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Retraction Map Categories and Their Applications to the Construction of Lambda Calculus Models

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This paper deals with categorical models of the λ-calculus. We generalize the inverse limit method Scott used for his construction of D_∞, and introduce order-enriched ccc's, retraction map categories and ε-categories. An order-enriched ccc is a cartesian closed category $\mathbf{C}$ equipped with a partial order relation $\leqslant$ on the set of the arrows. A retraction map category of $\mathbf{C}$ is $\mathbf{R} = (\mathbf{R}, \leqslant, i, j)$, where $\leqslant$ is a partial order relation on the set $|\mathbf{C}|$ of all the objects of $\mathbf{C}$, $\mathbf{R}$ is the category of the poset $(|\mathbf{C}|, \leqslant)$, and i and j are functors from $\mathbf{R}$ to $\mathbf{C}$ and from $\mathbf{R}^{\mathrm{op}}$ to $\mathbf{C}$ that satisfy the conditions: (1) $j[a, b] \circ i[a, b] \geqslant \mathrm{id}_a$ and (2) $i[a, b] \circ j[a, b] \leqslant \mathrm{id}_b$ for every arrow $[a, b]: a \to b$ in $\mathbf{R}$ (i.e., $a \leqslant b$). The ε-category $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$ of $\mathbf{C}$ w.r.t. $\mathbf{R}$ is the category whose objects are ideals of $(|\mathbf{C}|, \leqslant)$ and whose arrows are ideals of $(\mathbf{C}, \sqsubseteq)$, where $\leqslant$ is the partial order relation in $\mathbf{R}$ and $\sqsubseteq$ is the partial order relation defined by $f \sqsubseteq g$ iff $\mathrm{dom}(f) \leqslant \mathrm{dom}(g)$, $\mathrm{cod}(f) \leqslant \mathrm{cod}(g)$ in $\mathbf{R}$ and $f \leqslant j[a, b] \circ g \circ i(a, b]$ in $\mathbf{C}$. We show that every ε-category $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$ is also an order-enriched ccc. Moreover when $\mathbf{E}$ and $\mathbf{R}$ satisfy a particular condition, $\mathbf{E}(\mathbf{C}, \mathbf{R})$ has a reflexive object. For example, if there is an ideal U of $(|\mathbf{C}|, \leqslant)$ satisfying the following conditions, then U is isomorphic to U^U in $\mathbf{E}$ and a λ-algebra is constructed from $\mathbf{E}$ and U: (1) for every pair of $a, b \in U$, U contains b^a, and (2) for every $c \in U$, there are $a, b \in U$ such that $c \in b^a$. We reconstruct P_ω and D_∞ using ε-categories. © 1986 Academic Press, Inc.

1. INTRODUCTION

This paper deals with categorical models of the λ-calculus. We will generalize the construction of D_∞ by Scott (1972) and present a quite general method of constructing the models. As examples we will reconstruct D_∞ and P_ω using the method.

It is known that the λ-calculus is closely related to cartesian closed categories (abbreviated to ccc). See Koymans (1982, 1984), Barendregt (1984), Lambek and Scott (1982), and Lambek (1974, 1980). The definition of ccc will appear in Definition 2.1. Given a model $\mathfrak{M}$ of the λ-calculus, a ccc $\mathbf{P}(\mathfrak{M})$ can be constructed from $\mathfrak{M}$, and $\mathbf{P}(\mathfrak{M})$ has a particular object u, called reflexive object, and a pair of arrows $\Phi: u \to u^u$ and $\Psi: u^u \to u$ such that $\Phi \circ \Psi = \mathrm{id}_u u$. Conversely if a ccc $\mathbf{P}$ has a reflexive object u and a pair of arrows Φ and Ψ that satisfy the above conditions,

0019-9958/86 $3.00

then a model $\mathfrak{M}(\mathbf{P})$ of the λ-calculus is naturally defined. Furthermore, for every model $\mathfrak{N}$ of the λ-calculus, $\mathfrak{M}(\mathbf{P}(\mathfrak{N}))$ is essentially the same as $\mathfrak{N}$. More rigorously $\mathfrak{M}(\mathbf{P}(\mathfrak{N}))$ and $\mathfrak{N}$ are isomorphic. Therefore a model of the λ-calculus is completely characterized by a ccc with a reflexive object. We refer the reader to Koymans (1982).

In order to construct a model of the λ-calculus under the above discussion, we must find a ccc with a reflexive object. Indeed Scott (1972) chose the category **CL** whose objects are all continuous lattices and whose arrows are all continuous functions, and he was successful in constructing a reflexive object D_∞ by means of the so called inverse limit. As for the graph model P_ω (see Scott (1976)), P_ω is also another reflexive object in **CL**.

We briefly repeat the construction of D_∞. First we arbitrarily choose a continuous lattice D_0 and define the sequence $\{D_n\}$ of continuous lattices from D_0 as follows: $D_{n+1} = [D_n \to D_n]$, where $[D_n \to D_n]$ means the continuous lattice that consists of all continuous functions from D_n to D_n. Next we choose a pair of continuous functions $i_0: D_0 \to D_1$ and $j_0: D_1 \to D_0$ that satisfy the conditions: (1) $(j_0 \circ i_0)(x) = x$ for every $x \in D_0$, and (2) $(i_0 \circ j_0)(f) \leqslant f$ for every $f \in D_1$. And we define the sequence $\{(i_n, j_n)\}$ from (i_0, j_0):

$$i_n: D_n \to D_{n+1}, \qquad j_n: D_{n+1} \to D_n,$$

$$i_{n+1}(f) = i_n \circ f \circ j_n \qquad \text{for} \quad f \in D_{n+1},$$

$$j_{n+1}(g) = j_n \circ g \circ i_n \qquad \text{for} \quad g \in D_{n+2}.$$

Then D_∞ is defined as the inverse limit of the system $\{j_n\}$. Also D_∞ is the inductive limit of $\{i_n\}$.

In this paper, we intend to extend the above mechanism of generating D_∞. We will introduce an *order-enriched ccc* corresponding to **CL**. An order enriched-ccc is a ccc equipped with a partial order relation among the arrows. The category **CL** is an instance of order-enriched ccc's. On the other hand, for the system $\{(i_n, j_n)\}$ we will introduce a *retraction map category* of an order-enriched ccc **C**. A retraction map category is a category **R** equipped with a partial order relation $\leqslant$ among the objects. The objects of **R** are just the same as of **C**. For each pair of objects a and b, when $a \leqslant b$, **R** has a unique arrow $(i, j): a \to b$, where $i: a \to b$ and $j: b \to a$ are arrows of **C** that satisfy the conditions: (1) $j \circ i = \mathrm{id}_a$ (In Sect. 2, we will take the weaker condition $j \circ i \geqslant \mathrm{id}_a$.) and (2) $i \circ j \leqslant \mathrm{id}_b$. Informally speaking, each arrow $(i, j): a \to b$ in **R** means an embedding of a into b. The name "retraction" comes from "retract" in Scott (1972). Similar concepts to an order-enriched category and a retraction map category appear in Wand (1979). But cartesian closedness is not discussed there.

Next we will realize a work corresponding to the construction of D_∞ in the above situation. In the case of D_∞, the limit of $\{D_n\}$ is examined. In our method, $\{D_n\}$ is generalized under **R** to a directed set of objects of **C** w.r.t. the partial order relation $\leqslant$ of **R**. A directed set means a set A that satisfies the condition: for every pair of objects a and b in A, there exists c in A such that $a \leqslant c$ and $b \leqslant c$ in **R**. We will examine the limit of such A. As for D_∞, the inverse limit of $\{j_n\}$ exists in **CL**. In an arbitrary ccc, however, this is not generally possible. We intend to expand the original order-enriched ccc so that the inverse limit exists. For an order-enriched ccc **C** and a retraction map category **R** of **C**, we will define the category $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$ called the *ε-category* of **C** w.r.t. **R**. Roughly speaking, the objects of **E** are sets of objects of **C**, and the arrows of **E** are also sets of arrows of **C**. The ε-category **E** means a completion of **C** and **E** is a desired expansion of **C**. As a main theorem we will show that the ε-category also becomes an order-enriched ccc.

When **R** has a particular property, the ε-category **E** has a reflexive object. For example, we consider the case where **R** satisfies the condition: there exists a directed set U of objects of **C** w.r.t. $\leqslant$ of **R** that is closed under $(-)^{(-)}$. Namely (1) for every pair of $a, b \in U$ there is $c \in U$ such that $a \leqslant c$ and $b \leqslant c$ in **R**; and (2) if $a \in U$ and $b \in U$, then $b^a \in U$. Then, U becomes a reflexive object of **E**. This is a typical example. We will show that D_∞ and P_ω can be reconstructed by means of the ε-category method.

Furthermore we will construct a λ-algebra but not a λ-model (see Barendregt, 1984) using the ε-category method. It is known that the closed-term model is such a λ-algebra. It follows from the result by Plotkin (1976): the λ-calculus is ω-incomplete. The closed term model is a syntactical model, while our λ-algebra is a mathematical model. Our λ-algebra is constructed independently of Plotkin's result.

In Section 2, we give basic definitions of order-enriched ccc's, retraction map categories and ε-categories. In Section 3, we prove that every ε-category is an order-enriched ccc. In Sections 4 and 5, we examine properties of ε-categories. In Section 6, we show that relationship between ε-categories and models of the λ-calculus. In Section 7, we examine properties of retraction map categories. In Sections 8 and 9, we deal with D_∞ and P_ω as examples of ε-categories.

2. Order-Enriched ccc's, Retraction Map Categories and ε-Categories

In this section we will give basic definitions of order-enriched ccc's, retraction map categories, and ε-categories, which are foundations through the whole discussions.

2.1. DEFINITION. (i) Let $\mathbf{C}$ be a category. In general, the set of all the objects of $\mathbf{C}$ is denoted by $|\mathbf{C}|$, and for each pair of objects $a, b \in |\mathbf{C}|$ the set of all the arrows from a to b in $\mathbf{C}$ is denoted by $\mathbf{C}(a, b)$. When f is an arrow from a to b in $\mathbf{C}$, we define $\mathrm{dom}(f) = a$ and $\mathrm{cod}(f) = b$. For each object a the identity arrow on a is denoted by id_a. Sometimes $\mathbf{C}$ means the set of all arrows in $\mathbf{C}$.

(ii) A *cartesian closed category* (abbreviated to ccc) is a category equipped with an additional structure $(1, !_{(-)}, (-)\times(-), \pi_1^{(-),(-)}, \pi_2^{(-),(-)}, \langle(-), (-)\rangle, (-)^{(-)}, \mathrm{ev}^{(-),(-)}, \Lambda(-))$ as follows:

(a) $1 \in |\mathbf{C}|$ (called the terminal). For each $a \in |\mathbf{C}|$, $!_a \in \mathbf{C}(a, 1)$ and $!_a \circ f = !_b$ for any $f \in \mathbf{C}(b, a)$.

(b) For each pair of $a, b \in |\mathbf{C}|$, $a \times b \in |\mathbf{C}|$ (called the product of a and b), $\pi_1^{a,b} \in \mathbf{C}(a \times b, a)$ and $\pi_2^{a,b} \in \mathbf{C}(a \times b, b)$. For each pair of $f \in \mathbf{C}(c, a)$ and $g \in \mathbf{C}(c, b)$, $\langle f, g\rangle \in \mathbf{C}(c, a \times b)$ is defined. Moreover the following conditions must be satisfied:

$$\pi_1^{a,b} \circ \langle f, g\rangle = f \qquad \text{and} \qquad \pi_2^{a,b} \circ \langle f, g\rangle = g$$

for any pair of $f \in \mathbf{C}(c, a)$ and $g \in \mathbf{C}(c, b)$, and

$$\langle \pi_1^{a,b} \circ h, \pi_2^{a,b} \circ h\rangle = h \qquad \text{for any} \quad h \in \mathbf{C}(c, a \times b).$$

For each pair of $f \in \mathbf{C}(a, a')$ and $g \in \mathbf{C}(b, b')$, we abbreviate $\langle f \circ \pi_1^{a,b}, g \circ \pi_2^{a,b}\rangle \in \mathbf{C}(a \times b, a' \times b')$ to $f \times g$.

(c) For each pair of $a, b \in |\mathbf{C}|$, $b^a \in |\mathbf{C}|$ (called the exponential from a to b) and $\mathrm{ev}^{a,b} \in C(b^a \times a, b)$. For each pair of $c, a \in |\mathbf{C}|$ and each $f \in \mathbf{C}(c \times a, b)$, $\Lambda_{c,a}(f) \in \mathbf{C}(c, b^a)$ is defined. Whenever no confusion occurs, we omit the subscript c and a of $\Lambda_{c,a}(f)$. Moreover the following conditions must be satisfies:

$$\mathrm{ev}^{a,b} \circ (\Lambda(f) \times \mathrm{id}_c) = f \qquad \text{for any } f \in \mathbf{C}(c \times a, b)$$

$$\Lambda(\mathrm{ev}^{a,b} \circ (h \times \mathrm{id}_c)) = h \qquad \text{for any } h \in \mathbf{C}(c, b^a).$$

(iii) An *order-enriched ccc* is a ccc $\mathbf{C}$ equipped with an additional structure $\leqslant$, where $\leqslant$ is a partial order on $\mathbf{C}(a, b)$ for each pair of $a, b \in |\mathbf{C}|$ such that the following conditions are satisfied:

(1) for $f, f' \in \mathbf{C}(a, b)$, and $g, g' \in \mathbf{C}(b, c)$, if $f \leqslant f'$ and $g \leqslant g'$ then $g \circ f \leqslant g' \circ f'$;

(2) for $f, f' \in \mathbf{C}(c, a)$, and $g, g' \in \mathbf{C}(c, b)$, if $f \leqslant f'$ and $g \leqslant g'$ then $\langle f, g\rangle \leqslant \langle f', g'\rangle$; and

(3) for $f, f' \in \mathbf{C}(c \times a, b)$, if $f \leqslant f'$ then $\Lambda(f) \leqslant \Lambda(f')$.

Note that the following equations are satisfied in ccc's:

$$\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle \quad \text{and} \quad \Lambda(h \circ (g \times \mathrm{id})) = \Lambda(h) \circ g.$$

2.2. DEFINITION (Retraction pair). Let **C** be an order-enriched ccc. For each pair of arrows $i\colon a \to b$ and $j\colon b \to a$ in **C**, when $j \circ i \geqslant \mathrm{id}_a$ and $i \circ j \leqslant \mathrm{id}_b$, (i, j) is called a *retraction pair* from a to b in **C**.

For each retraction pair (i, j) from a to b in **C**,

(1) (i, j) is upper-injective iff $j \circ i = \mathrm{id}_a$, and

(2) (i, j) is lower-injective iff $i \circ j = \mathrm{id}_b$.

When (i, j) is a retraction pair from a to b, we write $(i, j)\colon a \to b$.

2.3. LEMMA. *Let C be an order-enriched* ccc. *If* $(i_a, j_a)\colon a \to a'$ *and* $(i_b, j_b)\colon b \to b'$ *are retraction pairs in* **C**, *then*

(1) $(i_a \times i_b, j_a \times j_b)\colon a \times b \to a' \times b'$ *and*

(2) $(\Lambda(i_b \circ \mathrm{ev}^{a,b} \circ (\mathrm{id}_{b^a} \times j_a)), \Lambda(j_b \circ \mathrm{ev}^{a',b'} \circ (\mathrm{id}_{b'^{a'}} \times i_a)))\colon b^a \to b'^{a'}$

are also retraction pairs in **C**. *If* $(i, j)\colon a \to b$ *and* $(i', j')\colon b \to c$ *are retraction pairs in* **C**, *then*

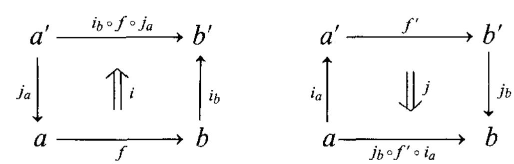
(3) $(i' \circ i, j \circ j')\colon a \to c$

is also a retraction pair in **C**. *We define* $(i_a, j_a) \times (i_b, j_b)$, $(i_b, j_b)^{(i_a, j_a)}$ *and* $(i', j') \circ (i, j)$ *as the above retraction pairs* (1), (2), *and* (3), *respectively.*

Furthermore if (i_a, j_a) *and* (i_b, j_b) *are both upper- (lower-) injective, then* $(i_a, j_a) \times (i_b, f_b)$ *and* $(i_b, j_b)^{(i_a, j_a)}$ *are also upper- (lower-) injective. Similarly if* (i, j) *and* (i', j') *are upper- (lower-) injective, then* $(i', j') \circ (i, j)$ *is so.*

Proof. By simple calculation. ∎

The following figures illustrate the intuitive meanings of $(i, j) = (i_b, j_b)^{(i_a, j_a)}$ in Lemma 2.3:



We will use the notations $(i_a, j_a) \times (i_b, j_b)$, $(i_b, j_b)^{(i_a, j_a)}$, and $(i', j') \circ (i, j)$ in the following discussions, too.

2.4. DEFINITION (Retraction map category). Let $\mathbf{C}$ be an order-enriched ccc. A *retraction map category* of $\mathbf{C}$ is $\mathbf{R} = (\mathbf{R}, \leqslant, i, j)$ defined as follows:

(a) $\leqslant$ is a partial order relation on $|\mathbf{C}|$ that satisfies the conditions: if $a \leqslant a'$ and $b \leqslant b'$, then $a \times b \leqslant a' \times b'$ and $b^a \leqslant b'^{a'}$.

(b) $\mathbf{R}$ is the category defined by the partially ordered set $(|\mathbf{C}|, \leqslant)$. Namely, the objects of $\mathbf{R}$ are just the same as of $\mathbf{C}$, and the arrows in $\mathbf{R}$ are all $[a, b]: a \to b$ such that $a \leqslant b$.

