MENAS' CONJECTURE AND GENERIC ULTRAPOWERS*

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We apply the technique of generic ultrapowers to study the splitting problem of stationary subsets of \( \mathcal{P}_\kappa \lambda \). We present some conditions which guarantee the splitting of stationary subsets of \( \mathcal{P}_\kappa \lambda \).

0. Introduction

Solovay proved the following well-known theorem: Any stationary subset of a regular cardinal \( \kappa \) can be decomposed into \( \kappa \) many disjoint stationary subsets. Menas [7] conjectured the following analogue of Solovay's theorem for \( \mathcal{P}_\kappa \lambda \).

Menas' conjecture. Any stationary subset of \( \mathcal{P}_\kappa \lambda \) can be decomposed into \( \lambda^{<\kappa} \) many disjoint stationary subsets.

In the first section, we review the basic definitions and survey some results concerning Menas' conjecture. Then in the second section, we study the generic ultrapowers over \( \mathcal{P}_\kappa \lambda \). The results stated in the first section and some new results are proved by means of generic ultrapowers.

Throughout this paper \( \kappa \) represents a regular uncountable cardinal and \( \lambda \) a cardinal above \( \kappa \).

1. Survey of the results related to Menas' conjecture

In [4] Jech extended the notion of cub and stationary sets to such sets in \( \mathcal{P}_\kappa \lambda \) and showed that many of their properties are preserved. In that paper the next theorem was proved.

Theorem 1 (Jech). If \( \kappa \) is a successor cardinal and \( \lambda \) a regular cardinal \( \kappa \), then every stationary subset of \( \mathcal{P}_\kappa \lambda \) can be split into \( \lambda^{<\kappa} \) many disjoint stationary subsets.

We now know that the regularity condition on \( \lambda \) can be dropped.

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Theorem 2. If $\kappa$ is a successor cardinal and $\lambda$ any cardinal $>\kappa$, then the conclusion of the above theorem holds. (See the remark following Theorem 12.)

Baumgartner and DiPrisco showed that a large cardinal hypothesis and the splitting problem are related.

Theorem 3 (Baumgartner, DiPrisco). If $0^\# \text{ does not exist}$, then for any regular $\kappa$, every stationary subset of $\mathcal{P}_\kappa \lambda$ splits into $\lambda$ many disjoint stationary subsets.

The proof of Theorem 3 is based on the next two lemmas.

Lemma 4 (DiPrisco). Every stationary subset of $\{s \in \mathcal{P}_\kappa \lambda : |s \cap \kappa| = |s|\}$ splits into $\lambda$ many disjoint stationary subsets.

Lemma 5 (Baumgartner). If $0^\# \text{ does not exist}$, then $\{s \in \mathcal{P}_\kappa \lambda : |s \cap \kappa| = |s|\}$ contains a cub subset of $\mathcal{P}_\kappa \lambda$.

Lemma 4 gives an example of a type of stationary subsets which readily splits into $\lambda$ many disjoint stationary subsets. We will give more examples of such stationary subsets.

Theorem 6. Assume $\lambda$ is a regular cardinal and let $A = \{s \in \mathcal{P}_\kappa \lambda : \text{cf}(\text{ot}(s)) < \text{ot}(s)\}$ where $\text{ot}(s)$ means the order type of $s$. Every stationary subset of $A$ splits into $\lambda$ many disjoint stationary subsets.

Corollary 7. If $X \subseteq \mathcal{P}_\kappa \lambda$ is a stationary set which does not split into $\lambda$ many disjoint stationary subsets, then $\{s \in X : \text{ot}(s) \text{ is a regular cardinal}\}$ is stationary.

Theorem 8. Assume $\lambda^{<\kappa} > \lambda$ and $\kappa$ is a weakly inaccessible cardinal. Then every stationary subset of $\{s \in \mathcal{P}_\kappa \lambda : |s|^{<\text{cf}(\text{ot}(s))} = |s|\}$ splits into $\lambda^{<\kappa}$ many disjoint stationary subsets.$^1$

Recently Gitik [3] proved the following surprising result.

