Discrete Mathematics 311 (2011) 1423–1447

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Locally finite graphs with ends: A topological approach, I. Basic theory

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ARTICLE INFO

Article history:
Available online 12 June 2010

Keywords:
Infinite graph
Locally finite graph
End
Freudenthal compactification
Topology
Boundary

ABSTRACT

This paper is the first of three parts of a comprehensive survey of a newly emerging field: a topological approach to the study of locally finite graphs that crucially incorporates their ends. Topological arcs and circles, which may pass through ends, assume the role played in finite graphs by paths and cycles. The first two parts of the survey together provide a suitable entry point to this field for new readers; they are available in combined form from the ArXiv [18]. They are complemented by a third part [28], which looks at the theory from an algebraic-topological point of view.

The topological approach indicated above has made it possible to extend to locally finite graphs many classical theorems of finite graph theory that do not extend verbatim. While the second part of this survey [19] will concentrate on those applications, this first part explores the new theory as such: it introduces the basic concepts and facts, describes some of the proof techniques that have emerged over the past 10 years (as well as some of the pitfalls these proofs have in stall for the naive explorer), and establishes connections to neighbouring fields such as algebraic topology and infinite matroids. Numerous open problems are suggested.

0. Introduction

The survey [18], of which this paper contains the introduction and Sections 2, 3 and 5, describes a topological framework in which many well-known theorems about finite graphs that appear to fail for infinite graphs have a natural infinite analogue. It has been realised in recent years that many such theorems, especially about paths and cycles, do work in a slightly richer setting: not in the (locally finite) graph itself, but in its compactification obtained by adding its *ends*. For example, the plane graph G in Fig. 0 has three ends. When we add these, we obtain a compact space $|G|$ in which the fat edges form a *circle*—a subspace homeomorphic to the standard topological circle S^1. Allowing such circles as 'infinite cycles', and allowing topological arcs through ends as 'infinite paths', we can restore the truth of many well-known theorems about finite graphs whose infinite analogues would fail if we allowed only the usual finite paths and cycles familiar from finite graphs.

The aim of [18] is to provide a reasonably complete but readable introduction to this new approach, offering a fast track to its current state of the art. It describes all the fundamental concepts, all the main results, and many open problems. Proofs of the most elementary results, listed here in Section 1, can be found in [20, Ch. 8.5]; the reader is encouraged to consult this source early, to get some more feel for the subject. For readers already familiar with those basic techniques, the current paper also describes some more advanced but fundamental proof techniques that cannot be found explicitly in [20].

We begin in Section 1 with the definition of the space $|G|$ and an overview of its *basic properties*. These properties mostly concern the topological analogues of familiar finite concepts involving paths and cycles, such as spanning trees, degrees (of ends), connectivity and so on. It is these analogues that we shall need in the place of their finite counterparts when we wish to extend theorems from finite graph theory to infinite graphs whose naive extension fails.

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.05.023

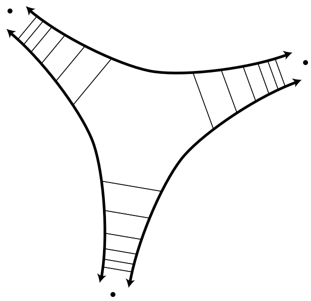

Fig. 0. A circle through three ends.

We continue in Section 2 with some *basic theory*, a body of results that are not yet applications of the new topological concepts but relate them to each other, much in the way their finite counterparts are related. Most results in this section concern the homology of a graph, i.e., its *cycle* and *cut space* and the way they interact. These homology aspects have, so far, been the prime field of application for those topological notions involving ends. But there are other applications too, and reading Section 2 will not be a prerequisite for reading the rest of the paper.

Section 3 explains some *proof techniques*, e.g. for the construction of topological arcs and circles as limits of finite paths and cycles, that have evolved over the past 10 years. These years have seen some considerable simplification of the techniques required to deal with the problems one usually encounters in this area. The aim of Section 3 is to describe the state of the art here, so as to equip those new to the field quickly with the main techniques now available.

Finally, there is an *outlook* in Section 4 to new horizons: extensions to graphs that are not locally finite, and implications of our findings in related fields, such as geometric group theory and infinite matroids. Homological aspects from an algebraic point of view are also indicated; these will be explored more fully in [28].

Open problems are not collected at the end but interspersed within the text. There are plenty of these here, as well as in [19] or the section on applications in [18]. More fundamentally, there is the overall quest to push the general approach further: to identify more theorems about paths and cycles in finite graphs that do not extend naively, and to find the correct topological analogue that does extend.

1. Concepts and basic facts

Throughout this section, let G be a fixed infinite, locally finite, connected graph. This section serves to introduce the concepts on which our topological approach to the study of such graphs is based: the space $|G|$ formed by G and its ends; topological paths, circles and spanning trees in this space; notions of connectivity in $|G|$. The style will be descriptive and informal, aiming for overall readability; should any technical points remain unclear, the reader is referred to [20, Ch. 8.5] for more formal definitions of the concepts introduced here, and to [20] in general for graph-theoretic terms and notation.

Terms such as 'path' or 'connected', which formally have different meanings in topology and in graph theory, will be used according to context: in the graph-theoretical sense for graphs, and in the usual topological sense for topological spaces. If the context is ambiguous, the two meanings will probably coincide, making a formal distinction unnecessary.

We call 1-way infinite paths *rays*, and 2-way infinite paths *double rays*. An *end* of G is an equivalence class of rays in G, where two rays are considered equivalent if no finite set of vertices separates them in G. The graph shown in Fig. 0 has three ends; the $\mathbb{Z} \times \mathbb{Z}$ grid has only one, the infinite binary tree has continuum many. We write $\Omega(G)$ for the set of ends of G.

Topologically, we view G as a cell complex with the usual topology. Adding its ends compactifies it, with the topology generated by the open sets of G and the following additional basic open sets. For every finite set S of vertices and every end ω, there is a unique component C of $G - S$ in which every ray of ω has a tail. We say that ω *lives in* C and write $C =: C(S, \omega)$. Now for every such S and every component C of $G - S$, we declare as open the union $\hat{C}$ of C with the set of ends living in C and with all the 'open' S–C edges of G (i.e., without their endpoints in S). We denote the space just obtained by $|G|$.[2]

Theorem 1.1 *([17,20]). The space $|G|$ is compact and metrizable.*

The space $|G|$ is known as the *Freudenthal compactification* of G. The main feature of its topology is that rays in G converge as they should: to the end of which they are an element. One can show [21, Prop. 4.5] that every Hausdorff topology on $G \cup \Omega(G)$ with (essentially) this feature (and which induces the 1-complex topology on G) refines the topology of $|G|$. This identifies $|G|$ as the unique most powerful Hausdorff topology on $G \cup \Omega(G)$, in a sense that can be made quite precise [21].[3]

[2] The open neighbourhoods of ends are defined slightly more generally in [20], but for locally finite graphs the two definitions are equivalent. Topologies for graphs with infinite degrees are discussed in Section 4.

[3] The identity to $|G|$ from $G \cup \Omega(G)$ with any finer topology is continuous, so in $|G|$ there will be at least as many arcs and circles (and possibly more). Arcs and circles in $|G|$ will be our main tools.

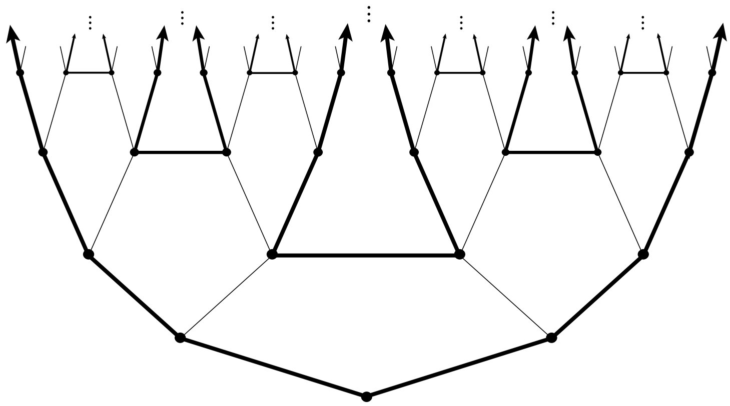

Fig. 1.1. The heavy edges form a *wild circle*.

Of the many natural aspects of this topology let us mention just two more, which relate it to better-known objects. Consider the binary tree T_2, and think of its rays from the root as 0–1 sequences. The resulting bijection between the ends of T_2 and these sequences is a homeomorphism between $\Omega(T_2)$, as a subspace of $|T_2|$, and $\{0, 1\}^{\mathbb{N}}$ with the product topology. Identifying pairs of ends whose sequences specify the same rational (one sequence ending on zeros, the other on 1s) turns this bijection into a homeomorphism from the resulting identification space of $\Omega(T_2)$ to $[0, 1]$. Without such identification, on the other hand, $\Omega(G)$ is always a subset of a Cantor set.

Instead of paths and cycles in G we can now consider *arcs* and *circles* in $|G|$: homeomorphic images of the real interval $[0, 1]$ and of the complex unit circle S^1. While paths and cycles are examples of arcs and circles, Fig. 0 shows a circle that is not a cycle. Arcs and circles that are not paths or cycles must contain ends. An arc containing uncountably many ends always induces the ordering of the rationals on a subset of its vertices [8]. Such arcs, and circles containing them, are called *wild* (Fig. 1.1), but they are quite common.

Arcs and circles are examples of a natural type of subspace of $|G|$: subspaces that are the closure in $|G|$ of some subgraph of G.[4] We call such a subspace X of $|G|$ a *standard subspace*, and write $V(X)$ and $E(X)$ for the set of vertices or edges it contains. Note that the ends in X are ends of G, not of the subgraph that gave rise to X; in particular, ends in X need not have a ray in that subgraph. Which ends of G are in X is determined just by $V(X)$: they are precisely the ends that are limits of vertices in X.

Given a standard subspace X and an end $\omega \in X$, the maximum number of arcs in X that end in ω but are otherwise disjoint is the *(vertex-) degree* of ω in X; the maximum number of edge-disjoint arcs in X ending in ω is its *edge-degree* in X. Both maxima are indeed attained, but it is non-trivial to prove this [12]. End degrees behave largely as expected; for example, the connected standard subspaces in which every vertex and every end has (vertex-) degree 2 are precisely the circles. (Use Lemma 1.2 to prove this.) In [18, Section 4.1] we define a third type of end degrees, their *relative degree*, which is useful for the application of end degrees to extremal-type problems about infinite graphs.

Standard subspaces have the important property that connectedness and arc-connectedness are equivalent for them. This will often be convenient: while connectedness is much easier to prove (see Lemma 1.5), it is usually arc-connectedness that we need.

Lemma 1.2 (*[20,23,50]*). *Connected standard subspaces of $|G|$ are locally connected and arc-connected.*

The proof that a connected standard subspace X is locally connected is not hard: an open neighbourhood $\hat{C} \cap X$ of an end ω will be connected if we choose the set S in its definition so as to minimize the number of C–S edges in X. But local connectedness is not a property we shall often use directly. Its role here is that it offers a convenient stepping stone towards the proof of arc-connectedness.[5] Direct proofs that X is arc-connected can be found in [23,30], and we shall indicate one in Section 3. Connected subspaces of $|G|$ that are neither open nor closed need not be arc-connected [30].

Corollary 1.3. *The arc-components of a standard subspace are closed.*

Proof. The closure in X of an arc-component of a standard subspace X is connected and itself standard, and hence arc-connected by Lemma 1.2. $\square$

The edge set $E(C)$ of any circle C will be called a *circuit*. Given any set F of edges in G, we write $\overline{F}$ for the closure of $\bigcup F$ in $|G|$, and call $\overline{F}$ the standard subspace *spanned by* F. (This is with slight abuse of our usual notation, in which we write $\overline{X}$

4 It takes a line of proof that arcs and circles do indeed have this property, i.e., that the union of their edges is dense in them. This is because any arc between distinct ends must contain an edge incident with finite set S that separates them; this will follow from Lemma 1.4 below.

5 By general 'continuum theory' [42], compact, connected and locally connected metric spaces are arc-connected.

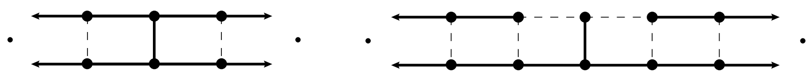

Fig. 1.2. An ordinary spanning tree of G (left), and a topological spanning tree of G (right).

for the closure in $|G|$ of a subset $X \subseteq |G|$.) Similarly, we write $\mathring{F}$ for the set of all inner points of edges in F (while usually we write $\mathring{X}$ for the interior of a subset $X \subseteq |G|$).

The set of edges of G across a partition $\{V_1, V_2\}$ of $V(G)$ is a *cut* of G; the sets V_1, V_2 are the *sides* of this cut. A minimal non-empty cut is a *bond*.

The following lemma is one of our basic tools for handling arcs. It says that an arc cannot 'jump across' a *finite* cut without containing an edge from it:

Lemma 1.4 (*Jumping Arc Lemma [20]*). *Let F be a cut of G with sides V_1, V_2. Let X be a standard subspace of $|G|$, and put $X_i := X \cap V_i$ $(i = 1, 2)$.*

(i) *If F is finite, then $\overline{V_1} \cap \overline{V_2} = \emptyset$, and there is no arc in $|G| \setminus \mathring{F}$ with one endpoint in V_1 and the other in V_2.*
(ii) *If $F \cap E(X)$ is infinite, then $\overline{X_1} \cap \overline{X_2} \neq \emptyset$, and there may be such an arc in X.*

The proof of Lemma 1.4(i) is straightforward from the definition of the topology of $|G|$: deleting the edges of a finite cut splits $|G|$ into two disjoint open sets. When $F \cap E(X)$ is infinite, an intersection point $\omega \in \overline{X_1} \cap \overline{X_2}$ can be obtained as the limit of two vertex sequences, one in X_1 and the other in X_2, that are joined by infinitely many cut edges of X.

Although the 'jumping arc' is a nice way to memorize Lemma 1.4, its main assertion is not about arcs but about connectedness. It implies that connectedness for standard subspaces can be characterized in graph-theoretical terms alone, without any explicit mention of ends or the topology of $|G|$:

Lemma 1.5 (*[20]*). *A standard subspace of $|G|$ is connected if and only if it contains an edge from every finite cut of G of which it meets both sides.*

We shall say that a standard subspace X of $|G|$ is *k-edge-connected* if the deletion of fewer than k edges will not make it disconnected. Similarly, X is *k-vertex-connected* if $V(X) > k$ and the deletion of fewer than k vertices and their incident edges does not leave a disconnected space. Note that for $k = 1$ both notions coincide with ordinary topological connectedness, and that for $X = |G|$ the space X is k-edge-connected or k-vertex-connected if and only if the graph G is k-edge-connected or k-connected (by Lemma 1.5).

How about deleting ends as well as vertices and/or edges? For $X = |G|$, this will never help to disconnect X: if deleting a finite set U of vertices and any set of ends disconnects $|G|$, then so does the deletion of U alone, and similarly for edges. For arbitrary standard subspaces X, however, deleting ends can make sense. It will normally result in a subspace that is no longer standard, but the main reason for primarily considering standard subspaces, that connectedness in them is equivalent to arc-connectedness (Lemma 1.2), is preserved.[6]

So let us call a subspace $X \subseteq |G|$ *substandard* if its closure in $|G|$ is standard (so that X contains no partial edges), and *k-connected* if after the deletion of fewer than k vertices, edges or ends it will still be arc-connected. This makes sense in the context of Menger's theorem, which Thomassen and Vella [50] proved for topological spaces X that include all subspaces of $|G|$: given any two points $a, b \in X$ and $k \in \mathbb{N}$, if for every set $S \subseteq X$ of fewer than k points there is an a–b arc in $X \setminus S$, then X contains k arcs from a to b that pairwise meet only in a and b.[7] Hence in a k-connected standard or substandard subspace any two vertices or ends can be linked by k independent arcs: a useful property that can fail in standard subspaces that are merely k-vertex-connected.