(c) i is a functor from $\mathbf{R}$ to $\mathbf{C}$ and j is a functor from $\mathbf{R}^{\mathrm{op}}$ to $\mathbf{C}$ such that $i(a) = j(a) = a$ for $a \in |\mathbf{R}| = |\mathbf{R}^{\mathrm{op}}|$, and $(i[a, b], j[a, b])$ is a retraction pair from a to b in $\mathbf{C}$ for every $[a, b]$ in $\mathbf{R}$. Here $\mathbf{R}^{\mathrm{op}}$ means the opposite category of $\mathbf{R}$.

(d) Moreover, when $a \leqslant a'$ and $b \leqslant b'$ in $\mathbf{R}$, the following equations must be satisfied:

$$(1)\quad (i[a \times b, a' \times b'], j[a \times b, a' \times b']) = (i[a, a'], j[a, a']) \times (i[b, b'], j[b, b']) \text{ and}$$

$$(2)\quad (i[b^a, b'^{a'}], j[b^a, b'^{a'}]) = (i[b, b'], j[b, b'])^{(i[a,a'], j[a,a'])}.$$

When no confusion occurs, we sometimes regard retraction pair $(i[a, b], j[a, b])$ as the unique arrow from a to b in $\mathbf{R}$ for each pair of $a, b \in |\mathbf{R}|$ such that $a \leqslant b$,

If $(i[a, b], j[a, b])$ is upper- (lower-) injective for all $[a, b]$ in $\mathbf{R}$, then we say that $\mathbf{R}$ is upper- (lower-) injective.

The following are the necessary and sufficient conditions for the above i and j to be functors:

(3) if $a \leqslant b$ and $b \leqslant c$ in $\mathbf{R}$, then
$$(i[a, c], j[a, c]) = (i(b, c] \circ i[a, b], j[a, b] \circ j[b, c]) \text{ in } \mathbf{C};$$

(4) for all $a \in |\mathbf{R}|$,
$$(i[a, a], j[a, a]) = (\mathrm{id}_a, \mathrm{id}_a).$$

Note that, by Lemma 2.3, conditions (d)(1) and (d)(2) in Definition 2.4 and the above (3) are well defined. And note that we do not claim the conditions: $a \leqslant a'$ and $b' \leqslant b'$ iff $a \times b \leqslant a' \times b'$ iff $b^a \leqslant b'^{a'}$. In Definition 7.3 we will examine retraction map categories that satisfy such stronger conditions.

In general, there are many different retraction map categories for each order-enriched ccc $\mathbf{C}$. And there always exists at least one retraction map category of $\mathbf{C}$. The following retraction may category is an example of such ones.

2.5. DEFINITION (Trivial retraction map category). Let **C** be an order-enriched ccc. The *trivial retraction map category* **IR(C)** of **C** is defined as follows: for each pair of objects a and b of **C**, $a \leqslant b$ iff $a = b$. Therefore **IR(C)** consists of only the identity arrows.

2.6. LEMMA. *Let* **C** *and* **R** *be an order-enriched ccc and a retraction map category of* **C**, *respectively. Let a, a', b, b', c, and c' be objects of* **C**. *Suppose that $a \leqslant a'$, $b \leqslant b'$, and $c \leqslant c'$ in* **R**.

(i) *For all pairs of arrows $f'\colon c' \to a'$ and $g'\colon c' \to b'$ in* **C**,

$$j[a \times b, a' \times b'] \circ \langle f', g' \rangle \circ i[c, c'] \\ = \langle j[a, a'] \circ f' \circ i[c, c'], j[b, b'] \circ g' \circ i[c, c'] \rangle.$$

(ii) *For all arrows $f'\colon c' \times a' \to b'$ in* **C**,

$$j[b^a, b'^{a'}] \circ \Lambda(f') \circ i[c, c'] \\ = \Lambda(j[b, b'] \circ f' \circ i[c \times a, c' \times a']).$$

Proof. By simple calculation. ▮

2.7. DEFINITION. Let **C** and **R** be an order-enriched ccc and a retraction map category of **C**, respectively. We define the partial order relation $\sqsubseteq$ among the arrows of **C** as follows: for each pair of arrows $f\colon a \to b$ and $g\colon a' \to b'$ in **C**,

$$f \sqsubseteq g \qquad \text{iff} \qquad a \leqslant a', b \leqslant b' \text{ and } f \leqslant j[b, b'] \circ g \circ i[a, a'].$$

Note that $f \leqslant j[b, b'] \circ g \circ i[a, a']$ iff $i[b, b'] \circ f \circ j[a, a'] \leqslant g$. The above $\sqsubseteq$ depends on both **C** and **R**. It can be easily proved that $\sqsubseteq$ is really a partial order relation.

The following are basic properties of $\sqsubseteq$.

2.8. LEMMA. *Let* **C** *and* **R** *be an order-enriched ccc and a retraction map category of* **C**, *respectively. For objects a, a', b, and b' of* **C**, *if $a \leqslant a'$ and $b \leqslant b'$ in* **R**, *then*

(i) $\pi_1^{a,b} \sqsubseteq \pi_1^{a',b'}$, $\pi_2^{a,b} \sqsubseteq \pi_2^{a',b'}$, *and*
(ii) $\mathrm{ev}^{a,b} \sqsubseteq \mathrm{ev}^{a',b'}$.

Proof. Clear. ▮

2.9. LEMMA. *Let* **C** *and* **R** *be an order-enriched ccc and a retraction map category of* **C**, *respectively. Let $a, b, c, a', b', c' \in |\mathbf{C}|$. Suppose that $a \leqslant a'$, $b \leqslant b'$, and $c \leqslant c'$ in* **R**,

(i) *For* $f \in C(a, b)$, $g \in C(b, c)$, $f' \in C(a', b')$, *and* $g' \in C(b', c')$, *if* $f \sqsubseteq f'$ *and* $g \sqsubseteq g'$, *then* $g \circ f \sqsubseteq g' \circ f'$.

(ii) *For* $f \in C(c, a)$, $g \in C(c, b)$, $f' \in C(c', a')$, *and* $g' \in C(c', b')$, *if* $f \sqsubseteq f'$ *and* $g \sqsubseteq g'$, *then* $\langle f, g \rangle \sqsubseteq \langle f', g' \rangle$.

(iii) *For* $f \in \mathbf{C}(c \times a, b)$ *and* $f' \in \mathbf{C}(c' \times a', b')$, *if* $f \sqsubseteq f'$, *then* $\Lambda(f) \sqsubseteq \Lambda(f')$.

Proof. (i) If $f \sqsubseteq f'$ and $g \sqsubseteq g'$, then

$$\begin{aligned} g \circ f &\leqslant j[c, c'] \circ g' \circ i[b, b'] \circ j[b, b'] \circ f' \circ i[a, a'] \\ &\leqslant j[c, c'] \circ g' \circ f' \circ i[a, a']. \end{aligned}$$

Hence $g \circ f \sqsubseteq g' \circ f'$.

(ii) If $f \sqsubseteq f'$ and $g \sqsubseteq g'$, then

$$\begin{aligned} \langle f, g \rangle &\leqslant \langle j[a, a'] \circ f' \circ i[c, c'], j[b, b'] \circ g' \circ i[c, c'] \rangle \\ &= j[a \times b, a' \times b'] \circ \langle f', g' \rangle \circ i[c, c'] \text{ (by Lemma 2.6(i))}. \end{aligned}$$

Hence $\langle f, g \rangle \sqsubseteq \langle f', g' \rangle$.

(iii) If $f \sqsubseteq f'$, then

$$\begin{aligned} \Lambda(f) &= \Lambda(j[b, b'] \circ f' \circ i[c \times a, c' \times a']) \\ &= j[b^a, b'^{a'}] \circ \Lambda(f') \circ i[c, c'] \text{ (by Lemma 2.6(ii))}. \end{aligned}$$

Hence $\Lambda(f) \sqsubseteq \Lambda(f')$. ∎

2.10. Definition. Let $(X, \leqslant)$ be a partially ordered set:

(i) For each subset Y of X we define

$$Y{\downarrow} = \{x \in X \mid (\exists y \in Y)(x \leqslant y)\}.$$

(ii) A nonempty subset $Y \subset X$ is *directed* iff for every pair of $x, y \in Y$ there exists $z \in Y$ such that $x \leqslant z$ and $y \leqslant z$.

(iii) A nonempty subset $Y \subset X$ is an ideal of $(X, \leqslant)$ iff $Y = Y{\downarrow}$ and Y is directed.

2.11. Definition (ε-category). Let $\mathbf{C}$ and $\mathbf{R}$ be an order-enriched ccc and a retraction map category of $\mathbf{C}$, respectively. The *ε-category* $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$ of $\mathbf{C}$ with respect to $\mathbf{R}$ is the category defined as follows:

(a) The objects of **E** are all ideals A of $(|\mathbf{C}|, \leqslant)$, where $\leqslant$ is the partial order relation in **R**. Namely,

(1) if $a \in A$ and $b \in A$, then there exists $c \in A$ such that $a \in c$ and $b \leqslant c$ in **R**; and

(2) if $a \in A$, $b \in |\mathbf{C}|$ and $b \leqslant a$ in **R**, then $b \in A$.

(b) The arrows in **E** are all ideals F of $(C, \sqsubseteq)$, where $\sqsubseteq$ is the partial order relation defined in Definition 2.7. Namely,

(1) if $f \in F$ and $g \in F$, then there is $h \in F$ such that $f \sqsubseteq h$ and $g \sqsubseteq h$; and

(2) if $f \in F$ and $g \sqsubseteq f$ then $g \in F$.

The domain and the codomain of F are defined by

$$\mathrm{dom}(F) = \{\mathrm{dom}(f) \mid f \in F\}$$

$$\mathrm{cod}(F) = \{\mathrm{cod}(f) \mid f \in F\}.$$

(Note that dom(F) and cod(F) are ideals of $(|C|, \leqslant)$ from the definition of $\sqsubseteq$.)

(c) For each pair of arrows $F: A \to B$ and $g: B \to C$ in **E**, the composite arrow $G \circ F: A \to C$ is defined by

$$G \circ F = \{g \circ f \mid f \in F,\ g \in G, \text{ and } \mathrm{cod}(f) = \mathrm{dom}(g)\}\downarrow,$$

where $\downarrow$ is the operation defined in Definition 2.7 with respect to $\sqsubseteq$.

(d) For each object A of **E**, the identity arrow ID_A on A is defined by $\mathrm{ID}_A = \{\mathrm{id}_a \mid a \in A\}\downarrow$. (In ε-categories we use the notation ID_A instead of id_A.)

3. Cartesian Closedness of ε-categories

Let **C** be an order-enriched ccc and let **R** be a retraction map category of **C**. We will prove that the ε-category $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$ is an order-enriched ccc if we define a suitable structure $(1, !_{(-)}, (-)\times(-), \pi_1^{(-),(-)}, \pi_2^{(-),(-)}, \langle(-),(-)\rangle, (-)^{(-)}, \mathrm{ev}^{(-),(-)}, \Lambda(-))$ for ccc, and a partial order relation $\leqslant$ in **E**.

In this section **C**, **R**, and **E** always mean an order-enriched ccc, a retraction map category of **C** and the ε-category **E**(**C**, **R**), respectively.

3.1. Lemma. *Let A, $B \in |\mathbf{E}|$ and let F be a subset of $\{f \mid f$ is an arrow of* **C**, dom(f) $\in A$, *and* cod(f) $\in B\}$. *If A, B, and F satisfy the following con-*

ditions (called the arrow conditions), *then* $F\!\downarrow$ *(w.r.t.* $\sqsubseteq$*) is an arrow from* A *to* B *in* **E**:

(1) *for every pair of* $a \in A$ *and* $b \in B$, *there is* $f \in F$ *such that* $a \leqslant \mathrm{dom}(f)$ *and* $b \leqslant \mathrm{cod}(f)$; *and*

(2) *for every pair of* $f, g \in F$, *there is* $h \in F$ *such that* $f \sqsubseteq h$ *and* $g \sqsubseteq h$.

Proof. From the arrow condition (1) and the definition of $F\!\downarrow$, $\mathrm{dom}(F) = A$ and $\mathrm{cod}(F) = B$. The condition (2) of (*b*) in Definition 2.11 follows from the definition of $F\!\downarrow$. If $f' \in F\!\downarrow$ and $g' \in F\!\downarrow$, then from the definition of $F\!\downarrow$ there are $f \in F$ and $g \in F$ such that $f' \sqsubseteq f$ and $g' \sqsubseteq g$. By the arrow condition (2) there is $h \in F$ such that $f \sqsubseteq h$ and $g \sqsubseteq h$. Thus the condition (1) of (b) in Definition 2.11 is satisfied. ∎

3.2. Lemma. *Let* $F: A \to B$ *be an arrow of* **E**. *For* $a, a' \in A$, $b, b' \in B$, *and* $f: a \to b \in F$, *if* $a \leqslant a'$ *and* $b \leqslant b'$, *then there exists* $f': a' \to b' \in F$ *such that* $f \sqsubseteq f'$.

Proof. Because F is an arrow of **E**, there exist $k: a' \to b' \in F$ and $h: a'' \to b'' \in F$ such that $f \sqsubseteq h$ and $k \sqsubseteq h$. So

$$\begin{aligned} f &\leqslant j[b, b''] \circ h \circ i[a, a''] \\ &= j[b, b'] \circ j[b', b''] \circ h \circ i[a', a''] \circ i[a, a']. \end{aligned}$$

Here note that $a \leqslant a' \leqslant a''$ and $b \leqslant b' \leqslant b''$. Thus $f \sqsubseteq j[b', b''] \circ h \circ i[a', a'']$, and $j[b', b''] \circ h \circ i[a', a''] \in F$ since $j[b', b''] \circ h \circ i[a', a''] \sqsubseteq h \in F$. ∎

3.3. Lemma. *Every* ε*-category* **E** *is really a category. That is, the following are satisfied*:

(i) *If* $F: A \to B$ *and* $G: B \to C$ *are arrows of* **E**, *then* $G \circ F: A \to C$ *is also an arrow of* **E**.

(ii) *For every object* A *of* E, $\mathrm{ID}_A: A \to A$ *is an arrow of* **E**.

(iii) *For every triple of arrows* $F: A \to B$, $G: B \to C$, *and* $H: C \to D$ *in* **E**, $H \circ (G \circ F) = (H \circ G) \circ F$.

(iv) *For every arrow* $F: A \to B$ *of* **E**, $F \circ \mathrm{ID}_A = F$, *and* $\mathrm{ID}_B \circ F = F$.

Proof. (i) From Lemma 3.1 it is enough to show that $\{g \circ f \mid f \in F, g \in G, \text{ and } \mathrm{cod}(f) = \mathrm{dom}(g)\}$ satisfies the arrow conditions. It is clear that the arrow condition (1) is satisfied. We will show that the arrow condition (2) is satisfied. Suppose that $f_1, f_2 \in F$, $g_1, g_2 \in G$, $\mathrm{cod}(f_1) = \mathrm{dom}(g_1)$, and $\mathrm{cod}(f_2) = \mathrm{dom}(g_2)$. Because F and G are directed, there are $f \in F$ and $g \in G$ such that $f_1 \sqsubseteq f$, $f_2 \sqsubseteq f$, $g_1 \sqsubseteq g$, and $g_2 \sqsubseteq g$. There is $b \in B$ such that $\mathrm{cod}(f) \leqslant b$ and $\mathrm{dom}(g) \leqslant b$. From Lemma 3.2 there are $f' \in F$ and $g' \in G$

such that $\mathrm{cod}(f') = \mathrm{dom}(g') = b$, $f \sqsubseteq f'$, and $g \sqsubseteq g'$. From Lemma 2.9(i), $g_1 \circ f_1 \sqsubseteq g' \circ f'$ and $g_2 \circ f_2 \sqsubseteq g' \circ f'$. Hence the arrow condition (2) is satisfied.

(ii) From Lemma 3.1 it is enough to show that $\{\mathrm{id}_a \mid a \in A\}$ satisfies the arrow conditions. It is clear that the arrow condition (1) is satisfied. If $a_1, a_2 \in A$, then there is $a \in A$ such that $a_1 \leqslant a$ and $a_2 \leqslant a$. Because $\mathrm{id}_{a_1} \leqslant j[a_1, a] \circ \mathrm{id}_a \circ i[a_1, a]$, $\mathrm{id}_{a_1} \sqsubseteq \mathrm{id}_a$. Similarly $\mathrm{id}_{a_2} \sqsubseteq \mathrm{id}_a$. Hence the arrow condition (2) is satisfied.

(iii) Let $k \in (H \circ G) \circ F$. Then there exist $k' \in H \circ G$ and $f \in F$ such that $\mathrm{dom}(k') = \mathrm{cod}(f)$ and $k \sqsubseteq k' \circ f$. There are $h \in H$ and $g \in G$ such that $\mathrm{dom}(h) = \mathrm{cod}(g)$ and $k' \sqsubseteq h \circ g$. From Lemma 3.2 there is $f' \in F$ such that $\mathrm{cod}(f') = \mathrm{dom}(h \circ g)$ and $f \sqsubseteq f'$. From Lemma 2.9(i), $k \sqsubseteq k' \circ f \sqsubseteq h \circ g \circ f' \in H \circ (G \circ F)$. Hence $k \in H \circ (G \circ F)$. Conversely, by a similar argument it can be proved that $k \in H \circ (G \circ F)$ implies $k \in (H \circ G) \circ F$. Hence $(H \circ G) \circ F = H \circ (G \circ F)$.

(iv) Let $h \in F \circ \mathrm{ID}_A$. Then there exist $f \in F$ and $k \in \mathrm{ID}_A$ such that $\mathrm{dom}(f) = \mathrm{cod}(k)$ and $h \sqsubseteq f \circ k$. There is $a \in A$ such that $k \sqsubseteq \mathrm{id}_a$. From Lemma 3.2 there is $f' \in F$ such that $\mathrm{dom}(f') = a$ and $f \sqsubseteq f'$, because $\mathrm{dom}(f) \leqslant a$. Since $h \sqsubseteq f \circ k \sqsubseteq f' \circ \mathrm{id}_a = f' \in F$ by Lemma 2.9(i), $h \in F$. Conversely if $f \in F$, then $f = f \circ \mathrm{id}_a \in F \circ \mathrm{ID}_A$, where $a = \mathrm{dom}(f)$. Hence $F \circ \mathrm{ID}_A = F$. Similarly $\mathrm{ID}_B \circ F = F$. ▮

3.4. Definition (Terminal in ε-categories). We define

(1) $1' = \{1\}\downarrow$, where 1 is the terminal object of **C**, and

(2) for each object A of E, $!_A = \{!_a \mid a \in A\}\downarrow$.

