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Liftings and quasi-liftings of DG modules

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ABSTRACT

We prove lifting results for DG modules that are akin to Auslander, Ding, and Solberg's famous lifting results for modules.

Introduction

Convention. Throughout this paper, let R be a commutative noetherian ring.

Hochster famously wrote that "life is really worth living" in a Cohen–Macaulay ring [7].[2] For instance, if R is Cohen–Macaulay and local with maximal regular sequence $\underline{t}$, then $R/(\underline{t})$ is artinian and

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2 We know of this quote from [5].

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the natural epimorphism $R \to R/(\underline{t})$ is nice enough to allow for transfer of properties between the two rings. Thus, if one can prove a result for artinian local rings, then one can (often) prove a similar result for Cohen–Macaulay local rings by showing that the desired conclusion descends from $R/(\underline{t})$ to R. When R is complete, then this is aided sometimes by the lifting result of Auslander, Ding, and Solberg [2, Propositions 1.7 and 2.6].

Theorem. *Let $\underline{t} \in R$ be an R-regular sequence, and let M be a finitely generated $R/(\underline{t})$-module. Assume that R is local and $(\underline{t})$-adically complete.*

(a) *If $\operatorname{Ext}^2_{R/(\underline{t})}(M, M) = 0$, then M is "liftable" to R, that is, there is a finitely generated R-module N such that $R/(\underline{t}) \otimes_R N \cong M$ and $\operatorname{Tor}_i^R(R/(\underline{t}), N) = 0$ for all $i \geqslant 1$.*
(b) *If $\operatorname{Ext}^1_{R/(\underline{t})}(M, M) = 0$, then M has at most one lift to R.*

In this paper, we are concerned with what happens when the sequence $\underline{t}$ is not R-regular. One would like a similar mechanism for reducing questions about arbitrary local rings to the artinian case.

It is well known that the map $R \to R/(\underline{t})$ is not nice enough in general to guarantee good descent/lifting behavior. Our perspective[3] in this matter is that this is not the right map to consider in general: the correct one is the natural map from R to the Koszul complex $K = K^R(\underline{t})$. This perspective requires one to make some adjustments. For instance, K is a differential graded R-algebra, so not a commutative ring in the traditional sense. This may cause some consternation, but the payoff can be handsome. For instance, in [8] we use this perspective to answer a question of Vasconcelos [9]. One of the tools for the proof of this result is the following version of Auslander, Ding, and Solberg's lifting result. Note that we do not assume that R is local in part (a) of this result.

Main Theorem. *Let $\underline{t} = t_1, \ldots, t_n$ be a sequence of elements of R, and assume that R is $\underline{t}R$-adically complete. Let D be a DG $K^R(\underline{t})$-module that is homologically bounded below and homologically degreewise finite.*

(a) *If $\operatorname{Ext}^2_{K^R(\underline{t})}(D, D) = 0$, then D is quasi-liftable to R, that is, there is a semi-free R-complex D' such that $D \simeq K^R(\underline{t}) \otimes_R D'$.*
(b) *Assume that R is local. If D is quasi-liftable to R and $\operatorname{Ext}^1_{K^R(\underline{t})}(D, D) = 0$, then any two homologically degreewise finite quasi-lifts of D to R are quasiisomorphic over R.*

This result is proved in Corollaries 3.7 and 3.12, which follow from more general results on liftings along morphisms of DG algebras. Note that it is similar to, but quite different from, some results of Yoshino [10].

We briefly describe the contents of the paper. Section 1 contains some background material on DG algebras and DG modules. Section 2 contains some structural results for DG modules and homomorphisms between them. Finally, Section 3 is where we prove our Main Theorem.

1. DG modules

We assume that the reader is familiar with the category of R-complexes. For clarity, we include a few definitions.

Definition 1.1. In this paper, complexes of R-modules ("R-complexes" for short) are indexed homologically:

$$M = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} \cdots.$$

[3] This perspective is not original to our work. We learned of it from Avramov and Iyengar.

The degree of an element $m \in M$ is denoted $|m|$. The *tensor product* of two R-complexes M, N is denoted $M \otimes_R N$, and the *Hom complex* is denoted $\mathrm{Hom}_R(M, N)$. A *chain map* $M \to N$ is a cycle of degree 0 in $\mathrm{Hom}_R(M, N)$. An R-complex M is *homologically bounded below* if $\mathrm{H}_i(M) = 0$ for $i \gg 0$; it is *bounded below* if $M_i = 0$ for $i \gg 0$.

Next, we begin our background material on DG algebras; see [1,3,4].

Definition 1.2. A *commutative differential graded R-algebra* (*DG R-algebra* for short) is an R-complex A equipped with a chain map $\mu^A : A \otimes_R A \to A$ with $ab := \mu^A(a \otimes b)$ that is:

associative: for all $a, b, c \in A$ we have $(ab)c = a(bc)$;
unital: there is an element $1 \in A_0$ such that for all $a \in A$ we have $1a = a$;
graded commutative: for all $a, b \in A$ we have $ab = (-1)^{|a||b|}ba$ and $a^2 = 0$ when $|a|$ is odd; and
positively graded: $A_i = 0$ for $i < 0$.

The map μ^A is the *product* on A. Given a DG R-algebra A, the *underlying algebra* is the graded commutative R-algebra $A^\natural = \bigoplus_{i=0}^{\infty} A_i$.

A *morphism* of DG R-algebras is a chain map $f : A \to B$ between DG R-algebras respecting products and multiplicative identities: $f(aa') = f(a)f(a')$ and $f(1) = 1$.

Example 1.3. The ring R, considered as a complex concentrated in degree 0, is a DG R-algebra. Given a DG R-algebra A, the map $R \to A$ given by $r \mapsto r \cdot 1$ is a morphism of DG R-algebras.

Fact 1.4. Let A be a DG R-algebra. The fact that the product on A is a chain map says that ∂^A satisfies the *Leibniz rule*:

$$\partial^A_{|a|+|b|}(ab) = \partial^A_{|a|}(a)b + (-1)^{|a|}a\partial^A_{|b|}(b).$$

It is straightforward to show that the R-module A_0 is an R-algebra. Moreover, the natural map $A_0 \to A$ is a morphism of DG R-algebras. The condition $A_{-1} = 0$ implies that A_0 surjects onto $\mathrm{H}_0(A)$ and that $\mathrm{H}_0(A)$ is an A_0-algebra. Furthermore, the R-module A_i is an A_0-module, and $\mathrm{H}_i(A)$ is an $\mathrm{H}_0(A)$-module for each i.

Given a second DG R-algebra K, the tensor product $K \otimes_R A$ is also a DG R-algebra with multiplication $(x \otimes a)(x' \otimes a') := (-1)^{|a||x'|}(xx') \otimes (aa')$.

Definition 1.5. Let A be a DG R-algebra. We say that A is *noetherian* if $\mathrm{H}_0(A)$ is noetherian and the $\mathrm{H}_0(A)$-module $\mathrm{H}_i(A)$ is finitely generated for all $i \geqslant 0$. When $(R, \mathfrak{m})$ is local, we say that A is *local* if it is noetherian and the ring $\mathrm{H}_0(A)$ is a local R-algebra[4] with maximal ideal $\mathfrak{m}_{\mathrm{H}_0(A)}$

Fact 1.6. Assume that $(R, \mathfrak{m})$ is local, and let A be a local DG R-algebra. The composition $A \to \mathrm{H}_0(A) \to \mathrm{H}_0(A)/\mathfrak{m}_{\mathrm{H}_0(A)}$ is a surjective morphism of DG R-algebras with kernel of the form $\mathfrak{m}_A = \cdots \xrightarrow{\partial^A_2} A_1 \xrightarrow{\partial^A_1} \mathfrak{m}_0 \to 0$ for some maximal ideal $\mathfrak{m}_0 \subsetneq A_0$. The quotient complex $A/\mathfrak{m}_A$ is A-isomorphic to $\mathrm{H}_0(A)/\mathfrak{m}_{\mathrm{H}_0(A)}$. Since $\mathrm{H}_0(A)$ is a local R-algebra, we have $\mathfrak{m}A_0 \subseteq \mathfrak{m}_0$.

Definition 1.7. If R is local and A is a local DG R-algebra, then the subcomplex $\mathfrak{m}_A$ is the *augmentation ideal* of A.

The following is a key example for this investigation.

[4] This means that $\mathrm{H}_0(A)$ is a local ring whose maximal ideal contains the ideal $\mathfrak{m}\mathrm{H}_0(A)$.

Example 1.8. Given a sequence $\underline{t} = t_1, \dots, t_n \in R$, the Koszul complex $K = K^R(\underline{t})$ is a DG R-algebra with product given by the wedge product. If $(R, \mathfrak{m})$ is local and $\underline{t} \in \mathfrak{m}$, then K is a local DG R-algebra with augmentation ideal $\mathfrak{m}_K = (0 \to R \to \cdots \to R^n \to \mathfrak{m} \to 0)$.

Definition 1.9. Let A be a DG R-algebra. A *DG A-module* is an R-complex M with a chain map $\mu^M : A \otimes_R M \to M$ such that the rule $am := \mu^M(a \otimes m)$ is associative and unital. The map μ^M is the *scalar multiplication* on M. A *morphism* of DG A-modules is a chain map $f : M \to N$ between DG A-modules that respects scalar multiplication: $f(am) = af(m)$. Isomorphisms in the category of DG A-modules are identified by the symbol $\cong$. Quasiisomorphisms in the category of DG A-modules are identified by the symbol $\simeq$; these are the morphisms that induce bijections on all homology modules. Two DG A-modules M and N are *quasiisomorphic*, written $M \simeq N$ if there is a finite sequence of quasiisomorphisms $M \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} N$.

Example 1.10. Consider the ring R as a DG R-algebra. A DG R-module is just an R-complex, and a morphism of DG R-modules is simply a chain map.

Fact 1.11. Let A be a DG R-algebra, and let M be a DG A-module. The fact that the scalar multiplication on M is a chain map says that ∂^M satisfies the *Leibniz rule*: $\partial^A_{|a|+|m|}(am) = \partial^A_{|a|}(a)m + (-1)^{|a|}a\partial^M_{|m|}(m)$. The R-module M_i is an A_0-module, and $\mathrm{H}_i(M)$ is an $\mathrm{H}_0(A)$-module for each i.