To work with arcs and circles in a way that resembles finite graph theory, we need one more addition to our topological toolkit: the notion of a topological spanning tree. A *topological spanning tree* of a connected standard subspace X of $|G|$ is an (arc-) connected standard subspace $T \subseteq X$ of $|G|$ that contains every vertex (and hence every end) of X but contains no circle. A topological spanning tree of $|G|$ will also be called a *topological spanning tree of G*.

The closure $\overline{T}$ of an ordinary spanning tree T of G is not normally a topological spanning tree: as soon as T contains disjoint rays from the same end, $\overline{T}$ will contain a circle. Conversely, the subgraph of G underlying a topological spanning tree need not be a graph-theoretical tree: it will be acyclic, of course, but it need not be connected. Fig. 1.2 shows examples of both these phenomena in the double ladder.

Ordinary spanning trees whose closures are topological spanning trees do always exist, however: all *normal* spanning trees have this property, and all countable connected graphs have normal spanning trees. (A spanning tree T of G is *normal* if, for a suitable choice of a root, the endvertices of every edge of G are comparable in the tree-order of T. See [20].) Often, therefore, normal spanning trees are the best choice of a spanning tree for our purposes.

More generally, we have the following existence lemma for connected standard subspaces X of $|G|$:

6 Georgakopoulos's characterization of the subspaces that are connected but not arc-connected [30] implies that any such space has uncountably many arc-components consisting of one end only. One clearly cannot obtain such a space by deleting finitely many ends from a connected standard subspace.

7 There is also version of Menger's theorem for disjoint arcs between sets $A, B \subseteq X$ in [50], which is easier to prove.

Lemma 1.6. *Every standard subspace $Z \subseteq X$ not containing a circle extends to a topological spanning tree of X.*

Proof. We begin by enumerating the edges in $E(X) \setminus E(Z)$. We then go through these edges one by one, considering each for deletion from X. We delete an edge if this does not disconnect the space $X \setminus \mathring{F}$, where F is the set of edges already deleted. Having considered every edge in $E(X) \setminus E(Z)$, we are left with a standard subspace T that contains $V(X)$ but contains no circle: this would have an edge in $E(X) \setminus E(Z)$, which was considered for deletion and should have been deleted. The space T is connected (cf. Lemma 1.5), and hence arc-connected (Lemma 1.2). By construction, $Z \subseteq T \subseteq X$ as desired. □

Unlike in finite graphs, it is considerably harder to construct a topological spanning tree 'from below' (maintaining acyclicity) than, as we did just now, 'from above' (maintaining connectedness, and using the non-trivial Lemma 1.2). A proof 'from below' will be indicated in the proof of Lemma 3.2.

The properties of topological spanning trees resemble those of spanning trees of finite graphs. For example:

Lemma 1.7 ([20]). *The following assertions are equivalent for a standard subspace T of $|G|$ contained in a standard subspace X:*

(i) *T is a topological spanning tree of X.*
(ii) *T is maximally acirclic, that is, it contains no circle but adding any edge of $E(X) \setminus E(T)$ creates one.*
(iii) *T is minimally connected, that is, it is connected but deleting any edge of T disconnects it.*

The proof of Lemma 1.7 needs Lemma 1.2: we use connectedness in the form of arc-connectedness, but prove only ordinary topological connectedness.

It is not hard to show that the arcs which a topological spanning tree T of X contains between any two of its points are unique. Hence every *chord* $e \in E(X) \setminus E(T)$ creates a well-defined *fundamental circuit* C_e in $T \cup e$, while every edge $f \in E(T)$ lies in a well-defined *fundamental cut* D_f of X, the set of edges in X between the two arc-components of $T \setminus \mathring{f}$.[8] Note that such fundamental cuts are finite by Lemma 1.4(ii), since the two arc-components of $T \setminus \mathring{f}$ together contain all the vertices of X and are closed (Corollary 1.3) but disjoint.

For $X = |G|$, topological spanning trees compare with ordinary spanning trees as follows:

Lemma 1.9. *Let G be a locally finite connected graph.*

(i) *The fundamental circuits of any ordinary spanning tree of G are finite, but its fundamental cuts may be infinite.*
(ii) *The fundamental circuits of any topological spanning tree of G may be infinite, but its fundamental cuts are finite.*
(iii) *The fundamental circuits and cuts of normal spanning trees of G are finite.*

The fact that the fundamental cuts of a topological spanning tree of G are finite implies by Lemma 1.4 that they are in fact bonds.

Fundamental circuits and cuts of topological spanning trees are subject to the same duality as for ordinary spanning trees:

Lemma 1.8. *Let T be a topological spanning tree of a standard subspace X of $|G|$, and let $f \in E(T)$ and $e \in E(X) \setminus E(T)$. Then $e \in D_f \Leftrightarrow f \in C_e$.* □

2. The topological cycle space

The way in which cycles and cuts interact in a graph can be described algebraically: in terms of its 'cycle space', its 'cut space', and the duality between them. In this section we show how the cycle space theory of finite graphs extends to locally finite graphs in a way that encompasses infinite circuits. The fact that this can be done, that our topological circuits, cuts and spanning trees interact in the same way as ordinary cycles, cuts and spanning trees do in a finite graph, is by no means clear but rather surprising. For example, there is nothing visibly topological about a finite cut in an infinite graph,[9] so the fact that the edge sets orthogonal to its finite cuts are precisely its topological circuits and their sums (Theorem 2.6) comes as a pleasant surprise: it provides a natural answer to a natural question, but not by design—it was not 'built into' the definition of a circle.

As it turns out, extending finite cycle space theory in this way is not only possible but also necessary: it is the 'topological cycle space' of a locally finite graph, not its usual finitary cycle space, that interacts with its other structural features, such as planarity, in the way we know it from finite graphs. This is discussed in some depth in [18, Section 4]. In Section 4 we shall see how our theory integrates with the broader topological context of (singular) homology in more general spaces.

As before, let G be a fixed infinite, connected, locally finite graph. We start by defining the 'topological cycle space' $\mathcal{C}(G)$ of G in analogy to the mod-2 (or 'unoriented') cycle space of a finite graph[10]: its elements will be sets of edges (that is to say,

[8] For $X = |G|$ this is a cut of G in the usual sense.

[9] With hindsight, of course, Lemma 1.5 shows that this impression is wrong.

[10] One can also define 'oriented' versions $\vec{\mathcal{C}}(G)$ of $\mathcal{C}(G)$, with integer or real coefficients [25]. Some of the theorems listed below have obvious oriented analogues, with very similar proofs. But the differences have not yet been investigated systematically and may well be worth further study; see Sections 2.3–2.4 and [18, Section 4.3] for some good problems.

maps $E(G) \to \mathbb{F}_2$, or formal sums of edges with coefficients in $\mathbb{F}_2$) generated from circuits by taking symmetric differences of edge sets. These edge sets, the circuits, and the sums may be infinite.

Let us make this more precise. Let the *edge space* $\mathcal{E}(G)$ of G be the $\mathbb{F}_2$-vector space of all maps $E(G) \to \mathbb{F}_2$, which we think of as subsets of $E(G)$ with symmetric difference as addition. The *vertex space* $\mathcal{V}(G)$ is defined likewise. Call a family $(D_i)_{i\in I}$ of elements of $\mathcal{E}(G)$ *thin* if no edge lies in D_i for infinitely many i. Let the *(thin) sum* $\sum_{i\in I} D_i$ of this family be the set of all edges that lie in D_i for an odd number of indices i. Given any subset $\mathcal{D} \subseteq \mathcal{E}(G)$, the edge sets that are sums of sets in $\mathcal{D}$ form a subspace of $\mathcal{E}(G)$. The *(topological) cycle space* $\mathcal{C}(G)$ of G is the subspace of $\mathcal{E}(G)$ consisting of the sums of circuits. The *cut space* $\mathcal{B}(G)$ of G is the subspace of $\mathcal{E}(G)$ consisting of all the cuts in G and the empty set. (Unlike the circuits, these already form a subspace.) We sometimes call the elements of $\mathcal{C}(G)$ *algebraic cycles* in G.

2.1. Generating sets

The sums of elements of $\mathcal{D} \subseteq \mathcal{E}(G)$, and the subspace of $\mathcal{E}(G)$ consisting of all those sums, are said to be *generated* by $\mathcal{D}$. For example, the cycle space of the graph in Fig. 0 is generated by its central hexagon and its squares, or by the infinite circuit consisting of the fat edges and all the squares. Bruhn and Georgakopoulos [10] proved that if $\mathcal{D}$ is thin, the space it generates is closed under thin sums. As we shall see, this applies to both $\mathcal{C}(G)$ and $\mathcal{B}(G)$.

As in Lemma 1.9, the duality between $\mathcal{C}(G)$ and $\mathcal{B}(G)$ – which we address more thoroughly later – is reflected by a switch between topological and ordinary spanning trees:

Theorem 2.1 ([20]).

(i) *Given an ordinary spanning tree of G, its fundamental cuts generate $\mathcal{B}(G)$, but its fundamental circuits need not generate $\mathcal{C}(G)$.*
(ii) *Given a topological spanning tree of G, its fundamental circuits generate $\mathcal{C}(G)$, but its fundamental cuts need not generate $\mathcal{B}(G)$.*
(iii) *Given a normal spanning tree of G, its fundamental circuits generate $\mathcal{C}(G)$, and its fundamental cuts generate $\mathcal{B}(G)$.*

To prove the first assertion in Theorem 2.1(ii), one shows that a given set $D \in \mathcal{C}(G)$ equals the sum $\sum C_e$ taken over all chords e of the topological spanning tree such that $e \in D$. One has to show that these C_e form a thin family (use Lemmas 1.8 and 1.9(ii)), but also that $D = \sum C_e$. While for finite G one just notes that $D + \sum C_e$ consists of tree edges and deduces that $D + \sum C_e = \emptyset$ (yielding $D = \sum C_e$), this last implication is now non-trivial: it is not clear that the tree, which by assumption contains no circuit, cannot contain a sum of circuits. The proof in [20] deduces this, i.e. that D and $\sum_e C_e$ also agree on tree edges, from Theorem 2.5(i) $\Rightarrow$ (iv) below, which in turn is an easy consequence of the jumping arc lemma.

For the second assertion in (ii), recall that the edge set of a topological spanning tree can miss an infinite cut (as in Fig. 1.2). Such a cut will not be a sum of fundamental cuts D_f, because any such sum contains all those tree-edges f.

The proof of assertion (i) is analogous to that of (ii), though in the first statement one also has to show that the fundamental cuts do not generate *more* than $\mathcal{B}(G)$: that a thin sum of cuts is again a cut. (See the discussion after Theorem 2.2.) For the second statement of (i), recall that the edges of an ordinary spanning tree may contain a circuit, which will not be a sum of fundamental circuits C_e because any such sum contains all those chords e.

Assertion (iii) follows from (i) and (ii).

By Lemmas 1.8 and 1.9, the fundamental circuits and cuts of normal spanning trees form thin families. Hence by the result of [10] cited earlier, the subspaces they generate in $\mathcal{E}(G)$ are closed under thin sums. By Theorem 2.1(iii) these are the spaces $\mathcal{C}(G)$ and $\mathcal{B}(G)$:

Theorem 2.2. (i) *$\mathcal{C}(G)$ is generated by finite circuits and is closed under infinite thin sums.*
(ii) *$\mathcal{B}(G)$ is generated by finite bonds and is closed under infinite thin sums.*

Since we already used the closure statement of (ii) for cuts in our proof of Theorem 2.1, our proof of this part of Theorem 2.2(ii) has been circular. But one does not need Theorem 2.1, or indeed [10], to prove that a sum of cuts is again a cut, a set of edges across a vertex partition. One way to do this directly is to construct that partition explicitly. The simplest way, however, is indirect: to show first that the cuts are precisely the edge sets that meet every finite circuit in an even number of edges,[11] and then to observe that the set of those edge sets is closed under taking sums.

But also the closure statement of (i), for $\mathcal{C}(G)$, is non-trivial. Indeed, if $\sum D_i$ is a thin sum of sets $D_i = \sum_j C_i^j \in \mathcal{C}(G)$, where the C_i^j are circuits, the combined sum $\sum_{i,j} C_i^j$ need not be thin. By Theorem 2.3 below one can assume for each i that its own C_i^j are disjoint, in which case the combined family (C_i^j) will be thin (because the family (D_i) is thin by assumption). But Theorem 2.3 is hard, and too big a tool for this proof. As for cuts, the quickest way to prove that $\mathcal{C}(G)$ is closed under thin sums is to use Theorem 2.6(i): that the elements of $\mathcal{C}(G)$ are precisely those sets of edges that meet every finite cut evenly, a class of edge sets that is obviously closed under sums. See also the discussion after Theorem 2.6.

A bond is *atomic* if it consists of all the edges at some fixed vertex. Following Tutte, let us call a circuit in $|G|$ *peripheral* if it contains every edge between its incident vertices (i.e., has no chord in G) and the set of these vertices does not separate G.

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11 This is Theorem 2.6(ii); its proof will be independent of what we are proving here.

Theorem 2.3. (i) *Every element of* $\mathcal{C}(G)$ *is a disjoint union of circuits.*
(ii) *If* G *is* 3*-connected, its peripheral circuits generate* $\mathcal{C}(G)$.
(iii) *Every element of* $\mathcal{B}(G)$ *is a disjoint union of bonds.*
(iv) *The atomic bonds of* G *generate* $\mathcal{B}(G)$.

Statement (i) was first established in [22], with a long and involved proof. As techniques developed, simpler and more elegant proofs were found [31,50,51]. We shall discuss these techniques, and the different proofs they lead to, in Section 3. Statement (ii), which extends a theorem of Tutte for finite graphs (see [20]), is due to Bruhn [4]. It fails if we allow only finite peripheral circuits or only finite sums. Statements (iii) and (iv) are easy, with proofs as for finite graphs.

Theorem 2.3 implies that, as in finite graphs, the circuits in $|G|$ are the minimal non-empty elements of $\mathcal{C}(G)$, while the bonds in G are the minimal non-empty elements of $\mathcal{B}(G)$.

Finally, there is a generating theorem for the cycle space of a finite graph whose extension to $|G|$ and $\mathcal{C}(G)$ requires an interesting twist. When G is finite, its cycle space is generated by the edge sets of its *geodetic* cycles: those that contain a shortest path in G between any two of their vertices [20]. This is still true for locally finite graphs (with 'arc' instead of 'path'), but only if we measure the length of an arc in the right way: rather than by counting its edges, we have to assign lengths to the edges of G, and then measure the length of an arc by adding up the lengths of its edges. By giving edges shorter lengths if they lie 'far out near the ends', it is possible to do this in such a way that the resulting metric on $|G|$ induces its original topology.[12] And then, no matter how exactly we choose the edge lengths, Georgakopoulos and Sprüssel proved that the geodetic circles do what they should:

Theorem 2.4 (*[35]*)**.** *Given any edge-length function* $\ell: E(G) \to [0, \infty)$ *whose resulting metric on* $|G|$ *induces the original topology of* $|G|$*, the circuits of the geodetic circles in* $|G|$ *generate* $\mathcal{C}(G)$.