Theorem 9 (Gitik). If the existence of a supercompact cardinal is consistent with ZFC, then it is consistent that for a regular cardinal $\kappa$ and some $\lambda > \kappa$ there is a stationary subset of $\mathcal{P}_\kappa \lambda$ which does not split into $\kappa^+$ many disjoint stationary subsets. Furthermore, we can make $\kappa$ in the resulting model to be the least inaccessible cardinal.

This result shows that Menas' conjecture as stated in the introduction cannot be

$^1$ The proof of Theorem 8 appears in [6].
proven without refuting the existence of a supercompact cardinal. The question whether \( P_\kappa \lambda \) itself can be decomposed into \( \lambda^\kappa \) many disjoint stationary subsets for every regular \( \kappa \) remains open. Our forthcoming [6] deals with this question.

2. Precipitous ideals on \( P_\kappa \lambda \)

In this section we will study precipitous ideals on \( P_\kappa \lambda \) and their applications to Menas' conjecture. We begin with some basic definitions.

**Definition.** \( I \) is an ideal on \( P_\kappa \lambda \) if \( I \) is a collection of subsets of \( P_\kappa \lambda \) such that
(i) \( \emptyset \in I \) and \( P_\kappa \lambda \notin I \).
(ii) If \( X, Y \subseteq P_\kappa \lambda \), \( X \in I \) and \( Y \subseteq X \), then \( Y \in I \).
(iii) If \( X \in I \) and \( Y \in I \), then \( X \cup Y \in I \).

An ideal \( I \) on \( P_\kappa \lambda \) is \( \delta \)-complete for a cardinal \( \delta \leq \kappa \) if \( I \) is closed under union of less than \( \delta \) many members.

An ideal \( I \) on \( P_\kappa \lambda \) is normal if for any \( \{X_\alpha : \alpha < \lambda \} \subseteq I \), we have \( \bigwedge_{\alpha < \lambda} X_\alpha \in I \) where \( \bigwedge_{\alpha < \lambda} X_\alpha \) is called the diagonal union of \( \{X_\alpha : \alpha < \kappa \} \) and defined as follows:

\[
s \in \bigwedge_{\alpha < \lambda} X_\alpha \iff s \in \bigcup_{\alpha \in s} X_\alpha
\]

An ideal \( I \) on \( P_\kappa \lambda \) is fine if for each \( \alpha < \lambda \), \( \{s \in P_\kappa \lambda : \alpha \notin s \} \in I \). For the sake of convenience, throughout this paper, by 'ideal' we mean 'fine ideal'.

A filter \( \mathcal{F} \) on \( P_\kappa \lambda \) and an ideal \( I \) on \( P_\kappa \lambda \) are dual to each other if the following holds:

\[
X \in \mathcal{F} \iff P_\kappa \lambda - X \in I \quad \text{for every } X \subseteq P_\kappa \lambda.
\]

**Definition.** Let \( \text{NS}(\kappa, \lambda) = \{X \subseteq P_\kappa \lambda : X \text{ is not stationary}\} \). So \( \text{NS}(\kappa, \lambda) \) is the dual ideal of the cub filter on \( P_\kappa \lambda \). \( \text{NS}(\kappa, \lambda) \) is called the non-stationary ideal on \( P_\kappa \lambda \).

By [4] we know the following:

**Theorem 10** (Jech). \( \text{NS}(\kappa, \lambda) \) is a \( \kappa \)-complete normal ideal on \( P_\kappa \lambda \).

**Definition.** We say a subset \( X \) of \( P_\kappa \lambda \) is \( I \)-positive for an ideal \( I \) on \( P_\kappa \lambda \) if \( X \notin I \). Given \( X, Y \subseteq P_\kappa \lambda \), we say \( X \) and \( Y \) are almost disjoint with respect to an ideal \( I \) if \( X \cap Y \in I \). An ideal \( I \) on \( P_\kappa \lambda \) is \( \delta \)-saturated for a cardinal \( \delta \) if there is no pairwise almost disjoint collection of size \( \delta \) of \( I \)-positive subsets of \( P_\kappa \lambda \).

In some cases, we can obtain a collection of disjoint \( I \)-positive subsets from a collection of almost disjoint \( I \)-positive subsets.

