that $P(m, n)$ is increasing in m and n for $m, n > 0$.

Remark 1.2. In a forthcoming paper, we will consider the local existence of the strong solution and also give a blow-up criterion for the $2D$ (or $3D$) viscous liquid-gas two-phase flow model (1.1), when there is initial vacuum, i.e., $m_0 \geqslant 0$ and $n_0 \geqslant 0$.

Just because of the similarity of the viscous liquid-gas two-phase flow model with the Navier–Stokes equation, so some ideas used to get the blow-up criterion of the strong solution for the Navier–Stokes equation will be applied to deal with the two-phase flow model. For the 2D compressible Navier–Stokes equations, Sun and Zhang in [12] obtained a blow-up criterion in terms of the upper bound of the density for the strong solution. For the 3D compressible Navier–Stokes equations, Sun, Wang and Zhang in [11] also obtained a blow-up criterion in terms of the upper bound of the density for the strong solution, when $\lambda < 7\mu$. In the both papers above, the initial vacuum was allowed and the domain included both the bounded smooth domain and $\mathbb{R}^N$, $N = 2, 3$. It also worths mentioning recent works [6,7], under the assumptions

$$N=2, \quad \mu+\lambda \geqslant 0, \quad \Omega = T^2;$$

$$N=3, \quad \lambda < 7\mu, \quad \Omega \text{ is a smooth domain including } \mathbb{R}^3,$$

Huang and Xin proved the following blow-up criterion: if $T^* < \infty$ is the maximal time of the existence of the strong solution, then

$$\lim_{T\to T^*} \int_0^T \left\|\nabla u(t)\right\|_{L^\infty(\Omega)} dt = \infty.$$

In our present paper, we want to obtain the same result for the viscous liquid-gas two-phase flow model. Because of the complexity of the pressure $P(m,n)$, we can only deal with the simpler case: the domain is smooth and bounded; there is no initial vacuum. We remark that the result also applies to 3D case at the end of this paper, see Theorem 4.1.

2. Preliminaries

In this section, we give some useful lemmas which will be used in the next two sections.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain with piecewise smooth boundaries. Then the following inequality is valid for every function $u \in W_0^{1,p}(\Omega)$ or $u \in W^{1,p}(\Omega)$, $\int_\Omega u\,dx = 0$:*

$$\|u\|_{L^q(\Omega)} \leqslant C_1 \|\nabla u\|_{L^p(\Omega)}^{\alpha} \|u\|_{L^r(\Omega)}^{1-\alpha}, \tag{2.1}$$

where $\alpha = (1/r - 1/q)(1/r - 1/p + 1/N)^{-1}$; moreover, if $p < N$, then $q \in [r, pN/(N-p)]$ for $r \leqslant pN/(N-p)$, and $q \in [pN/(N-p), r]$ for $r > pN/(N-p)$. If $p \geqslant N$, then $q \in [r, \infty)$ is arbitrary; moreover, if $p > N$, then inequality (2.1) *is also valid for $q = \infty$. The positive constant C_1 in inequality* (2.1) *depends on N, p, r, α and the domain Ω but independent of the function u.*

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded domain with piecewise smooth boundaries. Then the following inequality is valid for every function $u \in W^{1,p}(\Omega)$:*

$$\|u\|_{L^q(\Omega)} \leqslant C_2\big(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^p(\Omega)}^{\alpha} \|u\|_{L^r(\Omega)}^{1-\alpha}\big), \tag{2.2}$$

where N, p, r, q and α are the same as those in Lemma 2.1. *The positive constant C_2 in inequality* (2.2) *depends on N, p, r, α and the domain Ω but independent of the function u.*

The above two lemmas can be found in [9,13] and the references therein.

Next, we give some L^q ($q \in (1,\infty]$) regularity estimates for the solution of the following boundary problem:

$$\begin{cases} LU := \mu\Delta U + (\mu+\lambda)\nabla \operatorname{div} U = F & \text{in } \Omega, \\ U(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, L is the Lamé operator, $U = (U_1, U_2, \ldots, U_N)$, $F = (F_1, F_2, \ldots, F_N)$. From (1.4), we know that (2.3) is a strong elliptic system. If $F \in W^{-1,2}(\Omega)$, then there exists a unique weak solution $U \in H_0^1(\Omega)$. In the subsequent context, we will use $L^{-1}F$ to denote the unique solution U of the system (2.3) with F belonging to some suitable space such as $W^{-1,p}(\Omega)$. Sun, Wang and Zhang in [11,12] give the following estimates:

Lemma 2.3. *Let $q \in (1, \infty)$, and U be a solution of (2.3). Then there exists a constant C depending only on μ, λ, q, N and Ω such that*

(1) *if $F \in L^q(\Omega)$, then*

$$\|U\|_{W^{2,q}(\Omega)} \leqslant C\|F\|_{L^q(\Omega)}; \tag{2.4}$$

(2) *if $F \in W^{-1,q}(\Omega)$ (i.e., $F = \operatorname{div} f$ with $f = (f_{ij})_{N\times N}$, $f_{ij} \in L^q(\Omega)$), then*

$$\|U\|_{W^{1,q}(\Omega)} \leqslant C\|f\|_{L^q(\Omega)}; \tag{2.5}$$

(3) *if $F = \operatorname{div} f$ with $f_{ij} = \partial_k h^k_{ij}$ and $h^k_{ij} \in W^{1,q}_0(\Omega)$ for $i, j, k = 1, 2, \ldots, N$, then*

$$\|U\|_{L^q(\Omega)} \leqslant C\|h\|_{L^q(\Omega)}. \tag{2.6}$$

Lemma 2.4. *If $F = \operatorname{div} f$ with $f = (f_{ij})_{N\times N}$, $f_{ij} \in L^\infty(\Omega) \cap L^2(\Omega)$, then $\nabla U \in BMO(\Omega)$ and there exists a constant C depending only on μ, λ and Ω such that*

$$\|\nabla U\|_{BMO(\Omega)} \leqslant C\big(\|f\|_{L^\infty(\Omega)} + \|f\|_{L^2(\Omega)}\big). \tag{2.7}$$