2.2. Characterizations of algebraic cycles

There are various equivalent ways to describe the elements of $\mathcal{C}(G)$ and of $\mathcal{B}(G)$, each extending a similar statement about finite graphs. Let us list these now, beginning with $\mathcal{C}(G)$.

A closed topological path in a standard subspace X of $|G|$, based at a vertex, is a *topological Euler tour* of X if it traverses every edge in X exactly once. One can show that if $|G|$ admits a topological Euler tour it also has one that traverses every end at most once [32]. For arbitrary standard subspaces this is false: consider the closure of two disjoint double rays in the $\mathbb{Z} \times \mathbb{Z}$ grid.

Recall that, given a set D of edges, $\overline{D}$ denotes the closure of the union of all the edges in D, the standard subspace of $|G|$ *spanned by D*.

Theorem 2.5. *The following statements are equivalent for sets* $D \subseteq E(G)$*:*

(i) $D \in \mathcal{C}(G)$*, that is to say, D is a sum of circuits in* $|G|$.
(ii) *Every component of* $\overline{D}$ *admits a topological Euler tour.*
(iii) *Every vertex and every end has even (edge-) degree in* $\overline{D}$.
(iv) *D meets every finite cut in an even number of edges.*

The equivalence of (i) and (ii) was proved in [21] for $D = E(G)$ and extended to arbitrary D by Georgakopoulos [31]; we shall meet the techniques needed for the proof in Section 3. The equivalence with (iii) is a deep theorem, due to Bruhn and Stein [12] for $D = E(G)$ and to Berger and Bruhn [2] for arbitrary D. Note that (iii) assumes that end degrees have a parity even when they are infinite. Finding the right way to divide ends of infinite edge-degree into 'odd' and 'even' was one of the major difficulties to overcome for this characterization of $\mathcal{C}(G)$.

The equivalence of (i) with (iv), again from [21], is one of the cornerstones of topological cycle space theory: its power lies in the fact that the finitary statement in (iv) is directly compatible with compactness proofs (see Section 3). Its implication (i)→(iv) follows from the jumping arc lemma, applied to the circles whose circuits sum to the given set $D \in \mathcal{C}(G)$. For the converse implication one compares a given set D as in (iv) with the sum $\sum C_e$ of fundamental circuits of a topological spanning tree taken over all chords $e \in D$. It is clear that D agrees with this sum on chords. Using Lemmas 1.8 and 1.9 one proves that they also agree on tree edges.

2.3. Cycle-cut orthogonality

Given a set $\mathcal{F} \subseteq \mathcal{E}(G)$, let us write $\mathcal{F}_{\text{fin}}$ for the set of finite elements of $\mathcal{F}$. Call two sets $D, F \subseteq E(G)$ *orthogonal* if $|D \cap F|$ is finite and even, and put

$$\mathcal{F}^{\perp} := \{D \subseteq E(G) \mid D \text{ is orthogonal to every } F \in \mathcal{F}\}.$$

Let us abbreviate $\mathcal{V} := \mathcal{V}(G)$, $\mathcal{E} := \mathcal{E}(G)$, $\mathcal{C} := \mathcal{C}(G)$ and $\mathcal{B} := \mathcal{B}(G)$.

[12] This will be discussed in detail in Section 4.2.

The following theorem describes the duality between the topological cycle and cut space of G in terms of orthogonal sets. Its four assertions show some interesting symmetries: they can be summarized neatly as 'all equations containing each of the symbols $=$, $\mathcal{C}$, $\mathcal{B}$, $_{\text{fin}}$ and $^{\perp}$ exactly once (and no others)'.

Theorem 2.6. (i) $\mathcal{C} = \mathcal{B}_{\text{fin}}^{\perp}$.
(ii) $\mathcal{B} = \mathcal{C}_{\text{fin}}^{\perp}$.
(iii) *If G is 2-connected, then* $\mathcal{C}^{\perp} = \mathcal{B}_{\text{fin}}$.
(iv) $\mathcal{B}^{\perp} = \mathcal{C}_{\text{fin}}$.

Theorem 2.6(i) is just a reformulation of Theorem 2.5 (i) $\Leftrightarrow$ (iv).

Assertion (ii) is the dual (see Section 4.4) finitary characterization of the cut space. An interesting feature of its seemingly trivial inclusion $\mathcal{B} \subseteq \mathcal{C}_{\text{fin}}^{\perp}$ is that it requires an application of Theorem 2.3(i). Indeed, any cut F is clearly orthogonal to every finite circuit. But an arbitrary finite element D of $\mathcal{C}$ might come as an infinite sum of circuits (finite or infinite), and even if F is orthogonal to every term in the sum it need not be orthogonal to D. However, the circuits into which D decomposes *disjointly* by Theorem 2.3(i) will be finite and yield D in a finite sum, which preserves orthogonality. The proof of $\mathcal{B} \supseteq \mathcal{C}_{\text{fin}}^{\perp}$ is the same as for finite graphs [20]: contracting the edges not in a given set $D \in \mathcal{C}_{\text{fin}}^{\perp}$ leaves a graph with edge set D, whose finite circuits have to be even by definition of D, because they expand to finite circuits of G. This contracted bipartite graph defines a partition of $V(G)$ showing that D is a cut.

The inclusion $\mathcal{C}^{\perp} \supseteq \mathcal{B}_{\text{fin}}$ in (iii) is equivalent to $\mathcal{C} \subseteq \mathcal{B}_{\text{fin}}^{\perp}$ of (i). The converse inclusion is due to Richter and Vella [46]. Its interesting part is that an edge set orthogonal to every circuit cannot be infinite. (It will be a cut by (ii), since in particular it is orthogonal to every finite circuit.) It is here that 2-connectedness is needed; assuming 2-edge-connectedness, for example, is easily seen not to be enough.

In (iv), the weaker inclusion of $\mathcal{B}^{\perp} \subseteq \mathcal{C}$ follows from (i). To see that an edge set orthogonal to all cuts cannot be infinite, assume it is, pick infinitely many independent edges, and obtain a contradiction by extending these to a cut. The inclusion $\mathcal{B}^{\perp} \supseteq \mathcal{C}_{\text{fin}}$ is equivalent to the inclusion $\mathcal{B} \subseteq \mathcal{C}_{\text{fin}}^{\perp}$ of (i).

Theorem 2.6(ii) has a nice corollary (which we already used in our proofs of Theorems 2.1 and 2.2): that $\mathcal{C}(G)$ and $\mathcal{B}(G)$ are closed under infinite sums. While this is not immediate from the definition of either $\mathcal{C}(G)$ or $\mathcal{B}(G)$, the proof that $\mathcal{B}_{\text{fin}}^{\perp}$ $(=\mathcal{C}(G))$ and $\mathcal{C}_{\text{fin}}^{\perp}$ $(=\mathcal{B}(G))$ are closed under thin sums is immediate from the definition of these sets.

We remark further that, unlike in finite graphs, there can be edge sets that are orthogonal to all circuits but not to all elements of $\mathcal{C}$, and edge sets that are orthogonal to all bonds but not to all cuts [20] (even when G is 2-connected).

For finite G it is well known that $\dim(\mathcal{E}) = \dim(\mathcal{C}) + \dim(\mathcal{B})$ (see e.g. [20]), so $\mathcal{E}/\mathcal{B} \simeq \mathcal{C}$ and $\mathcal{E}/\mathcal{C} \simeq \mathcal{B}$. All this is still true for infinite G and our topological cycle space $\mathcal{C}$. But it is more instructive to find canonical such isomorphisms: to define an epimorphism $\sigma\colon \mathcal{E} \to \mathcal{C}$ with kernel $\mathcal{B}$, and an epimorphism $\tau\colon \mathcal{E} \to \mathcal{B}$ with kernel $\mathcal{C}$. This can be done with Theorem 2.6. Indeed, let T be a normal spanning tree[13] of G, and write C_e and D_f for its fundamental circuits and cuts. By Lemma 1.9, all these are finite. Given $E \subseteq E(G)$, define

$$\sigma(E) := \sum \{C_e : e \in E(G) \smallsetminus E(T) \text{ and } |E \cap C_e| \text{ is odd}\}$$

and

$$\tau(E) := \sum \{D_f : f \in E(T) \text{ and } |E \cap D_f| \text{ is odd}\}.$$

Corollary 2.7. (i) *The map σ is an epimorphism $\mathcal{E} \to \mathcal{C}$ with kernel $\mathcal{B}$.*
(ii) *The map τ is an epimorphism $\mathcal{E} \to \mathcal{B}$ with kernel $\mathcal{C}$.*

Proof. (i) To see that σ is onto, let $D \in \mathcal{C}$ be given and let E consist of those chords e whose fundamental circuits C_e sum to D. The kernel of σ consists of those sets E that meet every C_e evenly. These are precisely the sets E that meet every finite circuit evenly, since these are generated by finitely many C_e. By Theorem 2.6(ii), these sets E are precisely those in $\mathcal{B}$.

(ii) is analogous, using Theorem 2.6(i). □

By Theorem 2.6, subsets of sets in $\mathcal{C}$ cannot contain an odd cut, and subsets of sets in $\mathcal{B}$ cannot contain an odd circuit. These properties are the only obstructions to being a subset of a set in $\mathcal{C}$ or $\mathcal{B}$:

Theorem 2.8 ([20,31]). *Let $E \subseteq E(G)$ be any set of edges.*

(i) *E extends to an element of $\mathcal{B}$ if and only if E contains no odd circuit.*
(ii) *E extends to an element of $\mathcal{C}$ if and only if E contains no odd bond.*

13 For each of σ and τ defined below we shall need that T is both an ordinary and a topological spanning tree.

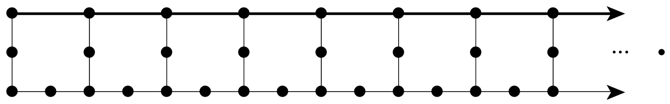

Fig. 2.1. An infinite pedestrian graph.

Proof. We prove the backward implications.

(i) Pick a spanning tree in each component of the graph $(V(G), E)$. Adding edges of $E(G) \setminus E$, extend the union of these trees to a spanning tree T of G. Consider the sum

$$F := \sum_{f \in E \cap E(T)} D_f \ \in \mathcal{B}$$

of fundamental cuts of T. (The sum is thin by Lemmas 1.8 and 1.9.) We claim that $E \subseteq F$. Let $e \in E$ be given. If $e \in E(T)$ then $e \in F$, because D_e is the only fundamental cut of T containing e. If e is a chord of T, then its fundamental circuit C_e lies in E, by construction of T. As C_e is finite and hence even (by assumption), Lemma 1.8 implies that $e \in D_f$ for an odd number of edges $f \in E \cap E(T)$, giving $e \in F$ as desired.

(ii) In every (arc-) component of the standard subspace $\overline{E(G) \setminus E}$ pick a topological spanning tree (Lemma 1.6). Use Lemma 1.6 again to extend their union to a topological spanning tree T of G; note that the additional edges will be in E, since adding them did not create a circle. Consider the sum

$$D := \sum_{e \in E \setminus E(T)} C_e \ \in \mathcal{C}$$

of fundamental circuits of T. (The sum is thin by Lemmas 1.8 and 1.9.) We claim that $E \subseteq D$. Let $d \in E$ be given. If $d \notin E(T)$ then $d \in D$, because C_d is the only fundamental circuit of T containing d. If $d \in E(T)$, then its fundamental cut D_d lies in E, by construction of T. As D_d is finite (Lemma 1.9) and hence even (by assumption; recall that fundamental cuts are in fact bonds), Lemma 1.8 implies that $d \in C_e$ for an odd number of edges $e \in E \setminus E(T)$, giving $d \in D$ as desired. □

2.4. Orthogonal decomposition

If we can write a set $E \subseteq E(G)$ as a sum $E = D + F$ with $D \in \mathcal{C}$ and $F \in \mathcal{B}$, we call this a *decomposition* of E. If the sets D and F are orthogonal (which for infinite sets is not automatic, their intersection could be infinite or even odd; cf. Lemma 1.4), we call this decomposition of E *orthogonal*.

Problem 2.9. (i) For which G does every set $E \in \mathcal{E}(G)$ have a decomposition?
(ii) For which G does every set $E \in \mathcal{E}(G)$ have an orthogonal decomposition?
(iii) For which G does every set $E \in \mathcal{E}(G)$ have a decomposition $E = D + F$ with either $D \in \mathcal{C}(G)$ and $F \in \mathcal{B}_{\text{fin}}(G)$ or $D \in \mathcal{C}_{\text{fin}}(G)$ and $F \in \mathcal{B}(G)$?
(iv) When are such decompositions unique?

Note that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) for G fixed, by Theorem 2.6.

If G is finite, a look at the dimensions of the subspaces involved shows that Problem 2.9 has a clear solution: the answer is 'yes' on all counts if $\mathcal{C}(G) \cap \mathcal{B}(G) = \{\emptyset\}$, while otherwise it is 'no' on all counts.

For G finite, non-empty elements of $\mathcal{C}(G) \cap \mathcal{B}(G)$ – edge sets that are both (algebraic) cycles and 'cocycles' (cuts) – are called *bicycles*. If G has no bicycle then, naturally, it is called *pedestrian*. We shall adopt these definitions verbatim for infinite graphs – Fig. 2.1 shows a pedestrian graph[14] from [9] – but remark that only the finite bicycles are orthogonal to themselves. In view of Theorem 2.6 we would, ideally, hope for a positive answer to Problem 2.9(ii) for all G that have no finite bicycles. As we shall see, this is too much to ask — but what exactly is possible is an open problem.

So what can we say when G is infinite? It turns out that neither (ii) nor (iii) of Problem 2.9 holds even for pedestrian graphs—except for those that we might call *essentially finite*: graphs whose spanning trees have only finitely many chords.[15]

Theorem 2.10 ([5,34]). *G satisfies the assertion of Problem 2.9(ii) if and only if G is pedestrian and essentially finite.*

Proof. If G is essentially finite, the union D of all its circuits is the edge set of a finite subgraph H. Its algebraic cycles coincide with those of H, and $\mathcal{B}(G)$ consists of those edge sets whose intersection with D lies in $\mathcal{B}(H)$. (To see this, consider a spanning tree of G that extends spanning trees of the components of H.) The assertion now follows easily from the corresponding assertion for H.

14 To see it's pedestrian, consider the leftmost edge of a hypothetical bicycle, and check on which sides of the cut that it defines the vertices near it come to lie.

15 It is easy to see that this property does not depend on the spanning tree chosen.

Before we consider the case that G is not essentially finite, let us outline some general conditions which an infinite set $E \subseteq E(G)$ must satisfy in order to have an orthogonal decomposition. We shall then show that if G is not essentially finite it will always contain an infinite set of edges violating those conditions. So let E have an orthogonal decomposition, $E = D + F$ say. Then $D \cap F$ is finite (and even), and E is the disjoint union of a set $E_{\mathcal{C}} := D \setminus (D \cap F)$ and a set $E_{\mathcal{B}} := F \setminus (D \cap F)$ that can be turned into elements of $\mathcal{C}(G)$ or $\mathcal{B}(G)$, respectively, by adding a finite set (the set $D \cap F$). Moreover, at least one of $E_{\mathcal{C}}$ and $E_{\mathcal{B}}$ must be infinite. Hence, for G to be a counterexample to Problem 2.9(ii), all it takes is an infinite set $E \subseteq E(G)$ that has no infinite subset $E_{\mathcal{C}}$ that can be turned into an algebraic cycle by adding finitely many edges, and no infinite subset $E_{\mathcal{B}}$ that can be turned into a cut by adding finitely many edges.