\(^2\) For \( \kappa = \aleph_1 \), Baumgartner and Taylor refuted Menas' conjecture [1].
Theorem 11. Assume \( I \) is a \( \kappa \)-complete normal ideal and \( \delta \) any cardinal \( \leq \lambda \). If there are \( \delta \) many almost disjoint \( I \)-positive subsets of \( \mathcal{P}_\kappa \lambda \), then there are \( \delta \) many disjoint \( I \)-positive subsets of \( \mathcal{P}_\kappa \lambda \).

Proof. Given \( \langle X_\alpha : \alpha < \delta \rangle \) a collection of pairwise almost disjoint \( I \)-positive subsets of \( \mathcal{P}_\kappa \lambda \), define \( W \subseteq \mathcal{P}_\kappa \lambda \) as follows:

\[
P \in W \iff (\exists \alpha, \beta \in s \cap \delta)(\alpha \neq \beta \wedge \alpha \in X_\alpha \wedge \beta \in X_\beta).
\]

By the normality and the fact \( X_\alpha \cap X_\beta \in I \) for each distinct \( \alpha, \beta < \delta \), we have \( W \in I \). For each \( \alpha < \delta \), let \( Y_\alpha = X_\alpha - \{s \in \mathcal{P}_\kappa \lambda : \alpha \notin s \} \cup W \). It is clear that \( \langle Y_\alpha : \alpha < \delta \rangle \) forms a collection of pairwise-disjoint \( I \)-positive subsets of \( \mathcal{P}_\kappa \lambda \). \[ \square \]

The following result is a \( \mathcal{P}_\kappa \lambda \) analogue of a theorem of Ulam.

Theorem 12 (Folk). If \( \kappa \) is a successor cardinal, then for any cardinal \( \lambda > \kappa \) there is no \( \kappa \)-complete \( \lambda \)-saturated ideal on \( \mathcal{P}_\kappa \lambda \).

Proof. Let \( \kappa = \delta^+ \) for some infinite cardinal \( \delta \). Assume \( I \) is a \( \kappa \)-complete \( \lambda \)-saturated ideal on \( \mathcal{P}_\kappa \lambda \). We first prove the following claim:

Claim. For every \( \gamma \leq \lambda \) and \( X \subseteq \mathcal{P}_\kappa \lambda \), if \( \text{cf}(\gamma) > \delta \) and \( X \) is \( I \)-positive, then \( X \) can be decomposed into \( \gamma \) many disjoint \( I \)-positive subsets.

Proof of Claim. For each \( s \in X \), fix an injection \( f_s : s \to \delta \). For each \( \alpha < \gamma \), \( \xi < \delta \) let \( X_\alpha^\xi = \{s \in X : \alpha \in s \text{ and } f_s(\alpha) = \xi \} \). Thus if \( \alpha < \beta < \gamma \), then \( X_\alpha^\xi \cap X_\beta^\xi = \emptyset \). For each \( \alpha < \gamma \), \( \bigcup_{\xi < \delta} X_\alpha^\xi \notin I \).

So \( \bigcup_{\xi < \delta} X_\alpha^\xi \notin I \). By \( \kappa \)-completeness of \( I \) there is some \( \xi_\alpha < \delta \) such that \( X_\alpha^{\xi_\alpha} \notin I \). \( \alpha \mapsto \xi_\alpha \) defines a map from \( \gamma \) into \( \delta \). Since \( \text{cf}(\gamma) > \delta \), there is some \( \xi < \delta \) such that \( |\{\alpha < \gamma : \xi_\alpha = \xi\}| = \gamma \). Clearly \( \{X_\alpha^{\xi} : \xi_\alpha = \xi\} \) can be extended to the desired collection of \( I \)-positive subsets. \[ \square \] (Claim)

We will complete the proof. If \( \text{cf}(\lambda) > \delta \), the above claim gives a proof. Assume \( \text{cf}(\lambda) \leq \delta \). First we split \( X \) into \( \text{cf}(\lambda) \) many disjoint stationary subsets. Then we split each piece into the appropriate number of disjoint stationary subsets to obtain \( \lambda \) many disjoint stationary subsets of \( X \). \[ \square \]