Here $BMO(\Omega)$ denotes the John–Nirenberg's space of bounded mean oscillation whose norm is defined by

$$\|f\|_{BMO(\Omega)} = \|f\|_{L^2(\Omega)} + [f]_{BMO(\Omega)},$$

with the semi-norm

$$[f]_{BMO(\Omega)} = \sup_{x\in\Omega, r\in(0,d)} \fint_{\Omega_r(x)} \big|f(y) - f_{\Omega_r(x)}\big|\, dy,$$

where $\Omega_r(x) = B_r(x) \cap \Omega$, $B_r(x)$ is the ball with center x and radius r and d is the diameter of Ω. For a measurable subset E of $\mathbb{R}^N$, $|E|$ denotes its Lebesgue measure and

$$f_{\Omega_r(x)} = \fint_{\Omega_r(x)} f(y)\, dy = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y)\, dy.$$

Lemma 2.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $f \in W^{1,q}(\Omega)$ with $q \in (N, \infty)$. Then there exists a constant C depending on q, N and the Lipschitz property of the domain Ω such that*

$$\|f\|_{L^\infty(\Omega)} \leqslant C\big(1 + \|f\|_{BMO(\Omega)} \ln\big(e + \|\nabla f\|_{L^q(\Omega)}\big)\big). \tag{2.8}$$

3. A priori estimates

Let (m, n, u) be a strong solution to the problem (1.1)–(1.3) in $[0, T)$ with the regularity stated in Theorem 1.1. We assume that the opposite of (1.8) holds, i.e., there exists a positive constant M, such that

$$\limsup_{T\to T^*} \|m\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant M < \infty. \tag{3.1}$$

In this section, we denote by C a general positive constant which may depend on μ, λ, Ω, m_0, n_0, u_0, M, and the parameters in the expression of P in (1.5).

Let

$$T_1^* = \sup\{T \in (0, T^*);\ m(x,t) > 0,\ \text{for all } (x,t) \in \Omega \times [0,T]\}. \tag{3.2}$$

At first, we give the estimate of $\frac{n(x,t)}{m(x,t)}$.

Lemma 3.1. *Under the conditions of Theorem* 1.2*, we have*

$$0 < \underline{s}_0 \leqslant \frac{n(x,t)}{m(x,t)} \leqslant \bar{s}_0 < \infty, \quad 0 \leqslant T < T_1^*, \tag{3.3}$$

where $\underline{s}_0 = \inf_{x\in\Omega} \frac{n_0}{m_0}$, $\bar{s}_0 = \sup_{x\in\Omega} \frac{n_0}{m_0}$.

Proof. Define the particle trajectories $x = X(t,y)$ given by:

$$\begin{cases} \dfrac{d}{dt} X(t,y) = u(X(t,y),t), \\ X(0,y) = y. \end{cases} \tag{3.4}$$

From $(1.1)_1$ and $(1.1)_2$, we have

$$\left(\frac{n}{m}\right)_t + u \cdot \nabla\left(\frac{n}{m}\right) = 0, \tag{3.5}$$

which implies

$$\frac{d}{dt}\left(\frac{n}{m}\right)(X(t,y),t) = 0,$$

i.e.,

$$\frac{n(x,t)}{m(x,t)} = \frac{n_0}{m_0}\big(X^{-1}(t,x)\big) := s_0 = s_0(x,t), \quad \text{for } t \in (0, T_1^*),$$

where X^{-1} denotes the inverse of X. It follows

$$0 < \underline{s}_0 \leqslant \min\left\{\frac{n_0}{m_0}\big(X^{-1}(t,x)\big)\right\} \leqslant \frac{n(x,t)}{m(x,t)} \leqslant \max\left\{\frac{n_0}{m_0}\big(X^{-1}(t,x)\big)\right\} \leqslant \bar{s}_0 < \infty. \qquad \square \tag{3.6}$$

Then, we give the basic energy estimate.

Proposition 3.1. *Assume*

$$\|m\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant M, \quad 0 \leqslant T < T_1^*. \tag{3.7}$$

Then we have

$$\|\sqrt{m}u\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \qquad \|\nabla u\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \quad 0 \leqslant T < T_1^*. \tag{3.8}$$

Proof. Let

$$A(t)=\int_\Omega\left\{\frac{1}{2}m|u|^2+Q(m,n)\right\}dx,\quad 0\leqslant t\leqslant T<T_1^*,$$

where $Q(m,n)=m\int_1^m\frac{P(s,\frac{n}{m}s)}{s^2}\,ds$. Then we have from (1.1) that