Now assume that G is not essentially finite. Let T be a normal spanning tree. Then T has an infinite independent set E of chords. Since algebraic cycles induce even degrees, we cannot add finitely many edges to an infinite subset E' of E to obtain an algebraic cycle. But neither can we add finitely many edges to E' to obtain a cut F: since E' is infinite it would take infinitely many fundamental cuts D_f to generate F (Lemma 1.9), and F would differ from E' by at least those (infinitely many) tree edges f. $\square$

Two alternative approaches to Problem 2.9 remain: to settle for (i), or to prove the assertions from (ii) and (iii) for specific edge sets $E \subseteq E(G)$. The entire set $E = E(G)$, for example, always has an orthogonal decomposition, even into disjoint sets: this is a theorem of Gallai for finite graphs, whose extension to locally finite graphs [9] has an easy compactness proof of the kind we shall discuss in Section 3.

For singleton sets $\{e\}$, the following is known:

Theorem 2.11 (*[11,14]*)**.** *Let e be any edge of G.*

(i) *Either $e \in E$ for some $E \in \mathcal{C} \cap \mathcal{B}$, or there are $D \in \mathcal{C}_{\text{fin}}$ and $F \in \mathcal{B}_{\text{fin}}$ such that $\{e\} = D + F$, but not both.*
(ii) *Either $e \in E$ for some $E \in \mathcal{C}_{\text{fin}} \cap \mathcal{B}_{\text{fin}}$, or there are $D \in \mathcal{C}$ and $F \in \mathcal{B}$ such that $\{e\} = D + F$, but not both.*

Theorem 2.11(i) implies a positive answer to Problem 2.9(i) for pedestrian graphs G: the sets from $\mathcal{C}_{\text{fin}}$ and $\mathcal{B}_{\text{fin}}$ needed to generate the singleton sets $\{e\}$ form a thin family [11, Prop. 13], and their sum over all $e \in E$ is exactly E.

Corollary 2.12. *If $\mathcal{C}(G) \cap \mathcal{B}(G) = \emptyset$, then very set $E \subseteq E(G)$ has a decomposition $E = D + F$ with $D \in \mathcal{C}(G)$ and $F \in \mathcal{B}(G)$.*

What can we say if all we assume is that G has no *finite* bicycle? Then every singleton set $\{e\}$ has a decomposition $\{e\} = D + F$ as in Theorem 2.11(ii). But unlike for pedestrian graphs, this cannot always be chosen orthogonal: there can be edges e all whose decompositions $\{e\} = D + F$ are such that both D and F (and hence also $D \cap F$) are infinite [11]. Moreover, the family of *all* the D and F that can be used in such singleton decompositions will not be thin, as soon as G has an infinite bicycle [5]. However we might still hope that for every concrete $E \subseteq E(G)$ we can choose these decompositions, one for every $e \in E$, so that the sets D and F used form a thin family. This would yield the result, stronger than Corollary 2.12, that every edge set in a graph without a finite bicycle has a decomposition.

Finally, uniqueness. For pedestrian graphs, any orthogonal decomposition $E = D + F$ of an edge set E will be unique: if $E = D' + F'$ was another, then $D - D' = F' - F$ would be a bicycle. For those pedestrian graphs that do have orthogonal decompositions, i.e., for the essentially finite ones, we thus have linear projections $\mathcal{E} \to \mathcal{C}$ and $\mathcal{E} \to \mathcal{B}$ mapping E to D and to F, respectively. There are explicit descriptions of these projections in terms of the number of spanning trees of the graph involved [3] (when the graph is finite)—not very tangible but perhaps unavoidably so.

Finally, homology theory appears to suggest the following related question:

Problem 2.13. For which G is there a natural 'boundary operator' $\partial : \mathcal{E}(G) \to \mathcal{V}(G)$ with kernel $\mathcal{C}(G)$, and a natural 'coboundary operator' $\delta : \mathcal{V}(G) \to \mathcal{E}(G)$ with image $\mathcal{B}_{\text{fin}}(G)$?

I suspect that this simplicial attempt to marry our topological cycle space to the usual homological setup is in fact inadequate, and that the way to do this lies not in changing the boundary operators but in restricting the chains. A singular approach that does capture the topological cycle space as its first homology group, just as in finite graphs, and where some infinite chains are allowed but not every subset (or sum) of edges is a legal 1-chain, has been constructed in [27]. It allows for some infinite chains, but not all; see Section 4 for more.

It may also be instructive to study how the answer to the questions in this section, in particular to Problem 2.9, changes if we replace our coefficient ring $\mathbb{F}_2$ with $\mathbb{Z}$ or $\mathbb{R}$. There will be no bicycles then, which simplifies things. But other problems occur. With integer coefficients, for example, 1-dimensional simplicial (co-) chains even of a finite graph need not decompose into an algebraic cycle and a cut (both oriented): as one can readily check, the triangle with one oriented edge mapped to 1 and the others to 0 is a counterexample.

For infinite graphs and real coefficients, let $\vec{\mathcal{E}}(G)$ consist of only those functions $\psi : \vec{E}(G) \to \mathbb{R}$ for which $(\psi(\vec{e}) = \psi(\overleftarrow{e})$ for all $\vec{e} \in \vec{E}(G)$ and)

$$\sum_{e \in E(G)} \psi^2(e) < \infty,$$

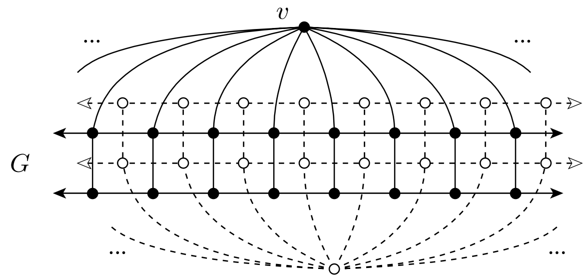

Fig. 2.2. A self-dual graph with infinite circuits but no added ends.

where $\psi^2(e) := \psi(\vec{e})^2$ $(= \psi(\overleftarrow{e})^2)$. For such $\psi, \psi' \in \vec{\mathcal{E}}(G)$ we may define

$$\langle \psi, \psi' \rangle := \sum_{e \in E(G)} \psi(\vec{\vec{e}})\psi'(\vec{\vec{e}}) \quad (<\infty),$$

where $\vec{\vec{e}}$ is the *natural orientation* of e, one of its two orientations that was picked and fixed once and for all when G was first defined. Now ψ and ψ' can be called *orthogonal* if $\langle \psi, \psi' \rangle = 0$, and we can study Problem 2.9.

2.5. Duality

For finite multigraphs G and G^* one calls G^* a *dual* of G if there is a bijection $e \mapsto e^*$ between their edge sets $E := E(G)$ and $E^* := E(G^*)$ such that every set $F \subseteq E$ satisfies

$$F \subseteq \mathcal{C}(G) \quad \text{if and only if} \quad F^* \in \mathcal{B}(G^*), \tag{$\dagger$}$$

where $F^* := \{e^* \in E^* \mid e \in F\}$. If G is 3-connected, there is at most one such G^* (up to isomorphism), which is again 3-connected. In this case we think of G^* as being *defined* by (†), given G. By the finite version of Theorem 2.6, condition (†) implies the corresponding condition with $\mathcal{C}$ and $\mathcal{B}$ swapped. Hence if G^* is a dual of G then G is a dual of G^*, and for 3-connected G we have $G^{**} = G$.

Can such duals exist also for locally finite infinite graphs? Geometric duals of plane graphs suggest not: the geometric dual of the $\mathbb{N} \times \mathbb{N}$ grid,[16] for example, has a vertex of infinite degree. And indeed, Thomassen [49] proved that if G is a locally finite 3-connected graph that (even only) has a locally finite 'finitary dual' – a graph G^* for which only the finite sets of edges in G and G^* are required to satisfy (†) – then every edge of G must lie in exactly two finite peripheral circuits. In particular, the $\mathbb{N} \times \mathbb{N}$ grid has no such dual.

Hence in order to be able to define duals for all locally finite graphs that should have a dual – that is, for the locally finite planar graphs – we shall have to allow as duals at least some graphs with infinite degrees. And if we want duals to have duals, we have to define duals also for such graphs with infinite degrees. As we shall see in Section 4, allowing arbitrary countable graphs with infinite degrees leads to considerable difficulties. But, fortunately, Thomassen [49] proved also that any graph with (even only) a finitary dual must satisfy another condition:

$$\text{Every two vertices can be separated by finitely many edges.} \tag{$*$}$$

Thus, no matter how we define duals: as soon as their finite edge sets are required to satisfy (†) – which is certainly the minimum we shall have to ask – these graphs will have to satisfy (∗). On the other hand, the class of graphs satisfying (∗) is not closed under taking finitary duals, and finitary duals are not unique [6,49]. Extending duality to infinite graphs thus seems to be fraught with problems.

In our topological setup, however, the problem has a most elegant solution. Let us say that vertex v *dominates* a ray R in G if G contains an infinite v–R fan (Fig. 4.1). It is easy to see that if v dominates one ray of an end it dominates all its rays; we then also say that its *dominates* that end ω. Assuming that G satisfies (∗), we can then define a space $\tilde{G}$ similar to $|G|$, but with the difference that new points at infinity are added only for the undominated ends, while the dominated ends are made to converge to the vertex dominating them.[17] Fig. 2.2 shows a graph G whose two ends are both dominated by the vertex v. Thus, $\tilde{G}$ has no added ends, but the double rays 'bend round' so that both their tails converge to v.

In this space $\tilde{G}$, duality works as if by magic. We define $\mathcal{C}(G)$ as the set of circuits (the edge sets of circles) in $\tilde{G}$, define $\mathcal{B}(G^*)$ as the set of cuts of G^* (as before), and call G^* a *dual* of G if there is a bijection $e \mapsto e^*$ between their edge sets satisfying (†) for all sets $F \subseteq E(G)$, finite or infinite. If G is 3-connected (and satisfies (∗)), then any dual G^* of G is unique and also satisfies (∗), and we have $G^{**} = G$ witnessed by the map $e^* \mapsto e =: e^{**}$ [6].

16 …naively defined just for this example; see Problem 2.16 below.

17 Note that, by (∗), this vertex is unique. We shall define $\tilde{G}$ more formally in Section 4.

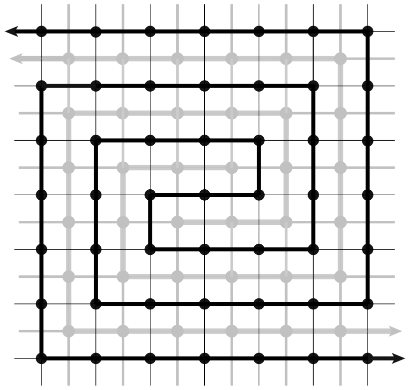

Fig. 2.3. Dual 'spanning trees': the grey tree contains the shared end, the black tree does not.

For the purpose only of the next theorem, let us call a (graph-theoretical) spanning tree of G satisfying $(*)$ *acirclic* if its closure in $\tilde{G}$ contains no circle (and similarly for G^*). Note that unless G has no dominated ends, normal spanning trees of G will *not* be acirclic: they contain a ray from every end, and a ray from a dominated ends that starts at its dominating vertex forms a circle in $\tilde{G}$. However, one can show that every connected graph satisfying $(*)$ has an acirclic spanning tree [6].

Theorem 2.14 ([6]). *Let $G = (V, E)$ and $G^* = (V^*, E^*)$ be connected multigraphs satisfying $(*)$, and let $^*\colon E \to E^*$ be a bijection. Then the following two assertions are equivalent:*

(i) *G and G^* are duals of each other, and this is witnessed by the map * and its inverse.*
(ii) *Given a set $F \subseteq E$, the graph (V, F) is an acirclic spanning tree of G if and only if $(V^*, E^* \setminus F^*)$ is an acirclic spanning tree of G^*.*

(For (ii), define $F^* := \{e^* \in E^* \mid e \in F\}$ as before.)

Can we generalize Theorem 2.14 to topological spanning trees that are not necessarily graph-theoretical trees? A positive answer to this question would be in line with the philosophy of our theory, but the answer is not obvious. Once we use ends as 'connectors' (as we do in a topological spanning tree), we should treat them on a par with edges: we should have a bijection between the ends of $\tilde{G}$ and those of $\tilde{G^*}$, and use each end from this common set of ends in precisely one of the two trees. But no matter how one defines the 'ends of $\tilde{G}$', whether one takes just the undominated ends or the undominated ends plus the dominating vertices (i.e., the set of all limit points of rays), there need not be a bijection between the ends of $\tilde{G}$ and those of $\tilde{G^*}$: the $3 \times \mathbb{Z}$ grid, for example, is a 3-connected graph with two ends, whose dual has no undominated end and only one dominating vertex.

Surprisingly, things work better if we, initially, continue to consider the ends of G and G^* in the original spaces $|G|$ and $|G^*|$. Bruhn and Stein [13] have shown that, if G is 2-connected, there is a homeomorphism between the subspaces Ω of $|G|$ and Ω^* of $|G^*|$ that is *compatible* with the duality map $e \mapsto e^*$ from E to E^* in the sense that, for every set $F \subseteq E$, an end $\omega \in \Omega$ is an accumulation point of $\bigcup F$ in $|G|$ if and only if ω^* is an accumulation point of $\bigcup F^*$ in $|G^*|$. The question, then, could be roughly as follows: is it true that no matter how we form a spanning tree in $\tilde{G}$ from some of the edges and ends of G (taken in $|G|$), the other edges and ends (taken in $|G^*|$) will form a spanning tree in $\tilde{G^*}$? And that, conversely, if this is the case for two graphs G and G^* then these form a dual pair?

Fig. 2.3 shows a dual pair of graphs illustrating this. The grey tree in G^* needs the 'shared' end in order to be connected, and adding the end to it does not create a circle. The dual black tree, however, is already connected (without the end), and adding the end there would create a circle. Hence there is a unique way of assigning the end to one of the two trees, so that both are spanning and acirclic in their respective graphs.

Let us make this precise in greater generality. Given a set $F \subseteq E$ of edges and a set $\Psi \subseteq \Omega$ of ends, let us say that the subspace $\bigcup F \cup \Psi$ of $|G|$ is *formed by* F and Ψ. Given a subspace X of $|G|$, let $\tilde{X}$ denote the quotient space of X obtained by identifying every vertex in X with all the ends in X that it dominates (in G). (For $X = G$ this yields precisely the definition of $\tilde{G}$ given informally earlier.) For example, if G is the fan shown on the left in Fig. 4.1, we might take as X the space formed by the set F of fat edges and the unique end of G. Then $\tilde{X}$ is a circle. However, if we take as X the space formed only by the set F of fat edges, without the end, then $\tilde{X}$ is homeomorphic to the half-open interval $[0, 1)$. Call $\tilde{X}$ a *spanning tree* of $\tilde{G}$ if it contains all the vertices of $\tilde{G}$, is arc-connected, and contains no circle.