3.5. Lemma. *In* **E**, *$1'$ is a terminal object and $!_A$ is the unique arrow from A to $1'$ for every object A.*

Proof. Clear. ▮

When we mention ε-categories, we use 1 instead of $1'$.

3.6. Definition (Product in ε-categories). We define

(1) $A \times B = \{a \times b \mid a \in A \text{ and } b \in B\}\downarrow$,

(2) $\Pi_1^{A,B} = \{\pi_1^{a,b} \mid a \in A \text{ and } b \in B\}\downarrow$,

(3) $\Pi_2^{A,B} = \{\pi_2^{a,b} \mid a \in A \text{ and } b \in B\}\downarrow$, and

(4) $\langle F, G\rangle = \{\langle f, g\rangle \mid f \in F,\ g \in G, \text{ and } \mathrm{dom}(f) = \mathrm{dom}(g)\}\downarrow$, where A, B, and C are objects of **E**, and $F\colon C \to A$ and $G\colon C \to B$ are arrows of **E**.

3.7. LEMMA. (i) *If A and B are objects of* $\mathbf{E}$, *then* $A \times B$ *is an object of* $\mathbf{E}$.

(ii) *If A and B are objects of* $\mathbf{E}$, *then* $\Pi_1^{A,B}$ *is an arrow from* $A \times B$ *to* A *in* $\mathbf{E}$ *and* $\Pi_2^{A,B}$ *is an arrow from* $A \times B$ *to* B *in* $\mathbf{E}$.

(iii) *If* $F: C \to A$ *and* $G: C \to B$ *are arrows of* $\mathbf{E}$, *then* $\langle F, G \rangle$ *is an arrow from* C *to* $A \times B$ *in* $\mathbf{E}$.

(iv) *For every pair of arrows* $F: C \to A$ *and* $G: C \to B$ *in* $\mathbf{E}$, $\Pi_1^{A,B} \circ \langle F, G \rangle = F$ *and* $\Pi_2^{A,B} \circ \langle F, G \rangle = G$.

(v) *For every arrow* $H: C \to A \times B$ *in* $\mathbf{E}$, $\langle \Pi_1^{A,B} \circ H, \Pi_2^{A,B} \circ H \rangle = H$.

Proof. (i) Let $d_1, d_2 \in A \times B$. Then there are $a_1 \in A$, $a_2 \in A$, $b_1 \in B$, and $b_2 \in B$ such that $d_1 \leqslant a_1 \times b_1$ and $d_2 \leqslant a_2 \times b_2$. Because A and B satisfy the condition (1) of (a) in Definition 2.11, there are $a \in A$ and $b \in B$ such that $a_1 \leqslant a$, $a_2 \leqslant a$, $b_1 \leqslant b$, $b_2 \leqslant b$, and $a \times b \in A \times B$. Since $d_1 \leqslant a_1 \times b_1 \leqslant a \times b$ and $d_2 \leqslant a_2 \times b_2 \leqslant a \times b$, $A \times B$ satisfies the condition (1) of (a) in Definition 2.11. And $A \times B$ clearly satisfies the condition (2) of (a) in Definition 2.11.

(ii) It is enough to show that $\{\pi_1^{a,b} \mid a \in A \text{ and } b \in B\}$ satisfies the arrow conditions. If $d \in A \times B$ and $a' \in A$, then there are $a \in A$ and $b \in B$ such that $d \leqslant a \times b$ and $a' \leqslant a$. So the arrow condition (1) is satisfied.

Next let $a_1 \in A$, $a_2 \in A$, $b_1 \in B$, and $b_2 \in B$. Then there are $a \in A$ and $b \in B$ such that $a_1 \leqslant a$, $a_2 \leqslant a$, $b_1 \leqslant b$, and $b_2 \leqslant b$. By Lemma 2.8(i), $\pi_1^{a_1,b_1} \sqsubseteq \pi_1^{a,b}$ and $\pi_1^{a_2,b_2} \sqsubseteq \pi_1^{a,b}$. Thus the arrow condition (2) is satisfied.

In the case of $\Pi_2^{A,B}$, it is similar.

(iii) It is enough to show that $\{\langle f, g \rangle \mid f \in F,\ g \in G, \text{ and } \mathrm{dom}(f) = \mathrm{dom}(g)\}$ satisfies the arrow conditions. It is clear that the arrow condition (1) is satisfied. We will show the arrow condition (2). Suppose $f_1 \in F$, $f_2 \in F$, $g_1 \in G$, $g_2 \in G$, $\mathrm{dom}(f_1) = \mathrm{dom}(g_1)$ and $\mathrm{dom}(f_2) = \mathrm{dom}(g_2)$. Then there are $f \in F$ and $g \in G$ such that $f_1 \sqsubseteq f$, $f_2 \sqsubseteq f$, $g_1 \sqsubseteq g$, and $g_2 \sqsubseteq g$. Because there is $c \in C$ upper than both $\mathrm{dom}(f)$ and $\mathrm{dom}(g)$, from Lemma 3.2 there exist $f' \in F$ and $g' \in G$ such that $\mathrm{dom}(f') = \mathrm{dom}(g') = c$, $f \sqsubseteq f'$, and $g \sqsubseteq g'$. By Lemma 2.9(ii), $\langle f_1, g_1 \rangle \sqsubseteq \langle f', g' \rangle$ and $\langle f_2, g_2 \rangle \sqsubseteq \langle f', g' \rangle$. Hence the arrow condition (2) is satisfied.

(iv) First we will show that $\Pi_1^{A,B} \circ \langle F, G \rangle \subset F$. Suppose $k \in \Pi_1^{A,B} \circ \langle F, G \rangle$. Then there are $h_1 \in \Pi_1^{A,B}$ and $h_2 \in \langle F, G \rangle$ such that $\mathrm{dom}(h_1) = \mathrm{cod}(h_2)$ and $k \sqsubseteq h_1 \circ h_2$. There are $a \in A$, $b \in B$, $f \in F$, and $g \in G$ such that $h_1 \sqsubseteq \pi_1^{a,b}$, $\mathrm{dom}(f) = \mathrm{dom}(g)$, and $h_2 \sqsubseteq \langle f, g \rangle$. There are $a' \in A$ and $b' \in B$ such that $a \leqslant a'$, $\mathrm{cod}(f) \leqslant a'$, $b \leqslant b'$, and $\mathrm{cod}(g) \leqslant b'$. By Lemma 2.8(i), $h_1 \sqsubseteq \pi_1^{a,b} \sqsubseteq \pi_1^{a',b'}$. By Lemma 3.2 there exist $f' \in F$ and $g' \in G$ such that $\mathrm{dom}(f')=\mathrm{dom}(g')$, $\mathrm{cod}(f')=a'$, $\mathrm{cod}(g')=b'$, $f \sqsubseteq f'$, and $g \sqsubseteq g'$. By Lemma 2.9(ii), $h_2 \sqsubseteq \langle f, g\rangle \sqsubseteq \langle f', g'\rangle$, and by Lemma 2.9(i),

$$k \sqsubseteq h_1 \circ h_2 \sqsubseteq \pi_1^{a',b'} \circ \langle f', g'\rangle = f' \in F.$$

So $k \in F$, Namely $\Pi_1^{A,B} \circ \langle F, G\rangle \subset F$.

Conversely, let $f\colon c \to a \in F$ be given. Because $\mathrm{dom}(G) = C \ni c$, there exists at least one arrow $g\colon c \to b \in G$. So $f = \pi_1^{a,b} \circ \langle f, g\rangle \in \Pi_1^{A,B} \circ \langle F, G\rangle$. Namely $F \subset \Pi_1^{A,B} \circ \langle F, G\rangle$.

Hence $\Pi_1^{A,B} \circ \langle F, G\rangle = F$ and similarly $\Pi_2^{A,B} \circ \langle F, G\rangle = G$.

(v) First we will show that $\langle \Pi_1^{A,B} \circ H, \Pi_2^{A,B} \circ H\rangle \subset H$. Let $k \in \langle \Pi_1^{A,B} \circ H, \Pi_2^{A,B} \circ H\rangle$ be given. Step by step we decompose the expression $\langle \Pi_1^{A,B} \circ H, \Pi_2^{A,B} \circ H\rangle$ by the definition.

There are $k_1 \in \Pi_1^{A,B} \circ H$ and $k_2 \in \Pi_2^{A,B} \circ H$ such that

$$\mathrm{dom}(k_1) = \mathrm{dom}(k_2) \quad \text{and} \quad k \sqsubseteq \langle k_1, k_2\rangle.$$

There are $l_1 \in \Pi_1^{A,B}$, $l_2 \in \Pi_2^{A,B}$, $h_1 \in H$, and $h_2 \in H$ such that

$$\mathrm{dom}(l_1) = \mathrm{cod}(h_1), \qquad \mathrm{dom}(l_2) = \mathrm{cod}(h_2),$$

$$k_1 \sqsubseteq l_1 \circ h_1 \quad \text{and} \quad k_2 \sqsubseteq l_2 \circ h_2.$$

There is $h \in H$ such that

$$h_1 \sqsubseteq h \quad \text{and} \quad h_2 \sqsubseteq h.$$

There are $a \in A$ and $b \in B$ such that

$$\mathrm{cod}(h) \leqslant a \times b.$$

There are $a_1 \in A$, $a_2 \in A$, $b_1 \in B$, and $b_2 \in B$ such that

$$l_1 \sqsubseteq \pi_1^{a_1,b_1} \quad \text{and} \quad l_2 \sqsubseteq \pi_2^{a_2,b_2}.$$

There are $a' \in A$ and $b' \in B$ such that

$$a, a_1, a_2 \leqslant a' \quad \text{and} \quad b, b_1, b_2 \leqslant b'.$$

By Lemma 3.2 there exists $h' \in H$ such that $\mathrm{cod}(h') = a' \times b'$ and $h \sqsubseteq h'$. By Lemma 2.8(i), $\pi_1^{a_1,b_1} \sqsubseteq \pi_1^{a',b'}$ and $\pi_2^{a_2,b_2} \sqsubseteq \pi_2^{a',b'}$. By Lemma 2.9(i) and (ii),

$$\begin{aligned} k &\sqsubseteq \langle k_1, k_2\rangle \sqsubseteq \langle l_1 \circ h_1, l_2 \circ h_2\rangle \\ &\sqsubseteq \langle \pi_1^{a',b'} \circ h', \pi_2^{a',b'} \circ h'\rangle = h' \in H. \end{aligned}$$

Hence $k \in H$. Because k is arbitrary, $\langle \Pi_1^{A,B} \circ H, \Pi_2^{A,B} \circ H\rangle \subset H$.

Conversely we will show that $H \subset \langle \Pi_1^{A,B} \circ H, \Pi_2^{A,B} \circ H \rangle$. Let $h \in H$. Then there are $a \in A$ and $b \in B$ such that $\mathrm{cod}(h) \leqslant a \times b$. By Lemma 3.2 there is $h' \in H$ such that $\mathrm{cod}(h') = a \times b$ and $h \sqsubseteq h'$. Since $h \sqsubseteq h' = \langle \pi_1^{a',b'} \circ h', \pi_2^{a',b'} \circ h' \rangle \in \langle \Pi_1^{A,B} \circ H, \Pi_2^{A,B} \circ H \rangle$, $h \in \langle \Pi_1^{A,B} \circ H, \Pi_2^{A,B} \circ H \rangle$. Hence we conclude that $\langle \Pi_1^{A,B} \circ H, \Pi_2^{A,B} \circ H \rangle = H$. ∎

3.8. DEFINITION (Exponentiation in ε-categories). We define

(1) $B^A = \{b^a \mid a \in A \text{ and } b \in B\}\downarrow$,

(2) $\mathrm{EV}^{A,B} = \{\mathrm{ev}^{a,b} \mid a \in A \text{ and } b \in B\}\downarrow$,

(3) $\Lambda_{C,A}(F) = \{\Lambda_{c,a}(f) \mid f \in F,\ c \in C,\ a \in A, \text{ and } \mathrm{dom}(f) = c \times a\}\downarrow$,

where A, B, and C are objects of **E**, and $F: C \times A \to B$ is an arrow of **E**.

3.9. LEMMA. (i) *If A and B are objects of* **E**, *then B^A is an object of E.*

(ii) *If A and B are object of* **E**, *then $\mathrm{EV}^{A,B}$ is an arrow from $B^A \times A$ to B in* **E**.

(iii) *If $F: C \times A \to B$ is an arrow of* **E**, *then $\Lambda(F)$ is an arrow from C to B^A in* **E**.

(iv) *For every arrow $F: C \times A \to B$ in* **E**, $\mathrm{EV}^{A,B} \circ (\Lambda(F) \times \mathrm{ID}_A) = F$.

(v) *For every arrow $H: C \to B^A$ in* **E**, $\Lambda(\mathrm{EV}^{A,B} \circ (H \times \mathrm{ID}_A)) = H$.

Proof. It is similar to the proof of Lemma 3.7. ∎

3.10. LEMMA. (i) *For arrows $F, F': A \to B$ and $G, G': B \to C$ in* **E**, *if $F \subset F'$ and $G \subset G'$, then $G \circ F \subset G' \circ F'$.*

(ii) *For arrows $F, F': C \to A$ and $G, G': C \to B$ in* **E**, *if $F \subset F'$ and $G \subset G'$ then $\langle F, G \rangle \subset \langle F', G' \rangle$.*

(iii) *For arrows $F, F': C \times A \to B$, if $F \subset F'$ then $\Lambda(F) \subset \Lambda(F')$.*

Proof. Clear from the definitions. ∎

3.11. THEOREM. *Let* **C** *be an order-enriched ccc and let* **R** *be a retraction map category of* **C**. *Then the ε-category* **E(C, R)** *equipped with the structure* $(1, !_{(-)}, (-) \times (-), \Pi_1^{(-),(-)}, \Pi_2^{(-),(-)}, \langle (-), (-) \rangle, (-)^{(-)}, \mathrm{EV}^{(-),(-)}, \Lambda(-))$ *and the set inclusion* $\subset$ *as the partial order relation among the arrows is an order-enriched ccc.*

Proof. It follows from Lemmas 3.3., 3.5, 3.7, 3.9, and 3.10. ∎

4. Complete Order-enriched ccc's and ε^*-categories

Let **C** and **R** be an order-enriched ccc and a retraction map category of **C**, respectively. Arrows of **C** are ordered and objects of **C** are also ordered by **R**. The construction of the ε-category **E(C, R)** carries out completion of both arrows and objects. We consider the case that the partially ordered set of arrows is already complete, that is, for every pair of $a, b \in |\mathbf{C}|$ and for every directed subset X of $\mathbf{C}(a, b)$, there exists the least upper bound of X in $\mathbf{C}(a, b)$. Such a ccc is called a complete order-enriched ccc. We intend to carry out completion w.r.t. objects starting from a complete order-enriched ccc, and we define ε^*-categories based on ε-categories.

4.1. Definition (Complete order-enriched ccc). Let **C** be an order-enriched ccc. Then **C** is *complete* iff the following conditions are satisfied:

(1) for every pair of objects $a, b \in |\mathbf{C}|$ and for every directed subset $F \subset \mathbf{C}(a, b)$, there exists the least upper bound $\bigsqcup F$ of F contained in $\mathbf{C}(a, b)$; and

(2) if F is a directed subset of $\mathbf{C}(a, b)$ and G is a directed subset of $\mathbf{C}(b, c)$, then $\bigsqcup G \circ \bigsqcup F = \bigsqcup \{ g \circ f \mid f \in F \text{ and } g \in G\}$, where $a, b, c \in |\mathbf{C}|$.

Note that an order-enriched ccc **C** is complete if $\mathbf{C}(a, b)$ is finite for any pair of $a, b \in |\mathbf{C}|$.

From Theorem 3.11 every ε-category is an order-enriched ccc. Moreover every ε-category is complete. In ε-categories, the partial order relation among arrows is the set inclusion $\subset$ and the least upper bound operation is the set union $\bigcup$. Let **E** be an ε-category and let P be a directed set of arrows from A to B in **E**. Then the least upper bound of P is

$$\bigcup P = \{f \mid (\exists F \in P)(f \in F)\},$$

and $\bigcup P$ is clearly an arrow from A to B in **E**. Namely **E** satisfies condition (1) in Definition 4.1. Moreover, if Q is a directed set of arrows from B to C in **E**, then

$$\bigcup Q \circ \bigcup P = \bigcup \{G \circ F \mid F \in P \text{ and } G \in Q\}.$$

Namely **E** satisfies condition (2) in Definition 4.1.