$$\begin{aligned}
A'(t)&=\int_\Omega\left\{\frac{1}{2}|u|^2m_t+mu\cdot u_t+Q_mm_t+Q_nn_t\right\}dx\\
&=\int_\Omega\biggl\{-\frac{1}{2}|u|^2\operatorname{div}(mu)+u^j\bigl[-mu\cdot\nabla u^j-\partial_jP(m,n)+\mu\Delta u^j+(\mu+\lambda)\partial_j\operatorname{div}u\bigr]\\
&\quad-Q_m\operatorname{div}(mu)-Q_n\operatorname{div}(nu)\biggr\}dx\\
&=\int_\Omega\biggl\{-\frac{1}{2}|u|^2\operatorname{div}(mu)+u^j\bigl[-mu\cdot\nabla u^j-\partial_jP(m,n)+\mu\Delta u^j+(\mu+\lambda)\partial_j\operatorname{div}u\bigr]\\
&\quad-Q_mu\cdot\nabla m-Q_nu\cdot\nabla n-Q_mm\operatorname{div}u-Q_nn\operatorname{div}u\biggr\}dx\\
&=\int_\Omega\biggl\{-\frac{1}{2}|u|^2\operatorname{div}(mu)-mu^ju^i\partial_iu^j-u\cdot\nabla P(m,n)-\mu\bigl(\partial_iu^j\bigr)^2-(\mu+\lambda)(\operatorname{div}u)^2\\
&\quad-\operatorname{div}uP-\operatorname{div}(uQ)\biggr\}dx\\
&=\int_\Omega\biggl\{\frac{1}{2}\nabla\bigl(|u|^2\bigr)\cdot(mu)-mu^ju^i\partial_iu^j-\operatorname{div}(uP)\\
&\quad-\operatorname{div}(uQ)-\mu\bigl(\partial_iu^j\bigr)^2-(\mu+\lambda)(\operatorname{div}u)^2\biggr\}dx.
\end{aligned}\tag{3.9}$$

Here we have used integration by parts, boundary conditions (1.3) and the following identity

$$mQ_m+nQ_n=Q+P,$$

which can be easily obtained from the expression of $Q(m,n)$. From (3.9), we get

$$\begin{aligned}
A'(t)+\int_\Omega\bigl\{\mu|\nabla u|^2+(\mu+\lambda)(\operatorname{div}u)^2\bigr\}dx&=\int_\Omega\left\{\frac{1}{2}\nabla\bigl(|u|^2\bigr)\cdot(mu)-mu^ju^i\partial_iu^j\right\}dx\\
&=\int_\Omega\left\{\frac{1}{2}\partial_i\bigl(\bigl(u^j\bigr)^2\bigr)mu^i-mu^ju^i\partial_iu^j\right\}dx\\
&=0,
\end{aligned}$$

which implies

$$\int_\Omega \left\{ \frac{1}{2} m|u|^2 + Q(m,n) \right\} dx + \int_0^t \int_\Omega \left\{ \mu |\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 \right\} dx\, d\tau$$
$$= \int_\Omega \left\{ \frac{1}{2} m_0 |u_0|^2 + Q(m_0,n_0) \right\} dx. \tag{3.10}$$

From (3.10), (3.3), (3.7) and the expression of $P(m,n)$, we get

$$\begin{aligned}
\int_\Omega m|u|^2\,dx + \int_0^t \int_\Omega |\nabla u|^2\,dx\,d\tau &\leqslant C\int_\Omega m_0|u_0|^2\,dx + C\left|\int_\Omega Q(m_0,n_0)\,dx\right| + C\left|\int_\Omega Q(m,n)\,dx\right| \\
&\leqslant C\int_\Omega m_0|u_0|^2\,dx + C\int_\Omega |m\ln m|\,dx + C\int_\Omega m\,dx + \int_\Omega \sqrt{m}\,dx \\
&\quad + C\int_\Omega |m_0 \ln m_0|\,dx + C\int_\Omega m_0\,dx + \int_\Omega \sqrt{m_0}\,dx \\
&\leqslant C.
\end{aligned} \tag{3.11}$$

This completes the proof of Proposition 3.1. □

The following arguments are similar to that in [11,12], which discussed the single-phase Navier–Stokes equations. We enclose its proof for the self-containedness of the present paper.

Proposition 3.2. *Under the condition* (3.7)*, we have for some $r > 2$ that*

$$\sup_{0\leqslant t\leqslant T} \int_\Omega m|u|^r\,dx \leqslant C, \quad 0 \leqslant T < T_1^*. \tag{3.12}$$

Proof. Multiplying $(1.1)_3$ by $r|u|^{r-2}u$, and integrating the resulting equation over Ω, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega m|u|^r\,dx + \int_\Omega \Big[r|u|^{r-2}\big(\mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2 + \mu(r-2)\big|\nabla|u|\big|^2\big) \\
&\quad + r(\lambda+\mu)\big(\nabla|u|^{r-2}\big)\cdot u \operatorname{div} u\Big]\,dx \\
&= r\int_\Omega \operatorname{div}\big(|u|^{r-2}u\big)P\,dx \leqslant C\int_\Omega m^{\frac{1}{2}}|u|^{r-2}|\nabla u|\,dx \\
&\leqslant \varepsilon \int_\Omega |u|^{r-2}|\nabla u|^2\,dx + \frac{C}{\varepsilon}\left(\int_\Omega m|u|^r\,dx\right)^{\frac{r-2}{r}} dx,
\end{aligned} \tag{3.13}$$

where we have used $P(m,n) \leqslant Cm^{\frac{1}{2}}$, which can be obtained from (3.3), (3.7) and the expression of P easily, where C depends only on M, $\bar{s}_0$ and the parameters in the expression of P.