Theorem 2.15 ([8]). *Let $G = (V, E, \Omega)$ and $G^* = (V^*, E^*, \Omega^*)$ be 2-connected multigraphs satisfying $(*)$, and let $^*\colon E \to E^*$ and $^*\colon \Omega \to \Omega^*$ be compatible bijections. Then the following two assertions are equivalent:*

(i) *G and G^* are duals of each other, and this is witnessed by the maps $E \to E^*$ and $\Omega \to \Omega^*$ and their inverses.*
(ii) *Whenever the subspace X of $|G|$ formed by subsets $F \subseteq E$ and $\Psi \subseteq \Omega$ is such that $\tilde{X}$ is a spanning tree of $\tilde{G}$, the sets $E^* \setminus F^*$ and $\Omega^* \setminus \Psi^*$ form a subspace Y of $|G^*|$ such that $\tilde{Y}$ is a spanning tree of $\tilde{G^*}$.*

Here, as before, $F^* := \{e^* \in E^* \mid e \in F\}$ and $\Omega^* := \{\omega^* \in \Omega^* \mid \omega \in \Omega\}$. The obvious implications for infinite matroids which Theorems 2.6, 2.14 and 2.15 suggest will be a topic in Section 4.

For finite graphs, the theory of algebraic duality outlined above corresponds to a theory of *geometric duality*: two connected graphs are duals of each other if and only if they can be drawn in the plane so that precisely the corresponding edges cross and vertices correspond bijectively to faces of the dual.[18]

For our graphs satisfying $(*)$, this is not so obvious: how, for instance, should we draw the two dual graphs of Fig. 2.2 so that their rays converge as they do in $\tilde{G}$ and $\tilde{G^*}$? In particular, where in such a drawing can we put the two vertices of infinite degree, the limit points of those rays? For now, setting up geometric duality in a satisfactory way remains an unsolved problem:

Problem 2.16. Define geometric duals for infinite graphs satisfying $(*)$ compatibly with their algebraic duals.

Perhaps the solution to Problem 2.16 lies in an approach similar to that for Theorem 2.15: to apply duality 'before identification' rather than after, that is to say, to look for geometrically dual embeddings of $|G|$ and $|G^*|$ rather than of $\tilde{G}$ and $\tilde{G^*}$. The drawback of such an approach is that $|G|$ and $|G^*|$ need not be compact as soon as we refine their topologies enough to make their embeddings in the plane continuous.[19] This, then, raises issues such as how exactly to define a face.

3. Proof techniques

Once more, let G be an infinite, locally finite, connected graph. The aim of this section is to bring the reader new to the field up to date with its collective memory of techniques. Although each of the deeper proofs of existing theorems naturally has its own difficulties and ways of overcoming them, there is by now a body of basic approaches that can be described—as well as some common pitfalls the novice might like to hear about before plummeting their depths themselves.

We shall concentrate on proofs that establish the existence of infinite substructures of $|G|$ such as arcs, circles, or topological spanning trees. Such a proof usually has two parts: the construction of this structure, usually by a limit process, and the proof that the object thus constructed does what it is intended to do.[20] We shall only discuss the construction part. If the construction is done well, it can happen that the proof of correctness becomes easy. This does not always happen, and if it does, it may simply mean that the problem is intricate enough that 'blind' constructions will fail: that any difficulties that may arise in the proof have to be anticipated by the construction. Still, the basic techniques that are common to many proofs tend to lie at the construction level, while difficulties arising at proof level tend to be individual to the problem. Our focus on the construction part of proofs reflects this phenomenon; it is not a deliberate restriction.

3.1. The direct approach

The direct, or naive, approach to the construction of standard subspaces with certain desired properties, such as an arc, a circle or a topological spanning tree, is to obtain them as the union of nested finite subgraphs defined explicitly, or as the intersection of a nested sequence of explicitly defined superspaces. For example, we might try to construct a spanning tree as a union of finite trees $T_0 \subseteq T_1 \subseteq \ldots$ eventually covering all the vertices of G. The union T of these T_n will be a tree, but its closure in $|G|$ may well contain a circle: unlike finite cycles, this will arise only at the limit step, and it may not be clear how to choose the T_n so that this does not happen.

If we want this naive approach to work, we have to express the statement to be proved in a finitary way—for example, as statements about finite sets of edges. The characterization of topological connectedness in Lemma 1.5 is a case in point; the characterization of cycle space elements in Theorem 2.6(i) is another. Unlike the circle in our example, finitary properties do not appear or disappear at limits, which makes it easy to prove such statements by induction or an application of Zorn's lemma.

For instance, we have already seen the construction of a topological spanning tree 'from above', in the proof of Lemma 1.6. In that proof we used the fact that connectedness of a standard subspace is a finitary property (Lemma 1.5), and that topological spanning trees are edge-minimal connected standard sub- spaces of $|G|$ containing all its vertices. The proof of the following lemma is very similar:

Lemma 3.1. *Let X be a standard subspace of G.*

(i) *If X is k-edge-connected, we can delete edges from X to obtain a standard subspace Y that is edge-minimally k-edge-connected.*
(ii) *If X is k-vertex-connected, we can delete edges from X to obtain a standard subspace Y that is edge-minimally k-vertex-connected.*

---

[18] In particular, graphs with a dual are planar. This is Whitney's theorem, which extends to locally finite graphs satisfying $(*)$ if and only if duality is defined as we did: with $\mathcal{C}$ as the topological cycle space of G based on $\tilde{G}$. See [18, Section 4.2].

[19] Since $|G|$ fails to be Hausdorff as soon as G has a dominated end, we need to refine our current topology, known as VTop, to a topology called MTop; see Sections 4.1 and 4.2. Under MTop, $|G|$ is compact only if G is locally finite (in which case MTop agrees with VTop).

[20] For example, we might be constructing a *Hamilton circle*, a circle in $|G|$ passing through all the vertices. The construction part might define a map $\sigma: [0, 1] \to |G|$ that is, by definition, injective at vertices and inner points of edges (except that $\sigma(0) = \sigma(1)$), while we might still have to show that σ is continuous and injective at ends.

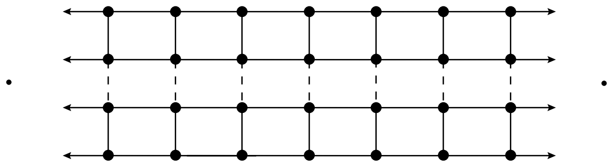

Fig. 3.1. The subspace obtained by deleting all the broken edges is no longer 3-connected.

Proof. (i) To obtain Y from X, we go through the edges of X in turn, deleting the edge under consideration if and only if this does not spoil the k-edge-connectedness of the current space. To show that Y is still k-edge-connected (in which case it will clearly be minimally so), we consider a hypothetical set F of fewer than k edges whose removal disconnects Y. By Lemma 1.5, the space $Y \smallsetminus \mathring{F}$ meets both sides of some finite cut of G in which it has no edge. But $X \smallsetminus \mathring{F}$ has an edge in this cut, because it is connected. All its edges in the cut, however, were deleted in the construction of Y, and the edge deleted last should not have been deleted.

(ii) The proof of the vertex-connectivity case is analogous: we again delete edges one by one, but instead of an edge set F and the space $Y \smallsetminus \mathring{F}$ we consider a separating set U of fewer than k vertices and the standard space obtained from Y by deleting U and its incident edges. □

The analogue of Lemma 3.1 for k-connected subspaces – those that cannot be made disconnected by the deletion of fewer than k vertices or ends – cannot be proved in the same way. For arbitrary substandard subspaces it is false (consider the double ladder, without the ends, for $k = 2$), but for standard subspaces it is an open problem.[21] For example, if we delete all the broken edges in the graph of Fig. 3.1 in ω steps, the space will lose its 3-connectedness (since deleting the two ends will then disconnect it), but does so only at the limit step.

Many applications of the object $|G|$ – see [18, Section 4] – have the following form: there is an existence statement about finite graphs that fails for infinite graphs – for example, a theorem asserting the existence of a Hamilton cycle – but a topological analogue works in $|G|$. In our example, it might be that the same conditions (such as being planar and 4-connected) that force a Hamilton cycle in a finite graph force a Hamilton circle in $|G|$. Rather than proving the infinite theorem from first principles, we might try to use the finite result for the proof of its infinite analogue. We would then want to express $|G|$ as a limit of finite graphs $G_0, G_1, \dots$ that satisfy the assumptions (such as planarity and 4-connectedness) if G does, use the finite theorem to find the desired structures in the G_n, and then take a limit of these to obtain the analogous structure in $|G|$.

Experience has shown that defining these G_n as induced subgraphs of G, e.g. those on its first n vertices, is *not* often a good approach: a badly chosen $G_n \subseteq G$ is too oblivious of how it lies inside G. (There might, for example, exist G_n-paths[22] in G between vertices of G_n that are easily separated in G_n.) However, there is a standard approach using finite 'nested' minors rather than subgraphs as G_n, which we define now.

Let $v_0, v_1, \dots$ be an enumeration of the vertices of G. For each $n \in \mathbb{N}$ let $S_n := \{v_0, \dots, v_n\}$, and write G_n for the minor of G obtained by contracting each component of $G - S_n$ to a vertex, deleting any loops that arise in the contraction but keeping multiple edges. The vertices of G_n contracted from components of $G - S_n$ will be called the *dummy vertices* of G_n.

Note that every cut of G_n is also a cut of G. Conversely, a cut of G is a cut of every G_n that contains all its edges. This is an important feature, which distinguishes these minors from a similar sequence of finite subgraphs exhausting G. A topologically connected standard subspace of $|G|$, for example, will induce a connected subgraph in each G_n (cf. Lemma 1.5), an element of the cycle space of G will induce a cycle space element of G_n (cf. Theorem 2.5), and so on. We shall appeal to these properties again in our discussion of homology in Section 4.3.

As a first application of these G_n let us show that they allow us to construct a topological spanning tree directly 'from below', using the direct approach.[23]

Lemma 3.2. *G has a topological spanning tree.*

Proof (*By Explicit Construction From Below*)**.** Pick spanning trees T_n of G_n recursively so that $E(T_{n+1}) \cap E(G_n) = E(T_n)$.[24] We claim that $T := \overline{\bigcup_n E(T_n)}$ is a topological spanning tree of G.

To show that T is connected (and hence arc-connected, by Lemma 1.2), we have to check that every finite cut of G contains an edge from T. It does, because it is also a cut of some G_n, and T_n has an edge in this cut.

21 We discuss this problem in [18, Section 4.1].

22 See [20] for the notion of an H-path in G for a subgraph $H \subseteq G$.

23 The proof of Lemma 3.2 can be generalized to a proof 'from below' of the full statement of Lemma 1.6 for the existence of topological spanning trees in standard subspaces [8].

24 There is a canonical way of doing this. Unless v_{n+1} forms a component of $G - S_n$ by itself (in which case $T_{n+1} = T_n$), the graph G_{n+1} is obtained from G_n by expanding the dummy vertex of G_n contracted from the component containing v_{n+1} to a star with centre v_{n+1} and some new dummy vertices of G_{n+1} as leaves. Note that while the vertices of this star are unique, its edges are not, since G_{n+1} has multiple edges. We obtain T_{n+1} from T_n by adding the edges of this star. In other words, we extend T_n to T_{n+1} in any way that does not add an edge at a vertex in S_n.

Now suppose that T contains a circle, C say. Pick vertices $u, v \in C$, and choose n large enough that $u, v \in S_n$. Let F be a fundamental cut of T_n separating u from v. Since this is also a cut of G and C meets both its sides, C has an edge in F (Lemma 1.5). Deleting this edge from C leaves an arc, which still has an edge in F (by the same argument). Hence $C \subseteq T$ has two edges in F, contradicting the fact that $E(T) \cap E(G_n) = E(T_n)$ and T_n has only one edge in F. □

Finally, there is a type of construction, very common in finite graph theory, which is rather naive in the context of $|G|$ but should not go unmentioned: the construction of paths or arcs by 'greedily moving along'. As an example, consider the proof of Theorem 2.3(i), that every edge set $D \in \mathcal{C}(G)$ is a disjoint union of circuits. Let us just try to find the first circuit. If G is finite, we start from any edge in D and 'greedily move along' further edges of D: since D has an even number of edges at every vertex, we cannot get stuck until we reach our starting vertex again. We stop, however, when we hit the first previously visited vertex, having found our first circuit. We then delete this and repeat, eventually decomposing D into circuits.

When G and D are infinite, this approach can fail quite spectacularly. For example, D might be the wild circuit of Fig. 1.1. By 'moving along' its edges we would run into an end, from which there would be no escape, not even by an inverse ray. The linear order structure of the double rays on the wild circle indicates that the only way to obtain it as a limit is to 'approach all its points at once'—which is exactly what we shall do using our sequence of G_n.

3.2. The use of compactness

A typical compactness proof, such as of the Erdős–De Bruijn theorem on colouring, is a way of making consistent choices for all the finite substructures of an infinite structure, and then deducing from the fact that all the finite substructures have a certain property that so does the entire structure.

In our context, only the simplest results have compactness proofs quite like this. But we often use compactness for the construction part of a proof, including the more difficult ones: to define a limit object (e.g., a subspace X) in such a way that all of certain finite substructures (e.g., the finite cuts of G) relate to the limit object in a certain desired way (contain an edge of X). This may be less than we actually want to prove (e.g. that X is arc-connected)—but it may be a good start.

The best examples for this are infinite analogues of theorems asserting for a finite graph the existence of a path or cycle that can be found by a process of 'focussing in' on some part of a graph: in such cases, the same process may converge to a limit object in $|G|$ with similar properties. Bruhn's proof [4] of Theorem 2.3(ii), for example, works in principle like Tutte's own proof of his finite theorem. Or think of the 'focussing in' on a component in Thomassen's proof that every finite 3-connected graph has an edge whose contraction preserves 3-connectedness. (See [20] for both.)

To illustrate how such a compactness proof typically works, or might fail, let us take another look at the proof of Theorem 2.3(i). We were trying to find a single circuit C in a given set $D \in \mathcal{C}(G)$ (which might be the wild circle, so that we cannot find C by 'moving along'). Our approach now is to construct C as a limit of circuits C_n in the G_n. Since D meets every cut of G_n evenly (because that cut is also a cut of G), we have $D_n := D \cap E(G_n) \in \mathcal{C}(G_n)$. By the trivial finite version of Theorem 2.3(i), we find a circuit $C_n \subseteq D_n$. Will these C_n, or at least a subsequence, tend to a limit $C \subseteq D$ that is a circle in $|G|$?

Not necessarily. For a start, the limit might exist[25] but be empty. Indeed, depending on the structure of D, we might have chosen the C_n so badly that no edge lies eventually in C_n.[26] But there is a simple way to prevent this: by insisting that every C_n contain some fixed edge $e \in D$.

Assuming this, let us use the infinity lemma to construct a limit of these C_n. To apply the lemma we need to define 'predecessors', which is easy: let the nth finite set in the setup of the lemma contain not only C_n but also all edge sets of the form $C_n^m := C_m \cap E(G_n)$ with $m > n$. Since the cuts of G_n are also cuts of G_m, these sets will meet every such cut evenly and hence lie in $\mathcal{C}(G_n)$. We may now define the predecessor relation for the infinity lemma as inclusion (of edge sets), and obtain the desired limit C as a union of certain nested sets $C_n^m \in \mathcal{C}(G_n)$ found by the lemma, one for each n. Then C, too, meets every finite cut of G evenly, because that cut is also a cut of some G_n and the selected C_n^m meets this cut evenly. Hence $C \in \mathcal{C}(G)$, by Theorem 2.6(i).