Remark. Theorem 12 provides a proof of Theorem 2. Let \( X \subseteq \mathcal{P}_\kappa \lambda \) be a stationary set where \( \kappa \) is a successor cardinal. By Theorem 12, we know that \( \text{NS}(\kappa, \lambda) \upharpoonright X \) cannot be \( \lambda \)-saturated. Thus there are \( \lambda \) many almost disjoint stationary subsets of \( X \). By the proof of Theorem 11, \( X \) splits into \( \lambda \) many disjoint stationary subsets.

\[ ^3 \text{We have learned this idea from Zwicker [10].} \]
We now study the method of generic ultrapowers. The method was first used significantly in Solovay [8] to study saturated ideals on cardinals. The method easily extends to the $\mathcal{P}_\kappa\lambda$ situation. The set-up of generic ultrapowers with respect to ideals on $\mathcal{P}_\kappa\lambda$ is completely analogous to that of generic ultrapowers with respect to ideals on a cardinal. For this reason we will just sketch the construction of generic ultrapowers. See Jech and Prikry [5] for a good reference on generic ultrapowers.

Let us view the universe as a ground model denoted by $V$. Consider the generic extension of $V$ given by the completion of the Boolean algebra $\mathcal{P}(\mathcal{P}_\kappa\lambda)/I$. In other words, we are forcing with $(\mathcal{P}_\kappa\lambda)/I$ where $\mathcal{P}_\kappa\lambda = \{X \subseteq \mathcal{P}_\kappa\lambda : X \notin I\}$ and $X \leq_I Y$ iff $X - Y \in I$.

Let us assume that $I$ is a $\kappa$-complete ideal on $\mathcal{P}_\kappa\lambda$ and $G$ is a generic filter on $P_I$. Then $G$ is a $V$-$\kappa$-complete $V$-ultrafilter on $\mathcal{P}_\kappa\lambda$. Now consider the class of all functions $f \in V$ with domain $\mathcal{P}_\kappa\lambda$, and let

- $f = * g$ iff $\{s \in \mathcal{P}_\kappa\lambda : f(s) = g(s)\} \in G$
- $f \in * g$ iff $\{s \in \mathcal{P}_\kappa\lambda : f(s) \in g(s)\} \in G$

The relation $=*$ is an equivalence relation. Let $[f]$ denote the equivalence class of functions represented by $f$. By $\text{Ult}(V, G)$ we denote the collection of all such equivalence classes. $(\text{Ult}(V, G), \epsilon^*)$ forms a model for the language of set theory. We will call this ultrapower a generic ultrapower modulo $G$. It is customary to denote $(\text{Ult}(V, G), \epsilon^*)$ by $\text{Ult}(V, G)$.

Just as one would expect the fundamental theorem holds; for any formula $\phi(V_1, \ldots, V_n)$ in the language of set theory,

$$\text{Ult}(V, G) \vDash \phi([f_1], \ldots, [f_n])$$

iff $\{s \in \mathcal{P}_\kappa\lambda : V \vDash \phi(f_1(s), \ldots, f_n(s))\} \in G$ for every $f_1, \ldots, f_n \in V \cap \mathcal{P}_\kappa\lambda V$.

So $\text{Ult}(V, G)$ is a model of ZFC, but not necessarily well-founded. Let $j : V \to \text{Ult}(V, G)$ be the natural embedding given by $j(x) = [C_x]$ where $C_x$ is the constant function on $\mathcal{P}_\kappa\lambda$ with value $x$. If $\text{Ult}(V, G)$ turns out to be well-founded, then we identify $\text{Ult}(V, G)$ with its transitive collapse. We are particularly interested in the case when all the generic ultrapowers are well founded.

**Definition** (following Jech and Prikry). If $I$ is a $\kappa$-complete ideal on $\mathcal{P}_\kappa\lambda$ we say $I$ is precipitous if $\text{Ult}(V, G)$ is well-founded for every generic filter $G$ on $\mathcal{P}(\mathcal{P}_\kappa\lambda)/I$.

It is known that all the $\kappa$-complete $\lambda^+$-saturated normal ideals on $\mathcal{P}_\kappa\lambda$ are precipitous.

**Theorem 13** (Foreman [2]). If $I$ is a $\kappa$-complete $\lambda^+$-saturated ideal on $\mathcal{P}_\kappa\lambda$, then $I$ is precipitous.
The next lemma provides proofs of Theorem 2, Lemma 4 and Theorem 6.