4.2. Lemma. *Let* **C** *be a complete order-enriched ccc. Let a, b, and c be objects of* **C**:

(i) *For every pair of directed subsets* $F \subset \mathbf{C}(c, a)$ *and* $G \subset \mathbf{C}(c, b)$,

$$\left\langle \bigsqcup F, \bigsqcup G \right\rangle = \bigsqcup \{\langle f, g\rangle \mid f \in F \text{ and } g \in G\}.$$

(ii) *For every directed subset* $F \subset \mathbf{C}(c \times a, b)$,

$$\Lambda\left(\bigsqcup F\right) = \bigsqcup \{\Lambda(f) \mid f \in F\}.$$

Proof. (i)

$$\begin{aligned}
&\bigsqcup \{\langle f, g\rangle \mid f \in F \text{ and } g \in G\}\\
&= \left\langle \pi_1^{a,b} \circ \bigsqcup \{\langle f, g\rangle \mid f \in F \text{ and } g \in G\},\right.\\
&\qquad \left.\pi_2^{a,b} \circ \bigsqcup \{\langle f, g\rangle \mid f \in F \text{ and } g \in G\}\right\rangle\\
&= \left\langle \bigsqcup F, \bigsqcup G\right\rangle.
\end{aligned}$$

(ii)

$$\begin{aligned}
&\bigsqcup \{\Lambda(f) \mid f \in F\}\\
&= \Lambda\left(\mathrm{ev}^{a,b} \circ \left\langle \bigsqcup \{\Lambda(f) \mid f \in F\} \circ \pi_1^{c,a}, \pi_2^{c,a}\right\rangle\right)\\
&= \Lambda\left(\bigsqcup \{\mathrm{ev}^{a,b} \circ \langle \Lambda(f) \circ \pi_1^{c,a}, \pi_2^{c,a}\rangle \mid f \in F\}\right)\\
&\qquad \text{(using (i) of this lemma)}\\
&= \Lambda\left(\bigsqcup F\right). \quad \blacksquare
\end{aligned}$$

4.3. Lemma. *Let* **C** *and* **R** *be an order-enriched ccc and a retraction map category of* **C**, *respectively. For every arrow* $F: A \to B$ *and for every pair of* $a \in A$ *and* $b \in B$, $\{f \mid f \in F, \operatorname{dom}(f) = a, \text{ and } \operatorname{cod}(f) = b\}$ *is directed.*

Proof. If $f: a \to b \in F$ and $g: a \to b \in F$, then there is $h: a' \to b' \in F$ such that $f \sqsubseteq h$ and $g \sqsubseteq h$. If we take $k = j[b, b'] \circ h \circ i[a, a']: a \to b$ then $f \leqslant k$, $g \leqslant k$, and $k \in F$. $\blacksquare$

4.4. Definition. Let **C** and **R** be a complete order-enriched ccc and a retraction map category of **C**, respectively. Let $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$:

(i) For each arrow $F: A \to B$ in **E** and for each pair of $a \in A$ and $b \in B$,

$$F(a, b) = \{f \in F \mid \operatorname{dom}(f) = a \text{ and } \operatorname{cod}(f) = b\}.$$

(ii) For each pair of arrows $F, G: A \to B$ in $\mathbf{E}$, $F \lesssim G$ iff $\bigsqcup F(a, b) \leqslant \bigsqcup G(a, b)$ for all pairs of $a \in A$ and $b \in B$, and $F \simeq G$ iff $F \lesssim G$ and $G \lesssim F$.

(iii) For each arrow $F: A \to B$ in $\mathbf{E}$,

$$[F] = \{G \in \mathbf{E}(A, B) \mid F \simeq G\}.$$

Note that $F(a, b)$ and $G(a, b)$ in (ii) above are both directed by Lemma 4.3. And note that $\lesssim$ is a preorder.

4.5. LEMMA. *Let* $\mathbf{C}$ *and* $\mathbf{R}$ *be an order-enriched ccc and a retraction map category of* $\mathbf{C}$, *respectively. Let* $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$:

(i) *Let* $F: A \to B$ *be an arrow of* $\mathbf{E}$. *For* $a, a' \in A$, $b, b' \in B$, *if* $a \leqslant a'$ *and* $b \leqslant b'$ *then*

$$F(a, b) = \{j[b, b'] \circ f' \circ i[a, a'] \mid f' \in F(a', b')\}\downarrow.$$

(ii) *Let* $F: A \to B$ *and* $G: B \to C$ *be arrows of* $\mathbf{E}$. *For any* $a \in A$ *and* $c \in C$,

$$(G \circ F)(a, c) = \{g \circ f \mid (\exists b \in B)(f \in F(a, b) \text{ and } g \in G(b, c))\}\downarrow.$$

(iii) *Let* $F: C \to A$ *and* $G: C \to B$ *be arrows of* $\mathbf{E}$. *For any* $a \in A$, $b \in B$, *and* $c \in C$,

$$\langle F, G\rangle(c, a \times b) = \{\langle f, g\rangle \mid f \in F(c, a) \text{ and } g \in G(c, b)\}.$$

(iv) *Let* $F: C \times A \to B$ *be an arrow of* $\mathbf{E}$. *For any* $a \in A$, $b \in B$, *and* $c \in C$,

$$\Lambda(F)(c, b^a) = \{\Lambda(f) \mid f \in F(c \times a, b)\}.$$

Proof. (i) Let $f \in F(a, b)$ be given. Then by Lemma 3.2 there exists $f' \in F(a', b')$ such that $f \sqsubseteq f'$. So $f \leqslant j[b, b'] \circ f' \circ i[a, a']$. Conversely let $f' \in F(a', b')$ be given. Then $j[b, b'] \circ f' \circ i[a, a'] \in F$ because $j[b, b'] \circ f' \circ i[a, a'] \sqsubseteq f'$. So $f' \in F(a, b)$.

(ii) It is clear that

$$(G \circ F)(a, c) \supset \{g \circ f \mid (\exists b \in B)(f \in F(a, b) \text{ and } g \in G(b, c))\}\downarrow.$$

Conversely let $h \in (G \circ F)(a, c)$ be given. Then there exist $a' \in A$, $c' \in C$, $b \in B$, $f' \in F(a', b)$, and $g' \in G(b, c')$ such that $a \leqslant a'$, $c \leqslant c'$, and $h \sqsubseteq g' \circ f'$. So $h \leqslant j[c, c'] \circ g' \circ f' \circ i[a, a']$. Because $f' \circ i[a, a'] \in F(a, b)$ and $j[c, c'] \circ g' \in G(b, c)$,

$$h \in \{g \circ f \mid (\exists b \in B)(f \in F(a, b) \text{ and } g \in G(b, c))\}\downarrow.$$

(iii) It is clear that

$$\langle F, G\rangle(c, a\times b) \supset \{\langle f, g\rangle \mid f\in F(c, a) \text{ and } g\in G(c, b)\}.$$

Conversely let $h\in\langle F, G\rangle(c, a\times b)$ be given. Then there exist $a'\in A$, $b'\in B$, $c'\in C$, $f'\in F(c', a')$, and $g'\in G(c', b')$ such that $a\times b\leqslant a'\times b'$, $c\leqslant c'$, and $h\sqsubseteq\langle f', g'\rangle$. There are $a''\in A$ and $b''\in B$ such that $a\leqslant a''$, $a'\leqslant a''$, $b\leqslant b''$, and $b'\leqslant b''$. So by Lemma 3.2 there exist $f''\in F(c', a'')$ and $g''\in G(c', b'')$ such that $f'\sqsubseteq f''$ and $g'\sqsubseteq g''$. By Lemma 2.9(ii), $h\sqsubseteq\langle f', g'\rangle\sqsubseteq\langle f'', g''\rangle$. Therefore

$$\begin{aligned} h &\leqslant j[a\times b, a''\times b'']\circ\langle f'', g''\rangle\circ i[c, c'] \\ &= \langle j[a, a'']\circ f''\circ i[c, c'], j[b, b'']\circ g''\circ i[c, c']\rangle \end{aligned}$$

by Lemma 2.6(i). Because

$$\pi_1^{a,b}\circ h\leqslant j[a, a'']\circ f''\circ i[c, c']$$

and

$$\pi_2^{a,b}\circ h\leqslant j[b, b'']\circ g''\circ i[c, c'],$$

$\pi_1^{a,b}\circ h\in F(c, a)$ and $\pi_2^{a,b}\circ h\in G(c, b)$. Hence

$$h=\langle\pi_1^{a,b}\circ h, \pi_2^{a,b}\circ h\rangle\in\{\langle f, g\rangle \mid f\in F(c, a) \text{ and } g\in G(c, b)\}.$$

(iv) It is similar to (iii). ▮

Note that $\downarrow$ is needed in (i) and (ii), but $\downarrow$ is not needed in (iii) or (iv). Indeed the following equations are satisfied:

$$\begin{aligned} &\{\langle f, g\rangle \mid f\in F(c, a) \text{ and } g\in G(c, b)\} \\ &\qquad =\{\langle f, g\rangle \mid f\in F(c, a) \text{ and } g\in G(c, b)\}\downarrow \end{aligned}$$

and

$$\begin{aligned} &\{\Lambda(f) \mid f\in F(c\times a, b)\} \\ &\qquad =\{\Lambda(f) \mid f\in F(c\times a, b)\}\downarrow. \end{aligned}$$

It is due to the fact that all arrows in $\mathbf{C}(c, a\times b)$ are completely determined by arrows in $\mathbf{C}(c, a)$ and $\mathbf{C}(c, b)$. So properties on F and G are preserved in $\{\langle f, g\rangle \mid f\in F(c, a) \text{ and } g\in G(c, b)\}$. Similarly all arrows in $\mathbf{C}(c, b^a)$ are determined by arrows in $\mathbf{C}(c\times a, b)$. In composition, however, it is not the case. There may be objects a, b, c, and an arrow $k\colon a\to c$ such that $k\neq g\circ f$ for any pair of $f\colon a\to b$ and $g\colon b\to c$.

4.6. COROLLARY. *Let* **C** *and* **R** *be a complete order-enriched ccc and a retraction map category of* **C**, *respectively. Let* $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$:

(i) *Let* $F: A \to B$ *be an arrow of* **E**. *For* $a, a' \in A$ *and* $b, b' \in B$, *if* $a \leqslant a'$ *and* $b \leqslant b'$, *then*

$$\bigsqcup F(a, b) = \bigsqcup \{ j[b, b'] \circ f' \circ i[a, a'] \mid f' \in F(a', b') \}.$$

(ii) *Let* $F: A \to B$ *and* $G: B \to C$ *be arrows of* **E**. *For any* $a \in A$ *and* $c \in C$,

$$\bigsqcup (G \circ F)(a, c) = \bigsqcup \left\{ \bigsqcup G(b, c) \circ \bigsqcup F(a, b) \mid b \in B \right\}.$$

(iii) *Let* $F: C \to A$ *and* $G: C \to B$ *be arrows of* **E**. *For any* $a \in A$, $b \in B$, *and* $c \in C$,

$$\bigsqcup \langle F, G \rangle (c, a \times b) = \left\langle \bigsqcup F(c, a), \bigsqcup G(c, b) \right\rangle.$$

(iv) *Let* $F: C \times A \to B$ *be an arrow of* **E**. *For any* $a \in A$, $b \in B$, *and* $c \in C$,

$$\bigsqcup \Lambda(F)(c, b^a) = \Lambda\left(\bigsqcup F(c \times a, b) \right).$$

Proof. (i) Clear from Lemma 4.5(i).

(ii)

$$\begin{aligned}
\bigsqcup (G \circ F)(a, c) &= \bigsqcup \{ g \circ f \mid (\exists b \in B)(f \in F(a, b) \text{ and } g \in G(b, c)) \} \\
&\qquad \text{(by Lemma 4.5(ii))} \\
&= \bigsqcup \left\{ \bigsqcup \{ g \circ f \mid f \in F(a, b) \text{ and } g \in G(b, c) \} \mid b \in B \right\} \\
&= \bigsqcup \left\{ \bigsqcup G(b, c) \circ \bigsqcup F(a, b) \mid b \in B \right\}.
\end{aligned}$$

(iii)

$$\begin{aligned}
\bigsqcup \langle F, G \rangle (c, a \times b) &= \bigsqcup \{ \langle f, g \rangle \mid f \in F(c, a) \text{ and } g \in G(c, b) \} \\
&\qquad \text{(by Lemma 4.5(iii))} \\
&= \left\langle \bigsqcup F(c, a), \bigsqcup G(c, b) \right\rangle \quad \text{(by Lemma 4.2(i))}.
\end{aligned}$$

(iv) Similar to (iii). ∎

4.7. LEMMA. *Let* **C** *and* **R** *be a complete order-enriched ccc and a retraction map category of* **C**, *respectively. Let* $\mathbf{E}=\mathbf{E}(\mathbf{C},\mathbf{R})$:

(i) *For arrows* $F, F' : A \to B$ *and* $G, G' : B \to C$ *in* **E**, *if* $F \lesssim F'$ *and* $G \lesssim G'$ *then* $G \circ F \lesssim G' \circ F'$.

(ii) *For arrows* $F, F' : C \to A$ *and* $G, G' : C \to B$ *in* **E**, *if* $F \lesssim F'$ *and* $G \lesssim G'$ *then* $\langle F, G\rangle \lesssim \langle F', G'\rangle$.

(iii) *For arrows* $F, F' : C \times A \to B$ *in* **E**, *if* $F \lesssim F'$ *then* $\Lambda(F) \lesssim \Lambda(F')$.

Proof. (i) For every pair of $a \in A$ and $c \in C$, if $F \lesssim F'$ then

$$\bigsqcup (G \circ F)(a, c) = \bigsqcup \left\{ \bigsqcup G(a, b) \circ \bigsqcup F(b, c) \mid b \in B \right\}$$

(by Corollary 4.6(ii)

$$\leqslant \bigsqcup \left\{ \bigsqcup G'(a, b) \circ \bigsqcup F'(b, c) \mid b \in B \right\}$$

$$= \bigsqcup (G' \circ F')(a, c).$$

(ii) Let $c \in C$ and $d \in A \times B$. Then there are $a \in A$ and $b \in B$ such that $d \leqslant a \times b$. If $F \lesssim F'$ and $G \lesssim G'$ then

$$\bigsqcup \langle F, G\rangle(c, d) = \bigsqcup \{ j[d, a \times b] \circ h \mid h \in \langle F, G\rangle(c, a \times b)\}$$

(by Corollary 4.6(i))

$$= j[d, a \times b] \circ \bigsqcup \langle F, G\rangle(c, a \times b)$$

$$= j[d, a \times b] \circ \left\langle \bigsqcup F(c, a), \bigsqcup G(c, b) \right\rangle$$

(by Corollary 4.6(iii))

$$\leqslant j[d, a \times b] \circ \left\langle \bigsqcup F'(c, a), \bigsqcup G'(c, b) \right\rangle$$

$$= \bigsqcup \langle F', G'\rangle(c, d).$$

(iii) Similar to (ii). ∎

4.8. DEFINITION (ε^*-category). Let **C** and **R** be a complete order-enriched ccc and a retraction map category of **C**, respectively. Let $\mathbf{E}=\mathbf{E}(\mathbf{C},\mathbf{R})$. Using **E**, we define the category $\mathbf{E}^*=\mathbf{E}^*(\mathbf{C},\mathbf{R})$ called the *ε^*-category* of **C** with respect to **R** as follows:

(a) The objects of $\mathbf{E}^*$ are just the same as of $\mathbf{E}$.

(b) For each pair of objects A and B of $\mathbf{E}^*$, the arrows from A to B in $\mathbf{E}^*$ are all $[F]$, where Fs are arrows from A to B in $\mathbf{E}$.

(c) For arrows $[F]: A \to B$ and $[G]: B \to C$ in $\mathbf{E}^*$, the composite arrow $[G] \circ [F]: A \to C$ is defined by $[G] \circ [F] = [G \circ F]$.

(d) For each objects of $\mathbf{E}^*$, the identity arrow on A in $\mathbf{E}^*$ is defined as $[\mathrm{ID}_A]$, where ID_A is the identity arrow on A in E.

Here note that Lemma 4.7(i) guarantees that the definition (c) of composition in the above is well defined.

4.9. THEOREM. *Let* $\mathbf{C}$ *and* $\mathbf{R}$ *be a complete order-enriched ccc and a retraction map category of C, respectively. Let* $\mathbf{E}^* = \mathbf{E}^*(\mathbf{C}, \mathbf{R})$. *We define* $\langle [-], [-] \rangle$, $\Lambda([-1])$, *and* $\leqslant$ *on* $\mathbf{E}^*$:

(1) *For each pair of arrows* $[F]: C \to A$ *and* $[G]: C \to B$ *in* $\mathbf{E}^*$, $\langle [F], [G] \rangle = [\langle F, g \rangle]$.

(2) *For each arrow* $[F]: C \times A \to B$ *in* $\mathbf{E}^*$, $\Lambda([F]) = [\Lambda(F)]$.

(3) *For each pair of arrows* $[F]$, $[G]: A \to B$ *in* $\mathbf{E}^*$, $[F] \leqslant [G]$ *iff* $F \lesssim G$.

Then $\mathbf{E}^*$ *with the structures* $(1, [!_{(-)}], (-) \times (-), [\Pi_1^{(-),(-)}], [\Pi_2^{(-),(-)}], \langle [-], [-] \rangle, (-)^{(-)}, [\mathrm{EV}^{(-),(-)}], \Lambda([-]))$, *and* $\leqslant$ *is a complete order-enriched ccc, where* 1, $!_{(-)}$, $(-) \times (-)$, $\Pi_1^{(-),(-)}$, $\Pi_2^{(-),(-)}$, $(-)^{(-)}$, *and* $\mathrm{EV}^{(-),(-)}$ *are the same as of the* ε*-category* $\mathbf{E}(\mathbf{C}, \mathbf{R})$.

Proof. By Theorem 3.11 and Lemma 4.7, $\mathbf{E}^*$ is an order-enriched ccc. Note that $\langle [-], [-] \rangle$ and $\Lambda([-])$ are well defined by Lemma 4.7(ii) and (iii).

Let P be a directed subset of $\mathbf{E}^*(A, B)$, where A and B are objects of $\mathbf{E}^*$. The least upper bound of P is

$$\bigsqcup P = \left[\bigcup \{F \mid [F] \in P\} \right],$$

which is an arrow from A to B in $\mathbf{E}^*$. Let Q be a directed subset of $\mathbf{E}^*(B, C)$. Then we calculate

$$\begin{aligned} \bigsqcup Q \circ \bigsqcup P &= \left[\bigcup \{G \mid [G] \in Q\} \right] \circ \left[\bigcup \{F \mid [F] \in P\} \right] \\ &= \left[\bigcup \{G \mid [G] \in Q\} \circ \bigcup \{F \mid [F] \in P\} \right] \end{aligned}$$

$$=\left[\bigcup\{G\circ F\mid[F]\in P\text{ and }[G]\in Q\}\right]$$

$$=\bigsqcup\{[G\circ F]\mid[F]\in P\text{ and }[G]\in Q\}$$

$$=\bigsqcup\{[G]\circ[F]\mid[F]\in P\text{ and }[G]\in Q\}.$$

Hence $\mathbf{E}^*$ satisfies condition (2) of Definition 4.1. ∎

Let **C** be an order-enriched ccc and let **R** be a retraction map category of **C**. The construction of the ε-category **E(C, IR(C))** means completion of **C** w.r.t. arrows, where **IR(C, R)** is the trivial retraction map category of **C** defined in Definition 2.5. The objects of **E**(**C**, **IR**(**C**)) are essentially the same as of **C**. Each object a of **C** corresponds to the object $\{a\}$ in **E**(**C**, **IR**(**C**)). Completion of **E(C, IR(C))** w.r.t. objects is accomplished by the ε*-category construction.