Note that $|\nabla|u|| \leqslant |\nabla u|$, we get that

$$\begin{aligned}
&r|u|^{r-2}\big(\mu|\nabla u|^2+(\lambda+\mu)(\operatorname{div}u)^2+\mu(r-2)\big|\nabla|u|\big|^2\big)+r(\lambda+\mu)\big(\nabla|u|^{r-2}\big)\cdot u\operatorname{div}u\\
&\quad\geqslant r|u|^{r-2}\big[\mu|\nabla u|^2+(\lambda+\mu)(\operatorname{div}u)^2\big]+r(\lambda+\mu)\big(\nabla|u|^{r-2}\big)\cdot u\operatorname{div}u\\
&\quad\geqslant r|u|^{r-2}\big[\mu|\nabla u|^2+(\lambda+\mu)(\operatorname{div}u)^2-(\lambda+\mu)(r-2)\big|\nabla|u|\big||\operatorname{div}u|\big]\\
&\quad\geqslant r|u|^{r-2}\left[\left(\mu-\frac{\lambda+\mu}{2}(r-2)\right)|\nabla u|^2+\frac{\lambda+\mu}{2}(4-r)|\operatorname{div}u|^2\right].
\end{aligned}$$

We can choose $2<r<4$ and $r-2$ small enough such that the last term is bounded from below by $C|u|^{r-2}|\nabla u|^2$, i.e.,

$$\begin{aligned}
&r|u|^{r-2}\big[\mu|\nabla u|^2+(\lambda+\mu)(\operatorname{div}u)^2+\mu(r-2)\big|\nabla|u|\big|^2\big]+r(\lambda+\mu)\big(\nabla|u|^{r-2}\big)\cdot u\operatorname{div}u\\
&\quad\geqslant C|u|^{r-2}|\nabla u|^2,
\end{aligned}\tag{3.14}$$

where (1.4) has been used. Inserting (3.14) into (3.13), and taking $\varepsilon=\frac{C}{2}$, we may apply Gronwall's inequality to conclude (3.12). □

Just as in [11,12], we introduce the quantity w, which is defined by

$$w=u-v,\qquad v=L^{-1}\nabla P(m,n),$$

where v is the solution of

$$\begin{cases}\mu\Delta v+(\lambda+\mu)\nabla\operatorname{div}v=\nabla P(m,n) & \text{in }\Omega,\\ v(x)=0 \quad \text{on }\partial\Omega.\end{cases}\tag{3.15}$$

From Lemma 2.3, for $p\in(1,\infty)$, we can get that

$$\begin{aligned}
&\|v\|_{W^{1,p}(\Omega)}\leqslant C\big\|P(m,n)\big\|_{L^p(\Omega)},\\
&\|v\|_{W^{2,p}(\Omega)}\leqslant C\big\|\nabla P(m,n)\big\|_{L^p(\Omega)}.
\end{aligned}\tag{3.16}$$

By using Eqs. (1.1), we find w satisfies

$$\begin{cases}m\partial_t w-\mu\Delta w-(\lambda+\mu)\nabla\operatorname{div}w=mF & \text{in }\Omega\times(0,T),\\ w(x,t)=0 \quad \text{on }\partial\Omega\times[0,T),\end{cases}\tag{3.17}$$

with $w(x,0):=w_0(x)=u_0(x)-v_0(x)$ and

$$\begin{aligned}
F&=-u\cdot\nabla u-L^{-1}\nabla\big(\partial_t P(m,n)\big)=-u\cdot\nabla u+L^{-1}\nabla\operatorname{div}\big[P(m,n)u\big]\\
&\quad+L^{-1}\nabla\big[\big(P_m m+P_n n-P(m,n)\big)\operatorname{div}u\big].
\end{aligned}$$

Lemma 3.2. *Under the condition* (3.7)*, we have*

$$P(m,n)\leqslant C,\qquad P_m(m,n)\leqslant C,\qquad P_n(m,n)\leqslant C.$$

Proof. From (3.3) and (1.9), we know that the bounds of $P(m,n)$ and $P_m(m,n)$ are obvious. So we only need to give the bound of $P_n(m,n)$. At first, we estimate the term $\frac{m+a_0n+k_0}{\sqrt{b^2+c}}$ in the expression of P_n as follows:

$$\left\{\frac{m+a_0n+k_0}{\sqrt{b^2+c}}\right\}^2 = \frac{k_0^2+(m+a_0n)^2+2k_0m+2k_0a_0n}{k_0^2+(m+a_0n)^2-2k_0m+2k_0a_0n}$$
$$= \frac{(k_0+m)^2+a_0^2s_0^2m^2+2a_0s_0m^2+2k_0a_0s_0m}{(k_0-m)^2+a_0^2s_0^2m^2+2a_0s_0m^2+2k_0a_0s_0m} =: I.$$

When $M < \frac{k_0}{2}$, which implies $m \leqslant M < \frac{k_0}{2}$, then

$$I \leqslant \frac{(k_0+M)^2+a_0^2s_0^2M^2+2a_0s_0M^2+2k_0a_0s_0M}{(k_0-M)^2}$$
$$\leqslant \frac{(k_0+M)^2+a_0^2\bar{s}_0^2M^2+2a_0\bar{s}_0M^2+2k_0a_0\bar{s}_0M}{(k_0-M)^2} =: C_1.$$

When $M \geqslant \frac{k_0}{2}$: Case 1: $m \leqslant \frac{k_0}{2} \leqslant M$, then

$$I \leqslant \frac{9k_0^2+a_0^2s_0^2k_0^2+2a_0s_0k_0^2+4a_0s_0k_0^2}{k_0^2} \leqslant \frac{9k_0^2+a_0^2\bar{s}_0^2k_0^2+2a_0\bar{s}_0k_0^2+4a_0\bar{s}_0k_0^2}{k_0^2} =: C_2.$$

Case 2: $\frac{k_0}{2} \leqslant m \leqslant M$, then

$$I \leqslant 1 + \frac{4k_0}{a_0^2s_0^2m+2a_0s_0m+2k_0a_0s_0} \leqslant 1 + \frac{8k_0}{a_0^2\underline{s}_0^2k_0+2a_0\underline{s}_0k_0+4k_0a_0\underline{s}_0} =: C_3.$$

Then we obtain

$$\frac{m+a_0n+k_0}{\sqrt{b^2+c}} \leqslant \min\{\sqrt{C_1}, \sqrt{C_2}, \sqrt{C_3}\},$$

which implies

$$P_n = C^0\left\{a_0 + \frac{a_0}{\sqrt{b^2+c}}(m+a_0n+k_0)\right\} \leqslant C. \qquad \square$$

Then, using the similar arguments as those in [11,12], we can get the regularity estimates about w, where we have used the estimates in Proposition 3.1 and Proposition 3.2.