**Lemma 14.** If $X$ is a stationary subset of $\mathcal{P}_\kappa \lambda$, then $X$ splits into $\lambda$ many disjoint stationary subsets, provided one of the following holds:

(i) $\kappa$ is a successor.
(ii) $X \cap \{s \in \mathcal{P}_\kappa \lambda : |s \cap \alpha| = |s|\}$ is stationary for some $\alpha < \kappa$.
(iii) $\lambda$ is regular and $X \cap \{s \in \mathcal{P}_\kappa \lambda : \text{cf}(\text{ot}(s)) < \text{ot}(s)\}$ is stationary.

**Proof.** If $X$ is a stationary subset of $\mathcal{P}_\kappa \lambda$ which cannot be decomposed into $\lambda$ many disjoint stationary subsets, we will show that each of (i)–(iii) fails. Let's consider the ideal

$$\text{NS}(\kappa, \lambda) | X = \{Y \subseteq \mathcal{P}_\kappa \lambda : X \cap Y \notin \text{NS}(\kappa, \lambda)\}.$$ 

It is easy to see that $\text{NS}(\kappa, \lambda) | X$ is a $\kappa$-complete normal ideal. Furthermore by Theorem 11 and the hypothesis on $X$, we know that $\text{NS}(\kappa, \lambda) | X$ is $\lambda$-saturated. Hence by Theorem 13, $\text{NS}(\kappa, \lambda) | X$ is a precipitous ideal. Let $G$ be a generic filter on $\mathcal{P}(\mathcal{P}_\kappa \lambda)/\text{NS}(\kappa, \lambda) | X$ and $j : V \rightarrow \text{Ult}(V, G)$ the canonical embedding. Note that by $\lambda$-saturatedness, cardinals $\leq \lambda$ are preserved in $V[G]$. For each $\alpha < \kappa$, define a function $f_\alpha$ on $\mathcal{P}_\kappa \lambda$ by $f_\alpha(s) = s \cap \alpha$. Let id denote the identity function on $\mathcal{P}_\kappa \lambda$. By the $\kappa$-completeness and the normality of our ideal, it is clear that $[f_\alpha] = j'' \alpha$, $[\text{id}] = j'' \lambda$ and $j(\alpha) = \alpha$ for any $\alpha < \kappa$.

Now let's assume that $\kappa$ is a successor cardinal, say $\kappa = \delta^+$ for some infinite cardinal $\delta$. It is easy to see that $\{s \in \mathcal{P}_\kappa \lambda : |s| = \delta\} \in G$. Hence we have $\text{Ult}(V, G) \models [\text{id}] = j(\delta)$. This implies $\text{Ult}(V, G) \models |\lambda| = \delta$ contradicting the fact that $\lambda$ is preserved. Thus $\kappa$ cannot be a successor cardinal.

Suppose $X \cap \{s \in \mathcal{P}_\kappa \lambda : |s \cap \alpha| = |s|\}$ is stationary. Without loss of generality we may assume $X \subseteq \{s \in \mathcal{P}_\kappa \lambda : |s \cap \alpha| = |s|\}$. We have $\text{Ult}(V, G) \models [[f_\alpha]] = [[\text{id}]]$. So $\text{Ult}(V, G) \models |\alpha| = |\lambda|$. Once again this is a contradiction. Thus (ii) also fails.

Now assume $X \cap \{s \in \mathcal{P}_\kappa \lambda : \text{cf}(\text{ot}(s)) < \text{ot}(s)\}$ is stationary and $\lambda$ is a regular cardinal. We may assume $X \subseteq \{s \in \mathcal{P}_\kappa \lambda : \text{cf}(\text{ot}(s)) < \text{of}(s)\}$. Then $\text{Ult}(V, G) \models \text{cf}(\lambda) < \lambda$ contradicting the $\lambda$-saturatedness, $\lambda$ being a regular cardinal. □

We know that the existence of a precipitous ideal on $\mathcal{P}_\kappa \lambda$ implies some large cardinal hypothesis. The next theorem is a generic version of a theorem of Vopěnka and Hrbáček [9].

**Theorem 15.** If there is a precipitous ideal on $\mathcal{P}_\kappa \lambda$, then for every bounded subset $b$ of $\lambda$, $b^*$ exists.