But is C a circuit? Unfortunately, it need not be: we cannot in general prevent the C_n^m from traversing dummy vertices more than once, in which case the closure $\overline{C}$ of their union can be a union of circles that are vertex-disjoint but meet at ends. Fig. 3.2 indicates an example where this happens: the sequence of C_n there is formed by alternately extending them in two different places, so that both extensions tend to the unique end, yielding a limit in which that end has degree 4.

So what did we achieve? As the limit of our circuits C_n we found just another algebraic cycle in $\mathcal{C}(G)$, but not a circuit. For our present problem this seems like next to nothing: after all, we started with an algebraic cycle $D \in \mathcal{C}(G)$, looking for a circuit $C \subseteq D$. But in general, the method does have its uses: it shows that from algebraic cycles $C_n \in \mathcal{C}(G_n)$ – we never used that they are actually circuits – we can always obtain an algebraic cycle $C \subseteq \bigcup_n C_n$ by compactness, even if the C_n were not induced by some given $D \in \mathcal{C}(G)$. And we can say a little more about C too: there is a subsequence of the C_n (consisting of those C_m that gave rise to one of the C_n^m in our nested sequence picked by the infinity lemma) such that every edge of C lies eventually on every C_n in this subsequence.

25 We have not made this precise yet.

26 For example, let G be the plane $\mathbb{Z} \times \mathbb{Z}$ grid, and choose as C_n the boundary circuit of the outer face of G_n.

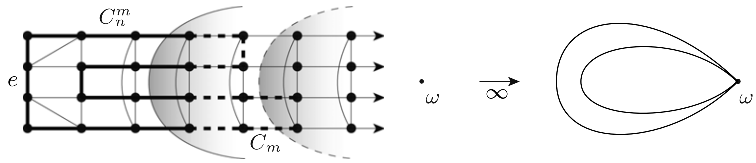

Fig. 3.2. Finite cycles tending to a double circle. The shaded regions contract to dummy vertices.

Moreover, the set $C \subseteq \mathcal{C}(G)$ we found (now using that the C_n are circuits) does have one advantage over D: the standard subspace $\overline{C} \setminus \mathring{e}$ is connected. Indeed, every $C_m \setminus \{e\}$ spans a connected subgraph in G_m, and hence so does $C_n^m \setminus \{e\}$ in G_n. The connectedness of $\overline{C} \setminus \mathring{e}$ now follows from $C_n^m \subseteq C$ and Lemma 1.5. We can therefore use Lemma 1.2 to find an arc in $\overline{C} \setminus \mathring{e}$ between the endvertices of e, which together with e will form the desired single circuit in D containing e—solving our problem.[27]

3.3. *Constructing arcs directly*

In our compactness proof discussed above, we obtained as a limit of finite cycles $C_n \subseteq G_n$ an edge set $C \in \mathcal{C}(G)$ with $\overline{C}$ connected, and we had to invoke Lemma 1.2 in order to find inside $\overline{C}$ an arc between the endpoints of our fixed edge e. Could we have obtained such an arc directly, by using compactness in a more subtle way—e.g. by choosing the edge sets to which we applied the infinity lemma more carefully?

Problem 3.3. Given two vertices $x, y \in G$ and, for all n large enough, an x–y path $P_n \subseteq G_n$, construct by a direct compactness argument an x–y arc in $|G|$ with edges in $\bigcup_n E(P_n)$.

This problem is still unsolved. What we can do is construct a topological x–y path in $|G|$ by direct compactness. This path will traverse vertices at most once, but it may be non-injective at ends (as in Fig. 3.2). In order to obtain the desired arc, we therefore still have to invoke a theorem from general topology: that the image of any x–y path in a Hausdorff space contains an x–y arc.

Before we apply this technique to obtain such a partial solution to Problem 3.3, let me illustrate it by an exercise. Let T be the infinite binary tree. The task is to construct a topological path $\sigma : [0, 1] \to |T|$ that starts and ends at the root and traverses every edge exactly twice, once in each direction. For $k \in \mathbb{N}$, let T^k denote the finite subtree of T consisting of its first k levels. We shall obtain σ as a limit of analogous paths σ_k around these T^k, as follows.

Let σ_0 be the constant map from $[0, 1]$ to the 1-vertex graph T^0. Assume inductively that we have defined for $i = 0, \ldots, k$ paths $\sigma_i : [0, 1] \to T^i$ such that the inverse image under σ_i of each leaf of T^i is a non-trivial interval, that the inverse image of any other point of T^i is a single point, and that σ_{i+1} agrees with σ_i on all points of $[0, 1]$ that σ_i maps to points of T^i other than leaves. (Thinking of σ_i as a journey with $[0, 1]$ measuring 'time', we thus ask that σ_i should pause for a while at every leaf of T^i, but not elsewhere.) In order to obtain σ_{k+1} from σ_k, we just expand every pause at a leaf t of T^k to a little path $tt'tt''t$ in T^{k+1} from t to its two successors t', t'' and back, pausing again at each of t' and t''.

To define our desired limit σ of these σ_k, we first look at all those $x \in [0, 1]$ for which the σ_k eventually agree, and let $\sigma(x) := \sigma_k(x)$ for these x (with k large enough). For any other x, every $\sigma_k(x)$ will be a leaf of T^k, and as $k \to \infty$ these $\sigma_k(x)$ form a ray in T; we let $\sigma(x)$ be the end of that ray. It is straightforward from the definition of the topology of $|T|$, but still instructive, to check that this map $\sigma : [0, 1] \to |T|$ is continuous at these latter points x mapped to ends. Indeed, every basic open neighbourhood of an end $\omega = \sigma(x)$ has the following form: for some vertex t on the ray R in ω starting at the root, it is the set of all points in $|T|$ strictly above t. Let t' be the upper neighbour of t on R. If k is the level of t' in T, then t' is a leaf of T^k, and all the points in the subinterval of $[0, 1]$ that σ_k maps to this leaf t' will be mapped by all σ_ℓ with $\ell > k$ to points of G that still lie above t. Hence σ will map all such points to the closure of this set (but not to t), i.e. to the open neighbourhood of ω we started with.

This technique for constructing a topological path directly as a limit of maps can be combined with compactness, as follows. Consider Problem 3.3. Every x–y walk in G_m, including the given path P_m, induces for every $n < m$ an x–y walk in G_n consisting of those of its edges that already lie in G_n, traversed in the same order and direction. In order to obtain an x–y arc in $|G|$, we first use the infinity lemma to select from these walks a sequence $(W_n)_{n \in \mathbb{N}}$ of x–y walks W_n each inducing the previous. We then parametrize these walks in turn for $n = 1, 2, \ldots$, as in our binary tree example: by topological paths σ_n 'pausing' at every dummy vertex of G_n (but only there), and so as to agree with σ_{n-1} at all points that σ_{n-1} does not map to a dummy vertex. The limit of these σ_n will be a topological x–y path σ in $|G|$, defined as earlier, since sequences of dummy vertices of the G_n ($n \to \infty$) that correspond to nested components of $G - S_n$ define a unique end, and these components form a neighbourhood basis for that end.

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[27] The proof of Theorem 2.3(i) follows easily now: we just collect circuits $C \subseteq D$ inductively and delete them, to eventually decompose D, starting each new circuit with the next uncovered edge of D in some fixed enumeration.

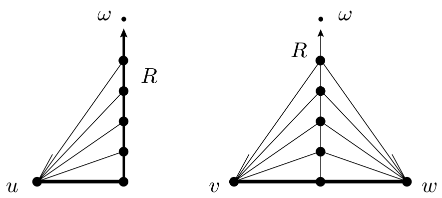

Fig. 4.1. Should the fat edge sets be circuits?

Note that if we insist that the lengths of pauses tend to zero as n tends to infinity, the inverse image of an end under the limit path σ will be a totally disconnected subset of $[0, 1]$. But I do not know how to set up the infinity lemma in such a way that σ will be injective at ends: we have found an x–y path in $|G|$, but not the desired arc.

The construction of a topological Euler tour in $|G|$ for the proof of Theorem 2.5(ii) works exactly like this, too. Georgakopoulos has bundled the essence of this approach into an out-of-the-box tool [31, Lemma 1.4] that provides a well-defined limit path for a suitable subsequence of a given sequence of x–y paths. It deals once and for all with the construction and continuity of the limit path, and can be used in many situations.

4. Outlook

The topological approach to studying locally finite graphs discussed in this paper can be taken further, in various directions. In this section we address some of these:

- graphs that are not locally finite;
- locally finite graphs with finer 'boundaries' than ends, such as boundaries of (Cayley graphs of) hyperbolic groups in the sense of Gromov;
- homology of non-compact spaces other than graphs;
- infinite matroids.

4.1. Graphs with infinite degrees

Consider the two graphs shown in Fig. 4.1, the *fan* and the *double fan*. In each of theses graphs there is a ray R that is dominated by a vertex (respectively, by two vertices). [28]

In both these graphs, the sum (mod-2) of all the triangle circuits equals the set of heavy edges. These edges do not look like circuits: they are the edge sets of a ray, or of a path of length 2. However, as thin sums of finite circuits, these edge sets ought to be elements of the cycles space $\mathcal{C}(G)$: remember that we need to allow infinite thin sums in $\mathcal{C}$ in order to make important applications of $\mathcal{C}$ work, such as MacLane's theorem. But if these sets of edges lie in $\mathcal{C}$, they ought to be disjoint unions of circuits (if we want Theorem 2.3(i) to extend to these graphs). Hence we feel compelled to admit those edge sets as circuits—or to abandon much of the cycle space theory we just established for locally finite graphs.

So can we make sense of the notion that those edge sets should be circuits? Indeed it seems we can; and the answer has to do with what topology we choose for $|G|$ when G is not locally finite.

Let us be more formal. Let G be any connected graph. The ends of G are defined as earlier, as equivalence classes of rays. As the topology on G itself we take slightly fewer basic open sets than the identification topology for a 1-complex does [29]: around every vertex v we take as basic open sets only the open stars $E_\epsilon(v)$ of length ϵ for arbitrary but fixed $\epsilon < 0$, the same ϵ for every edge at v. (Then the sets $E_{1/n}(v)$ form a countable neighbourhood basis of v. In the 1-complex topology, vertices of infinite degree have no countable neighbourhood basis.) As basic open sets around ends ω we take for every finite set S of vertices and every ϵ with $0 < \epsilon \leqslant 1$ the open ϵ-collar $\hat{C}_\epsilon(S, \omega)$ of $\overline{C(S, \omega)}$.[30] Note that, as for vertices, these ϵ-collars have uniform width.

Let us call this topology MTop.[31] Under this topology, $|G|$ is clearly Hausdorff, in fact normal [48]. However, unless G is locally finite, $|G|$ will no longer be compact. (Indeed, consider an open cover that consists the middle half of e for every edge e at some fixed vertex v, and one further open set to cover the rest of $|G|$ but none of the mid-points of edges at v. If v has infinite degree, this cover has no finite subcover.)

In order to give $|G|$ a chance to be compact, we have to take a coarser topology still: we take the same open stars $E_\epsilon(v)$ around vertices, but around ends ω we take only the open sets $\hat{C}_\epsilon(S, \omega)$ with $\epsilon = 1$, our original sets $\hat{C}(S, \omega)$. This topology is called VTop.

28 See the end of Section 2 for the formal definition of 'dominate'.

29 That topology leads to a topology for $|G|$ called Top. Historically, this was the first topology for $|G|$ to be considered, but it has few advantages over the topologies MTop and VTop discussed here. In particular, $|G|$ under Top is neither compact nor metrizable as soon as a vertex has infinite degree. See [17,21] for more.

30 More formally, let $\hat{C}_\epsilon(S, \omega)$ be obtained from the set $\hat{C}(S, \omega)$ defined in Section 1 by replacing each open C–S edge $\mathring{e} = (c, s)$ with its initial open segment of length ϵ, assuming that e itself has length 1.

31 The 'M' comes from fact that MTop makes $|G|$ metrizable, at least for countable connected G (see below).

Theorem 4.1 (*[17]*)**.** *For every graph $G = (V, E)$, not necessarily connected or locally finite, the following statements are equivalent under* VTop*:*

(i) *$|G|$ is compact.*
(ii) *For every finite $S \subseteq V$, the graph $G - S$ has only finitely many components.*
(iii) *Every closed set of vertices is finite.*

For G countable and connected, Theorem 4.1 implies that $|G|$ is compact under VTop if and only if G has a locally finite spanning tree, which can be chosen normal [17]. Note that since VTop agrees with Top and MTop on its closed subspace $\Omega(G)$, condition (ii) of Theorem 4.1 characterizes, for any of these topologies, the graphs whose end space Ω is compact.

We complete our excursion to the various topologies of $|G|$ by mentioning one more, known as ETop. Strictly speaking, this is not a topology on $|G|$,[32] but on a similar space obtained by adding to G its *edge-ends* rather than its ends: the equivalence classes of rays under which two rays are equivalent if no finite set of *edges* separates them. The basic open sets of this space are the components of $G-X$ left by deleting a finite set X of inner points of edges (and an edge-end belongs to the component in which all its rays have a tail).[33] If G is connected, as we assume, then this space is compact [47].

On the set of vertices and edge-ends, being separated by such a finite set X (i.e., by a finite set of edges) is an equivalence relation. The open sets in ETop cannot distinguish such equivalent points. But identifying equivalent points (including different vertices, such as the vertices v and w in Fig. 4.1) yields a compact Hausdorff space, which is metrizable if G was countable. See [37,47,33] for more on edge-ends and ETop.

4.2. The identification topology

Let us turn back to our two graphs from Fig. 4.1. Under MTop, the sets of fat edges are still far from being circuits. Under VTop and ETop, however, they almost are: these topologies cannot distinguish an end from vertices dominating it,[34] and if we identified the end with their dominating vertices, the fat edge sets would indeed become circuits. In the single fan on the left, this does indeed look right: why introduce, when forming $|G|$ from G, a new limit point for the ray R if a natural limit point, the vertex u, already exists? In the double fan on the right, however, one may have more qualms: identifying both v and w with the end ω would result in the identification of two vertices, changing the graph.

So shall we identify ends with their dominating vertices, or shall we not? Before we describe a radical and intriguing answer to this question due to Georgakopoulos [33], let us deal with the case when the dilemma does not arise: the case that no end is dominated by more than one vertex. This is the case, for example, for the graphs satisfying condition $(*)$ of Section 2: a natural class of graphs that occurs quite independently of this problem. For such graphs G, let $\tilde{G}$ denote the identification space obtained from $|G|$ by identifying each vertex with all the ends it dominates. This is a Hausdorff space—even under VTop when $|G|$ fails to be Hausdorff [23]. Other features of the topology of $\tilde{G}$, such as compactness, depend on the topology used for $|G|$. If VTop was used on $|G|$, the identification topology on $\tilde{G}$ is known as ITop.