Definition 1.12. Let A be a DG R-algebra, and let i be an integer. The ith *suspension* of a DG A-module M is the DG A-module $\Sigma^i M$ defined by $(\Sigma^i M)_n := M_{n-i}$ and $\partial_n^{\Sigma^i M} := (-1)^i \partial^M_{n-i}$. The scalar multiplication on $\Sigma^i M$ is defined by the formula $\mu^{\Sigma^i M}(a \otimes m) := (-1)^{i|a|}\mu^M(a \otimes m)$.

Definition 1.13. Let A be a DG R-algebra. A DG A-module M is *homologically degreewise finite* if $\mathrm{H}_i(M)$ is finitely generated over $\mathrm{H}_0(A)$ for all i; it is *homologically finite* if it is homologically degreewise finite and $\mathrm{H}_i(M) = 0$ for $|i| \gg 0$.

Definition 1.14. Let A be a DG R-algebra, and let M, N be DG A-modules. The *tensor product* $M \otimes_A N$ is the quotient $(M \otimes_R N)/U$ where U is the subcomplex generated by all elements of the form $(am) \otimes n - (-1)^{|a||m|} m \otimes (an)$. Given an element $m \otimes n \in M \otimes_R N$, we denote the image in $M \otimes_A N$ as $m \otimes n$.

Fact 1.15. Let A be a DG R-algebra, and let M, N be DG A-modules. The tensor product $M \otimes_A N$ is a DG A-module via the action

$$a(m \otimes n) := (am) \otimes n = (-1)^{|a||m|} m \otimes (an).$$

Fact 1.16. Let $A \to B$ be a morphism of DG R-algebras. Given a DG A-module M, the "base changed" complex $B \otimes_A M$ has the structure of a DG B-module by the action $b(b' \otimes m) := (bb') \otimes m$. This structure is compatible with the DG A-module structure via restriction of scalars.

Definition 1.17. Let A be a DG R-algebra, and let M be a DG A-module. The *underlying $A^\natural$-module* associated to M is the $A^\natural$-module $M^\natural = \bigoplus_{i=-\infty}^{\infty} M_i$. A subset E of M is called a *semi-basis* if it is a basis of the underlying $A^\natural$-module $M^\natural$. If M is bounded below, then M is called *semi-free* if it has a semi-basis.[5] A *semi-free resolution* of a DG A-module N is a quasiisomorphism $F \xrightarrow{\simeq} N$ of DG A-modules such that F is semi-free.

Assume that R and A are local. A *minimal semi-free resolution* of M is a semi-free resolution $F \xrightarrow{\simeq} M$ such that F is *minimal*, i.e., each (equivalently, some) semi-basis of F is finite in each degree and the differential on $(A/\mathfrak{m}_A) \otimes_A F$ is 0.

[5] As is noted in [4], when M is not bounded below, the definition of "semi-free" is significantly more technical.

Fact 1.18. Let A be a DG R-algebra. Let M be a homologically bounded below DG A-module. Then M has a semi-free resolution over A by [4, Theorem 2.7.4.2].

Assume that A is noetherian, and let j be an integer. If $\mathrm{H}_i(M)$ is finitely generated over $\mathrm{H}_0(A)$ for all i, and $\mathrm{H}_i(M) = 0$ for $i < j$, then M has a semi-free resolution $F \xrightarrow{\simeq} M$ such that $F^\natural \cong \bigoplus_{i=j}^{\infty} \Sigma^i (A^\natural)^{\beta_i}$ with $\beta_i \in \mathbb{Z}$ for all i and $F_i = 0$ for all $i < j$; see [1, Proposition 1]. In particular, homologically finite DG A-modules admit such "degreewise finite, bounded below" semi-free resolutions. Note that the condition $F^\natural \cong \bigoplus_{i=j}^{\infty} \Sigma^i (A^\natural)^{\beta_i}$ says that the degree-i piece of the semi-basis $E_i = E \cap F_i$ is finite for each i, and $E_i = \emptyset$ for $i < j$.

Assume that R and A are local. If $\mathrm{H}_i(M)$ is finitely generated over $\mathrm{H}_0(A)$ for all i, and $\mathrm{H}_i(M) = 0$ for $i < j$, then M has a minimal semi-free resolution $F \xrightarrow{\simeq} M$ such that $F_i = 0$ for all $i < j$; see [1, Proposition 2]. In particular, homologically finite DG A-modules admit minimal semi-free resolutions.

Definition 1.19. Let A be a DG R-algebra, and let M, N be DG A-modules. Given an integer i, a *DG A-module homomorphism of degree i* is a homomorphism $f : M \to N$ of the underlying R-complexes such that $f(am) = (-1)^{i|a|} a f(m)$ for all $a \in A$ and $m \in M$. We write $|f| = i$. The (graded) submodule of $\mathrm{Hom}_R(M, N)$ consisting of all DG A-module homomorphisms $M \to N$ is denoted $\mathrm{Hom}_A(M, N)$. A homomorphism $f \in \mathrm{Hom}_A(M, N)_i$ is *null-homotopic* if it is in $\mathrm{Im}(\partial_{i+1}^{\mathrm{Hom}_A(M,N)})$. Two homomorphisms $M \to N$ are *homotopic* if their difference is null-homotopic.

Fact 1.20. Let A be a DG R-algebra, and let M, N be DG A-modules. The complex $\mathrm{Hom}_A(M, N)$ is a DG A-module via the action

$$(af)(m) := a\big(f(m)\big) = (-1)^{|a||f|} f(am).$$

Definition 1.21. Let $A \to B$ be a morphism of DG R-algebras, and let M, M' be DG A-modules. Given $f \in \mathrm{Hom}_A(M, M')_i$, define $B \otimes_A f : B \otimes_A M \to B \otimes_B M'$ by the formula $(B \otimes_A f)(b \otimes m) = (-1)^{i|b|} b \otimes f(m)$.

Fact 1.22. Let $A \to B$ be a morphism of DG R-algebras, and let M, M' be DG A-modules. Given a homomorphism $f \in \mathrm{Hom}_A(M, M')_i$, the function $B \otimes_A f$ is B-linear, that is, an element of $\mathrm{Hom}_B(B \otimes_A M, B \otimes_A M')_i$. Furthermore, if f is a cycle in $\mathrm{Hom}_A(M, M')_i$, then $B \otimes_A f$ is a cycle in $\mathrm{Hom}_B(B \otimes_A M, B \otimes_A M')_i$.

Definition 1.23. Let A be a DG R-algebra, and let M, N be DG A-modules. Given a semi-free resolution $F \xrightarrow{\simeq} M$, we set $\mathrm{Ext}_A^i(M, N) = \mathrm{H}_{-i}(\mathrm{Hom}_A(F, N))$ for each integer i.

Fact 1.24. Let A be a DG R-algebra, and let M, N be DG A-modules. For each i, the module $\mathrm{Ext}_A^i(M, N)$ is independent of the choice of semi-free resolution of M, and we have $\mathrm{Ext}_A^i(M, N) \simeq \mathrm{Ext}_A^i(M', N')$ whenever $M \simeq M'$ and $N \simeq N'$; see [3, Propositions 1.3.1–1.3.3]. Given a semi-free resolution $F \simeq M$ and an integer i, the elements of $\mathrm{Ext}_A^i(M, N)$ are by definition the homotopy equivalence classes of morphisms of DG A-modules $F \to \Sigma^{-i} N$.

2. Structure of semi-free DG modules and DG homomorphisms

The proof of our Main Theorem involves the manipulation of the differentials on certain DG modules to construct isomorphisms that are amenable to lifting. For this, we need a concrete understanding of these differentials and the homomorphisms between these DG modules. This concrete understanding is the goal of this section. We begin by establishing some notation to be used for much of the paper.

Notation 2.1. Let A be a DG R-algebra such that each A_i is free over R of finite rank. Given an element $t \in R$, let $K = K^R(t)$ denote the Koszul complex $0 \to K_1 \xrightarrow{t} K_0 \to 0$ with $K_1 \cong R \cong K_0$ and basis elements $1 \in K_0$ and $e \in K_1$. We fix a basis $\{\gamma_{i,1}, \ldots, \gamma_{i,r_i}\}$ for A_i. Let B denote the DG R-algebra $K^R(t) \otimes_R A$, which has the following form

$$B \cong \cdots \xrightarrow{\partial^B_{i+1}} A_{i-1} \oplus A_i \xrightarrow{\partial^B_i} A_{i-2} \oplus A_{i-1} \xrightarrow{\partial^B_{i-1}} \cdots \xrightarrow{\partial^B_2} A_0 \oplus A_1 \xrightarrow{\partial^B_1} 0 \oplus A_0 \to 0.$$

This uses the isomoprhism $B_i = (K_1 \otimes_R A_{i-1}) \oplus (K_0 \otimes_R A_i) \cong A_{i-1} \oplus A_i$. We identify B_i with $A_{i-1} \oplus A_i$ for the remainder of this paper. Under this identification, the sum $e \otimes a_{i-1} + 1 \otimes a_i \in B_i$ corresponds to the column vector $\left[\begin{smallmatrix} a_{i-1} \\ a_i \end{smallmatrix}\right] \in A_{i-1} \oplus A_i$. The use of column vectors allows us to identify the differential of B as the matrix

$$\partial^B_i = \begin{bmatrix} -\partial^A_{i-1} & 0 \\ t & \partial^A_i \end{bmatrix}.$$

Remark 2.2. In Notation 2.1, the algebra structure on B translates to the formula

$$\begin{bmatrix} a_{i-1} \\ a_i \end{bmatrix} \begin{bmatrix} c_{j-1} \\ c_j \end{bmatrix} = \begin{bmatrix} a_{i-1}c_j + (-1)^i a_i c_{j-1} \\ a_i c_j \end{bmatrix}$$

where $\left[\begin{smallmatrix} a_{i-1} \\ a_i \end{smallmatrix}\right] \in B_i$ and $\left[\begin{smallmatrix} c_{j-1} \\ c_j \end{smallmatrix}\right] \in B_j$. This uses the fact that $e^2 = 0$ in K.