**Proof.** Let $I$ be a precipitous ideal on $\mathcal{P}_\kappa \lambda$. Fix a bounded subset $b$ of $\lambda$ and we will show that $b^*$ exists. Let $\delta$ be the cardinal such that $\delta = \max[(\text{sup} \ b)^+, \kappa^+]$ where $(\text{sup} \ b)^+$ denotes the least cardinal $>\text{sup} \ b$. Let $G$ be a generic ultrafilter on $\mathcal{P}(\mathcal{P}_\kappa \lambda)/I$. As usual we can build $\text{Ult}(V, G)$ the transitive collapse of the
generic ultrapower. We let \( j: V \rightarrow \text{Ult}(V, G) \) be the canonical elementary embedding.

Now we consider another version of ultrapower, denoted by \( \text{Ult}^-(V, G) \). To form \( \text{Ult}^-(V, G) \), we just consider functions \( f: \mathcal{P}_\lambda \rightarrow V \) such that \( |\text{range } f| < \delta \). It is clear that \( \text{Ult}^-(V, G) \) inherits its well-foundedness from the usual ultrapower. By abusing our notation, denote the transitive collapse of \( \text{Ult}^-(V, G) \) by \( \text{Ult}^-(V, G) \). Let \( i: V \rightarrow \text{Ult}^-(V, G) \) be the canonical embedding. We denote by \( [f]^- \) the element of \( \text{Ult}^-(V, G) \) represented by \( f \). It is easy to see that \( i \) is an elementary embedding. We need a few claims to complete this proof.

**Claim 1.** For any function \( f: \mathcal{P}_\lambda \rightarrow \delta \) with \( |\text{range}(f)| < \delta \), we have \( [f] = [f]^- \).

**Proof of Claim 1.** A straightforward induction argument on the rank of \( [f] \).

**Claim 2.** If \( x \) is a bounded subset of \( \delta \), then \( i(x) = j(x) \). In particular, \( i(b) = j(b) \) and \( i(\alpha) = j(\alpha) \) for each \( \alpha < \delta \).

**Proof of Claim 2.** Let \( x \) be a bounded subset of \( \delta \). Pick \( [f]^- \in i(x) \). By Claim 1, we have \( [f]^- = [f] \). We have \( [f] \in j(x) \). Thus \( i(x) \subseteq j(x) \). Conversely assume \( [g] \in j(x) \). We may assume that \( |\text{range}(g)| < \delta \). Again by Claim 1, \( [g] = [g]^- \). Also \( [g]^- \in i(x) \), so \( j(x) \subseteq i(x) \).

**Claim 3.** \( i(\delta) < j(\delta) \).

**Proof of Claim 3.** First we will show \( i(\delta) = \sup_{\alpha < \delta} i(\alpha) \). Clearly \( i(\delta) \geq \sup_{\alpha < \delta} i(\alpha) \). Pick \( [f]^- \in i(\delta) \). Since \( \text{range}(f) \) is bounded in \( \delta \), we have \( [f]^- \in \sup_{\alpha < \delta} i(\alpha) \). Thus \( i(\delta) = \sup_{\alpha < \delta} i(\alpha) \). Now define \( g^*: \mathcal{P}_\lambda \rightarrow \delta \) as follows: \( g^*(s) = \sup (s \cap \delta) \). Note: \( j(\delta) > [g^*] \geq \sup_{\alpha < \delta} j(\alpha) \). But by Claim 2, \( \sup_{\alpha < \delta} j(\alpha) = \sup_{\alpha < \delta} i(\alpha) \). Thus we have \( j(\delta) > i(\delta) \).

We now define \( k: \text{Ult}^-(V, G) \rightarrow \text{Ult}(V, G) \) as follows: \( k([f]^-) = [f] \). It is easy to check that \( k \) is a well-defined elementary embedding. Moreover, the following diagram commutes.

\[
\begin{array}{ccc}
V & \xrightarrow{i} & \text{Ult}^-(V, G) \\
\downarrow j & & \downarrow k \\
\text{Ult}(V, G) & & \\
\end{array}
\]

By Claim 3 we know that \( k \) moves \( i(\delta) \). Claim 2 together with the commutativity of the above diagram gives that \( k(i(b)) = j(b) = i(b) \). Thus \( k \upharpoonright L[i(b)] \) is a nontrivial elementary embedding of \( L[i(b)] \) into itself. Thus \( i(b)^* \) exists. Since all the reals are in \( \text{Ult}(V, G) \), \( \text{Ult}(V, G) \vdash j(b)^* \) exists. By the elementarity, \( V \vdash b^* \) exists. \( \Box \).
From Theorem 13 and Theorem 15 we can derive the following result which extends the theorem of Baumgartner and DiPrisco.