When G is 2-connected, then $(*)$ implies that G is countable, and that no finite set of vertices separates G into infinitely many components. Then $|G|$ is compact under VTop (Theorem 4.1), and hence so is $\tilde{G}$ under ITop. Answering a question of Sprüssel [48], Richter and Vella [46] extended this by showing that $\tilde{G}$ is in fact 'Peano':

Theorem 4.2 (*[46]*)**.** *Let G be a 2-connected graph in which every two vertices can be separated by finitely many edges.*[35] *Then $\tilde{G}$ under* ITop *is a compact, connected, locally connected, metric space.*

For more on $\tilde{G}$, see [23,46,50]*.*

4.3. Compactification versus metric completion

As mentioned already in Theorem 1.1, the compactification $|G|$ of a connected and locally finite graph G is metrizable. When G is not locally finite, $|G|$ with our usual topology, VTop, fails to be even Hausdorff (as soon as there is a dominated end), and no vertex of infinite degree has a countable neighbourhood basis in the usual 1-complex topology. However, if we consider $|G|$ with MTop, we have the following result:

Theorem 4.3 (*[17]*)**.** *For a connected graph G, and $|G|$ endowed with* MTop*, the space $|G|$ is metrizable if and only if G has a normal spanning tree.*

Unless otherwise mentioned, the space $|G|$ will in this section always carry the topology MTop.

32 …unless G is locally finite, in which case its edge-ends coincide with its usual ends.

33 Such a 'component' is a graph-theoretical component left by deleting from G the edges containing points from X, together with the segments of these edges (after deleting X) that have an endpoint in such a component.

34 More generally, it is easy to see that a space $|G|$ is Hausdorff under VTop if and only if no end of G is dominated.

35 This assumption can be weakened: it is only necessary to assume that no two vertices can be linked by infinitely many independent paths [46]. Under this weaker assumption, however, $\tilde{G}$ can contain arcs and circles consisting entirely of vertices or ends. Such spaces were studied in [46].

Theorem 4.3 is interesting in both directions: as a characterization of the graphs G for which $|G|$ is metrizable, but also as an unexpected topological characterization of the graphs admitting a normal spanning tree. These are known to include all countable graphs, and they have been characterized by two types of forbidden substructure [24]. But these substructures themselves are not fully understood, and classifying them more accurately remains a challenging open problem.

The metric which a normal spanning tree T of G induces on $|G|$ is easy to describe. We first assign length $\ell(e) := 2^{-n}$ to every tree-edge e from level $n-1$ to level n, the root being at level 0. This defines a metric on $V(G) \cup \Omega(G)$, via

$$d_\ell(x, y) := \sum_{e \in xTy} \ell(e),$$

where xTy is the path, ray or double ray in $\overline{T}$ from x to y. We then extend this metric to one on $|G|$ by mapping every edge xy homeomorphically to a real interval of length $d_\ell(x, y)$. As one can check [17], this metric induces the topology MTop on $|G|$.

Instead of defining distances between ends and other points explicitly, we could also have defined them only for points of G itself (as above), and then taken the completion of this metric space (G, d_ℓ). Since two Cauchy sequences of points in (G, d_ℓ) are equivalent if and only if in $|G|$ they converge to the same end, this would have yielded the same result: the complete metric space on $G \cup \Omega(G)$ we defined explicitly above, and whose metric induces MTop on $|G|$.

Georgakopoulos [33] showed that this is not just a feature of our particular metric. Given any function $\ell: E(G) \to \mathbb{R}^+$ assigning positive 'edge lengths', defining distances between vertices u, v via

$$d_\ell(u, v) = \inf\Big\{\sum_{e \in P} \ell(e) \mid P \text{ an } u\text{–}v \text{ path in } G\Big\}$$

and identifying any u, v with $d_\ell(u, v) = 0$ defines a metric on the resulting quotient space of $V(G)$, which extends to a metric d_ℓ on the corresponding quotient of the entire graph G once we fix homeomorphisms between the edges and real intervals of their respective lengths. Let us denote this metric space as (G, d_ℓ), and the topology it induces on its completion as ℓ-Top. The metric subspace induced by the completion points we added is the ℓ-Top *boundary* of G.

Theorem 4.4 *([33]). If G is any countable connected graph and $\ell: E(G) \to \mathbb{R}^+$ satisfies $\sum_{e \in G} \ell(e) < \infty$, then completing the metric space (G, d_ℓ) yields the edge-ends of G as completion points and* ETop *as the induced topology of the completion. If G is locally finite, this coincides with $|G|$.*

In view of our earlier normal spanning tree example one should expect that, for G locally finite, the finiteness condition in Theorem 4.4 can be relaxed considerably without losing that the metric completion of G coincides with $|G|$:

Problem 4.5. Given a countable connected graph G, characterize the functions $\ell: E(G) \to \mathbb{R}^+$ for which the completion of (G, d_ℓ) coincides with $|G|$.

For functions ℓ not satisfying $\sum_{e \in G} \ell(e) < \infty$, the metric-completion approach opens up a wide range of new possibilities over the purely topological compactification approach, even for locally finite graphs. For example, the *hyperbolic compactification* of a locally finite hyperbolic graph G, as introduced by Gromov [36], is sometimes defined in a somewhat roundabout way: in an intermediate step one endows G with a different (bounded) *Gromov metric* (which on G itself is somewhat odd[36]), whose sole purpose is that its completion has a compact boundary that can then be combined with G in its *original* topology to yield an interesting compact space.[37] But Gromov [36] also proved that the hyperbolic compactification of G can, alternatively, be obtained as the completion of (G, d_ℓ) for a natural function ℓ of edge lengths (see [15] for a detailed proof, or [33] for an indication), a result that makes its use – e.g. for the study of hyperbolic groups – more accessible to a graph-theoretical approach. Fig. 4.2 shows a hyperbolic graph G with an assignment ℓ of edge lengths for which the ℓ-Top boundary of (G, d_ℓ) is its hyperbolic boundary.

The flexibility of the metric-completion approach to defining a boundary of an infinite graph G allows us to keep our options open regarding the opening question of Section 4.1, the problem of how to define circuits – and the resulting homology – in a graph with infinite degrees. Depending on the graphs under investigation, we can choose our edge length function ℓ in whatever way seems best to describe those graphs: we can *choose* whether or not we want two rays from the same end to converge to a common point at infinity or not, or not even to converge at all.

For example, for the fan G shown on the left in Fig. 4.1 we could let ℓ assign length 2^{-n} to the (one or) two edges down or sideways from the nth vertex of the vertical ray. Then this ray converges in (G, d_ℓ) to the vertex u, so (G, d_ℓ) is complete and induces ITop on G. The double fan of Fig. 4.1, with a similar function ℓ of edge lengths, will have the two vertices of infinite degree identified in (G, d_ℓ) (which is again already complete), the vertical ray will converge to this identified new vertex, and the two bottom edges will form a circle.

As another example, consider the powers T^2 and T^3 of the $\aleph_0$-regular tree T. While T^2 still has all the 'original' ends of T (and even some new ones corresponding to the levels of T), the graph T^3 does not. Indeed, since every vertex v has infinitely

36 For example, it induces the discrete topology on G, including edges!

37 For example, this *hyperbolic boundary* refines the 'end boundary' $\Omega(G)$ of G induced by $|G|$, and unlike $\Omega(G)$ it can have non-trivial connected components.

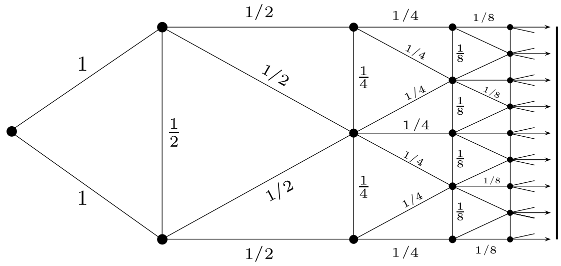

Fig. 4.2. A hyperbolic graph whose boundary is an arc (the vertical bar on the right).

many neighbours at a lower level than its own, or at level 1 (the upper neighbours of the root), no finite set S of vertices can separate v from the infinitely many vertices at level 1, which form a complete subgraph in T^3. Hence $T^3 - S$ has only one component, and T^3 has only one end! In the context of hamiltonicity problems for graph powers, however, we may well want that G^3 retains the end structure of G^2, but that its vertices and boundary points still form a compact set (so as to allow for Hamilton circles; see Section [18, Section 4.1]). Both these can be achieved by choosing ℓ appropriately and considering the metric completion of (G^3, d_ℓ) instead of the Freudenthal compactification $|G^3|$ of G^3.

But this flexibility comes at a cost: in order to benefit from our existing cycle space theory and its applications in the case of locally finite graphs, we need a homology theory that works in the generality of all metric spaces that can arise as such completions, while defaulting to our topological cycle space when the graph considered is locally finite. This homology theory will not just be a minor adaptation of the definition of $\mathcal{C}(G)$, since the spaces that can arise as completions need not look like graphs. [38] For example, the boundary of the hyperbolic graph of Fig. 4.2 is an arc that contains no edge. In particular, there may be circles in this completion in which the edges they contain are not dense; we shall therefore not be able to represent homology classes by sets of edges, as we did in the case of $\mathcal{C}(G)$.

Problem 4.6. . Extend the topological cycle space theory of locally finite graphs to a homology theory for metric spaces that can arise as completions of spaces of the form (G, d_ℓ).

As a step towards this goal, it will help to recast the topological cycle space theory of Sections 1 and 2, even for locally finite graphs, as a homology theory in the usual terms of algebraic topology [27]. We shall look at this problem next. Some first steps towards Problem 4.6 itself are already taken in [33].

4.4. Homology of locally compact spaces with ends

In this section, any undefined graph G will again be an arbitrary connected and locally finite graph.

When the topological cycle space was first introduced [16,21], the motivation was to extend the standard notion of the cycle space of a finite graph in such a way that it could play a similar role for infinite graphs. In particular, the various theorems relating $\mathcal{C}(G)$ to other structural properties of G, such as planarity, should extend to the new notion. The topological cycle space $\mathcal{C}(G)$, for locally finite G, has achieved this goal superbly – see [18, Section 4] – but at a price: due to the ad-hoc manner of its definition, this new $\mathcal{C}(G)$, unlike $\mathcal{C}_{\text{fin}}(G)$, is no longer obviously a special case of the first homology of either G or $|G|$ in any sense that would be standard in algebraic topology.

On the other hand, it seems desirable to recast our notion of $\mathcal{C}(G)$ in terms of standard homology. This is for two reasons: our understanding of $\mathcal{C}(G)$ might benefit from the vast body of knowledge available for standard homology theories; and conversely, the ideas that went into the notion of $\mathcal{C}(G)$ might throw a new light on the homology of more general spaces once we can rephrase $\mathcal{C}(G)$ in such a way that it makes sense when G is not a graph.

For a finite or infinite graph G, its first simplicial homology group is the same as, in our notation, $\mathcal{C}_{\text{fin}}(G)$. The elements of this group or vector space are finite sets of edges.[39] Hence as soon as G contains an infinite circuit, $\mathcal{C}(G)$ differs from $\mathcal{C}_{\text{fin}}(G)$. But $\mathcal{C}(G)$ also differs from the simplicial homology we would obtain from just allowing infinite (thin) sums of edges as 1-chains: in this homology, the edges of a double ray would form an algebraic cycle, since every vertex lies on two such edges and hence the boundary of this edge set is zero, but this is not a cycle in our topological cycle space. Thus, $\mathcal{C}(G)$ cannot be described by an extension of simplicial homology obtained by just allowing infinite chains.[40]

Another way to extend the simplicial homology of finite complexes to infinite ones is to take limits. In our situation, we could use the techniques from Section 3 to do this. Indeed, recall that for the finite minors $G_0 \preccurlyeq G_1 \preccurlyeq \cdots$ of G defined in

38 Gromov [36] observed that every compact metric space arises as the hyperbolic boundary of a hyperbolic graph. Georgakopoulos [33] showed that a metric space arises as the ℓ-Top boundary of a locally finite graph if and only if it is complete and has a countable dense subset.

39 We still take all our coefficients from $\mathbb{F}_2$, for easier compatibility with our earlier treatment of $\mathcal{C}(G)$. However all the results we describe hold with integer coefficients too, which is indeed the setting in the papers we refer to.

40 By contrast, $\mathcal{B}(G)$, the set of all cuts, remains the image of the coboundary operator δ^0 if we allow arbitrary infinite 0-cochains.

Section 3.1 the algebraic cycles of G_m induce algebraic cycles in G_n, for all $n < m$. The first simplicial homology groups of the G_n thus form an inverse system, whose limit is indeed $\mathcal{C}(G)$[41]:

Theorem 4.7. *The maps $D \mapsto D \cap E(G_n)$ define a group isomorphism from $\mathcal{C}(G)$ to $\varprojlim(\mathcal{C}_{\text{fin}}(G_n))_{n\in\mathbb{N}}$.*

While Theorem 4.7 points out an interesting aspect of $\mathcal{C}(G)$, it does not capture all its relevant aspects. For even if we assume that representing $\mathcal{C}(G)$ as $\varprojlim \mathcal{C}(G_n)$ captures all its relevant features as a group, the (abstract) homology group of a graph does not capture all the relevant aspects of its homology.[42] For example, recall that if G is 3-connected, then its peripheral circuits generate $\mathcal{C}(G)$ (Theorem 2.3(ii)), and G is planar if and only if these circuits form a sparse set [18, Thm. 20]. In order to prove this for infinite G, we thus need to know when a given edge set $D \in \mathcal{C}(G)$ is a peripheral circuit. If all we know of D are its intersections with the edge sets of the finite minors G_n (as we would if we viewed D as an element of $\varprojlim(\mathcal{C}_{\text{fin}}(G_n))_{n\in\mathbb{N}}$), this would be hard to decide; indeed it would not be obvious from this information whether D is a circuit at all (see Section 3.2).

A more subtle approach, which has been pursued in [25,27], is to see to what extent $\mathcal{C}(G)$ can be captured by the singular homology of $|G|$. After all, $\mathcal{C}(G)$ was defined via (the edge sets of) circles in $|G|$, which are just injective singular loops. Can we extend this correspondence between injective loops and circuits to one between $H_1(|G|)$ (singular) and $\mathcal{C}(G)$?

There are two things to notice about $H_1(|G|)$. The first is that we can subdivide a 1-simplex, or concatenate two 1-simplices into one, by adding a boundary. Indeed, if $\sigma: [0, 1] \to |G|$ is a path in $|G|$ from x to y, say, and z is a point on that path, there are paths σ' from x to z and σ'' from z to y such that $\sigma' + \sigma'' - \sigma$ is the boundary of a singular 2-simplex 'squeezed' on to the image of σ. The second fact to notice is that inverse paths cancel in pairs: if σ^+ is an x–y path in $|G|$, and σ^- an y–x path with the same image as σ^+, then $[\sigma^+ + \sigma^-] = 0 \in H_1$.[43] These two facts together imply that every homology class in H_1 is represented by a single loop: given any 1-cycle, we first add pairs of inverse paths between the endpoints of its simplices to make its image connected in the right way, and then use Euler's theorem to concatenate the 1-simplices of the resulting chain into a single loop σ.

To establish the desired correspondence between $H_1(|G|)$ and $\mathcal{C}(G)$, we would like to assign to a homology class in $H_1(|G|)$, represented by a single loop σ, an edge set $f([\sigma]) \in \mathcal{C}(G)$. Intuitively, we do this by counting for each edge e of G how often σ traverses it entirely (which, since the domain of σ is compact, is a finite number of times), and let $f([\sigma])$ be the set of those edges e for which this number is odd. Using the usual tools of homology theory, one can make this precise in such a way that f is clearly a well-defined homomorphism $H_1(|G|) \to \mathcal{E}(G)$,[44] and whose image is easily seen to be $\mathcal{C}(G)$. What is not clear at once is whether f is 1–1 and onto.