Note that a basis of B_i is

$$\left\{ \begin{bmatrix} \gamma_{i-1,1} \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} \gamma_{i-1,r_{i-1}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \gamma_{i,1} \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \gamma_{i,r_i} \end{bmatrix} \right\}.$$

Also, note that the assumptions on A imply that A and B are noetherian. From the explicit description of ∂^B, it follows that $\mathrm{H}_0(B) \cong \mathrm{H}_0(A)/t\mathrm{H}_0(A)$.

Assume that R and A are local. Then B is also local. Moreover, given the augmentation ideal $\mathfrak{m}_A = \cdots \xrightarrow{\partial^A_2} A_1 \xrightarrow{\partial^A_1} \mathfrak{m}_0 \to 0$ it is straightforward to show that the augmentation ideal of B is

$$\mathfrak{m}_B = \cdots \xrightarrow{\partial^B_{i+1}} A_{i-1} \oplus A_i \xrightarrow{\partial^B_i} A_{i-2} \oplus A_{i-1} \xrightarrow{\partial^B_{i-1}} \cdots \xrightarrow{\partial^B_2} A_0 \oplus A_1 \xrightarrow{\partial^B_1} 0 \oplus \mathfrak{m}_0 \to 0$$

and we have $B/\mathfrak{m}_B \cong A/\mathfrak{m}_A$.

Notation 2.3. We work in the setting of Notation 2.1. Let $\{\beta_i\}_{i=-\infty}^{\infty}$ be a set of cardinal numbers such that $\beta_i = 0$ for $i \ll 0$. For each integer i, set

$$M_i = \bigoplus_{j=0}^{\infty} A_j^{(\beta_{i-j})}$$

where $A_j^{(\beta_{i-j})}$ is a direct sum of copies of A_j indexed by β_{i-j}. Identify each β_i with a basis of $A_0^{(\beta_i)}$ over A_0, and set $\beta = \bigcup_i \beta_i$ considered as a subset of the disjoint union $\bigsqcup_i M_i$. Define scalar multiplication on M over A using the scalar multiplication on each $A^{(\beta_i)}$.

Consider R-module homomorphisms

$$\xi_i : M_i \to M_{i-1}, \qquad \tau_i : M_i \to M_i, \qquad \delta_i : M_i \to M_{i-2}, \quad \text{and} \quad \alpha_i : M_i \to M_{i-1}.$$

For each i, set

$$N_i = M_{i-1} \oplus M_i \quad \text{and} \quad \partial_i^N = \begin{bmatrix} \xi_{i-1} & \delta_i \\ \tau_{i-1} & \alpha_i \end{bmatrix} : N_i \to N_{i-1}.$$

We consider the sequences

$$M = \cdots \xrightarrow{\alpha_{i+1}} M_i \xrightarrow{\alpha_i} M_{i-1} \xrightarrow{\alpha_{i-1}} \cdots$$

and

$$N = \cdots \xrightarrow{\partial_{i+1}^N} N_i \xrightarrow{\partial_i^N} N_{i-1} \xrightarrow{\partial_{i-1}^N} \cdots.$$

Given elements $\left[\begin{smallmatrix} a_{i-1} \\ a_i \end{smallmatrix}\right] \in B_i$ and $\left[\begin{smallmatrix} m_{j-1} \\ m_j \end{smallmatrix}\right] \in N_j$, we define

$$\begin{bmatrix} a_{i-1} \\ a_i \end{bmatrix} \begin{bmatrix} m_{j-1} \\ m_j \end{bmatrix} = \begin{bmatrix} a_{i-1}m_j + (-1)^i a_i m_{j-1} \\ a_i m_j \end{bmatrix}.$$

For each $\beta_{i,j} \in \beta_i$ we set $e_{i,j} = \left[\begin{smallmatrix} 0 \\ \beta_{i,j} \end{smallmatrix}\right] \in N_i$. For each i, set $E_i = \{e_{i,j}\}_j$. Let $E = \bigcup_i E_i$ considered as a subset of the disjoint union $\bigsqcup_i N_i$.

Remark 2.4. In Notation 2.3, the sequences M and N may not be complexes. Note that the scalar multiplications defined on M and N make $\bigoplus_i M_i$ and $\bigoplus_i N_i$ into graded free modules over $A^\natural$ and $B^\natural$, respectively.

The next result is a straightforward consequence of the definitions in Notation 2.3.

Lemma 2.5. *We work in the setting of Notations 2.1 and 2.3. The following conditions are equivalent.*

(i) *The sequence M is a semi-free DG A-module*;
(ii) *The sequence M is a DG A-module*; *and*
(iii) *For all integers i and j we have*

$$\alpha_{i-1}\alpha_i = 0, \qquad \alpha_{i+j}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i \gamma_{i,s}\alpha_j(m_j) \tag{2.5.1}$$

for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$.

Next, we give a similar result for the sequence N.

Lemma 2.6. *We work in the setting of Notations 2.1 and 2.3. The following conditions are equivalent.*

(i) *The sequence N is a semi-free DG B-module*;
(ii) *The sequence N is a DG B-module*; *and*
(iii) *For all integers i and j we have*

$$\xi_i = -\alpha_i, \qquad \tau_i = t, \tag{2.6.1}$$

$$\alpha_{i-1}\alpha_i = -t\delta_i, \qquad \delta_i\alpha_{i+1} = \alpha_{i-1}\delta_{i+1}, \tag{2.6.2}$$

$$\delta_{i+j}(\gamma_{i,s}m_j) = \gamma_{i,s}\delta_j(m_j), \tag{2.6.3}$$

$$\alpha_{i+j}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i \gamma_{i,s}\alpha_j(m_j) \tag{2.6.4}$$

for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$.

In particular, if N is a DG B-module, then

$$\partial_i^N = \begin{bmatrix} -\alpha_{i-1} & \delta_i \\ t & \alpha_i \end{bmatrix}. \tag{2.6.5}$$

Proof. (ii) $\Longrightarrow$ (iii) Assume that N is a DG B-module. Then the scalar multiplication defined in Notation 2.3 must satisfy the Leibniz rule. The Leibniz rule for products of the form $\left[\begin{smallmatrix} 0 \\ \gamma_{i,s} \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ m_j \end{smallmatrix}\right]$, where $1 \leqslant s \leqslant r_i$ and $m_j \in M_j$, is equivalent to the following relations:

$$\delta_{i+j}(\gamma_{i,s} m_j) = \gamma_{i,s}\delta_j(m_j), \tag{2.6.6}$$

$$\alpha_{i+j}(\gamma_{i,s} m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j(m_j). \tag{2.6.7}$$

The Leibniz rule for products of the form $\left[\begin{smallmatrix} \gamma_{i,s} \\ 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ m_j \end{smallmatrix}\right]$ is equivalent to the following:

$$\tau_{i+j}(\gamma_{i,s} m_j) = t\gamma_{i,s} m_j, \tag{2.6.8}$$

$$\xi_{i+j}(\gamma_{i,s} m_j) = -\bigl(\partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j(m_j)\bigr). \tag{2.6.9}$$

The Leibniz rule for products of the form $\left[\begin{smallmatrix} 0 \\ \gamma_{i,s} \end{smallmatrix}\right]\left[\begin{smallmatrix} m_j \\ 0 \end{smallmatrix}\right]$ is equivalent to the following:

$$\tau_{i+j}(\gamma_{i,s} m_j) = \gamma_{i,s}\tau_j(m_j), \tag{2.6.10}$$

$$\xi_{i+j}(\gamma_{i,s} m_j) = -\partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\xi_j(m_j). \tag{2.6.11}$$

The Leibniz rule for $\left[\begin{smallmatrix} \gamma_{i,s} \\ 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} m_j \\ 0 \end{smallmatrix}\right] = 0$ is equivalent to the following:

$$(-1)^i t\gamma_{i,s} m_j + (-1)^{i+1}\gamma_{i,s}\tau_j(m_j) = 0. \tag{2.6.12}$$

Eq. (2.6.3) is the same as (2.6.6), and Eq. (2.6.4) is the same as (2.6.7). Comparing Eqs. (2.6.7) and (2.6.9) with $\gamma_{0,1} = 1$, we find $\xi_i = -\alpha_i$. Using Eq. (2.6.8) also with $\gamma_{0,1} = 1$, we see that $\tau_i = t$. This explains (2.6.2) and (2.6.5). It also shows that (2.6.12) is trivial. Since N is an R-complex, we have $\partial_i^N\partial_{i+1}^N = 0$ which gives the equations in (2.6.2). This completes the proof of the implication.

The implication (iii) $\Longrightarrow$ (ii) is handled similarly, and the equivalence (i) $\Longleftrightarrow$ (ii) is straightforward. $\square$

Our next two results characterize semi-free DG modules over A and B. The first one is straightforward.

Lemma 2.7. *We work in the setting of Notations 2.1 and 2.3. If F is a bounded below semi-free DG A-module with semi-basis G, then $F \cong M$ for some appropriate choice of α_i satisfying (2.5.1) for all i and j where $\beta_i = |G \cap F_i|$.*

Lemma 2.8. *We work in the setting of Notations 2.1 and 2.3. If F is a bounded below semi-free DG B-module with semi-basis G, then $F \cong N$ for some appropriate choices of ξ_i, τ_i, α_i, and δ_i satisfying (2.6.1)–(2.6.4) for all i and j where $\beta_i = |G \cap F_i|$.*

Proof. Let F be a bounded below semi-free DG B-module with semi-basis G. For each i, set $\beta_i = |G \cap F_i|$. Since F is semi-free, it is straightforward to show that $F_i \cong \bigoplus_{j=0}^{\infty} B_j^{(\beta_{i-j})}$. Decomposing B_j as $A_{j-1} \oplus A_j$, we see that $F_j \cong N_j$ for each j. Since the R-module homomorphisms $N_j \to N_{j-1}$ are necessarily of the form $\left[\begin{smallmatrix} \xi_{i-1} & \delta_i \\ \tau_{i-1} & \alpha_i \end{smallmatrix}\right]$, it follows that there are appropriate choices of ξ_i, τ_i, α_i, and δ_i such that $F \cong N$. Finally, the fact that F is a DG B-module implies that the maps ξ_i, τ_i, α_i, and δ_i satisfy (2.6.1)–(2.6.4), by Lemma 2.6. $\square$

The next result indicates how a semi-free DG B-module should look in order to be liftable to A. See Section 3 for more about this.