Proposition 3.3. *Under the condition* (3.7)*, we have*

$$\|\nabla w\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \qquad \left\|m^{\frac{1}{2}}\partial_t w\right\|_{L^2(0,T;L^2(\Omega))} \leqslant C,$$
$$\left\|\nabla^2 w\right\|_{L^2(0,T;L^2(\Omega))} \leqslant C, \quad 0 \leqslant T < T_1^*. \tag{3.18}$$

From Proposition 3.3, (3.16), Sobolev embedding theorem, Poincaré inequality and Lemmas 2.1–2.2, we can get the following regularity estimates about u.

Corollaty 3.1. *Under the condition* (3.7), *we have for any* $p \in (1,\infty)$ *that*

$$\|\nabla u\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \qquad \|u\|_{L^\infty(0,T;L^p(\Omega))} \leqslant C, \qquad \|\nabla u\|_{L^2(0,T;L^p(\Omega))} \leqslant C, \quad 0 \leqslant T < T_1^*.$$

Next, we give high order regularity estimates of w, the proof of which are due to [5,11] for the single-phase Navier–Stokes equations.

Proposition 3.4. *Under the condition* (3.7), *we have for any* $2 < p < \infty$ *that*

$$\|\nabla w\|_{L^2(0,T;W^{1,p}(\Omega))} \leqslant C, \quad 0 \leqslant T < T_1^*. \tag{3.19}$$

Proof. We rewrite Eq. $(1.1)_3$ as

$$m\dot{u} + \nabla P(m,n) - Lu = 0, \quad \text{i.e.,} \quad m\dot{u}^i + \partial_i P(m,n) - \mu\Delta u^i - (\lambda+\mu)\partial_i \operatorname{div} u = 0,$$

where we define the material derivative $\frac{D}{Dt}$ by $\frac{Dg}{Dt} = \dot{g} = g_t + u\cdot\nabla g$ for function $g(x,t)$. Taking the material derivative to the above equation and using the fact $\dot{f} = f_t + \operatorname{div}(fu) - f\operatorname{div} u$, we have

$$\begin{aligned} m\dot{u}_t^i + mu^j\partial_j\dot{u}^i + \partial_i P_t + \partial_j\big(\partial_i P u^j\big) &= \mu\big[\Delta u_t^i + \partial_j\big(\Delta u^i u^j\big)\big] \\ &\quad + (\lambda+\mu)\big[\partial_i \operatorname{div} u_t + \partial_j\big((\partial_i \operatorname{div} u)u^j\big)\big]. \end{aligned} \tag{3.20}$$

Multiplying (3.20) by $\dot{u}^i$ and integrating the resulting equation over Ω, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \frac{1}{2}m|\dot{u}|^2\,dx - \mu\int_\Omega \big(\Delta u_t^i + \partial_j\big(\Delta u^i u^j\big)\big)\dot{u}^i\,dx - (\lambda+\mu)\int_\Omega \big(\partial_i \operatorname{div} u_t + \partial_j\big((\partial_i \operatorname{div} u)u^j\big)\big)\dot{u}^i\,dx \\ &\quad = \int_\Omega \big(P_t \operatorname{div}\dot{u} + \partial_i P u^j \partial_j \dot{u}^i\big)\,dx. \end{aligned} \tag{3.21}$$

By using the integration by parts and boundary conditions (1.3), the μ-term, $(\lambda+\mu)$-term and the right-hand side of (3.21) can be estimated as follows:

$$\begin{aligned} &-\mu\int_\Omega \big(\Delta u_t^i + \partial_j\big(\Delta u^i u^j\big)\big)\dot{u}^i\,dx \\ &\quad = \mu\int_\Omega \big(\partial_j u_t^i \partial_j \dot{u}^i + \Delta u^i u^j \partial_j \dot{u}^i\big)\,dx \\ &\quad = \mu\int_\Omega \big[\partial_j\big(\dot{u}^i - u^k\partial_k u^i\big)\partial_j\dot{u}^i + \Delta u^i u^j \partial_j \dot{u}^i\big]\,dx \\ &\quad = \mu\int_\Omega \big[|\nabla\dot{u}|^2 - \partial_j u^k \partial_k u^i \partial_j \dot{u}^i - u^k\partial_k\partial_j u^i \partial_j \dot{u}^i - \partial_k u^i \partial_k\big(u^j\partial_j\dot{u}^i\big)\big]\,dx \\ &\quad = \mu\int_\Omega \big[|\nabla\dot{u}|^2 - \partial_j u^k \partial_k u^i \partial_j \dot{u}^i + \operatorname{div} u\,\partial_j u^i \partial_j \dot{u}^i + u^k\partial_j u^i \partial_k\partial_j \dot{u}^i - \partial_k u^i \partial_k\big(u^j\partial_j\dot{u}^i\big)\big]\,dx \end{aligned}$$