The next theorem shows that the obtained ε*-category of **E(C, IR(C))** coincides with the ε-category **E(C, R)** of **C**. In other words, successively carrying out the completion w.r.t. arrows, then w.r.t. objects, is the same as carrying them out "in parallel" by the ε-category construction.

4.10. THEOREM. *Let* **C** *and* **R** *be an order-enriched ccc and a retraction map category of* **C**, *respectively. Let* $\mathbf{E}'=\mathbf{E}(\mathbf{C},\mathbf{IR}(\mathbf{C}))$. *We define the retraction map category* $\mathbf{R}'$ *of* $\mathbf{E}'$ *as follows*:

(1) $\{a\}\leqslant\{b\}$ *in* $\mathbf{R}'$ *iff* $a\leqslant b$ *in* **R**,

(2) $i[\{a\},\{b\}]=\{i[a,b]\}\downarrow$, *and*

(3) $j[\{a\},\{b\}]=\{j[a,b]\}\downarrow$.

Then **E(C, R)** *is isomorphic to* $\mathbf{E}^*(\mathbf{E}',\mathbf{R}')$. (*Note that* $\mathbf{E}'$ *is a complete order-enriched ccc*).

Proof. Let $\mathbf{E}=\mathbf{E}(\mathbf{C},\mathbf{R})$ and $\mathbf{E}^*=\mathbf{E}^*(\mathbf{E}',\mathbf{R}')$. First note that the objects of $\mathbf{E}'$ are all $\{a\}$, where $a\in|\mathbf{C}|$. And the arrows from $\{a\}$ to $\{b\}$ in $\mathbf{E}'$ are all ideals of $(\mathbf{C}(a,b),\leqslant)$.

We define the pair of functors $K\colon\mathbf{E}\to\mathbf{E}^*$ and $L\colon\mathbf{E}^*\to\mathbf{E}$ as follows:

$$
\begin{aligned}
K(A)&=\{\{a\}\mid a\in A\} &&\text{for } A\in|\mathbf{E}|,\\
K(F)&=[K'(F)] &&\text{for } F\in\mathbf{E}(A,B),\\
L(A')&=\{a\mid\{a\}\in A'\} &&\text{for } A'\in|\mathbf{E}^*|,\text{ and}\\
L([F'])&=L'(F') &&\text{for } [F']\in\mathbf{E}^*(A',B'),
\end{aligned}
$$

where $K'(F) = \{P \mid P \text{ is an arrow of } \mathbf{E}', (\exists f \in F)(P \sqsubseteq \{f\}\downarrow)\}$, and

$$L'(F') = \bigcup F'.$$

Here $\{f\}\downarrow$ means $\{g \in \mathbf{C}(a, b) \mid g \leqslant f\}$ for each arrow $f: a \to b$ in $\mathbf{C}$. First note that $\bigsqcup F'(\{a\}, \{b\}) = (\bigcup F')(a, b)$ for any arrow $F': A' \to B'$ in $\mathbf{E}(\mathbf{E}', \mathbf{R}')$, $\{a\} \in A'$ and $\{b\} \in B'$. It follows that L is well defined, because for every pair of arrows F', $G': A' \to B'$ in $\mathbf{E}(\mathbf{E}', \mathbf{R}')$, $L'(F') = L'(G')$ if $F' \simeq G'$. And note that, for any arrow $F: A \to B$ in $\mathbf{E}$, and any pair of $a \in A$ and $b \in B$,

$$\begin{aligned}
&\bigsqcup K'(F)(\{a\}, \{b\}) \\
&\quad = \bigcup \{P \in \mathbf{E}'(\{a\}, \{b\}) \mid (\exists a' \in A)(\exists b' \in B)(\exists f' \in F(a', b')) \\
&\qquad\qquad (a \leqslant a', b \leqslant b', \text{and } P \subset \{j[b, b'] \circ f' \circ i[a, a']\}\downarrow)\} \\
&\quad = \{j[b, b'] \circ f' \circ i[a, a'] \mid a' \in A, b' \in B, a \leqslant a', \\
&\qquad\qquad b \leqslant b', \text{and } f' \in F(a', b')\}\downarrow \\
&\quad = F(a, b) \qquad \text{(by Lemma 4.5(i))}.
\end{aligned}$$

Clearly $L(K(A)) = A$ for every $A \in |\mathbf{E}|$ and $K(L(A')) = A'$ for every $A' \in |\mathbf{E}^*|$. For every arrow $[F']: A' \to B'$ in $\mathbf{E}^*$, $K(L([F'])) = [F']$ because $\bigsqcup K'(L'(F'))(\{a\}, \{b\}) = (\bigcup F')(a, b) = \bigsqcup F'(\{a\}, \{b\})$ for all pairs of $\{a\} \in A'$ and $\{b\} \in B'$. And for every arrow $F: A \to B$ in $\mathbf{E}$, $L(K(F)) = F$ because $L'(K'(F)) = \bigcup (K'(F)) = F$.

Next we will show that K and L are really functors. For all arrows $F: A \to B$ and $G: B \to C$ in $\mathbf{E}$ and all pairs of $a \in A$ and $c \in C$,

$$\begin{aligned}
&\bigsqcup (K'(G) \circ K'(F))(\{a\}, \{c\}) \\
&\quad = \bigsqcup \left\{ \bigsqcup K'(G)(\{b\}, \{c\}) \circ \bigsqcup K'(F)(\{a\}, \{b\}) \mid b \in B \right\} \\
&\qquad\qquad \text{(by Corollary 4.6(ii))} \\
&\quad = \bigcup \{G(b, c) \circ F(a, b) \mid b \in B\} \\
&\quad = \{g \circ f \mid (\exists b \in B)(f \in F(a, b) \text{ and } g \in G(b, c))\}\downarrow \\
&\quad = (G \circ F)(a, c) \qquad \text{(by Lemma 4.5(ii))} \\
&\quad = \bigsqcup K'(G \circ F)(\{a\}, \{c\}).
\end{aligned}$$

So $K([G]) \circ K([F]) = K([G] \circ [F])$, that is, K preserves the composition $\circ$. For all $A \in |\mathbf{E}|$ and all pairs of $a, b \in A$,

$$\begin{aligned}
&\bigsqcup \mathrm{ID}_{K(A)}(\{a\}, \{b\}) \\
&\quad = \bigsqcup \{P \in E'(\{a\}, \{b\} \mid (\exists c \in C)(a \leqslant c, b \leqslant c, \text{and } P \sqsubseteq \{\mathrm{id}_c\}\downarrow\} \\
&\quad = \bigcup \{P \in E'(\{a\}, \{b\}) \mid (\exists c \in C)(a \leqslant c, b \leqslant c, \\
&\qquad\qquad \text{and } P \subset \{j[b, c] \circ i[a, c]\}\downarrow\} \\
&\quad = \{j[b, c] \circ i[a, c] \mid (\exists c \in C)(a \leqslant c \text{ and } b \leqslant c)\}\downarrow \\
&\quad = \mathrm{ID}_A(a, b) \\
&\quad = \bigsqcup K'(\mathrm{ID}_A)(\{a\}, \{b\}).
\end{aligned}$$

So $K(\mathrm{ID}_A) = [\mathrm{ID}_{K(A)}]$, that is, K preserves the identity arrow. Hence K is a functor. Because L is the inverse of K, L is also a functor.

From the above we conclude that $\mathbf{E}$ and $\mathbf{E}^*$ are isomorphic. ∎

4.11. *Remark.* Let $(X, \leqslant)$ be a partially ordered set. In general, $(X, \leqslant)$ can be expanded so that every directed subset of X has the least upper bound. Such a partially ordered set is called a cpo. We define a partially ordered set $(Y, \subset)$ as follows:

$$Y = \{(A^*)_* \mid A \text{ is a directed subset of } X\},$$

where

$$V^* = \{x \in X \mid (\forall v \in V)(v \leqslant x)\}$$
$$V_* = \{x \in X \mid (\forall v \in V)(x \leqslant v)\}.$$

Then $(Y, \subset)$ is a cpo, and $(X, \leqslant)$ is embedded into $(Y, \subset)$ by the map $f: X \to Y$ defined by $f(x) = \{x\}\downarrow$. The map f is one-to-one, and f preserves all least upper bounds of directed subsets if they exist. Note that all the elements of Y are ideals of $(X, \leqslant)$.

In the construction of ε-categories, the completion of an order-enriched ccc $\mathbf{C}$ with a retraction map category $\mathbf{R}$ of $\mathbf{C}$ is slightly different from the above completion of partially ordered sets. In the ε-category $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$, arrows in $\mathbf{E}$ are ideals of $(\mathbf{C}, \sqsubseteq)$, and objects of $\mathbf{E}$ are also ideals of $(|\mathbf{C}|, \leqslant)$. In this type of completion, the least upper bounds of directed subsets are not generally preserved by f defined above. Define Z as the set of all ideals of $(X, \leqslant)$. Let A be a directed subset of X. Suppose that there exists the least upper bound $\bigsqcup A$ of A in $(X, \leqslant)$ and $\bigsqcup A \notin A$. Then

$$\bigsqcup \{f(a) \mid a \in A\} = A\downarrow \neq f\left(\bigsqcup A\right).$$

5. Induced Retraction Map Categories

We have constructed an order-enriched ccc **E(C, R)** called ε-category from an order-enriched ccc **C** and a retraction map category **R** of **C**. In this section we will show that a particular retraction map category **RE(C, R)** of **E(C, R)** is naturally defined. We will construct **RE(C, R)** expanding **R**. We call **RE(C, R)** the *induced retraction map category* of **E(C, R)**.

5.1. Definition. Let **C** and **R** be an order-enriched ccc and a retraction map category of **C**, respectively. Let $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$. For each pair of objects A and B of **E**, when $A \subset B$, we define the pair of arrows $I[A, B] \in \mathbf{E}(A, B)$ and $J[A, B] \in \mathbf{E}(B, A)$ by

$$I[A, B] = \{i[a, b] \mid a \in A, b \in B, \text{ and } a \leqslant b \text{ in } R\}\downarrow,$$

$$J[A, B] = \{j[a, b] \mid a \in A, b \in B, \text{ and } a \leqslant b \text{ in } R\}\downarrow.$$

5.2. Lemma. *Let* **C** *and* **R** *be an order-enriched ccc and a retraction map category of* **C**, *respectively. Let* $\mathbf{E} = \mathbf{E}(\mathbf{C},\mathbf{R})$ *and let* A, B, A', B', *and* C *be objects of* **E**:

(i) *If* $A \subset B$ *then* $I[A, B]$ *and* $J[A, B]$ *are arrows of* **E**.

(ii) *If* $A \subset B$ *then* $J[A, B] \circ I[A, B] \supset \mathrm{ID}_A$ *and* $I[A, B] \circ J[A, B] \subset \mathrm{ID}_B$. *That is,* $(I[A, B], J[A, B])$ *is a retraction pair from* A *to* B *in* **E**.

(iii) *If* $A \subset B \subset C$ *then*

$$(I[A, C], J[A, C]) = (I[B, C] \circ I[A, B], J[A, B] \circ J[B, C]).$$

(iv) *If* $A \subset A'$ *and* $B \subset B'$ *then* $A \times B \subset A' \times B'$ *and*

$$(I[A \times B, A' \times B'], J[A \times B, A' \times B']) = (I[A, A'], J[A, A']) \times (I[B, B'], J[B, B']).$$

(v) *If* $A \subset A'$ *and* $B \subset B'$ *then* $B^A \subset B'^{A'}$ *and*

$$(I[B^A, B'^{A'}], J[B^A, B'^{A'}]) = (I[B, B'], J[B, B'])^{(I[A, A'], J[A, A'])}.$$

(vi) $(I[A, A], J[A, A]) = (\mathrm{ID}_A, \mathrm{ID}_A)$.

(vii) *If* **R** *is upper- (lower-) injective, then* $(I[A, B], J[A, B])$ *is upper- (lower-) injective for every pair of* A *and* B *such that* $A \subset B$.

Proof. In general, for all a, a', b, and $b' \in |\mathbf{C}|$, if $a \leqslant a' \leqslant b'$ and $a \leqslant b \leqslant b'$ then

$$j[b, b'] \circ i[a', b'] \circ i[a, a'] = j[b, b'] \circ i[b, b'] \circ i[a, b] \geqslant i[a, b].$$

Namely $i[a, b] \sqsubseteq i[a', b']$. Similarly $j[a, b] \sqsubseteq j[a', b']$. We will prove the lemma, using these facts.

(i) We will show that $I[A, B]$ is really an arrow of **E**. From Lemma 3.1, it is enough to show that $\{i[a, b] \mid a \in A \wedge b \in B \wedge a \leqslant b\}$ satisfies the arrow conditions. For every pair of $a \in A$ and $b \in B$ there is $b' \in B$ such that $a \leqslant b'$ and $b \leqslant b'$, because $A \subset B$. So the arrow condition (1) is satisfied.

Let a_1, $a_2 \in A$ and b_1, $b_2 \in B$ be given and suppose $a_1 \leqslant b_1$ and $a_2 \leqslant b_2$. Then there are $a \in A$ and $b \in B$ such that $a_1 \leqslant a$, $a_2 \leqslant a$, $b_1 \leqslant b$, $b_2 \leqslant b$, and $a \leqslant b$, because $A \subset B$. So $i[a_1, b_1] \sqsubseteq i[a, b]$ and $i[a_2, b_2] \sqsubseteq i[a, b]$. Hence the arrow condition (2) is satisfied.

Similarly it can be proved that $J[A, B]$ is really an arrow of **E**.

(ii) Because $A \subset B$, $\mathrm{id}_a = j[a, a] \circ i[a, a] \in J[A, B] \circ I[A, B]$ for all $a \in A$. Thus $\mathrm{ID}_A \subset J[A, B] \circ I[A, B]$. Next let $k \in I[A, B] \circ J[A, B]$ be given. Then there are $k_1 \in I[A, B]$ and $k_2 \in J[A, B]$ such that $\mathrm{dom}(k_1) = \mathrm{cod}(k_2)$ and $k \sqsubseteq k_1 \circ k_2$. There are a_1, $a_2 \in A$ and b_1, $b_2 \in B$ such that $a_1 \leqslant b_1$, $a_2 \leqslant b_2$, $k_1 \sqsubseteq i[a_1, b_1]$, and $k_2 \sqsubseteq j[a_2, b_2]$. There are $a \in A$ and $b \in B$ such that $a_1 \leqslant a$, $a_2 \leqslant a$, $b_1 \leqslant b$, $b_2 \leqslant b$, and $a \leqslant b$. So $i[a_1, b_1] \sqsubseteq i[a, b]$ and $j[a_2, b_2] \sqsubseteq j[a, b]$. By Lemma 2.9(i),

$$k \sqsubseteq k_1 \circ k_2 \sqsubseteq i[a, b] \circ j[a, b] \leqslant \mathrm{id}_b \in \mathrm{ID}_B.$$

Hence $k \in \mathrm{ID}_B$ and $I[A, B] \circ J[A, B] \subset \mathrm{ID}_B$.

(iii) We will show that $I[A, C] = I[B, C] \circ I[A, B]$. For every pair of $a \in A$ and $c \in C$, if $a \leqslant c$, then $i[a, c] = i[a, c] \circ i[a, a] \in I[B, C] \circ I[A, B]$ because $A \subset B$. So $I[A, C] \subset I[B, C] \circ I[A, B]$. Conversely let $k \in I[B, C] \circ I[A, B]$ be given. Then there are $k_1 \in I[B, C]$ and $k_2 \in I[A, B]$ such that $\mathrm{dom}(k_1) = \mathrm{cod}(k_2)$ and $k \sqsubseteq k_1 \circ k_2$. There are $a \in A$, $b_1 \in B$, $b_2 \in B$, and $c \in C$ such that $b_1 \leqslant c$, $a \leqslant b_2$, $k_1 \sqsubseteq i[b_1, c]$, and $k_2 \sqsubseteq i[a, b_2]$. There are $b \in B$ and $c' \in C$ such that $b_1 \leqslant b$, $b_2 \leqslant b$, $b \leqslant c'$, and $c \leqslant c'$. Because $i[a, b_1] \sqsubseteq i[a, b]$ and $i[b_2, c] \sqsubseteq i[b, c']$,

$$k \sqsubseteq k_1 \circ k_2 \sqsubseteq i[b, c'] \circ i[a, b] = i[a, c'] \in I[A, C]$$

by Lemma 2.9(i). So $k \in I[A, C]$ and $I[B, C] \circ I[A, B] \subset I[A, C]$.

Similarly it can be proved that $J[A, C] = J[A, B] \circ J[B, C]$.

(iv) Suppose $A \subset A'$ and $B \subset B'$. It is clear that $A \times B \subset A' \times B'$. We will show that $I[A \times B, A' \times B'] = I[A, A'] \times I[B, B']$. Let $k \in I[A \times B, A' \times B']$ be given. Then there are $d \in A \times B$ and $d' \in A' \times B'$ such that $d \leqslant d'$ and $k \sqsubseteq i[d, d']$. There are $a \in A$, $b \in B$, $a' \in A'$ and $b' \in B'$ such that $d \leqslant a \times b$ and $d' \leqslant a' \times b'$. There are $a'' \in A'$ and $b'' \in B'$ such that $a \leqslant a''$, $a' \leqslant a''$, $b \leqslant b''$, and $b' \leqslant b''$. Because $d \leqslant a \times b \leqslant a'' \times b''$ and $d' \leqslant a' \times b' \leqslant a'' \times b''$,

$$k \sqsubseteq i[d, d'] \sqsubseteq i[a \times b, a'' \times b'']$$
$$= i[a, a''] \times i[b, b''] \in I[A, A'] \times I[B, B'].$$

So $k \in I[A, A'] \circ I[B, B']$ and $I[A \times B, A' \times B'] \subset I[A, A'] \times I[B, B']$.