Lemma 2.9. *We work in the setting of Notations 2.1 and 2.3. If M is a semi-free DG A-module, then $B \otimes_A M$ is a semi-free DG B-module, identified with a DG B-module N with*

$$\partial_i^N = \begin{bmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{bmatrix}.$$

Proof. Using the isomorphisms

$$B \otimes_A M \cong \left(K^R(t) \otimes_R A\right) \otimes_A M \cong K^R(t) \otimes_R M$$

the result follows directly from the definitions in Notation 2.3 with Lemmas 2.5 and 2.6. $\square$

Next, we describe DG module homomorphisms over A and B. Again, the proof of the first of these results is straightforward.

Lemma 2.10. *We work in the setting of Notations 2.1 and 2.3. Assume that M is a semi-free DG A-module, and let M' be a second semi-free DG A-module. Fix an integer p. A sequence of R-module homomorphisms $\{u_i : M_i \to M'_{i+p}\}$ is a DG A-module homomorphism $M \to M'$ of degree p if and only if it is a degree-p homomorphism $M \to M'$ of the underlying R-complexes and*

$$u_{i+j}(\gamma_{i,s} m_j) = (-1)^{pi} \gamma_{i,s} u_j(m_j)$$

for $s = 1, \dots, r_i$ and for all $m_j \in M_j$ for each integer j.

Lemma 2.11. *We work in the setting of Notations 2.1 and 2.3. Assume that N is a semi-free DG B-module. Let N' be a second semi-free DG B-module built from modules M'_i and maps ξ'_i, τ'_i, δ'_i, and α'_i as in Notation 2.3. Fix an integer p. A sequence of R-module homomorphisms $\{S_i : N_i \to N'_{i+p}\}$ is a DG B-module homomorphism $N \to N'$ of degree p if and only if it is a degree-p homomorphism $N \to N'$ of the underlying R-complexes such that for all integers i we have $S_i = \left[\begin{smallmatrix} (-1)^p z_{i-1} & v_i \\ 0 & z_i \end{smallmatrix}\right]$ for some $z_i : M_i \to M'_{i+p}$ and $v_i : M_i \to M'_{i+p-1}$ and*

$$v_{i+j}(\gamma_{i,s} m_j) = (-1)^{i(p+1)} \gamma_{i,s} v_j(m_j), \tag{2.11.1}$$

$$z_{i+j}(\gamma_{i,s} m_j) = (-1)^{ip} \gamma_{i,s} z_j(m_j) \tag{2.11.2}$$

for $s = 1, \dots, r_i$ and for all $m_j \in M_j$ for each integer j.

Proof. Fix a sequence of R-module homomorphisms $S = \{S_i : N_i \to N'_{i+p}\}$. By assumption, we have $N_i = M_{i-1} \oplus M_i$ and $N'_i = M'_{i-1} \oplus M'_i$, so the maps S_i have the form

$$S_i = \begin{bmatrix} u_{i-1} & v_i \\ y_{i-1} & z_i \end{bmatrix} : M_{i-1} \oplus M_i \to M'_{i+p-1} \oplus M'_{i+p}$$

where $u_{i-1} : M_{i-1} \to M'_{i+p-1}$, $v_i : M_i \to M'_{i+p-1}$, $y_{i-1} : M_{i-1} \to M'_{i+p}$, and $z_i : M_i \to M'_{i+p}$.

Assume that S is a DG B-module homomorphism $N \to N'$ of degree p. For each $b_q \in B_q$ and $n_d \in N_d$, we have

$$S_{q+d}(b_q n_d) = (-1)^{pq} b_q S_d(n_d). \tag{2.11.3}$$

Therefore, given integers i and j, for $s = 1, \dots, r_i$ and for all $m_j \in M_j$, by writing Eq. (2.11.3) for the elements $\left[\begin{smallmatrix} \gamma_{i,s} \\ 0 \end{smallmatrix}\right] \in B_{i+1}$ and $\left[\begin{smallmatrix} 0 \\ m_j \end{smallmatrix}\right] \in N_j$ we have

$$y_{i+j}(\gamma_{i,s} m_j) = 0, \tag{2.11.4}$$

$$u_{i+j}(\gamma_{i,s} m_j) = (-1)^{(i+1)p} \gamma_{i,s} z_j(m_j). \tag{2.11.5}$$

Using the elements $\left[\begin{smallmatrix} 0 \\ \gamma_{i,s} \end{smallmatrix}\right] \in B_i$ and $\left[\begin{smallmatrix} 0 \\ m_j \end{smallmatrix}\right] \in N_j$ we have

$$v_{i+j}(\gamma_{i,s} m_j) = (-1)^{i(p+1)} \gamma_{i,s} v_j(m_j), \tag{2.11.6}$$

$$z_{i+j}(\gamma_{i,s} m_j) = (-1)^{ip} \gamma_{i,s} z_j(m_j). \tag{2.11.7}$$

Similar equations arise using the elements $\left[\begin{smallmatrix} \gamma_{i,s} \\ 0 \end{smallmatrix}\right] \in B_{i+1}$ and $\left[\begin{smallmatrix} m_j \\ 0 \end{smallmatrix}\right] \in N_{j+1}$ and the elements $\left[\begin{smallmatrix} 0 \\ \gamma_{i,s} \end{smallmatrix}\right] \in B_i$ and $\left[\begin{smallmatrix} m_j \\ 0 \end{smallmatrix}\right] \in N_{j+1}$.

By comparing Eqs. (2.11.5) and (2.11.7) we conclude that $z_i = (-1)^p u_i$. Eq. (2.11.4) with $\gamma_{0,1} = 1$ implies that $y_i = 0$ for all i. Therefore, S_i has the desired form $S_i = \left[\begin{smallmatrix} (-1)^p z_{i-1} & v_i \\ 0 & z_i \end{smallmatrix}\right]$. Also, Eqs. (2.11.6) and (2.11.7) are exactly (2.11.1) and (2.11.2). This completes the proof of the forward implication. The converse is established similarly. □

The last result of this section describes some homomorphisms between semi-free DG B-modules that are liftable to A.

Lemma 2.12. *We work in the setting of Notations 2.1 and 2.3. Let M and M' be semi-free DG A-modules, and let $f \in \operatorname{Hom}_A(M, M')_j$. If $(B \otimes_A M)_i$ is identified with $M_{i-1} \oplus M_i$ and similarly for $(B \otimes_A M')_i$, then the map $(B \otimes_A f)_i : (B \otimes_A M)_i \to (B \otimes_A M')_{i+j}$ is identified with the matrix $\left[\begin{smallmatrix} (-1)^j f_{i-1} & 0 \\ 0 & f_i \end{smallmatrix}\right]$.*

Proof. This follows directly from Definition 1.19. □

3. Liftings and quasi-liftings of DG modules

In this section we prove our Main Theorem, starting with the definitions of our notions of liftings in the DG arena.

Definition 3.1. Let $T \to S$ be a morphism of DG R-algebras, and let D be a DG S-module. Then D is *quasi-liftable* to T if there is a semi-free DG T-module D' such that $D \simeq S \otimes_T D'$; in this case D' is called a *quasi-lifting* of D to T. We say that D is *liftable* to T if there is a DG T-module D' such that $D \cong S \otimes_T D'$; in this case D' is called a *lifting* of D to T.

Remark 3.2. In the definition of "quasi-liftable" we require that D' is semi-free in order to avoid the need for derived categories. If one prefers, one can remove the semi-free assumption and require that $D \simeq S \otimes^{\mathbf{L}}_T D'$ instead.

Our next result is a technical lemma for use in the proof of our Main Theorem.

Lemma 3.3. *We work in the setting of Notations 2.1 and 2.3. Let $n \geqslant 1$, and let $N^{(n-1)}$ be a semi-free DG B-module*

$$N^{(n-1)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1}^{(n-1)} & t^{n-1}\delta_i^{(n-1)} \\ t & \alpha_i^{(n-1)} \end{smallmatrix}\right]} M_{i-2} \oplus M_{i-1} \to \cdots$$

whose semi-basis over B is finite in each degree. In the case $n \geqslant 2$, assume that for each index i there are R-module homomorphisms $v_i^{(n-2)} : M_i \to M_{i-2}$ and $z_i^{(n-2)} : M_i \to M_{i-1}$ such that

$$\alpha_i^{(n-1)} = \alpha_i^{(n-2)} + t^{n-1} z_i^{(n-2)}, \tag{3.3.1}$$

$$\delta_i^{(n-1)} = v_i^{(n-2)} - t^{n-2} z_{i-1}^{(n-2)} z_i^{(n-2)}, \tag{3.3.2}$$

$$v_{i+j}^{(n-2)}(\gamma_{i,s} m_j) = \gamma_{i,s} v_j^{(n-2)}(m_j), \tag{3.3.3}$$

$$z_{i+j}^{(n-2)}(\gamma_{i,s} m_j) = (-1)^i \gamma_{i,s} z_j^{(n-2)}(m_j), \tag{3.3.4}$$

$$\alpha_{i-2}^{(n-2)} z_{i-1}^{(n-2)} + z_{i-2}^{(n-2)} \alpha_{i-1}^{(n-2)} + t v_{i-1}^{(n-2)} = \delta_{i-1}^{(n-2)}, \tag{3.3.5}$$

$$-\alpha_{i-2}^{(n-2)} v_i^{(n-2)} + t^{n-2} \delta_{i-1}^{(n-2)} z_i^{(n-2)} - t^{n-2} z_{i-2}^{(n-2)} \delta_i^{(n-2)} + v_{i-1}^{(n-2)} \alpha_i^{(n-2)} = 0, \tag{3.3.6}$$

for $s = 1, \ldots, r_i$, for all $m_j \in M_j$, and for each integer j.