$$
\begin{aligned}
&= \mu \int_\Omega \left[|\nabla \dot u|^2 - \partial_j u^k \partial_k u^i \partial_j \dot u^i + \operatorname{div} u \, \partial_j u^i \partial_j \dot u^i - \partial_k u^i \partial_k u^j \partial_j \dot u^i \right] dx \\
&\geqslant \frac{3\mu}{4} \int_\Omega |\nabla \dot u|^2 \, dx - C \int_\Omega |\nabla u|^4 \, dx, \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
-(\lambda+\mu) &\int_\Omega \left(\partial_i \operatorname{div} u_t + \partial_j \left((\partial_i \operatorname{div} u) u^j \right) \right) \dot u^i \, dx \\
&= (\lambda+\mu) \int_\Omega \left[\operatorname{div} \dot u \operatorname{div} u_t + \operatorname{div} \dot u (u \cdot \nabla \operatorname{div} u) - \operatorname{div} u \, \partial_i u^j \partial_j \dot u^i + \operatorname{div} \dot u (\operatorname{div} u)^2 \right] dx \\
&= (\lambda+\mu) \int_\Omega \left[|\operatorname{div} \dot u|^2 - \operatorname{div} \dot u \, \partial_i u^j \partial_j u^i - \operatorname{div} u \, \partial_i u^j \partial_j \dot u^i + \operatorname{div} \dot u (\operatorname{div} u)^2 \right] dx \\
&\geqslant \frac{\lambda+\mu}{2} \int_\Omega |\operatorname{div} \dot u|^2 \, dx - \varepsilon(\lambda+\mu) \int_\Omega |\nabla \dot u|^2 \, dx - \frac{C}{\varepsilon}(\lambda+\mu) \int_\Omega |\nabla u|^4 \, dx \\
&\quad - C(\lambda+\mu) \int_\Omega |\nabla u|^4 \, dx, \qquad (3.23)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega \left(P_t \operatorname{div} \dot u + \partial_i P u^j \partial_j \dot u^i \right) dx \\
&\quad = \int_\Omega \left[(P_m m_t + P_n n_t) \operatorname{div} \dot u + \partial_i P u^j \partial_j \dot u^i \right] dx \\
&\quad = \int_\Omega \left[(-m P_m - n P_n) \operatorname{div} u \operatorname{div} \dot u - u \cdot \nabla P(m,n) \operatorname{div} \dot u + \partial_i P u^j \partial_j \dot u^i \right] \\
&\quad = \int_\Omega \left[(-m P_m - n P_n) \operatorname{div} u \operatorname{div} \dot u + P \operatorname{div}(u \operatorname{div} \dot u) - P \operatorname{div}(u \cdot \nabla \dot u) \right] \\
&\quad = \int_\Omega \left[(-m P_m - n P_n) \operatorname{div} u \operatorname{div} \dot u + P \left(\operatorname{div} u \operatorname{div} \dot u - \partial_i u^j \partial_j \dot u^i \right) \right] \\
&\quad \leqslant C \|\nabla u\|_{L^2(\Omega)} \|\nabla \dot u\|_{L^2(\Omega)} \leqslant C \|\nabla \dot u\|_{L^2(\Omega)} \leqslant C + \frac{\mu}{4} \|\nabla \dot u\|^2_{L^2(\Omega)}. \qquad (3.24)
\end{aligned}
$$

Substituting (3.22)–(3.24) into (3.21) and choosing $\varepsilon = \frac{\mu}{8(\lambda+\mu)}$, if $\lambda + \mu > 0$, we have

$$
\frac{d}{dt} \int_\Omega m |\dot u|^2 \, dx + \mu \int_\Omega |\nabla \dot u|^2 \, dx + (\lambda+\mu) \int_\Omega |\operatorname{div} \dot u|^2 \, dx \leqslant C \int_\Omega |\nabla u|^4 \, dx + C.
$$

If $\lambda + \mu = 0$, there is no need to estimate the $(\lambda+\mu)$-term, we still have

$$\frac{d}{dt}\int_\Omega m|\dot{u}|^2\,dx+\mu\int_\Omega|\nabla\dot{u}|^2\,dx\leqslant C\int_\Omega|\nabla u|^4\,dx+C.$$

Then, for all $\mu>0$ and $\lambda+\mu\geqslant 0$, we have

$$\frac{d}{dt}\int_\Omega m|\dot{u}|^2\,dx+\mu\int_\Omega|\nabla\dot{u}|^2\,dx\leqslant C\int_\Omega|\nabla u|^4\,dx+C. \tag{3.25}$$

In the following, we estimate the term $\int_\Omega|\nabla u|^4\,dx$. From Eqs. $(1.1)_3$ and (3.15), we know that w satisfies

$$\begin{cases}\mu\Delta w+(\lambda+\mu)\nabla\operatorname{div}w=m\dot{u} & \text{in }\Omega,\\ w(x)=0 & \text{on }\partial\Omega.\end{cases} \tag{3.26}$$

From Lemma 2.3, we get

$$\|w\|_{H^2(\Omega)}\leqslant C\|m\dot{u}\|_{L^2(\Omega)}\leqslant C\|\sqrt{m}\dot{u}\|_{L^2(\Omega)},$$

which together with the interpolation inequality, Sobolev embedding theorem, (3.16) and Corollary 3.1 yield

$$\begin{aligned}\|\nabla u\|^4_{L^4(\Omega)}\leqslant\|\nabla u\|_{L^2(\Omega)}\|\nabla u\|^3_{L^6(\Omega)}&\leqslant C\|\nabla u\|_{L^6(\Omega)}\|\nabla u\|^2_{L^6(\Omega)}\\ &\leqslant C\|\nabla u\|^2_{L^6(\Omega)}\big(\|\nabla w\|_{L^6(\Omega)}+\|\nabla v\|_{L^6(\Omega)}\big)\\ &\leqslant C\|\nabla u\|^2_{L^6(\Omega)}\big(1+\|\nabla w\|_{H^1(\Omega)}\big)\\ &\leqslant C\|\nabla u\|^2_{L^6(\Omega)}\big(1+\|\nabla^2 w\|_{L^2(\Omega)}\big)\\ &\leqslant C\|\nabla u\|^2_{L^6(\Omega)}\big(1+\|\sqrt{m}\dot{u}\|_{L^2(\Omega)}\big).\end{aligned} \tag{3.27}$$

Substituting (3.27) into (3.25) and noticing $\|\nabla u\|^2_{L^6(\Omega)}\in L^1(0,T)$, which was shown in Corollary 3.1, we get by Gronwall's inequality that

$$\int_\Omega m|\dot{u}|^2\,dx+\int_0^T\int_\Omega|\nabla\dot{u}|^2\,dx\,dt\leqslant C. \tag{3.28}$$

From (3.28), (3.26), Sobolev embedding theorem and Poincaré inequality, we can get the following high regularity estimates about w:

$$\|\nabla w\|_{L^2(0,T;W^{1,p}(\Omega))}\leqslant C, \tag{3.29}$$

and this completes the proof of Proposition 3.4. □

In the following, we give the estimates of derivative of the liquid and gas masses.