Conversely let $h \in I[A, A'] \times I[B, B']$ be given. Because $I[A, A'] \times I[B, B'] = \langle I[A, A'] \circ \Pi_1^{A,B}, I[B, B'] \circ \Pi_2^{A,B} \rangle$, there are $h_1 \in I[A, A'] \circ \Pi_1^{A,B}$ and $h_2 \in I[B, B'] \circ \Pi_2^{A,B}$ such that $\mathrm{dom}(h_1) = \mathrm{dom}(h_2)$ and $h \sqsubseteq \langle h_1, h_2 \rangle$. There are a_1, $a_2 \in A$, $a_1' \in A'$, b_1, $b_2 \in B$ and $b_2' \in B'$ such that $a_1 \leqslant a_1'$, $b_2 \leqslant b_2'$, $h_1 \sqsubseteq i[a_1, a_1'] \circ \pi_1^{a_1,b_1}$ and $h_2 \sqsubseteq i[b_2, b_2'] \circ \pi_2^{a_2,b_2}$. There are $a \in A$, $a' \in A'$, $b \in B$, and $b' \in B'$ such that $a_1 \leqslant a$, $a_2 \leqslant a$, $b_1 \leqslant b$, $b_2 \leqslant b$, $a \leqslant a'$, $a_1' \leqslant a'$, $b \leqslant b'$, and $b_2' \leqslant b'$. So $i[a_1, a_1'] \sqsubseteq i[a, a']$, $i[b_2, b_2'] \sqsubseteq i[b, b']$, $\pi_1^{a_1,b_1} \sqsubseteq \pi_1^{a,b}$, and $\pi_2^{a_2,b_2} \sqsubseteq \pi_2^{a,b}$ by Lemma 2.8(i). By Lemma 2.9(i) and (ii),

$$i[a_1, a_1'] \circ \pi_1^{a_1,b_1} \sqsubseteq i[a, a'] \circ \pi_1^{a,b},$$
$$i[b_2, b_2'] \circ \pi_2^{a_2,b_2} \sqsubseteq i[b, b'] \circ \pi_2^{a,b},$$

and

$$h \sqsubseteq \langle i[a, a'] \circ \pi_1^{a,b}, i[b, b'] \circ \pi_2^{a,b} \rangle$$
$$= i[a, a'] \times i[b, b']$$
$$= i[a \times b, a' \times b'] \in I[A \times B, A' \times B'].$$

Hence $h \in I[A \times B, A' \times B']$ and $I[A, A'] \times I[B, B'] \subset I[A \times B, A' \times B']$.

Similarly it can be proved that

$$J[A \times B, A' \times B'] = J[A, A'] \times J[B, B'].$$

(v) Similar to (iv).

(vi) It is clear that $\mathrm{ID}_A \subset I[A, A]$. Conversely for every pair or a, $a' \in A$, if $a \leqslant a'$, then $i[a, a'] \sqsubseteq i[a', a'] = \mathrm{id}_{a'} \in \mathrm{ID}_A$. So $I[A, A] \subset \mathrm{ID}_A$. Similarly $\mathrm{ID}_A = J[A, A]$.

(vii) If **R** is upper injective, by the similar way of the proof for $I[A, B] \circ J[A, B] \subset \mathrm{ID}_B$ it can be proved that $J[A, B] \circ I[A, B] \subset \mathrm{ID}_A$. So $J[A, B] \circ I[A, B] = \mathrm{ID}_A$. If **R** is lower injective, clearly $I[A, B] \circ J[A, B] \supset \mathrm{ID}_B$. ∎

5.3. THEOREM. *Let* **C** *and* **R** *be an order-enriched ccc and a retraction map category of* **C**, *respectively. Then we can define the retraction map category* **RE(C, R)** *of* **E(C, R)**, *whose partial order relation among the objects is the set inclusion* $\subset$ *and whose unique retraction from* A *to* B *in* **E(C, R)** *is* $(I[A, B], J[A, B])$ *for each pair of objects* A *and* B *such that* $A \subset B$.

Moreover if **R** *is upper- (lower-) injective, then* **RE(C, R)** *is upper- (lower-) injective.*

Proof. By Lemma 5.2. ▌

5.4. COROLLARY. *Let* **C** *and* **R** *be a complete order-enriched ccc and a retraction map category of* **C**, *respectively. Then we can define the retraction map category* **RE*(C, R)** *of* **E*(C, R)** *whose order relation among the objects is the set inclusion* $\subset$ *and whose unique retraction pair from* A *to* B *in* **E*(C, R)** *is* $([I[A, B]], [J[A, B]])$ *for each pair of objects* A *and* B *such that* $A \subset B$.

Proof. It follows from Theorem 5.3. Note that for every pair of arrows F and G of **E(C, R)**, if $F \subset G$, then $F \lesssim G$. ▌

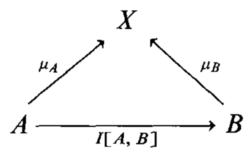
5.5. *Remark.* Let **C** be a complete order-enriched ccc and let **R** be a retraction map category of **C**. We have proved that the ε^*-category **E*(C, R)** is a complete order-enriched ccc, too. Moreover we have defined the induced retraction map category **RE*(C, R)** of **E*(C, R)**. So we can define the ε^*-category **E** = **E*(E*(C, R), RE*(C, R))**. However **E** is essentially the same as **E*(C, R)**. Indeed it can be proved that **E** and **E*(C, R)** are equivalent in the category theoretical sense.

In the construction of D_∞, D_∞ is a colimit (direct limit) of the system $\{i_n\}$ and also a limit (inverse limit) of $\{j_n\}$, as stated in Section 1. We examine colimits and limits in ε-categories based on induced retraction map categories.

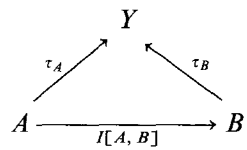
5.6. THEOREM. *Let* **C** *and* **R** *be an order-enriched ccc and a retraction map category of* R, *respectively. Let* **D** *be a full subcategory of* **RE(C, R)** *such that* $|\mathbf{D}|$ *is a directed subset of* $(|\mathbf{RE(C, R)}|, \subset)$. *Define* $I \upharpoonright \mathbf{D}: \mathbf{D} \to \mathbf{E(C, R)}$ *and* $J \upharpoonright \mathbf{D}^{op}: \mathbf{D}^{op} \to \mathbf{E(C, R)}$ *as the functors obtained from* I *and* J *by restricting their domains to* **D** *and* $\mathbf{D}^{op}$, *respectively. Let* $X = \bigcup\{A \mid A \in |\mathbf{D}|\}$, *which is an object of* **E(C, R)**.

Then X *is a colimit of* $I \upharpoonright \mathbf{D}$ *together with a universal cone* μ *defined by* $\mu_A = I[A, X]: A \to X$ *for* $A \in |\mathbf{D}|$. *At the same time,* X *is a limit of* $J \upharpoonright \mathbf{D}^{op}$ *together with a limiting cone* υ *defined by* $\upsilon_A = J[A, X]: X \to A$ *for* $A \in |\mathbf{D}^{op}|$.

Proof. We deal only with colimit. As for limit, it is similar. For every pair of A, $B \in |\mathbf{D}|$, if $A \subset B$, then $I[B, X] \circ I[A, B] = I[A, X]$ from Lemma 5.2(iii). Namely the following diagram commutes:

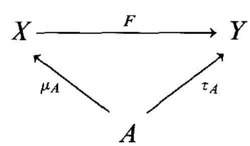


Next we assume that there is an object Y of $\mathbf{E(C, R)}$ and that there is an object Y of $\mathbf{E(C, R)}$ and that, for each $C \in |\mathbf{D}|$, there is an arrow $\tau_C\colon (I \restriction \mathbf{D})(C) \to Y$ in $\mathbf{E(C, R)}$ such that the following diagram commutes for every arrow $[A, B]$ in $\mathbf{D}$.



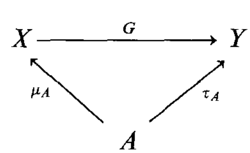
Namely $\tau_B \circ I[A, B] = \tau_A$. Define $F = \bigcup\{\tau_C \mid C \in |\mathbf{D}|\}$. Then F is an arrow from X to Y in $\mathbf{E(C, R)}$. Here note that $\tau_A = \tau_B \circ I[A, B] \subset \tau_B \circ I[B, B] = \tau_B$ if $A \subset B$.

We show that the following diagram commutes for every $A \in |\mathbf{D}|$.



Namely, $F \circ \mu_A = \tau_A$. Clearly $F \circ \mu_A \supset \tau_A \circ \mathrm{ID}_A = \tau_A$. Conversely suppose that $k \in F \circ \mu_A$. Then there are $C \in |\mathbf{D}|$, $f \in \tau_C$ and $h \in I[A, X]$ such that $\mathrm{dom}(f) = \mathrm{cod}(h)$ and $k \sqsubseteq f \circ h$. Because $h \in I[A, X]$, there are $a \in A$ and $b \in X$ such that $a \leqslant b$ in $\mathbf{R}$ and $h \sqsubseteq i[a, b]$. Because $b \in X$, there is $B \in |\mathbf{D}|$ such that $b \in B$. Since $|\mathbf{D}|$ is directed, there is $A' \in |\mathbf{D}|$ such that $A \subset A'$, $B \subset A'$, and $C \subset A'$. Therefore $f \in \tau_C \subset \tau_{A'}$, $h \sqsubseteq i[a, b] \in I[A, A']$, and $k \sqsubseteq f \circ h \in \tau_{A'} \circ I[A, A'] = \tau_A$. Hence $F \circ \mu_A = \tau_A$.

Finally assume that there is an arrow $G\colon X \to Y$ in $\mathbf{E(C, R)}$ such that the following diagram commutes for every $A \in |\mathbf{D}|$.



Namely $G \circ \mu_A = \tau_A$. Then

$$\begin{aligned} F &= \bigcup \{\tau_A \mid A \in |\mathbf{D}|\} \\ &= G \circ \bigcup \{\mu_A \mid A \in |\mathbf{D}|\} \\ &= G \circ I[X, X] = G. \end{aligned}$$

We conclude that X is a colimit of $I \restriction \mathbf{D}$ together with a universal cone μ. ▌

5.7. *Remark.* Let $\mathbf{C}$ be an order-enriched ccc and let $\mathbf{R}$ be a retraction map category of $\mathbf{C}$. Let $\mathbf{E} = \mathbf{E}(\mathbf{C}, \mathbf{R})$ and $\mathbf{RE} = \mathbf{RE}(\mathbf{C}, \mathbf{R})$. We define a functor $K: \mathbf{C} \to \mathbf{E}$ by

$$\begin{array}{ll} K(a) = \{a\}\downarrow & \text{for object } a \text{ of } \mathbf{C}, \text{ and} \\ K(f) = \{f\}\downarrow & \text{for arrow } f \text{ of } \mathbf{C}, \end{array}$$

and define a functor $L: \mathbf{R} \to \mathbf{RE}$ by

$$\begin{array}{ll} L(a) = \{a\}\downarrow & \text{for object } a \text{ of } \mathbf{R}, \text{ and} \\ L([a, b]) = [\{a\}\downarrow, \{b\}\downarrow] & \text{for arrow } [a, b] \text{ of } \mathbf{R}. \end{array}$$

Then the following diagrams commute:

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{i} & \mathbf{C} \\ {\scriptstyle L}\downarrow & & \downarrow{\scriptstyle K} \\ \mathbf{RE} & \xrightarrow[I]{} & \mathbf{E} \end{array} \qquad \begin{array}{ccc} \mathbf{R}^{\mathrm{op}} & \xrightarrow{j} & \mathbf{C} \\ {\scriptstyle L^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle K} \\ \mathbf{RE}^{\mathrm{op}} & \xrightarrow[J]{} & \mathbf{E} \end{array}$$

Namely $K \circ i = I \circ L$ and $K \circ j = J \circ L^{\mathrm{op}}$, where L^{op} is a functor from $\mathbf{R}^{\mathrm{op}}$ to $\mathbf{RE}^{\mathrm{op}}$ defined by

$$\begin{array}{ll} L^{\mathrm{op}}(a) = \{a\}\downarrow & \text{for object } a \text{ of } \mathbf{R}^{\mathrm{op}}, \text{ and} \\ L^{\mathrm{op}}([a, b]) = [\{a\}\downarrow, \{b\}\downarrow] & \text{for arrow } [a, b] \text{ of } \mathbf{R}^{\mathrm{op}}. \end{array}$$

The above functor K, L, and L^{op} are one-to-one maps both w.r.t. objects and w.r.t. arrows. Thus $\mathbf{C}$ is embedded into $\mathbf{E}$, and $\mathbf{R}$ with i and j is embedded into $\mathbf{RE}$ with I and J.

By Theorem 5.6, for every full subcategory $\mathbf{D}$ of $\mathbf{R}$, if $|\mathbf{D}|$ is directed, then $X = \{a \mid (\exists x \in |\mathbf{D}|)(a \leqslant x)\}$ is an object of $\mathbf{E}$ and X is a colimit of $(I \circ L) \restriction \mathbf{D}$. Conversely, for every object X' of $\mathbf{E}$, X' is a colimit of $(I \circ L) \restriction \mathbf{D}'$, where $\mathbf{D}'$ is the full subcategory of $\mathbf{R}$ defined by $|\mathbf{D}'| = X'$. Thus, informally speaking, the ε-category **E(C, R)** is an expansion of **C** so that $i \restriction \mathbf{D}$ has a colimit and $j \restriction \mathbf{D}^{\mathrm{op}}$ has a limit for every full subcategory **D** of **R** such that $|\mathbf{D}|$ is directed.

6. Models of the λ-calculus

In this section we will examine the relationship between models of the λ-calculus and ε*-categories. For the definition of lambda-calculus models, we refer the reader to Barendregt (1984). Also see Meyer (1982) and Hindley and Longo (1980). Let **P** be a ccc. We say that an object u of **P** is reflexive if there are arrows $\Phi: u \to u^u$ and $\Psi: u^u \to u$ in **P** such that $\Phi \circ \Psi = \mathrm{id}_{u^u}$. Many authors pointed out the relation between models of λ-calculus and ccc's equipped with reflexive objects. In this paper we call such (u, Φ, Ψ) a *reflexive structure.*

6.1. Facts. (i) Every ccc **P** equipped with a reflexive structure (u, Φ, Ψ) defines a λ-algebra $\mathfrak{M}(\mathbf{P}, u, \Phi, \Psi)$.

(ii) (Scott) Every λ-algebra $\mathfrak{N}$ defines a ccc $\mathbf{P}(\mathfrak{N})$ with a reflexive structure (u, Φ, Ψ).

(iii) (Koymans) For every λ-algebra $\mathfrak{N}$, $\mathfrak{M}(\mathbf{P}(\mathfrak{N}), u, \Phi, \Psi)$ is isomorphic to $\mathfrak{N}$, where (u, Φ, Ψ) is the reflexive structure defined in (ii).

We refer the reader to Koymans (1982) and Barendregt (1984), in which the exact meanings of the above facts and their proofs appear. Also see Lambek (1974), Scott (1980), and Koymans (1984).

6.2. Definition. Let **P** be a ccc and let a and b be objects of **P**. Then **P** has *enough points* at a w.r.t. b iff the following condition is satisfied: for every pair of arrows $f, g \in \mathbf{P}(a, b)$, if $f \neq g$, then there exists an arrow $h \in \mathbf{P}(1, a)$ such tat $f \circ h \neq g \circ h$.

Similarly when **C** is an order-enriched ccc, we say that **C** has enough points at a w.r.t. b iff the following condition is satisfied: for every pair of arrows $f, g \in \mathbf{C}(a, b)$, if $f \nleq g$, there is an arrow $h \in \mathbf{C}(1, a)$ such that $f \circ h \nleq g \circ h$.

6.3. Fact. Let **P** be a ccc with a reflexive structure (u, Φ, Ψ). Then **P** has enough points at u w.r.t. u iff $\mathfrak{M}(\mathbf{P}, u, \Phi, \Psi)$ is a λ-model (i.e., a weakly extensional λ-algebra).

We refer the reader to Koymans (1982).

6.4. DISCUSSION. Our method of constructing λ-algebras is based on Facts 6.1 and 6.3. If we can construct a ccc **P** with a reflexive structure (u, Φ, Ψ), then we get a λ-algebra $\mathfrak{M}(\mathbf{P}, u, \Phi, \Psi)$ by Fact 6.1. So we are reduced to construct various kinds of ccc's with reflexive structures. The ε-category method can be used for constructing such ccc's.

Let **C** be an order-enriched ccc. In general there exist many retraction map categories of **C**. Consider the case that we can eventually choose an upper-injective retraction map category **R** of **C** that satisfies the condition: there exists a directed subset $U' \subset |\mathbf{C}|$ w.r.t. $\leqslant$ of **R** such that $b^a \in U'$ for every pair of $a \in U'$ and $b \in U'$. Let U be the object $U'\downarrow$ of **E(C, R)**. Because $U^U \subset U$ and **R** is upper injective, $(I[U^U, U], J[U^U, U])$ is an upper-injective retraction pair. So $J[U^U, U] \circ I[U^U, U] = \mathrm{ID}_{U^U}$ and namely U is a reflexive object of **E(C, R)**. By Fact 6.1 we can get a λ-algebra $\mathfrak{M}(\mathbf{E}(\mathbf{C}, \mathbf{R}), U, J[U^U, U], I[U^U, U])$. By such method we can construct various kinds of λ-algebras. The above construction is an example. We will give concrete examples in Sections 8 and 9. However there is not always a retraction map category **R** such that **E(C, R)** has a reflexive object.