If $\operatorname{Ext}_B^2(N^{(n-1)}, N^{(n-1)}) = 0$, then there are a semi-free DG B-module $N^{(n)}$ and an isomorphism of DG B-modules

$$\begin{array}{ccccccc}
N^{(n-1)} = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1}^{(n-1)} & t^{n-1}\delta_i^{(n-1)} \\ t & \alpha_i^{(n-1)} \end{smallmatrix}\right]} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots \\
 & & \Big\downarrow {\scriptstyle \left[\begin{smallmatrix} 1 & -t^{n-1} z_i^{(n-1)} \\ 0 & 1 \end{smallmatrix}\right]} & & \Big\downarrow {\scriptstyle \left[\begin{smallmatrix} 1 & -t^{n-1} z_{i-1}^{(n-1)} \\ 0 & 1 \end{smallmatrix}\right]} & & \\
N^{(n)} = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1}^{(n)} & t^{n}\delta_i^{(n)} \\ t & \alpha_i^{(n)} \end{smallmatrix}\right]} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots.
\end{array}$$

such that for each index i there are R-module homomorphisms $v_i^{(n-1)} : M_i \to M_{i-2}$ and $z_i^{(n-1)} : M_i \to M_{i-1}$ such that

$$\alpha_i^{(n)} = \alpha_i^{(n-1)} + t^{n} z_i^{(n-1)}, \tag{3.3.7}$$

$$\delta_i^{(n)} = v_i^{(n-1)} - t^{n-1} z_{i-1}^{(n-1)} z_i^{(n-1)}, \tag{3.3.8}$$

$$v_{i+j}^{(n-1)}(\gamma_{i,s} m_j) = \gamma_{i,s} v_j^{(n-1)}(m_j), \tag{3.3.9}$$

$$z_{i+j}^{(n-1)}(\gamma_{i,s} m_j) = (-1)^i \gamma_{i,s} z_j^{(n-1)}(m_j), \tag{3.3.10}$$

$$\alpha_{i-2}^{(n-1)} z_{i-1}^{(n-1)} + z_{i-2}^{(n-1)} \alpha_{i-1}^{(n-1)} + t v_{i-1}^{(n-1)} = \delta_{i-1}^{(n-1)}, \tag{3.3.11}$$

$$-\alpha_{i-2}^{(n-1)} v_i^{(n-1)} + t^{n-1} \delta_{i-1}^{(n-1)} z_i^{(n-1)} - t^{n-1} z_{i-2}^{(n-1)} \delta_i^{(n-1)} + v_{i-1}^{(n-1)} \alpha_i^{(n-1)} = 0, \tag{3.3.12}$$

for $s = 1, \ldots, r_i$, for all $m_j \in M_j$, and for each integer j.

Proof. By Lemma 2.6, we conclude that for all integers i, j we have

$$\alpha_{i-1}^{(n-1)}\alpha_i^{(n-1)} = -t^n\delta_i^{(n-1)}, \qquad t^{n-1}\delta_i^{(n-1)}\alpha_{i+1}^{(n-1)} = \alpha_{i-1}^{(n-1)}t^{n-1}\delta_{i+1}^{(n-1)}, \tag{3.3.13}$$

$$t^{n-1}\delta_{i+j}^{(n-1)}(\gamma_{i,s}m_j) = \gamma_{i,s}t^{n-1}\delta_j^{(n-1)}(m_j), \tag{3.3.14}$$

$$\alpha_{i+j}^{(n-1)}(\gamma_{i,s}m_j) = \partial_i^A(\gamma_{i,s})m_j + (-1)^i\gamma_{i,s}\alpha_j^{(n-1)}(m_j) \tag{3.3.15}$$

for $s = 1, \dots, r_i$ and for all $m_j \in M_j$.

Note that the sequence $\{\left[\begin{smallmatrix}\delta_{i-1}^{(n-1)} & 0\\ 0 & \delta_i^{(n-1)}\end{smallmatrix}\right] : M_{i-1} \oplus M_i \to M_{i-3} \oplus M_{i-2}\}$ is a cycle of degree -2 in the complex $\operatorname{Hom}_B(N^{(n-1)}, N^{(n-1)})$. Indeed, in the case $n = 1$, this follows from Eqs. (3.3.13)–(3.3.14); in the case $n \geqslant 2$, this follows from Eqs. (3.3.1)–(3.3.6). The assumption $\operatorname{Ext}_B^2(N^{(n-1)}, N^{(n-1)}) = 0$ implies that this cycle is null-homotopic, that is, there is a DG B-module homomorphism $S^{(n-1)} = \{S_i^{(n-1)}\} : N^{(n-1)} \to N^{(n-1)}$ of degree -1 such that

$$\begin{bmatrix}\delta_{i-1}^{(n-1)} & 0\\ 0 & \delta_i^{(n-1)}\end{bmatrix} = \begin{bmatrix}-\alpha_{i-2}^{(n-1)} & t^{n-1}\delta_{i-1}^{(n-1)}\\ t & \alpha_{i-1}^{(n-1)}\end{bmatrix} S_i^{(n-1)} + S_{i-1}^{(n-1)}\begin{bmatrix}-\alpha_{i-1}^{(n-1)} & t^{n-1}\delta_i^{(n-1)}\\ t & \alpha_i^{(n-1)}\end{bmatrix}. \tag{3.3.16}$$

Lemma 2.11 implies that each $S_i^{(n-1)}$ is of the form

$$S_i^{(n-1)} = \begin{bmatrix}-z_{i-1}^{(n-1)} & v_i^{(n-1)}\\ 0 & z_i^{(n-1)}\end{bmatrix}$$

where $v_i^{(n-1)} : M_i \to M_{i-2}$ and $z_i^{(n-1)} : M_i \to M_{i-1}$, and for $s = 1, \dots, r_i$, and for all $m_j \in M_j$ for each integer j we have

$$v_{i+j}^{(n-1)}(\gamma_{i,s}m_j) = \gamma_{i,s}v_j^{(n-1)}(m_j), \tag{3.3.17}$$

$$z_{i+j}^{(n-1)}(\gamma_{i,s}m_j) = (-1)^i\gamma_{i,s}z_j^{(n-1)}(m_j). \tag{3.3.18}$$

Hence for every i the equality (3.3.16) implies that we have

$$\alpha_{i-2}^{(n-1)}z_{i-1}^{(n-1)} + z_{i-2}^{(n-1)}\alpha_{i-1}^{(n-1)} + tv_{i-1}^{(n-1)} = \delta_{i-1}^{(n-1)}, \tag{3.3.19}$$

$$-\alpha_{i-2}^{(n-1)}v_i^{(n-1)} + t^{n-1}\delta_{i-1}^{(n-1)}z_i^{(n-1)} - t^{n-1}z_{i-2}^{(n-1)}\delta_i^{(n-1)} + v_{i-1}^{(n-1)}\alpha_i^{(n-1)} = 0. \tag{3.3.20}$$

Now let

$$\alpha_i^{(n)} = \alpha_i^{(n-1)} + t^nz_i^{(n-1)}, \qquad \delta_i^{(n)} = v_i^{(n-1)} - t^{n-1}z_{i-1}^{(n-1)}z_i^{(n-1)} \tag{3.3.21}$$

and

$$N^{(n)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\left[\begin{smallmatrix}-\alpha_{i-1}^{(n)} & t^n\delta_i^{(n)}\\ t & \alpha_i^{(n)}\end{smallmatrix}\right]} M_{i-2} \oplus M_{i-1} \to \cdots.$$

Note that the conclusions (3.3.7)–(3.3.12) follow directly from (3.3.17)–(3.3.21).

Since $N^{(n-1)}$ is a DG B-module, Eqs. (3.3.13), (3.3.19), and (3.3.21) give the following equation for all i:

$$\alpha_{i-1}^{(n)}\alpha_i^{(n)} + t^{n+1}\delta_i^{(n)} = 0. \tag{3.3.22}$$

By Eqs. (3.3.13), (3.3.19), (3.3.20), and (3.3.21), for all i we have

$$-\alpha_{i-1}^{(n)} t^n \delta_{i+1}^{(n)} + t^n \delta_i^{(n)} \alpha_{i+1}^{(n)} = 0. \tag{3.3.23}$$

For $s = 1, \dots, r_i$, and for all $m_j \in M_j$, Eqs. (3.3.14), (3.3.17), (3.3.18), and (3.3.21) give the following equality:

$$t^n \delta_{i+j}^{(n)}(\gamma_{i,s} m_j) = \gamma_{i,s} t^n \delta_j^{(n)}(m_j). \tag{3.3.24}$$

Also, by Eqs. (3.3.15), (3.3.18), and (3.3.21) we conclude that

$$\alpha_{i+j}^{(n)}(\gamma_{i,s} m_j) = \partial_i^A(\gamma_{i,s}) m_j + (-1)^i \gamma_{i,s} \alpha_j^{(n)}(m_j). \tag{3.3.25}$$

Therefore, Eqs. (3.3.22)–(3.3.25) and Lemma 2.6 imply that $N^{(n)}$ is a semi-free DG B-module. Eqs. (3.3.18)–(3.3.19) and (3.3.21) provide the next morphism of DG B-modules:

$$\begin{array}{ccccccc}
N^{(n-1)} = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1}^{(n-1)} & t^{n-1}\delta_i^{(n-1)} \\ t & \alpha_i^{(n-1)} \end{smallmatrix}\right]} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots \\
 & & \Big\downarrow {\scriptstyle \left[\begin{smallmatrix} 1 & -t^{n-1} z_i^{(n-1)} \\ 0 & 1 \end{smallmatrix}\right]} & & \Big\downarrow {\scriptstyle \left[\begin{smallmatrix} 1 & -t^{n-1} z_{i-1}^{(n-1)} \\ 0 & 1 \end{smallmatrix}\right]} & & \\
N^{(n)} = \cdots & \longrightarrow & M_{i-1} \oplus M_i & \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1}^{(n)} & t^{n}\delta_i^{(n)} \\ t & \alpha_i^{(n)} \end{smallmatrix}\right]} & M_{i-2} \oplus M_{i-1} & \longrightarrow & \cdots.
\end{array}$$

Similar reasoning shows that the sequence $\{\left[\begin{smallmatrix} 1 & t^{n-1} z_i^{(n-1)} \\ 0 & 1 \end{smallmatrix}\right]\}$ is a morphism of DG B-modules, and it is straightforward to show that these sequences are inverse isomorphisms. □

Part (a) of our Main Theorem is a consequence of the next result.