Surprisingly, f is indeed surjective—and this is not even hard to show. Indeed, let an edge set $D \in \mathcal{C}(G)$ be given. Our task is to find a loop σ that traverses every edge in D an odd number of times, and every other edge of G an even number of times. As a first approximation, we let σ_0 be a path that traverses every edge of some fixed normal spanning tree of G exactly twice, once in each direction; see Section 3.3 for how to construct such a loop. Moreover, we construct σ_0 in such a way that it pauses at every vertex v—more precisely, so that $\sigma_0^{-1}(v)$ is a union of finitely many closed intervals at least one of which is non-trivial. Next, we write D as a thin sum $D = \sum_i C_i$ of circuits; such a representation of D exists by definition of $\mathcal{C}(G)$. For each of these C_i we pick a vertex $v_i \in \overline{C_i}$, noting that no vertex of G gets picked more than finitely often, because it has only finitely many incident edges and the C_i form a thin family. Finally, we turn σ_0 into the desired loop σ by expanding the pause at each vertex v to a loop going once round every $\overline{C_i}$ with $v = v_i$. Using the methods from Section 3.3 it is not hard to show that σ is continuous [25], and clearly it traverses every edge of G the desired number of times.

Equally surprisingly, perhaps, f is usually not injective (see below). In summary, therefore, the topological cycle space $\mathcal{C}(G)$ of G is related to the first singular homology group of G as follows:

Theorem 4.8 (*[25]*)**.** *The map $f: H_1(|G|) \to \mathcal{E}(G)$ is a group homomorphism onto $\mathcal{C}(G)$, which has a non-trivial kernel if and only if G contains infinitely many (finite) circuits.*

An example of a non-null-homologous loop in $|G|$ whose homology class maps to the empty set $\emptyset \in \mathcal{C}(G)$ is easy to describe. Let G be the one-way infinite ladder L (with its end on the right), and define a loop ρ in L, as follows. We start at time 0 at the top-left vertex, v_0 say, and begin by going round the first square of L in a clockwise direction. This takes us back to v_0. We then move along the horizontal edge incident with v_0, to its right neighbour v_1. From here, we go round the second square in a clockwise direction, back to v_1 and on to its right neighbour v_2. We repeat this move until we reach the end ω of L on the right, say at time $\frac{1}{2} \in [0, 1]$. So far, we have traversed the first vertical edge and every bottom horizontal

41 This is tantamount to taking the *Čech homology* of G or of $|G|$; see [25].

42 This is reflected by the fact that, as abstract groups, $\mathcal{C}(G)$ hardly depends on G: by Theorem 2.1, it is the direct product of as many copies of the coefficient ring as a topological spanning tree of G has chords—i.e., of $\aleph_0$ copies for most G.

43 To see that this sum is a boundary, subtract the constant 1-simplex σ with value x: there is an obvious singular 2-simplex of which $\sigma^+ + \sigma^- - \sigma$ is the boundary. Subtracting σ is allowed, since $\sigma = \sigma + \sigma - \sigma$, too, is a boundary: of the constant 2-simplex with value x.

44 For each edge e, let $f_e: |G| \to S^1$ be a map wrapping e once round S^1 and mapping all of $|G| \setminus \mathring{e}$ to one point of S^1. Let π denote the group isomorphism $H_1(S^1; \mathbb{Z}_2) \to \mathbb{Z}_2$. Given $h \in H_1(|G|)$, let $f(h) := \{ e \mid (\pi \circ (f_e)_*)(h) = 1 \in \mathbb{Z}_2 \}$. See [25] for details.

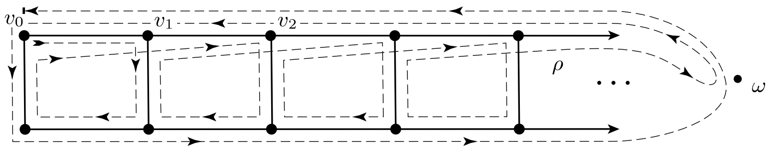

Fig. 4.3. The loop ρ is not null-homologous.

edge once (in the direction towards v_0), every other vertical edge twice (once in each direction), and every top horizontal edge twice in the direction towards the end. From there, we now use the remaining half of our time to go round the infinite circle formed by the first vertical edge and all the horizontal edges one and a half times, in such a way that we end at time 1 back at v_0 and have traversed every edge of L equally often in each direction. Clearly, f maps (the homology class of) this loop ρ to $0 \in \mathcal{C}(G)$.

The loop ρ is indeed not null-homologous [25], but it seems non-trivial to show this (Fig. 4.3). To see why this is hard, let us compare ρ to a loop winding round a finite ladder in a similar fashion, traversing every edge once in each direction. Such a loop σ is still not null-homotopic, but it is null-homologous. To see this, we subdivide it into single edges: we find a finite collection of 1-simplices σ_i, two for every edge, such that $[\sigma] = [\sum_i \sigma_i]$ and every σ_i just traverses its edge. Next, we pair up these σ_i into cancelling pairs: if σ_i and σ_j traverse the same edge e (in opposite directions), then $[\sigma_i + \sigma_j] = 0$. Hence $[\sigma] = [\sum_i \sigma_i] = 0$, as claimed. But we cannot imitate this proof for ρ and our infinite ladder L, because homology classes in $H_1(|G|)$ are still finite chains: we cannot add infinitely many boundaries to subdivide ρ infinitely often.

As it happened, the proof of the seemingly simple fact that ρ is not null-homologous took a detour via the solution of a much more fundamental problem: the problem of understanding the fundamental group of $|L|$, or more generally, of $|G|$ for a locally finite graph G. In order to distinguish ρ from boundaries, we looked for a numerical invariant Λ of 1-chains that was non-zero on ρ but both linear and additive (so that $\Lambda(\sigma_1\sigma_2) = \Lambda(\sigma_1 + \sigma_2) = \Lambda(\sigma_1) + \Lambda(\sigma_2)$ for concatenations of 1-simplices σ_1, σ_2) and invariant under homotopies (so that $\Lambda(\sigma_1\sigma_2) = \Lambda(\sigma)$ when $\sigma \sim \sigma_1\sigma_2$). Then, given a 2-simplex τ with boundary $\partial\tau = \sigma_1 + \sigma_2 - \sigma$, we would have $\Lambda(\partial\tau) = \Lambda(\sigma_1\sigma_2) - \Lambda(\sigma) = 0$, so Λ would vanish on all boundaries but not on ρ. We did not quite find such an invariant Λ, but a collection of similar invariants which, together, can distinguish loops like ρ from boundaries.

In order to find such functions on 1-chains that are invariant under homotopies, it was necessary to find a combinatorial description for the homotopy types of loops—that is, for the fundamental group of $|G|$. Such a combinatorial characterization is given in [26], where $\pi_1(|G|)$ is characterized as a group of (infinite) words of chords of a topological spanning tree—in the spirit of the usual description of $\pi_1(G)$ for a finite graph as the free group generated by such chords—and as a subgroup of the inverse limit of these finitely generated free groups. (The group $\pi_1(|G|)$ itself is not free, unless G is essentially finite.)

Not surprisingly, our topological cycle space $\mathcal{C}(G)$ can, as a group, be viewed as the *infinite abelianization* of $\pi_1(|G|)$: the factor group of $\pi_1(|G|)$ obtained by declaring two (reduced) words as equivalent if each letter occurs in both the same number of times (regardless of position).

Let us return to our original goal: to see to what extent $\mathcal{C}(G)$ can be captured by the singular homology of $|G|$. In view of Theorem 4.8, the goal might be phrased as follows:

Problem 4.9. Devise a singular-type homology theory for locally compact spaces with ends that coincides with $\mathcal{C}(G)$ when applied to $|G|$ in dimension 1.

Some first steps in this direction were already taken in [25]. The approach there was to allow only singular simplices whose vertices lie in G (i.e., are not ends); to allow infinite chains that are locally finite at points of G – i.e., every point in G has a neighbourhood meeting only finitely many simplices in the given chain – but not necessarily at ends (reflecting the fact that, for example, the union of all the vertical double rays in the grid defines an algebraic cycle of which infinitely many 1-simplices contain the end); and to restrict the set of cycles to those chains in the kernel of the boundary operator that could be written as a (possibly infinite) sum of finite cycles.

This approach solves Problem 4.9 in an ad-hoc sort of way: it permits the definition of homology groups for arbitrary locally compact spaces with ends, it defaults to the topological cycle space for graphs in dimension 1—but it is not a homology theory in the sense of the usual axioms [38,29]. To achieve the latter, one has to find a way of implementing the required restrictions as conditions on chains rather than on cycles. This was done in [27]. However, this is no more than a beginning, and more translation work remains to be done—for example, of the duality theory indicated for $\mathcal{C}(G)$ in Section 2.

4.5. Infinite matroids

Traditionally, infinite matroids are defined like finite ones, with the additional axiom that an infinite set is independent as soon as all its finite subsets are independent. This reflects the notion of linear independence in vector spaces, and also the absence of the usual (finite) cycles in a graph: the bases of the cycle matroid of an infinite graph are then the edge sets of its (ordinary) spanning trees. We shall call such matroids *finitary*. Note that the circuits in a finitary matroid, the minimal dependent sets, are necessarily finite.

An important and regrettable feature of such finitary matroids is that the additional axiom restricting the infinite independent sets spoils duality, one of the key features of matroid theory. For example, every bond of a graph would be a circuit in any dual of its cycle matroid: a set of edges that is minimal with the property of not lying in the complement of a spanning tree, i.e. of containing an edge from every spanning tree. Since finitary matroids have no infinite circuits, the cycle matroid of a graph with an infinite bond thus cannot have a finitary dual.

Our theory, however, suggests an obvious solution to this problem: shouldn't infinite matroids be defined in such a way that infinite circuits in a graph can become matroid circuits, and topological spanning trees become bases? Indeed, infinite circuits are not contained in the edge set of any topological spanning tree (although they *are* contained in the edge set of an ordinary spanning tree), while if we delete any edge from an infinite circuit, then its remaining edges can be extended to a topological spanning tree by Lemma 1.6.

There are two main challenges in devising axioms for such a non-finitary theory of infinite matroids: to avoid the mention of cardinalities, and to take care of limits.[45] In [7] such a theory has been proposed. It can be stated in terms of any of five equivalent sets of axioms: independence, basis, circuit, closure or rank axioms. They are shown to be equivalent to the 'B-matroids' explored in the late 1960 s by Higgs [39–41], who had defined them in terms of a different and rather more complicated set of closure requirements. As just one of a plethora of alternatives for a possible concept of infinite matroids considered at the time, these 'B-matroids' had gone largely unnoticed, although a workable combination of independence and exchange axioms was later found by Oxley [43,44].

With any of the said five sets of axioms, duality works as expected from finite matroids: it is exemplified by dual planar graphs (see below), there is a well-defined notion of minors with contraction and deletion as dual operations, and so on.

So what non-finitary matroids are there in graphs? As had been our motivation, the circuits (finite or infinite) of a locally finite graph G form the circuits of a matroid in this theory, as do the finite circuits of G. Let us denote these matroids by $M_{\mathrm{C}}(G)$ and $M_{\mathrm{FC}}(G)$, respectively. Similarly, the bonds (finite or infinite) of G form the circuits of a matroid $M_{\mathrm{B}}(G)$, just as its finite bonds form the circuits of a matroid $M_{\mathrm{FB}}(G)$. Clearly, $M_{\mathrm{FC}}(G)$ and $M_{\mathrm{B}}(G)$ form a pair of dual matroids, and by Lemmas 1.5 and 1.7 so do $M_{\mathrm{C}}(G)$ and $M_{\mathrm{FB}}(G)$.

The same is true for the slightly more general identification spaces $\tilde{G}$ under ITop, defined in Section 4.2. Thus if G and G^* are a pair of dual finitely separable graphs, and we take as circuits the edge sets of circles in the spaces $\tilde{G}$ and $\tilde{G}^*$, then $M_{\mathrm{FC}}(G)$ and $M_{\mathrm{C}}(G^*)$ form a dual pair of matroids, as do $M_{\mathrm{FB}}(G)$ and $M_{\mathrm{B}}(G^*)$.

Call a (finite or infinite) matroid *graphic* if it is the cycle matroid $M_{\mathrm{C}}(G)$ of a finitely separable graph G. The infinite version of Whitney's theorem can now be restated in matroid terms:

Theorem 4.10 ([7]). *A finitely separable graph G is planar if and only if its finite-cycle matroid $M_{\mathrm{FC}}(G)$ has a graphic dual.*

Much of the attractiveness of finite matroids stems from the fact that they provide a unified framework for some essential common aspects of otherwise disparate branches of mathematics. Whether or not the same can be said for infinite matroids, axiomatized in this way, will depend on concrete examples that have yet to be found—if possible, from as different areas of mathematics as possible.

There is no doubt that non-finitary matroids are plentiful. Indeed, a finitary matroid has a finitary dual only if it is the direct sum of finite matroids [52]. Since all our matroids have duals, the duals of all the other finitary matroids (e.g., of all connected matroids [1]) thus form a large class of non-finitary matroids. However, perhaps there are natural 'primary' matroids that are non-finitary and have therefore gone unnoticed—for example, in the context of Banach or Hilbert spaces?

Here is an interesting concrete problem. An example in [7] shows that there are matroids with both infinite circuits and cocircuits, indeed matroids in which all these are infinite. However, we do not know the answer to the following:

Problem 4.11. Is the intersection of a circuit and a cocircuit always finite?

Theorem 2.15 seems to suggest that, given a locally finite graph G (or a finitely separable one), there might be a matroid on the set $E(G) \cup \Omega(G)$ in which the sets $F \cup \Psi$ with $F \subseteq E(G)$ and $\Psi \subseteq \Omega(G)$ for which $\bigcup F \cup \Psi$ contains no circle form the independent sets. However, as soon as we try to apply basis or circuit elimination axioms to ends, we see that this fails. For example, if two circles meet in exactly one end, we cannot delete the end and find another circle in the rest (the union of all the edges of the two circles). Similarly, in the double ladder we could choose as a basis B_1 the union of one double ray, all the rungs, and both ends, and as another basis B_2 all the edges that are not rungs and one of the two ends. If we delete the other end, ω say, from B_1, we cannot find an element of $B_2 \setminus B_1$ (which would be an edge) that we could add to $B_1 - \omega$ to form another basis.

More generally, the duality which Theorem 2.15 expresses for graphs with ends cannot be expressed in terms of matroid duality. Indeed, suppose that, with the notation of Theorem 2.15, there is a matroid M on $E \cup \Omega$ whose bases are the sets $F \cup \Psi$ (with $F \subseteq E$ and $\Psi \subseteq \Omega$) that form the subspaces X of $|G|$ for which $\tilde{X}$ is a spanning tree of $\tilde{G}$. Then $M/\Omega = M_{\mathrm{C}}(G)$ and $M \setminus \Omega = M_{\mathrm{FC}}(G)$. But one can show that, given any 2-connected finitely separable graph G, there is no matroid M on any set $E(G) \cup X$ with $X \cap E(G) = \emptyset$ such that $M/X = M_{\mathrm{C}}(G)$ and $M \setminus X = M_{\mathrm{FC}}(G)$. See [8] for details.

Richter et al. used a non-finitary matroid similar to $M_{\mathrm{C}}(G)$ for their proof of an extension of Whitney's theorem to 'graph-like spaces'; see [45].

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45 If one wants to have basis and circuit axioms, one has to ensure that maximal independent sets and minimal dependent sets exist: with infinite sets, this is no longer clear.

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