**Corollary 16.** If there is a stationary subset of $P_\kappa \lambda$ which does not split into $\lambda$ many disjoint stationary subsets, then $b^\#$ exists for every bounded subset of $\lambda$.

Knowing Gitik's result, i.e., Theorem 9, one might conjecture that once $\kappa$ is not a successor cardinal, we cannot expect to find any condition of $\kappa$ and $\lambda$ which will guarantee the splitting of stationary subsets of $P_\kappa \lambda$. The next result shows that is not the case.

**Theorem 17.** If either (i) $\kappa$ is not a limit of weakly inaccessible cardinals and $\lambda$ is a singular limit of weakly inaccessible cardinals or (ii) $\kappa$ is not a limit of weakly Mahlo cardinals and $\lambda$ is a singular limit of weakly Mahlo cardinals, then there is no $\kappa$-complete $\lambda$-saturated normal ideal on $P_\kappa \lambda$. In particular every stationary subset of $P_\kappa \lambda$ splits into $\lambda$ many disjoint stationary subsets.

In order to prove Theorem 18, we first prove the following lemma:

**Lemma 18.** If either (i) $\kappa$ is not a limit of weakly inaccessible cardinals and $\lambda$ is a weakly inaccessible cardinal or (ii) $\kappa$ is not a limit of weakly Mahlo cardinals and $\lambda$ is a weakly Mahlo cardinal, then for any $\delta < \lambda$, there is no $\kappa$-complete $\delta$-saturated normal ideal on $P_\kappa \lambda$. In particular every stationary subset of $P_\kappa \lambda$ splits into $\delta$ many disjoint stationary subsets.

**Proof.** We will prove the contrapositive. Assume that $I$ is a $\kappa$-complete normal ideal which is $\gamma$-saturated for some $\gamma < \lambda$. By Theorem 13, $I$ is a precipitous ideal. Let $G$ be a generic filter on $P(P_\kappa \lambda)/I$ and $j: V \rightarrow \text{Ult}(V, G)$ be the canonical embedding.

Suppose (i) holds. By the condition on $\kappa$, we have $\{s \in P_\kappa \lambda : |s| \text{ is not weakly inaccessible}\} \in G$. Thus $\text{Ult}(V, G) \models \lnot \lnot \langle \lambda \text{ is not weakly inaccessible} \rangle$. But by the $\gamma$-saturatedness, the cardinal $\gamma$ are preserved and $\lambda$ remains regular. Hence $\text{Ult}(V, G) \models \langle \lambda \text{ is weakly inaccessible} \rangle$. This shows that (i) cannot hold.

Suppose (ii) holds. As above, we see that $\text{Ult}(V, G) \models \lnot \lnot \langle \lambda \text{ is not weakly Mahlo} \rangle$. But by the argument above, $\lambda$ remains as a weakly inaccessible cardinal in $\text{Ult}(V, G)$. Thus $\text{Ult}(V, G) \models \langle \text{there is a cub } C \subseteq \lambda \text{ such that for each } \alpha \in C, \alpha \text{ is not a regular cardinal} \rangle$. By the $\gamma$-saturatedness, we can find a subset $C^*$ of $C = (\gamma + 1)$ such that $C^* \in V$ and $C^*$ is cub in $\lambda$. Hence $V \models \langle \lambda \text{ is not weakly Mahlo} \rangle$, a contradiction. \[\Box\]