Theorem 3.4. *We work in the setting of Notations 2.1 and 2.3. Assume that R is tR-adically complete and that N is semi-free such that its semi-basis over B is finite in each degree. If $\operatorname{Ext}_B^2(N,N) = 0$, then N is liftable to A with semi-free lifting.*

Proof. Set $N^{(0)} = N$. Here, we use a natural variation of Notation 2.3; see Eq. (2.6.5):

$$N^{(0)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1}^{(0)} & \delta_i^{(0)} \\ t & \alpha_i^{(0)} \end{smallmatrix}\right]} M_{i-2} \oplus M_{i-1} \to \cdots.$$

Because of our assumptions, each M_i is a finitely generated free R-module.

Lemma 3.3 implies that for each $n \geqslant 1$ there are a semi-free DG B-module $N^{(n)}$ and an isomorphism of DG B-modules

$$\begin{tikzcd}[column sep=large]
N^{(n-1)} = \cdots \arrow[r] & M_{i-1} \oplus M_i \arrow[r, "{\left[\begin{smallmatrix} -\alpha_{i-1}^{(n-1)} & t^{n-1}\delta_i^{(n-1)} \\ t & \alpha_i^{(n-1)} \end{smallmatrix}\right]}"] \arrow[d, "{\left[\begin{smallmatrix} 1 & -t^{n-1}z_i^{(n-1)} \\ 0 & 1 \end{smallmatrix}\right]}"'] & M_{i-2} \oplus M_{i-1} \arrow[r] \arrow[d, "{\left[\begin{smallmatrix} 1 & -t^{n-1}z_{i-1}^{(n-1)} \\ 0 & 1 \end{smallmatrix}\right]}"] & \cdots \\
N^{(n)} = \cdots \arrow[r] & M_{i-1} \oplus M_i \arrow[r, "{\left[\begin{smallmatrix} -\alpha_{i-1}^{(n)} & t^{n}\delta_i^{(n)} \\ t & \alpha_i^{(n)} \end{smallmatrix}\right]}"] & M_{i-2} \oplus M_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that for each index i there are R-module homomorphisms $v_i^{(n-1)} : M_i \to M_{i-2}$ and $z_i^{(n-1)} : M_i \to M_{i-1}$ such that

$$\alpha_i^{(n)} = \alpha_i^{(n-1)} + t^n z_i^{(n-1)}, \tag{3.4.1}$$

$$\delta_i^{(n)} = v_i^{(n-1)} - t^{n-1} z_{i-1}^{(n-1)} z_i^{(n-1)}, \tag{3.4.2}$$

$$v_{i+j}^{(n-1)}(\gamma_{i,s} m_j) = \gamma_{i,s} v_j^{(n-1)}(m_j), \tag{3.4.3}$$

$$z_{i+j}^{(n-1)}(\gamma_{i,s} m_j) = (-1)^i \gamma_{i,s} z_j^{(n-1)}(m_j), \tag{3.4.4}$$

$$\alpha_{i-2}^{(n-1)} z_{i-1}^{(n-1)} + z_{i-2}^{(n-1)} \alpha_{i-1}^{(n-1)} + t v_{i-1}^{(n-1)} = \delta_{i-1}^{(n-1)}, \tag{3.4.5}$$

$$-\alpha_{i-2}^{(n-1)} v_i^{(n-1)} + t^{n-1} \delta_{i-1}^{(n-1)} z_i^{(n-1)} - t^{n-1} z_{i-2}^{(n-1)} \delta_i^{(n-1)} + v_{i-1}^{(n-1)} \alpha_i^{(n-1)} = 0, \tag{3.4.6}$$

for $s = 1, \ldots, r_i$, for all $m_j \in M_j$, and for each integer j. This follows by induction on n; note that this uses the isomorphism $N^{(n-1)} \cong N$ in the induction step to conclude that $\mathrm{Ext}_B^2(N^{(n-1)}, N^{(n-1)}) \cong \mathrm{Ext}_B^2(N, N) = 0$.

A straightforward calculation shows that

$$\prod_{j=0}^{n-1} \begin{bmatrix} 1 & -t^j z_i^{(j)} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\sum_{j=0}^{n-1} t^j z_i^{(j)} \\ 0 & 1 \end{bmatrix}.$$

Hence, the composite isomorphism $N^{(0)} \to N^{(n)}$ has the following form:

$$\begin{tikzcd}[column sep=large]
N^{(0)} = \cdots \arrow[r] & M_{i-1} \oplus M_i \arrow[r, "{\left[\begin{smallmatrix} -\alpha_{i-1}^{(0)} & \delta_i^{(0)} \\ t & \alpha_i^{(0)} \end{smallmatrix}\right]}"] \arrow[d, "{\left[\begin{smallmatrix} 1 & -\sum_{j=0}^{n-1} t^j z_i^{(j)} \\ 0 & 1 \end{smallmatrix}\right]}"'] & M_{i-2} \oplus M_{i-1} \arrow[r] \arrow[d, "{\left[\begin{smallmatrix} 1 & -\sum_{j=0}^{n-1} t^j z_i^{(j)} \\ 0 & 1 \end{smallmatrix}\right]}"] & \cdots \\
N^{(n)} = \cdots \arrow[r] & M_{i-1} \oplus M_i \arrow[r, "{\left[\begin{smallmatrix} -\alpha_{i-1}^{(n)} & t^{n}\delta_i^{(n)} \\ t & \alpha_i^{(n)} \end{smallmatrix}\right]}"] & M_{i-2} \oplus M_{i-1} \arrow[r] & \cdots.
\end{tikzcd}$$

Furthermore, Eq. (3.4.1) shows that

$$\alpha_i^{(n)} = \alpha_i^{(0)} + \sum_{j=0}^{n-1} t^{j+1} z_i^{(j)}.$$

We now define $N^{(\infty)}$ as follows:

$$N^{(\infty)} = \cdots \to M_{i-1} \oplus M_i \xrightarrow{\left[\begin{smallmatrix} -\alpha_{i-1}^{(\infty)} & 0 \\ t & \alpha_i^{(\infty)} \end{smallmatrix}\right]} M_{i-2} \oplus M_{i-1} \to \cdots$$

where

$$\alpha_i^{(\infty)} = \alpha_i^{(0)} + \sum_{j=0}^{\infty} t^{j+1} z_i^{(j)}.$$

Note that $\alpha_i^{(\infty)}$ is well-defined because R is tR-adically complete and the modules M_i and M_{i-1} are finitely generated free R-modules.

We claim that $N^{(\infty)}$ is a semi-free DG B-module. For all indices i and n, set

$$\zeta_i^{(n)} = \sum_{j=0}^{\infty} t^j z_i^{(j+n)} \tag{3.4.7}$$

and notice that

$$\alpha_i^{(\infty)} = \alpha_i^{(n)} + t^{n+1} \zeta_i^{(n)}. \tag{3.4.8}$$

Using (3.3.22), it follows that

$$\alpha_i^{(\infty)} \alpha_{i+1}^{(\infty)} = t^{n+1}\big(-\delta_{i+1}^{(n)} + \alpha_i^{(n)} \zeta_{i+1}^{(n)} + \zeta_i^{(n)} \alpha_{i+1}^{(n)} + t^{n+1} \zeta_i^{(n)} \zeta_{i+1}^{(n)}\big). \tag{3.4.9}$$

It follows that $\alpha_i^{(\infty)} \alpha_{i+1}^{(\infty)} \in \bigcap_{n=1}^{\infty} t^{n+1} \operatorname{Hom}_R(M_i, M_{i-1}) = 0$, by Krull's Intersection Theorem, so we have

$$\alpha_i^{(\infty)} \alpha_{i+1}^{(\infty)} = 0. \tag{3.4.10}$$

Furthermore, for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$ Eqs. (3.3.25), (3.4.4), (3.4.7), and (3.4.8) imply that

$$\alpha_{i+j}^{(\infty)}(\gamma_{i,s} m_j) = \partial_i^A(\gamma_{i,s}) m_j + (-1)^i \gamma_{i,s} \alpha_j^{(\infty)}(m_j). \tag{3.4.11}$$

Thus by Eqs. (3.4.10), (3.4.11) and Lemma 2.6 we conclude that $N^{(\infty)}$ is a DG B-module.

Now consider the chain map $\varphi = \{\varphi_i\} : N^{(0)} \to N^{(\infty)}$ defined by

$$\varphi_i = \prod_{j=0}^{\infty} \begin{bmatrix} 1 & -t^j z_i^{(j)} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\sum_{j=0}^{\infty} t^j z_i^{(j)} \\ 0 & 1 \end{bmatrix}.$$

This map is well-defined because R is tR-adically complete and N_i is finitely generated over R. Using these assumptions with Eq. (3.4.4) we conclude that

$$\left(-\sum_{l=0}^{\infty} t^l z_{i+j}^{(l)}\right)(\gamma_{i,s} m_j) = (-1)^i \gamma_{i,s} \left(-\sum_{l=0}^{\infty} t^l z_j^{(l)}\right)(m_j)$$

for $s = 1, \ldots, r_i$, and for all $m_j \in M_j$ for each integer j. Thus φ is B-linear and satisfies the assumptions of Lemma 2.11, so φ is a morphism of DG B-modules. Similar reasoning shows that the sequence $\{\left[\begin{smallmatrix} 1 & \sum_{j=0}^{\infty} t^j z_i^{(j)} \\ 0 & 1 \end{smallmatrix}\right]\}$ is a morphism of DG B-modules, and it is easy to show that these sequences are inverse isomorphisms.

On the other hand, by Eqs. (3.4.10)–(3.4.11) and Lemma 2.5, the sequence

$$M^{(\infty)} = \cdots \to M_i \xrightarrow{\alpha_i^{(\infty)}} M_{i-1} \to \cdots$$

is a semi-free DG A-module. Now Lemma 2.9 implies that $M^{(\infty)}$ is a lifting of $N^{(\infty)}$ to A. Since $N \cong N^{(\infty)}$ by definition, we conclude that $M^{(\infty)}$ is a lifting of N to A, so N is liftable to A. $\square$

Corollary 3.5. *We work in the setting of Notation 2.1. Assume that R is tR-adically complete. Let D be a DG B-module that is homologically both bounded below and degreewise finite. If $\operatorname{Ext}_B^2(D, D) = 0$, then D is quasi-liftable to A.*

Proof. Fact 1.18 implies that D has a semi-free resolution $N \simeq D$ over B such that the semi-basis for N is finite in each degree. Since $\operatorname{Ext}_B^2(N, N) \cong \operatorname{Ext}_B^2(D, D) = 0$, Theorem 3.4 implies that N is liftable to A, with semi-free lifting M. Thus, we have $B \otimes_A M \cong N \simeq D$, so D is quasi-liftable to A. $\square$

The proof of our Main Theorem uses induction on n, the length of the sequence $\underline{t}$. The next result is useful for the induction step in this proof.