6.5. PROPOSITION. *Let* **C** *and* **R** *be a complete order-enriched ccc and a retraction map category of* **C**, *respetively. Let A and B be objects of* **E*(C, R)**. *Then condition* (1) *below implies condition* (2):

(1) **C** *has enough points at a w.r.t. b for every pair of $a \in A$ and $b \in B$*;

(2) **E*(C, R)** *has enough points at A w.r.t. B.*

Moreover, if **R** *is upper-injective, the above conditions are equivalent.*

Proof. Let $\mathbf{E}^* = \mathbf{E}^*(\mathbf{C}, \mathbf{R})$.

First we show that condition (1) implies (2). Let $[F], [G] \in \mathbf{E}^*(A, B)$ and suppose that $[F] \nleqslant [G]$. Then there are $a_0 \in A$ and $b_0 \in B$ such that $\bigsqcup F(a_0, b_0) \nleqslant \bigsqcup G(a_0, b_0)$. From condition (1) there is $h_0 \in \mathbf{C}(1, a_0)$ such that $\bigsqcup F(a_0, b_0) \circ h_0 \nleqslant \bigsqcup G(a_0, b_0) \circ h_0$. We define $[H] \in \mathbf{E}^*(1, A)$ by

$$H = \{i[a_0, a] \circ h_0 \mid a \in A \text{ and } a_0 \leqslant a\}\downarrow$$

Then

$$\begin{aligned}
&\bigsqcup (F \circ H)(1, b_0) \\
&= \bigsqcup \left\{ \bigsqcup F(a, b_0) \circ \bigsqcup H(1, a) \,\middle|\, a \in A \right\} \qquad \text{(by Corollary 4.6(ii))} \\
&= \bigsqcup \Big\{ \bigsqcup F(a, b_0) \circ \bigsqcup \{ j[a, a'] \circ i[a_0, a'] \circ h_0 \mid \\
&\qquad a' \in A,\ a \leqslant a',\ a \leqslant a', \text{ and } a_0 \leqslant a' \} \,\Big|\, a \in A \Big\}
\end{aligned}$$

$$= \bigsqcup \left\{ \bigsqcup F(a, b_0) \circ j[a, a'] \circ i[a_0, a'] \,\middle|\, a \in A,\ a' \in A,\ a \leqslant a', \text{ and } a_0 \leqslant a' \right\} \circ h_0$$

$$= \bigsqcup \left\{ \bigsqcup F(a', b_0) \circ i[a, a'] \circ j[a, a'] \circ i[a_0, a'] \,\middle|\, a \in A,\ a' \in A,\ a \leqslant a', \text{ and } a_0 \leqslant a' \right\} \circ h_0$$

(by Corollary 4.6(i))

$$= \bigsqcup \left\{ \bigsqcup F(a', b_0) \circ i[a_0, a'] \,\middle|\, a' \in A \text{ and } a_0 \leqslant a' \right\} \circ h_0$$

$$= \bigsqcup F(a_0, b_0) \circ h_0 \qquad \text{(by Corollary 4.6(i))}$$

$$\nleqslant \bigsqcup G(a_0, b_0) \circ h_0$$

$$= \bigsqcup (G \circ F)(1, b_0).$$

Therefore $[F] \circ [H] \nleqslant [G] \circ [H]$.

Next we show that condition (2) implies (1), assuming that **R** is upper injective. Let $a \in A$, $b \in B$, $f \in \mathbf{C}(a, b)$, and $g \in \mathbf{C}(a, b)$, and suppose that $f \nleqslant g$. We define two arrows $[F]$, $[G] \in \mathbf{E}^*(A, B)$ by

$$F = \{ i[b, b'] \circ f \circ j[a, a'] \mid a' \in A,\ b' \in B,\ a \leqslant a', \text{ and } b \leqslant b' \}\downarrow$$

and

$$G = \{ i[b, b'] \circ g \circ j[a, a'] \mid a' \in A,\ b' \in B,\ a \leqslant a', \text{ and } b \leqslant b' \}\downarrow.$$

Because $f = \bigsqcup F(a, b)$ and $g = \bigsqcup G(a, b)$ from the assumption, $[F] \nleqslant [G]$. By condition (2), there is $[H] \in \mathbf{E}^*(1, A)$ such that $[F] \circ [H] \nleqslant [G] \circ [H]$. Namely there are $c_0 \in 1 = \{1\}\downarrow$ and $b_0 \in B$ such that $\bigsqcup(F \circ H)(c_0, b_0) \nleqslant \bigsqcup(G \circ H)(c_0, b_0)$. There is $b' \in B$ such that $b_0 \leqslant b'$ and $b \leqslant b'$. Define $h_0 = \bigsqcup H(1, a) \in \mathbf{C}(1, a)$. Then

$$\bigsqcup (F \circ H)(c_0, b')$$

$$= \bigsqcup \left\{ \bigsqcup F(a', b') \circ \bigsqcup H(c_0, a') \,\middle|\, a' \in A \right\} \qquad \text{(by Corollary 4.6(ii))}$$

$$= \bigsqcup \left\{ \bigsqcup F(a', b') \circ \bigsqcup H(c_0, a') \,\middle|\, a' \in A \text{ and } a \leqslant a' \right\}$$

$$= \bigsqcup \left\{ i[b, b'] \circ f \circ j[a, a'] \circ \bigsqcup H(c_0, a') \,\middle|\, a' \in A \text{ and } a \leqslant a' \right\}$$

(from the assumption)

$$= i[b, b'] \circ f \circ \bigsqcup H(c_0, a) \qquad \text{(by Corollary 4.6(i))}$$

$$= i[b, b'] \circ f \circ h_0 \circ i[c_0, 1] \qquad \text{(by Corollary 4.6(i))}$$

and similarly $\bigsqcup(G \circ H)(c_0, b') = i[b, b'] \circ g \circ h_0 \circ i[c_0, 1]$. Therefore $f \circ h_0 \nleqslant g \circ h_0$. Indeed, if $f \circ h_0 \leqslant g \circ h_0$, then

$$\bigsqcup (F \circ H)(c_0, b_0)$$

$$= j[b_0, b'] \circ \bigsqcup (F \circ H)(c_0, b') \qquad \text{(by Corollary 4.6(i))}$$

$$= j[b_0, b'] \circ i[b, b'] \circ f \circ h_0 \circ i[c_0, 1]$$

$$\leqslant j[b_0, b'] \circ i[b, b'] \circ g \circ h_0 \circ i[c_0, 1]$$

$$= \bigsqcup (G \circ H)(c_0, b_0),$$

which is a contradiction. ▮

7. Generation of Retraction Map Categories

In order to construct various kinds of λ-calculus models, we will examine the method of generating retraction map categories that satisfy some desired properties. In general, there are many retraction map categories of a given order-enriched ccc. We want particular retraction map categories of them.

We intend to construct a retraction map category from a basic set of retraction pairs adding the needed retraction pairs to them. We can naturally expand a given set of retraction pairs to a retraction map category, if the set satisfies a comfortable property. However we do not always accomplish the expansion. We will give a sufficient condition that the expansion of a set of retraction pairs really becomes a retraction map category. The method in this section will be used, when we give examples of λ-algebras based on ε-categories in Sections 8 and 9.

7.1. Definition (Retractive closure). Let R be a set of retraction pairs of an order-enriched ccc **C**. The *retractive closure* of R, denoted by close(R), is the set of retraction pairs of **C** inductively defined as follows:

$$R_0 = R \bigcup \{(\mathrm{id}_a, \mathrm{id}_a) \mid a \in |\mathbf{C}|\},$$

$$R_{n+1} = \{(i' \circ i, j \circ j') \mid (i, j) \in R_n, (i', j') \in R_n, \text{ and } \mathrm{cod}(i) = \mathrm{dom}(i')\}$$
$$\bigcup \{(i_a, j_a) \times (i_b, j_b), \quad (i_b, j_b)^{(i_a, j_a)} \mid (i_a, j_a) \in R_n \text{ and } (i_b, j_b) \in R_n\} \qquad (n \geqslant 0)$$

$$\mathrm{close}(R) = \bigcup_{n=0}^{\infty} R_n.$$

7.2. *Remarks.* The retractive closure of R satisfies the following conditions:

(1) for every object a, $(\mathrm{id}_a, \mathrm{id}_a)$: $a \to a \in \mathrm{close}(R)$;

(2) if (i, j): $a \to b$ and (i', j'): $b \to c$ are contained in close(R), then $(i' \circ i, j \circ j')$: $a \to c$ is also contained in close(R);

(3) if (i_a, j_a): $a \to a'$ and (i_b, j_b): $b \to b'$ are contained in close(R), then

$$(i_a, j_a) \times (i_b, j_b): a \times b \to a' \times b'$$
$$(i_b, j_b)^{(i_a, j_a)}: b^a \to b'^{a'}$$

are also contained in close(R).

We define a binary relation $\leqslant$ among the objects of **C** by

$$a \leqslant b \qquad \text{iff} \qquad \text{there is } r: a \to b \in \mathrm{close}(R).$$

The above binary relation $\leqslant$ is a preorder. If $\leqslant$ is a partial order relation and close(R) has at most one arrow $r: a \to b$ for every pair of objects a and b, then close(R) together with $\leqslant$ determines a retraction map category of **C**.

7.3. Definition (Canonical order-enriched ccc). Let **C** be an order-enriched ccc. Then **C** is *canonical* iff the following conditions are satisfied:

(1) if $a \times b = a' \times b'$ then $a = a'$ and $b = b'$;

(2) if $b^a = b'^{a'}$ then $a = a'$ and $b = b'$; and

(3) $a \times b \neq b'^{a'}$ for all a, b, a', and b'.

An object c of **C** is *atomic* iff c is neither $a \times b$ nor b^a for any pair of a, $b \in |\mathbf{C}|$. When **C** is canonical, a retraction map category **R** of **C** is canonical iff **R** satisfies the conditions: for all objects a, a', b, b', of **C**, $a \leqslant a'$, and $b \leqslant b'$ iff $a \times b \leqslant a' \times b'$ iff $b^a \leqslant b'^{a'}$.

7.4. Definition (Primitive set of retraction pairs). Let $\mathbf{C}$ be a canonical order-enriched ccc and let P be a set of retraction pairs of $\mathbf{C}$. Then P is *primitive* iff the following conditions are satisfied:

(1) if $r\colon a \to b \in P$ then a and b are atomic;

(2) for all atomic objects p of $\mathbf{C}$, $(\mathrm{id}_p, \mathrm{id}_p)\colon p \to p \in P$;

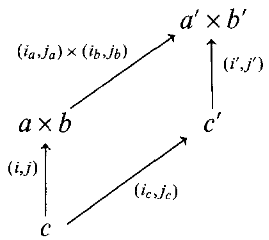
(3) if $(i, j)\colon a \to b \in P$ and $(i', j')\colon b \to c \in P$ then $(i' \circ i, j \circ j')\colon a \to c \in P$;

(4) if $r\colon a \to b \in P$ and $r'\colon a \to b \in P$ then $r = r'$;

(5) if $r\colon a \to b \in P$ and $r'\colon b \to a \in P$ then $a = b$.

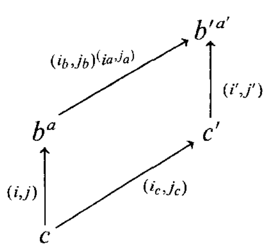
7.5. Theorem. *Let* $\mathbf{C}$ *be a canonical order-enriched ccc and let* P *be a primitive set of retraction pairs of* $\mathbf{C}$. *Let* Q *be a set of retraction pairs of* $\mathbf{C}$ *in the form of either* $r\colon c \to a \times b$ *or* $r\colon c \to b^a$, *where* a, b, *and* c *are atomic objects. If the following conditions are satisfied, then the retractive closure* $\mathrm{close}(R)$ *of* $R = P \cup Q$ *satisfies the conditions of retraction map categories of* $\mathbf{C}$. *Let* $(i_a, j_a)\colon a \to a' \in P$ *and* $(i_b, j_b)\colon b \to b' \in P$ *be given*:

(1) *If* $(i, j)\colon c \to a \times b \in Q$ *then there are* $(i_c, j_c)\colon c \to c' \in P$ *and* $(i', j')\colon c' \to a' \times b' \in Q$ *such that the following diagram commutes*:



That is, $(i_a, j_a) \times (i_b, j_b) \circ (i, j) = (i', j') \circ (i_c, j_c)$.

(2) *If* $(i, j)\colon c \to b^a \in Q$ *then there are* $(i_c, j_c)\colon c \to c' \in P$ *and* $(i', j')\colon c' \to b'^{a'} \in Q$ *such that the following diagram commutes*:



That is, $(i_b, j_b)^{(i_a, j_a)} \circ (i, j) = (i', j') \circ (i_c, j_c)$.

(3) *If* $r\colon c \to d \in Q$ *and* $r'\colon c' \to d \in Q$ *then* $c = c'$ *and* $r = r'$.

Moreover the generated retraction map category from close(R) *is canonical.*

Condition (3) in the above says that, for every nonatomic object d, there exists at most one retraction pair $r \in Q$ such that $\mathrm{cod}(r) = d$. Conditions (1) and (2) require that P and Q are chosen without conflict. Theorem 7.5 gives a method of generating retraction map categories. We will show examples in Sections 8 and 9.

For the proof we need lemmas.

7.6. LEMMA. (i) *If* $r\colon a \times b \to d \in \mathrm{close}(R)$, *then* d *is in the form of* $a' \times b'$ *and there are* $(i_a, j_a)\colon a \to a'$ *and* $(i_b, j_b)\colon b \to b'$ *in* close(R) *such that* $r = (i_a, j_a) \times (i_b, j_b)$.

(ii) *If* $r\colon b^a \to d \in \mathrm{close}(R)$, *then* d *is in the form of* $b'^{a'}$ *and there are* $(i_a, j_a)\colon a \to a'$ *and* $(i_b, j_b)\colon b \to b'$ *in* close(R) *such that* $r = (i_b, j_b)^{(i_a, j_a)}$.

(iii) *If* $r\colon a \to b \in \mathrm{close}(R)$ *and* b *is atomic, then* a *is atomic and* $r \in P$.

Proof. It is clear from the difinition of close(R). ∎

7.7. LEMMA. *Let* c *be an atomic object of* C:

(i) *If* $r\colon c \to a \times b \in \mathrm{close}(R)$, *then there are atomic objects* a', b', *and* c', *and retraction pairs*

$$r_1 : c \to c' \in P, \qquad r_2 : c' \to a' \times b' \in Q,$$

$$(i_a, j_a)\colon a' \to a \in \mathrm{close}(R),$$

and

$$(i_b, j_b)\colon b' \to b \in \mathrm{close}(R)$$

such that

$$r = (i_a, j_a) \times (i_b, j_b) \circ r_2 \circ r_1.$$

(ii) *If* $r\colon c \to b^a \in \mathrm{close}(R)$, *there are atomic objects* a', b', *and* c', *and retraction pairs*

$$r_1 : c \to c' \in P, \qquad r_2 : c' \to b'^{a'} \in Q,$$

$$(i_a, j_a)\colon a' \to a \in \mathrm{close}(R),$$

and

$$(i_b, j_b)\colon b' \to b \in \mathrm{close}(R)$$

such that

$$r = (i_b, j_b)^{(i_a, j_a)} \circ r_2 \circ r_1.$$

Proof. Let $r \in R_n$. We use induction on n:

(i) When $n = \varnothing$, $r \in Q$ and it is clear. Suppose that that $n \geqslant 1$. Then there are an object d, retraction pairs $r'\colon c \to d \in R_{n-1}$, and $r''\colon d \to a \times b \in R_{n-1}$ such that $r = r'' \circ r'$. From Lemma 7.6, d must be either atomic or in the form of $a'' \times b''$.

If d is atomic, then by the induction hypothesis there are atomic objects a', b', and c', and retraction pairs

$$r_1'\colon d \to c' \in P, \qquad r_2\colon c' \to a' \times b' \in Q,$$
$$(i_a, j_a)\colon a' \to a \in \operatorname{close}(R),$$

and

$$(i_b, j_b)\colon b' \to b \in \operatorname{close}(R)$$

such that $r'' = (i_a, j_a) \times (i_b, j_b) \circ r_2 \circ r_1'$. By the condition of P, $r_1' \circ r'$: $c \to c' \in P$. So the lemma is satisfied.

On the other hand, if d is $a'' \times b''$, then by Lemma 7.6 there are retraction pairs

$$(i_a', j_a')\colon a'' \to a \in \operatorname{close}(R)$$
$$(i_b', j_b')\colon b'' \to b \in \operatorname{close}(R)$$

such that $r'' = (i_a', j_a') \times (i_b', j_b')$.

By the induction hypothesis there are atomic objects a', b', and c', and retraction pairs

$$r_1\colon c \to c' \in P, \qquad r_2\colon c' \to a' \times b' \in Q,$$
$$(i_a'', j_a'')\colon a' \to a'' \in \operatorname{close}(R)$$
$$(i_b'', j_b'')\colon b' \to b'' \in \operatorname{close}(R)$$

such that $r' = (i_a'', j_a'') \times (i_b'', j_b'') \circ r_2 \circ r_1$. Let

$$(i_a, j_a) = (i_a' \circ i_a'', j_a'' \circ j_a')$$
$$(i_b, j_b) = (i_b' \circ i_b'', j_b'' \circ j_b').$$

Then $(i_a, j_a)\colon a' \to a$ and $(i_b, j_b)\colon b' \to b$ are contained in close(R), and

$$r = r'' \circ r' = (i_a, j_a) \times (i_b, j_b) \circ r_2 \circ r_1.$$

Hence the lemma is satisfied.