**Proof of Theorem 17.** Assume (i) or (ii) holds. Let $I$ be an arbitrary $\kappa$-complete normal ideal on $P_\kappa \lambda$. Since $\lambda$ is singular, it suffices to show that $I$ is not $\gamma$-saturated for each $\gamma < \lambda$. Pick $\gamma < \lambda$. By the hypothesis on $\lambda$, we can find $\delta$ such
that \( \gamma < \delta < \lambda \) and \( \delta \) is either weakly inaccessible or weakly Mahlo depending on (i) or (ii). Now define \( I_\delta \) as follows: \( I_\delta = \{ A \subseteq P. \delta : \tilde{A} \in I \} \) where \( \tilde{A} = \{ s \in P. \lambda : s \cap \delta \in a \} \). We know that \( I_\delta \) is a \( \kappa \)-complete normal ideal. By Lemma 19, \( I_\delta \) is not \( \gamma \)-saturated. Thus by Theorem 11, there is a family \( \langle A_\alpha : \alpha < \gamma \rangle \) of disjoint \( I_\delta \)-positive subsets of \( P. \delta \). Consequently \( \langle \tilde{A}_\alpha : \alpha < \gamma \rangle \) is a collection of disjoint \( I \)-positive subsets of \( P. \lambda \). □

In [5], Jech and Prikry showed that the size of \( 2^\kappa \) may be determined by the behavior of the continuum function below \( \kappa \) assuming the existence of a certain saturated ideal on \( \kappa \). An analogous result holds for \( P. \lambda \).

**Theorem 19.** If GCH holds below \( \kappa \) and \( P. \lambda \) carries a \( \kappa \)-complete \( \lambda^+ \)-saturated normal ideal, then \( 2^\lambda = \lambda^+ \). Furthermore if this ideal is \( \lambda \)-saturated then \( 2^{< \lambda} = \lambda \).

**Proof.** Let \( I \) be a \( \kappa \)-complete \( \lambda^+ \)-saturated normal ideal on \( P. \lambda \). Let \( G \) be a generic filter on \( P(P. \lambda)/I \). Since \( \{ s \in P. \lambda : 2^{\|s\|} = \|s^+\| \} \subseteq G \), we must have \( V[G] \models 2^\lambda = \lambda^+ \). Note: \( \lambda^+ \) in \( \text{Ult}(V, G) \) and \( \lambda^+ \) in \( V \) are the same by the \( \lambda^+ \)-saturatedness. Hence \( V[G] \models |P(\lambda) \cap \text{Ult}(V, G)| \leq (\lambda^+)^V \).

Now for each \( x \in P(\lambda) \cap V \), define \( f_x : P. \lambda \rightarrow P. \lambda \) by \( f_x(s) = s \cap x \). Thus \( [f_x] \in P([\text{id}]) \cap \text{Ult}(V, G) \). Furthermore, for each distinct \( x, y \in P(\lambda) \cap V \), \( [f_x] \neq [f_y] \). Using the fact \( [\text{id}] = \beta^+ \), we get \( V[G] \models |P(\lambda) \cap V| \leq |P(\lambda) \cap \text{Ult}(V, G)| \).

Combining the last two paragraphs, we obtain \( V[G] \models |P(\lambda) \cap V| \leq |(\lambda^+)^V| \). This implies \( V \models 2^\lambda = \lambda^+ \).

Now assume that \( I \) is a \( \kappa \)-complete \( \lambda \)-saturated normal ideal. Let \( G \) be a generic filter on \( P(P. \lambda)/I \). By an argument similar to the one given above, we have for each \( \gamma < \lambda \), \( V[G] \models 2^{\|\gamma\|} = |\gamma|^+ \). Since \( \lambda \) remains a cardinal by \( \lambda \)-saturatedness, \( V[G] \models |P(\gamma) \cap \text{Ult}(V, G)| \leq \lambda \). Once again by an argument similar to the one given above, \( V[G] \models |P(\gamma) \cap V| \leq |P(\gamma) \cap \text{Ult}(V, G)| \). So \( V[G] \models |P(\gamma) \cap V| \leq \lambda \). Hence \( V \models 2^\gamma \leq \lambda \). Thus \( V \models 2^{< \lambda} = \lambda \). □

The following is an immediate consequence.

**Corollary 20.** Suppose GCH holds below \( \kappa \) and either \( 2^\lambda \neq \lambda^+ \) or \( 2^{< \lambda} \neq \lambda \). Then every stationary subset of \( P. \lambda \) splits into \( \lambda \) many disjoint stationary subsets.

**References**