Proposition 3.6. *We work in the setting of Notation 2.1. Assume that R is tR-adically complete, and let D be a DG B-module that is homologically bounded below and homologically degreewise finite such that $\operatorname{Ext}_B^d(D, D) = 0$ for some integer d. If M is a quasi-lifting of D to A, then $\operatorname{Ext}_A^d(M, M) = 0$.*

Proof. Assume without loss of generality that M is degreewise finite and bounded below; see Fact 1.18. Lemma 2.7 shows that M has the shape dictated by Notation 2.3. Since M is a quasi-lifting of D to A, we see that $N = B \otimes_A M \cong K^R(t) \otimes_R M$ is a semi-free resolution of D over B; see Lemma 2.9. To show that $\operatorname{Ext}_A^d(M, M) = 0$, let $f = \{f_i : M_i \to M_{i-d}\}$ be a cycle in $\operatorname{Hom}_A(M, M)_{-d}$; we need to show that f is null-homotopic. The fact that f is a cycle says that for every i we have $f_i \alpha_{i+1} = (-1)^d \alpha_{i+1-d} f_{i+1}$. For each i set $v_i^{(-1)} = f_i$

Claim. *For all $n \geqslant 0$ and for all $i \in \mathbb{Z}$, there are maps $v_i^{(n)} : M_i \to M_{i-d}$ and $z_i^{(n)} : M_i \to M_{i+1-d}$ such that for $s = 1, \ldots, r_i$, and for all $m_j \in M_j$ for each j*

$$v_{i+j}^{(n)}(\gamma_{i,s} m_j) = (-1)^{-id} \gamma_{i,s} v_j^{(n)}(m_j), \tag{3.6.1}$$

$$z_{i+j}^{(n)}(\gamma_{i,s} m_j) = (-1)^{i(1-d)} \gamma_{i,s} z_j^{(n)}(m_j), \tag{3.6.2}$$

$$(-1)^d z_{i-2}^{(n)} \alpha_{i-1} + t v_{i-1}^{(n)} + \alpha_{i-d} z_{i-1}^{(n)} = v_{i-1}^{(n-1)}, \tag{3.6.3}$$

$$(-1)^d v_{i-1}^{(n)} \alpha_i - \alpha_{i-d} v_i^{(n)} = 0. \tag{3.6.4}$$

To prove the claim, we proceed by induction on n. We verify the base case and the inductive step simultaneously. Let $n \geqslant 0$ and assume that for each i there exists $v_i^{(n-1)} : M_i \to M_{i-d}$ such that for $s = 1, \ldots, r_i$, and for all $m_j \in M_j$ for each integer j, we have

$$v_{i+j}^{(n-1)}(\gamma_{i,s} m_j) = (-1)^{id} \gamma_{i,s} v_j^{(n-1)}(m_j), \tag{3.6.5}$$

$$v_{i-1}^{(n-1)} \alpha_i - (-1)^d \alpha_{i-d} v_i^{(n-1)} = 0. \tag{3.6.6}$$

Thus, the sequence $\{\left[\begin{smallmatrix} (-1)^d v_{i-1}^{(n-1)} & 0 \\ 0 & v_i^{(n-1)} \end{smallmatrix}\right]: M_{i-1} \oplus M_i \to M_{i-d-1} \oplus M_{i-d}\}$ is a cycle in $\operatorname{Hom}_B(N, N)_{-d}$, by Lemma 2.11. As $\operatorname{Ext}_B^d(D, D) = 0$, this morphism is null-homotopic. Thus there exists a DG B-module homomorphism $S^{(n)} = \{S_i^{(n)}\} \in \operatorname{Hom}_B(N, N)_{1-d}$ such that for every i we have

$$\begin{bmatrix} -\alpha_{i-d} & 0 \\ t & \alpha_{i-d+1} \end{bmatrix} S_i^{(n)} - (-1)^{1-d} S_{i-1}^{(n)} \begin{bmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{bmatrix} = \begin{bmatrix} (-1)^d v_{i-1}^{(n-1)} & 0 \\ 0 & v_i^{(n-1)} \end{bmatrix}. \qquad (3.6.7)$$

Lemma 2.11 implies that each $S_i^{(n)}$ is of the form

$$S_i^{(n)} = \begin{bmatrix} (-1)^{1-d} z_{i-1}^{(n)} & v_i^{(n)} \\ 0 & z_i^{(n)} \end{bmatrix} : N_i \to N_{i-d+1}$$

where $v_i^{(n)} : M_i \to M_{i-d}$ and $z_i^{(n)} : M_i \to M_{i+1-d}$, and for $s = 1, \ldots, r_i$, and for all $m_j \in M_j$ for each integer j we have

$$v_{i+j}^{(n)}(\gamma_{i,s} m_j) = (-1)^{-id} \gamma_{i,s} v_j^{(n)}(m_j), \qquad (3.6.8)$$

$$z_{i+j}^{(n)}(\gamma_{i,s} m_j) = (-1)^{i(1-d)} \gamma_{i,s} z_j^{(n)}(m_j). \qquad (3.6.9)$$

Hence for each i, Eq. (3.6.7) implies that we have

$$(-1)^d z_{i-2}^{(n)} \alpha_{i-1} + t v_{i-1}^{(n)} + \alpha_{i-d} z_{i-1}^{(n)} = v_{i-1}^{(n-1)}, \qquad (3.6.10)$$

$$(-1)^d v_{i-1}^{(n)} \alpha_i - \alpha_{i-d} v_i^{(n)} = 0. \qquad (3.6.11)$$

This completes the proof of the claim.

Eq. (3.6.3) implies the following equality for each i:

$$f_i = (-1)^d \left[\sum_{j=0}^{n} t^j z_{i-1}^{(j)} \right] \alpha_i + \alpha_{i+1-d} \left[\sum_{j=0}^{n} t^j z_i^{(j)} \right] + t^{n+1} v_i^{(n)}.$$

Since R is tR-adically complete and each M_i is finitely generated over R, each series $\eta_i = \sum_{j=0}^{\infty} t^j z_i^{(j)}$ converges in $\operatorname{Hom}_R(M_i, M_{i+1-d})$, and for every i we have

$$f_i = (-1)^d \eta_{i-1} \alpha_i + \alpha_{i+1-d} \eta_i. \qquad (3.6.12)$$

By Eq. (3.6.2), we conclude that $\eta_{i+j}(\gamma_{i,s} m_j) = (-1)^{i(1-d)} \gamma_{i,s} \eta_j(m_j)$ for all i, j, s. Thus, Lemma 2.10 implies that $\eta = \{\eta_i\} \in \operatorname{Hom}_A(M, M)$ is a DG A-module homomorphism of degree $1-d$. Eq. (3.6.12) implies that $f = \{f_i\}$ is null-homotopic, as desired. $\square$

The next result contains part (a) of our Main Theorem.

Corollary 3.7. *Let $\underline{t} = t_1, \ldots, t_n$ be a sequence of elements of R, and assume that R is $\underline{t}R$-adically complete. Let D be a DG $K^R(\underline{t})$-module that is homologically bounded below and homologically degreewise finite. If $\operatorname{Ext}^2_{K^R(\underline{t})}(D, D) = 0$, then D is quasi-liftable to R.*

Proof. By induction on n, using Corollary 3.5 and Proposition 3.6. $\square$

The version of our Main Theorem used in [8] requires a bit more terminology.

Definition 3.8. Let A be a DG R-algebra, and let M be a DG A-module. For each $a \in A$ the *homothety* (i.e., multiplication map) $\mu^{M,a} : M \to M$ given by $m \mapsto am$ is a homomorphism of degree $|a|$. The *homothety morphism* $X^A_M : A \to \operatorname{Hom}_A(M, M)$ is given by $X^A_M(a) = \mu^{M,a}$, i.e., $X^A_M(a)(m) = am$.

If A is noetherian and M is semi-free, then M is a *semidualizing* DG A-module if M is homologically finite and the homothety morphism $X^A_M : A \to \operatorname{Hom}_A(M, M)$ is a quasiisomorphism. We say that an R-complex is *semidualizing* provided that it is semidualizing as a DG R-module. Let $\mathfrak{S}(A)$ be the set of shift-quasiisomorphism classes of semidualizing DG A-modules, that is, the set of equivalence classes of semidualizing DG A-modules where $C \sim C'$ if there is an integer n such that $C' \simeq \Sigma^n C$.

Remark 3.9. Let A be a DG R-algebra, and let M, M' be semi-free DG A-modules. It is straightforward to show that if $M \sim M'$, then M is semidualizing if and only if M' is semidualizing. Note that our semi-free assumption in the definition of semidualizing is only made in order to avoid the need for the derived category $\mathcal{D}(A)$. If one prefers to work in $\mathcal{D}(A)$, then a homologically finite DG A-module N is semidualizing if (and only if) the induced map $\chi^A_N : A \to \mathbf{R}\operatorname{Hom}_A(N, N)$ is an isomorphism in $\mathcal{D}(A)$.

Corollary 3.10. *Let $\underline{t} = t_1, \dots, t_n$ be a sequence of elements of R, and assume that R is $\underline{t}R$-adically complete. If D is a semidualizing DG $K^R(\underline{t})$-module, then there exists a semidualizing R-complex C which is a quasi-lifting of D to R. Moreover, the base-change operation $C \mapsto K^R(\underline{t}) \otimes_R C$ induces a bijection $\mathfrak{S}(R) \xrightarrow{\cong} \mathfrak{S}(K^R(\underline{t}))$.*

Proof. Note that the fact that R is $\underline{t}R$-adically complete implies that $\underline{t}R$ is contained in the Jacobson radical of R. Using this, one checks readily that the conclusions of [6, Lemma A.3] hold in our setting. The existence of an R-complex C that is a quasi-lifting of D to R follows from Corollary 3.7; and C is semidualizing over R by [6, Lemma A.3(a)]. This says that the base-change map $\mathfrak{S}(R) \to \mathfrak{S}(K^R(\underline{t}))$ is surjective; it is injective by [6, Lemma A.3(b)]. $\square$

Part (b) of our Main Theorem is a consequence of the next result.