Proposition 3.5. *Under the condition* (3.7)*, we have for $q\in(2,\infty)$ that*

$$\sup_{t\in[0,T]}\big\|(\nabla m,\nabla n)(t)\big\|_{L^q(\Omega)}\leqslant C,\quad 0\leqslant T<T_1^*. \tag{3.30}$$

Proof. Differentiating Eq. $(1.1)_1$ with respect to x_i, then multiplying both sides of the resulting equation by $q|\partial_i m|^{q-2}\partial_i m$, we get

$$\partial_t|\partial_i m|^q + \operatorname{div}\big(|\partial_i m|^q u\big) + (q-1)|\partial_i m|^q \operatorname{div} u + qm|\partial_i m|^{q-2}\partial_i m\partial_i \operatorname{div} u \\ + q|\partial_i m|^{q-2}\partial_i m\partial_i u\cdot\nabla m = 0. \tag{3.31}$$

Integrating the above equality over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla m|^q\,dx &\leqslant C\int_\Omega |\nabla u||\nabla m|^q\,dx + q\int_\Omega m|\nabla \operatorname{div} u||\nabla m|^{q-1}\,dx \\ &\leqslant C\|\nabla u\|_{L^\infty(\Omega)}\|\nabla m\|_{L^q(\Omega)}^q + C\big\|\nabla^2 u\big\|_{L^q(\Omega)}\|\nabla m\|_{L^q(\Omega)}^{q-1}.\end{aligned} \tag{3.32}$$

Similarly,

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla n|^q\,dx &\leqslant C\int_\Omega |\nabla u||\nabla n|^q\,dx + q\int_\Omega n|\nabla \operatorname{div} u||\nabla n|^{q-1}\,dx \\ &\leqslant C\|\nabla u\|_{L^\infty(\Omega)}\|\nabla n\|_{L^q(\Omega)}^q + C\big\|\nabla^2 u\big\|_{L^q(\Omega)}\|\nabla n\|_{L^q(\Omega)}^{q-1}.\end{aligned} \tag{3.33}$$

Applying Lemma 2.3 to (3.15), we obtain

$$\big\|\nabla^2 v\big\|_{L^q(\Omega)} \leqslant C\big(\|\nabla m\|_{L^q(\Omega)} + \|\nabla n\|_{L^q(\Omega)}\big), \tag{3.34}$$

then, by using Lemma 2.4 and Lemma 2.5, we get

$$\begin{aligned}\|\nabla v\|_{L^\infty(\Omega)} &\leqslant C\big(1 + \|\nabla v\|_{BMO(\Omega)}\ln\big(e + \big\|\nabla^2 v\big\|_{L^q(\Omega)}\big)\big) \\ &\leqslant C\big(1 + \|P\|_{L^\infty\cap L^2(\Omega)}\ln\big(e + \|\nabla P\|_{L^q(\Omega)}\big)\big) \\ &\leqslant C\big(1 + \ln\big(e + \|\nabla m\|_{L^q(\Omega)} + \|\nabla n\|_{L^q(\Omega)}\big)\big),\end{aligned} \tag{3.35}$$

where we have used Lemma 3.2 in the above two estimates.

From (3.33)–(3.35), we get

$$\begin{aligned}&\frac{d}{dt}\big(\|\nabla m\|_{L^q(\Omega)} + \|\nabla n\|_{L^q(\Omega)}\big) \\ &\quad\leqslant C\big(1 + \|\nabla w\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)}\big)\big(\|\nabla m\|_{L^q(\Omega)} + \|\nabla n\|_{L^q(\Omega)}\big) + C\big\|\nabla^2 w\big\|_{L^q(\Omega)} \\ &\quad\leqslant C\big(1 + \|\nabla w\|_{W^{1,q}(\Omega)} + \ln\big(e + \|\nabla m\|_{L^q(\Omega)} + \|\nabla n\|_{L^q(\Omega)}\big)\big)\big(\|\nabla m\|_{L^q(\Omega)} + \|\nabla n\|_{L^q(\Omega)}\big) \\ &\qquad + C\big\|\nabla^2 w\big\|_{L^q(\Omega)}.\end{aligned} \tag{3.36}$$

Note that $\|\nabla w\|_{W^{1,q}(\Omega)} \in L^2(0,T)$ by (3.19). Then by Gronwall's inequality, we obtain (3.30). This completes the proof of Proposition 3.5. □

From (3.7), (3.28) and Proposition 3.5, we can obtain the bound of $\|\nabla^2 u\|_{L^2(\Omega)}$.