Theorem 3.11. *We work in the setting of Notation 2.1. Assume that R is tR-adically complete, and assume that R, A, and A_0 are local, and let D be a DG B-module that is homologically bounded below and homologically degreewise finite. If D is quasi-liftable to A and $\operatorname{Ext}^1_B(D, D) = 0$, then any two homologically degreewise finite quasi-lifts of D to A are quasiisomorphic over A.*

Proof. The assumption that R is tR-adically complete and local implies that t is in the maximal ideal $\mathfrak{m} \subset R$.

Let C and C' be two homologically degreewise finite semi-free DG A-modules such that $B \otimes_A C \simeq D \simeq B \otimes_A C'$. Let $M \xrightarrow{\simeq} C$ and $M' \xrightarrow{\simeq} C'$ be minimal semi-free resolutions of C and C' over A. Lemma 2.7 shows that M and M' have the shape dictated by Notation 2.3. Since C and C' are quasi-liftings of D to A, we see that $N := B \otimes_A M \cong K^R(t) \otimes_R M$ and $N' := B \otimes_A M' \cong K^R(t) \otimes_R M'$ are semi-free resolutions of D over B; see Lemma 2.9. Furthermore, from Remark 2.2, we have the isomorphism $B/\mathfrak{m}_B \cong A/\mathfrak{m}_A$, which implies that

$$B/\mathfrak{m}_B \otimes_B (B \otimes_A M) \cong A/\mathfrak{m}_A \otimes_A M.$$

Since M is minimal over A, the differential on this complex is 0, so $B \otimes_A M$ is minimal over B, and similarly for $B \otimes_A M'$.

From [4, Theorem 2.12.5.2 and Example 2.12.5.4] there exists an isomorphism $\Upsilon : N \xrightarrow{\cong} N'$. Lemma 2.11 implies that Υ has the following form

$$
\begin{tikzcd}[ampersand replacement=\&]
N = \cdots \arrow[r] \& M_{i-1} \oplus M_i \arrow[rr, "{\left[\begin{smallmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{smallmatrix}\right]}"] \arrow[d, "{\left[\begin{smallmatrix} z_{i-1} & v_i \\ 0 & z_i \end{smallmatrix}\right]}"'] \& \& M_{i-2} \oplus M_{i-1} \arrow[r] \arrow[d, "{\left[\begin{smallmatrix} z_{i-2} & v_{i-1} \\ 0 & z_{i-1} \end{smallmatrix}\right]}"] \& \cdots \\
N' = \cdots \arrow[r] \& M'_{i-1} \oplus M'_i \arrow[rr, "{\left[\begin{smallmatrix} -\alpha'_{i-1} & 0 \\ t & \alpha'_i \end{smallmatrix}\right]}"] \& \& M'_{i-2} \oplus M'_{i-1} \arrow[r] \& \cdots
\end{tikzcd}
\tag{3.11.1}
$$

and that we have

$$v_{i+j}(\gamma_{i,s} m_j) = (-1)^i \gamma_{i,s} v_j(m_j), \tag{3.11.2}$$

$$z_{i+j}(\gamma_{i,s} m_j) = \gamma_{i,s} z_j(m_j) \tag{3.11.3}$$

for all i, for $s = 1, \ldots, r_i$ and for all $m_j \in M_j$ for each integer j. As the diagram (3.11.1) commutes, we have

$$z_{i-1}\alpha_i = t v_i + \alpha'_i z_i \tag{3.11.4}$$

for all i. Since Υ is an isomorphism, it follows that the z_i's are isomorphisms.

The condition (3.11.2) implies that $v = \{v_i\} \in \operatorname{Hom}_A(M, M')_{-1}$; cf. Lemma 2.10. As the diagram (3.11.1) commutes, we have $v_{i-1}\alpha_i = -\alpha'_{i-1} v_i$ for all i, so v is a cycle in $\operatorname{Hom}_A(M, M')_{-1}$. This yields a cycle $B \otimes_A v \in \operatorname{Hom}_B(N, N')_{-1}$ which has the form $\{\left[\begin{smallmatrix} -v_{i-1} & 0 \\ 0 & v_i \end{smallmatrix}\right] : M_{i-1} \oplus M_i \to M_{i-d-1} \oplus M_{i-d}\}$; see Fact 1.20 and Lemma 2.12. Since $B \otimes_A v$ is a cycle, our Ext-vanishing assumption implies that there is a DG B-module homomorphism

$$T^{(0)} = \left\{ \begin{bmatrix} u^{(0)}_{i-1} & p^{(0)}_i \\ 0 & u^{(0)}_i \end{bmatrix} \right\} \in \operatorname{Hom}_B\bigl(N, N'\bigr)_0$$

such that for every i we have

$$\begin{bmatrix} -\alpha'_{i-1} & 0 \\ t & \alpha'_i \end{bmatrix} T^{(0)}_i - T^{(0)}_{i-1} \begin{bmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{bmatrix} = \begin{bmatrix} -v_{i-1} & 0 \\ 0 & v_i \end{bmatrix}.$$

Therefore for all $i \in \mathbb{Z}$ we obtain the following equations:

$$-u^{(0)}_{i-1}\alpha_i + t p^{(0)}_i + \alpha'_i u^{(0)}_i = v_i,$$

$$u^{(0)}_{i+j}(\gamma_{i,s} m_j) = \gamma_{i,s} u^{(0)}_j(m_j),$$

$$p^{(0)}_{i+j}(\gamma_{i,s} m_j) = (-1)^i \gamma_{i,s} p^{(0)}_i(m_j),$$

$$p^{(0)}_{i-1}\alpha_i = -\alpha'_{i-1} p^{(0)}_i.$$

The process repeats using $p^{(0)} = \{p^{(0)}_i\}$ in place of $p^{(-1)} = v$. Inductively, for each $n \geqslant 0$ one can construct a DG B-module homomorphism

$$T^{(n)} = \left\{ \begin{bmatrix} u^{(n)}_{i-1} & p^{(n)}_i \\ 0 & u^{(n)}_i \end{bmatrix} \right\} \in \operatorname{Hom}_B\bigl(N, N'\bigr)_0$$

such that for every i we have

$$\begin{bmatrix} -\alpha'_{i-1} & 0 \\ t & \alpha'_i \end{bmatrix} T_i^{(n)} - T_{i-1}^{(n)} \begin{bmatrix} -\alpha_{i-1} & 0 \\ t & \alpha_i \end{bmatrix} = \begin{bmatrix} -p_{i-1}^{(n-1)} & 0 \\ 0 & p_i^{(n-1)} \end{bmatrix}.$$

Therefore for all $i \in \mathbb{Z}$ and $n \geqslant 0$ we get the following equations:

$$\begin{aligned} -u_{i-1}^{(n)}\alpha_i + tp_i^{(n)} + \alpha'_i u_i^{(n)} &= p_i^{(n-1)}, \\ u_{i+j}^{(n)}(\gamma_{i,s} m_j) &= \gamma_{i,s} u_j^{(n)}(m_j) \end{aligned} \tag{3.11.5}$$

and hence

$$v_i = p_i^{(-1)} = \alpha'_i \left[\sum_{j=0}^{n} t^j u_i^{(j)} \right] + t^{n+1} p_i^{(n)} - \left[\sum_{j=0}^{n} t^j u_{i-1}^{(j)} \right] \alpha_i. \tag{3.11.6}$$

Since R is tR-adically complete, the next series converges for each i

$$\xi_i = \sum_{j=0}^{\infty} t^j u_i^{(j)}$$

and Eq. (3.11.6) implies that

$$v_i = \alpha'_i \xi_i - \xi_{i-1} \alpha_i. \tag{3.11.7}$$

Combining Eqs. (3.11.4) and (3.11.7), for each i we have

$$(z_{i-1} + t\xi_{i-1})\alpha_i = \alpha'_i (z_i + t\xi_i). \tag{3.11.8}$$

This shows that the sequence $z + t\xi : M \to M'$ is a degree-0 homomorphism of the underlying R-complexes. Combining Eqs. (3.11.3) and (3.11.5), we see that

$$(z + t\xi)_{i+j}(\gamma_{i,s} m_j) = \gamma_{i,s}(z + t\xi)_j(m_j)$$

for all i, for $s = 1, \dots, r_i$ and for all $m_j \in M_j$ for each j. So, Lemma 2.10 implies that $z + t\xi$ is a cycle in $\operatorname{Hom}_A(M, M')_0$. Since each z_i is bijective and $t \in \mathfrak{m}$, Nakayama's Lemma implies that for every i, the map $z_i + t\xi_i$ is also bijective. Hence $z + t\xi$ is an isomorphism $M \xrightarrow{\cong} M'$, so $C \simeq M \cong M' \simeq C'$, as desired. □

Here is Main Theorem (b) from the introduction.

Corollary 3.12. *Let $\underline{t} = t_1, \dots, t_n$ be a sequence of elements of R, and assume that R is local and $\underline{t}R$-adically complete. Let D be a DG $K^R(\underline{t})$-module that is homologically bounded below and homologically degreewise finite. If D is quasi-liftable to R and $\operatorname{Ext}^1_{K^R(\underline{t})}(D, D) = 0$, then any two homologically degreewise finite quasi-lifts of D to R are quasiisomorphic over R.*

Proof. By induction on n, using Theorem 3.11 and Proposition 3.6. $\square$

We conclude the paper with an example showing that the quasi-lifts in the previous two results must be homologically degreewise finite.

Example 3.13. Let $(R, \mathfrak{m})$ be a local integral domain that is not a field. Let $Q(R)$ be the field of fractions of R, and let $0 \neq t \in \mathfrak{m}$. If F is an R-free resolution of $Q(R)$, then F and 0 are both quasi-lifts of 0 from $K^R(t)$ to R.

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