Corollaty 3.2. *Under the condition* (3.7), *we have*

$$\|u\|_{L^\infty(0,T;H^2(\Omega))} \leqslant C, \quad 0 \leqslant T < T_1^*.$$

Proof. We rewrite Eq. $(1.1)_3$ as

$$\mu \Delta u + (\lambda + \mu)\nabla \operatorname{div} u = m\dot{u} + \nabla P(m, n),$$

then by Lemma 2.3, we have

$$\begin{aligned}\|u\|_{H^2(\Omega)} &\leqslant C\big(\|m\dot{u}\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)}\big)\\ &\leqslant C\left(\big\|m^{\frac{1}{2}}\dot{u}\big\|_{L^2(\Omega)} + \|\nabla m\|_{L^2(\Omega)} + \|\nabla n\|_{L^2(\Omega)}\right) \leqslant C. \qquad \square\end{aligned}$$

Finally, with the help of the above regularity estimates for w, ∇m and ∇n, we can give the lower bound estimates of the liquid mass m and gas mass n, since (3.3), we only need to get the lower bound of m.

Proposition 3.6. *Under the condition* (3.7), *we have*

$$m(x,t) \geqslant C, \qquad n(x,t) \geqslant C, \quad t \in [0,T],\ 0 \leqslant T < T_1^*. \tag{3.37}$$

Proof. From Proposition 3.4, we get

$$\|\nabla w\|_{L^2(0,T;L^\infty(\Omega))} \leqslant C. \tag{3.38}$$

From (3.16) and Proposition 3.5, we obtain

$$\begin{aligned}\|\nabla v\|_{L^2(0,T;L^\infty(\Omega))} &\leqslant C\|\nabla v\|_{L^2(0,T;W^{1,p}(\Omega))} \leqslant C\|\nabla P\|_{L^2(0,T;L^p(\Omega))}\\ &\leqslant C\big\{\|\nabla m\|_{L^2(0,T;L^p(\Omega))} + \|\nabla n\|_{L^2(0,T;L^p(\Omega))}\big\} \leqslant C,\end{aligned} \tag{3.39}$$

where $2 < p < \infty$. Then (3.38) and (3.39) imply

$$\begin{aligned}\|\nabla u\|_{L^1(0,T;L^\infty(\Omega))} &\leqslant C\|\nabla u\|_{L^2(0,T;L^\infty(\Omega))}\\ &\leqslant C\big(\|\nabla w\|_{L^2(0,T;L^\infty(\Omega))} + \|\nabla v\|_{L^2(0,T;L^\infty(\Omega))}\big) \leqslant C.\end{aligned} \tag{3.40}$$

Along the particle trajectories $x = X(t, y)$ defined by (3.4), we differentiate Eq. $(1.1)_1$ with respect to t, and get

$$\frac{dm}{dt}\big(X(t,y),t\big) = -m \operatorname{div} u\big(X(t,y),t\big),$$

which implies

$$m\big(X(t,y),t\big) = m_0(y)\exp\left\{-\int_0^t \operatorname{div} u\big(X(\tau,y),\tau\big)\,d\tau\right\},$$

then we have

$$m(x,t) \geqslant \inf_{y\in\Omega} m_0 \exp\left\{-\int_0^T \left\|\nabla u(t)\right\|_{L^\infty(\Omega)}\,dt\right\} \geqslant C,$$

and this completes the proof of Proposition 3.6. □

From Proposition 3.6 and the classical continuation method, we have

$$T_1^* = T^*. \tag{3.41}$$

4. Proof of Theorem 1.2

The estimates in Corollaries 3.1–3.2, Propositions 3.5–3.6 will be enough to extend the strong solution (m,n,u) beyond $t \geqslant T_1^* = T^*$.

In fact, in view of Corollaries 3.1–3.2 and Propositions 3.5–3.6, the functions $(m,n,u)|_{t=T^*} = \lim_{t\to T^*}(m,n,u)$ satisfy the conditions imposed on the initial data (1.6) at the time $t = T^*$. Therefore, we can take $(m,n,u)|_{t=T^*}$ as the initial data and apply the local existence theorem (Theorem 1.1) to extend the local strong solution beyond T^*. This contradicts the assumption on T^*, and it completes the proof of Theorem 1.2.

We can use the similar arguments in the present paper and [11] to deal with the 3D case. The corresponding result is given as follows:

Theorem 4.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^3$, $q \in (3,\infty)$, and (1.4) is replaced by $\mu > 0$, $3\lambda + 2\mu \geqslant 0$. Assume that the initial data m_0, n_0, u_0 satisfy*

$$0 < \underline{m}_1 \leqslant \inf_x m_0 \leqslant \sup_x m_0 \leqslant \overline{m}_1 < \infty, \qquad 0 < \underline{n}_1 \leqslant \inf_x n_0 \leqslant \sup_x n_0 \leqslant \bar{n}_1 < \infty,$$
$$m_0, n_0 \in W^{1,q}(\Omega), \qquad u_0 \in H_0^1(\Omega) \cap H^2(\Omega). \tag{4.1}$$

Then, there exist a $T_2 > 0$ and a unique strong solution $(m,n,u)(x,t)$ to the problem (1.1)–(1.3), such that

$$m,n > 0, \quad m,n \in C\big([0,T_2], W^{1,q_0}(\Omega)\big), \qquad m_t, n_t \in C\big([0,T_2], L^{q_0}(\Omega)\big),$$
$$u \in C\big([0,T_2], H_0^1(\Omega) \cap H^2(\Omega)\big) \cap L^2\big(0,T_2; W^{2,q}(\Omega)\big),$$
$$u_t \in L^\infty\big(0,T_2; L^2(\Omega)\big) \cap L^2\big(0,T_2; H_0^1(\Omega)\big),$$

where $q_0 = \min(6,q)$. Furthermore, under the additional assumption $\lambda < 7\mu$, we have the following blow-up criterion: If $T^* < \infty$ is the maximal existence time for strong solution $(m,n,u)(x,t)$ to the problem (1.1)–(1.3), then

$$\limsup_{T\to T^*} \|m\|_{L^\infty(0,T;L^\infty(\Omega))} = \infty. \tag{4.2}$$

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