(ii) Similar to (i). ∎

7.8. DEFINITION, We inductively define the subset $W \subset |\mathbf{C}|$, the atomic object $w(c)$ and the retraction pair

$$w_r(c) = (w_i(c), w_j(c)) \colon w(c) \to c \in \text{close}(R)$$

for each object $c \in W$ as follows:

(1) every atomic object c is contained in W, and $w(c) = c$ and $w_r(c) = (\text{id}_c, \text{id}_c)$;

(2) if $a \in W$, $b \in W$, and there is $r\colon d \to w(a) \times w(b) \in Q$, then $a \times b \in W$, $w(a \times b) = d$ and

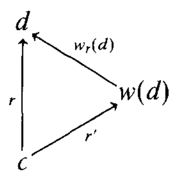
$$w_r(a \times b) = (w_r(a) \times w_r(b)) \circ r;$$

(3) if $a \in W$, $b \in W$, and there is $r\colon d \to w(b)^{w(a)} \in Q$, then $b^a \in W$, $w(b^a) = d$, and

$$w_r(b^a) = w_r(b)^w\, r^{(a)} \circ r.$$

Note that $w(a)$ and $w_r(a)$ are uniquely determined for each object $a \in W$, which follows from condition (3) of Q. The meaning of Definition 7.8 will be clarified by the next lemma.

7.9. LEMMA. *If $r\colon c \to d \in \text{close}(R)$ and c is atomic, then $d \in W$ and there is $r'\colon c \to w(d) \in P$ such that the following diagram commutes*:



That is, $r = w_r(d) \circ r'$.

Proof. We use induction on the structure of d:

(1) If d is atomic, then $w(d) = d$ and $w_r(d) = (\text{id}_d, \text{id}_d)$.

(2) If d is $a \times b$, then by Lemma 7.7 there are atomic objects a', b', and c', and retraction pairs

$$r_1 \colon c \to c' \in P, \qquad r_2 \colon c' \to a' \times b' \in Q,$$
$$(i_a, j_a)\colon a' \to a \in \text{close}(R)$$

and

$$(i_b, j_b)\colon b' \to b \in \text{close}(R)$$

such that $r = (i_a, j_a) \times (i_b, j_b) \circ r_2 \circ r_1$.

By the induction hypothesis, $a \in W$, $b \in W$, and there are

$$(i'_a, j'_a): a' \to w(a) \in P$$
$$(i'_b, j'_b): b' \to w(b) \in P$$

such that $(i_a, j_a) = w_r(a) \circ (i'_a, j'_a)$ and $(i_b, j_b) = w_r(b) \circ (i'_b, j'_b)$.

By condition (1) in Theorem 7.5, there are atomic object c'' and retraction pairs

$$r_3: c' \to c'' \in P$$

and

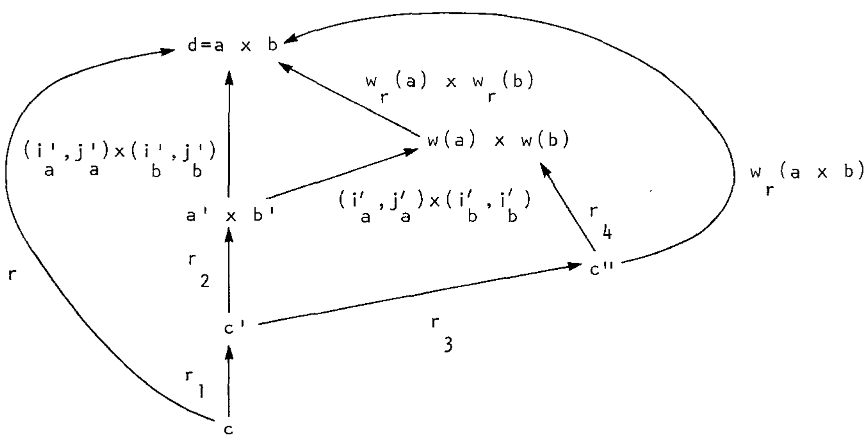
$$r_4: c'' \to w(a) \times w(b) \in Q$$

such that $(i'_a, j'_a) \times (i'_b, j'_b) \circ r_2 = r_4 \circ r_3$.

By the definitions of W, $w(-)$, and $w_r(-)$, $a \times b \in W$, $w(a \times b) = c''$, and

$$w_r(a \times b) = w_r(a) \times W_r(b) \circ r_4.$$

So the following diagram commutes:



Especially $r = w_r(a \times b) \circ (r_3 \circ r_1)$. Hence the lemma is satisfied.

(3) When d is b^a, it is similar to (2). ∎

Note that $w_r(d)$ is uniquely determined only by d and that $w_r(d)$ does not depend on $r: c \to d$. From the conditions in Theorem 7.5, $r': c \to w(d)$ is uniquely determined, because c and $w(d)$ are atomic objects. Therefore $r: c \to d$ is also uniquely determined for each pair of c and d, if it exists.

7.10. *Proof of Theorem* 7.5. First we show that close(R) has at most one retraction pair from c to d for every pair of objects c and d. We use induction on the structure of c. Let $r_1, r_2 : c \to d \in \text{close}(R)$:

(1) If c is atomic, then by Lemma 7.9, $d \in W$, and there are $r_1' : c \to w(d) \in P$ and $r_2' : c \to w(d) \in P$ such that $r_1 = w_r(d) \circ r_1'$ and $r_2 = w_r(d) \circ r_2'$. Because $r_1' = r_2'$ from the conditions of P, $r_1 = r_2$.

(2) If c is $a \times b$, then by Lemma 7.6, d is in the form of $a' \times b'$ and there are (i_a, j_a): $a \to a'$, (i_b, j_b): $b \to b'$, (i_a', j_a'): $a \to a'$, and (i_b', j_b'): $b \to b'$ in close (R) such that

$$r_1 = (i_a, j_a) \times (i_b, j_b)$$

$$r_2 = (i_a', j_a') \times (i_b', j_b').$$

Because $(i_a, j_a) = (i_a', j_a')$ and $(i_b, j_b) = (i_b', {}_b')$ by the induction hypothesis, $r_1 = r_2$.

(3) When c is in the form of b^a, it is similar to (2).

Next we show that the preorder $\leqslant$ defined in Remarks 7.2 is a partial order relation. Suppose that close(R) has $r: c \to d$ and $r': d \to c$. By induction on the structure of d, we show that $c = d$.

If d is atomic, then by Lemma 7.6(iii), c is atomic and $r \in P$. Also $r' \in P$. From condition (5) of primitive set P, $c = d$.

If d is $a \times b$, then by Lemma 7.6(i), c is in the form $a' \times b'$ and there are (i_a, j_a): $a' \to a$ and (i_b, j_b): $b' \to b$ in close(R). Similarly there are (i_a', j_a'): $a \to a'$ and (i_b', j_b'): $b \to b'$ in close(R). By the induction hypothesis, $a = a'$ and $b = b'$. Hence $c = a' \times b' = a \times b = d$.

If d is b^a, then it is similar to the case $d = a \times b$.

Hence close(R) determines a retraction map category of **C** by Remarks 7.2. Moreover it follows from Lemma 7.6(i), (ii), that the generated retraction map category is canonical. ▌

8. D_∞ AND ε^*-CATEGORIES

We will construct D_∞ by the ε-category method. The ε-category method is generalization of Scott's inverse limit method used for the D_∞-construction of λ-calculus models. In order to construct D_∞ we will define a complete order-enriched ccc **SP** and a retraction map category **RSP** of **SP** such that the ε^*-category $\mathbf{E} = \mathbf{E^*(SP, RSP)}$ has a reflexive object V. We will show that $V = V^V$ and $D_\infty = \mathfrak{M}(E, V, [\text{ID}_V], [\text{ID}_V])$. Note that $I[V^V, V] = J[V^V, V] = \text{ID}_V$ because $V = V^V$.

Furthermore, extending **SP** and **RSP**, we will define a complete order-enriched ccc **WP** and a retraction map category **RWP** of **WP**, and show that the generated λ-algebra from **E*(WP, RWP)** is not λ-model (that is, not weakly extensional).

8.1. DEFINITION. (i) A *complete partially ordered set* (abbreviated to cpo) is a partially ordered set $(X, \leqslant)$ such that for every directed subset $Y \subset X$ there is the least upper bound $\bigsqcup Y \in X$.

(ii) Let X and Y be two cpo's. A function from X to Y is *continuous* iff for every directed subset $Z \subset X$, $\{f(z) \mid z \in Z\}$ is a directed subset of Y and

$$f\left(\bigsqcup_X Z\right) = \bigsqcup_Y \{f(z) \mid z \in Z\}.$$

8.2. LEMMA. *We define the category* **CPO** *with partial order* $\leqslant$ *among the arrows as follows*:

(a) *the objects of* **CPO** *are all cpo's*;

(b) *the arrows of* **CPO** *are all continuous functions among cpo's*; *and*

(c) *for each pair of* f, g: $a \to b$, $f \leqslant g$ *iff* $f(x) \leqslant g(x)$ *for all* $x \in a$.

Then, **CPO** *is a complete order-enriched ccc if we define a suitable structure for ccc.*

Proof. Easy. ▮

Rigorously speaking, we must slightly modify the definition of complete order-enriched ccc's because the collection of all cpo's is not a set but a proper class.

8.3. DEFINITION. Let p_0 be a cpo that has the bottom (denoted by $\perp$) and at least two distinct elements. We arbitrarily choose such p_0 and fix it. Then **SP** is the subcategory of **CPO** that is obtained from **CPO** by restricting the objects of **CPO** as follows: 1 (a singleton set) $\in |\mathbf{SP}|$; $p_0 \in |\mathbf{SP}|$; and if $a \in |\mathbf{SP}|$ and $b \in |\mathbf{SP}|$, then $a \times b \in |\mathbf{SP}|$ and $b^a \in |\mathbf{SP}|$. Here we use induction. We can assume that **SP** is canonical.

8.4. LEMMA. *Let* $\phi \in \mathbf{SP}(p_0, p_0^{p_0})$ *and* $\psi \in \mathbf{SP}(p_0^{p_0}, p_0)$ *be defined by*

$$\phi(x)(y) = x \qquad \text{for} \quad x, y \in p_0$$

$$\psi(f) = f(\perp) \qquad \text{for} \quad f \in p_0^{p_0}.$$

Then (ϕ, ψ): $p_0 \to p_0^{p_0}$ *is a retraction pair of* **SP**, *and* $P = \{\mathrm{id}_1, \mathrm{id}_1),$ $(\mathrm{id}_{p_0}, \mathrm{id}_{p_0})\}$ *and* $Q = \{(\phi, \psi)\}$ *satisfy the conditions of Theorem* 7.5. *So we can define the generated retraction map category* **RSP** *of* **SP** *from* $P \cup Q$.

Proof. Clear. ▮

8.5. PROPOSITION. *Let V be the set of objects of* **SP** *inductively defined as follows: $p_0 \in V$; and if $a \in V$ and $b \in V$, then $b^a \in V$. Then V is an object of* **E*(SP, RSP)** *and $V = V^V$. Moreover D_∞ is isomorphic to $\mathfrak{M}$(**E*(SP, RSP)**, V, $[\mathrm{ID}_V]$, $[\mathrm{ID}_V]$), where D_∞ is the λ-model constructed from p_0 and (ϕ, ψ) by means of Scott's inverse limit method.*

Proof. First we show that V is directed, namely, for every pair of a, $b \in V$ there exists $c \in V$ such that $a \leqslant c$ and $b \leqslant c$. We use induction on the structures of a and b. For every $d \in V$, $p_0 \leqslant d$, which is proved by induction on the structure of d. So it is clear when $a = p_0$ or $b = p_0$. Suppose that $a = a_2^{a_1}$, $b = b_2^{b_1}$, and a_1, a_2, b_1, $b_2 \in V$. Then, by the induction hypothesis, there are c_1, $c_2 \in V$ such that $a_1 \leqslant c_1$, $b_1 \leqslant c_1$, $a_2 \leqslant c_2$, and $b_2 \leqslant c_2$. Therefore $a \leqslant c_2^{c_1}$, $b \leqslant c_2^{c_1}$, and $c_2^{c_1} \in V$.

Next we show that $V = V\!\downarrow$. We prove that $a \leqslant b \in V$ implies $a \in V$, using induction on the structure of a. When a is atomic, $a = p_0 \in V$. Note that **SP** is a canonical order-enriched ccc. It is impossible that a is of the form $a_1 \times a_2$, because b is either atomic or of the form $b_2^{b_1}$ by the definition of V. Suppose $a = a_2^{a_1}$. Then b must be of the form $b_2^{b_1}$, and $a_1 \leqslant b_1$ and $a_2 \leqslant b_2$. By the definition of V, $b_1 \in V$ and $b_2 \in V$. Thus, by the induction hypothesis, $a_1 \in V$ and $a_2 \in V$, and so $a = a_2^{a_1} \in V$.

Hence V is really an object of **E(SP, RSP)**. It is clear that $V = V^V$.

The construction of $\mathfrak{M}$(**E*(SP, RSP)**, V, $[\mathrm{ID}_V]$, $[\mathrm{ID}_V]$) is the same as D_∞. (The detailed proof is omitted, because the precise construction of D_∞ is needed.) ∎

8.6. PROPOSITION. *We define the category* **WCPO** *equipped with partial order $\leqslant$ among the arrows as follows:*

(a) *The objects of* **WCPO** *are all pairs of cpo's a and a' such that $a \subset a'$ and $\bigsqcup_a X = \bigsqcup_{a'} X \in a$ for every directed subset $X \subset a$.*

(b) *The arrows from (a, a') to (b, b') are all continuous functions f from a' to b' such that $f(x) \in b$ for every $x \in a$.*

(c) *For each pair of arrows f, g: $(a, a') \to (b, b')$, $f \leqslant g$ iff $f(x) \leqslant g(x)$ for every $x \in a'$.*

Then **WCPO** *is a complete order-enriched ccc with a suitable structure for ccc.*

Proof. We define the structure for ccc in **WCPO**:

(1) The terminal object is $(1, 1)$, where 1 is an arbitrary singleton set. For each object a, $!_{(a,a')}(x) = z_0$ for $x \in a'$, where $\{z_0\} = 1$.

(2) For each pair of objects (a, a') and (b, b'),

$$(a, a') \times (b, b') = (a \times b, a' \times b'),$$

$$\pi_1^{(a,a'),(b,b')}(\langle x, y \rangle) = x,$$

$$\pi_2^{(a,a'),(b,b')}(\langle x, y \rangle) = y \qquad \text{for} \quad \langle x,y \rangle \in a' \times b'.$$

For each pair of arrows $f\colon (c, c') \to (a, a')$ and $g\colon (c, c') \to (b, b')$, $\langle f, g \rangle\colon (c, c') \to (a \times b, a' \times b')$ is the same arrow as in **CPO**.

(3) For each pair of objects (a, a') and (b, b'),

$$(b, b')^{(a,a')} = (\alpha, b'^{a'}),$$

$$\mathrm{ev}^{(a,a'),(b,b')}(\langle f, x \rangle) = f(x) \qquad \text{for} \quad f \in b'^{a'} \text{ and } x \in a',$$

where α is the set of all continuous functions f from a' to b' such that $f(x) \in b$ for every $x \in a$. For each arrow $f\colon (c, c') \times (a, a') \to (b, b')$, $\Lambda(f)\colon (c, c') \to (b, b')^{(a,a')}$ is the same as in **CPO**.

It is clear that the above structure satisfies the conditions of complete order-enriched ccc. ∎

8.7. Definition. Let (p_0, p_0') be a pair of nonempty cpo's that satisfies the following conditions:

(1) (p_0, p_0') is an object of **WCPO**. That is, $p_0 \subset p_0'$ and $\bigsqcup_{p_0} X = \bigsqcup_{p_0'} X \in p_0$ for every directed subset $X \subset p_0$.

(2) p_0 and p_0' have the same least element.

(3) There is a pair of continuous functions $f, g\colon p_0' \to p_0'$ such that $f \neq g$ and $f(x) = g(x)$ for every $x \in p_0$.

We arbitrarily choose such p_0 and p_0' and fix them. For example, if we set $p_0 = \{\bot\}$ and $p_0' = \{\bot, \top\}$ $(\bot < \top)$, then (p_0, p_0') satisfies the above conditions.

Then we define **WP** as the subcategory of **WCPO** obtained from **WCPO** by restricting the objects as follows:

$$(1, 1) \in |\mathbf{WP}|, \qquad (p_0, p_0') \in |\mathbf{WP}|,$$

if $(a, a') \in |\mathbf{WP}|$ and $(b, b') \in |\mathbf{WP}|$ then $(a, a') \times (b, b') \in |\mathbf{WP}|$ and $(b, b')^{(a,a')} \in |\mathbf{WP}|$.

8.8. Lemma. *We define two arrows* $\phi'\colon (p_0, p_0') \to (p_0, p_0')^{(p_0, p_0')}$ *and* $\psi'\colon (p_0, p_0')^{(p_0, p_0')} \to (p_0, p_0')$ *in* **WP** *by*

$$\phi'(x)(y) = x \qquad \text{for} \quad x, y \in p_0',$$

$$\psi'(f) = f(\bot) \qquad \text{for} \quad f \in p_0'^{p_0'}.$$

Then $(\phi', \psi'): (p_0, p'_0) \to (p_0, p'_0)^{(p_0, p'_0)}$ *is a retraction pair, and* $P = \{(\mathrm{id}_{(1,1)}, \mathrm{id}_{(1,1)}), (\mathrm{id}_{(p_0, p'_0)}, \mathrm{id}_{(p_0, p'_0)})\}$ *and* $Q = \{(\phi', \psi')\}$ *satisfy the conditions of Theorem* 7.5. *So we can define the generated retraction map category* **RWP** *of* **WP** *from* $P \cup Q$.

Proof. Clear. ∎

8.9. Proposition. *Let* V *be the set of objects of* **WP** *inductively defined as follows:* $(p_0, p'_0) \in V$; *if* $(a, a') \in V$ *and* $(b, b') \in V$ *then* $(b, b')^{(a, a')} \in V$. *Then* V *is an object of* **E*(WP, RWP)** *and* $V = V^V$. *Moreover the* λ-*algebra* $\mathfrak{M}$(**E*(WP, RWP)**, V, $[\mathrm{ID}_V]$, $[\mathrm{ID}_V]$) *is not a* λ-*model.*

Proof. Let $\mathfrak{M} = \mathfrak{M}$(**E*(WP, RWP)**, V, $[\mathrm{ID}_V]$, $[\mathrm{ID}_V]$). Suppose $\mathfrak{M}$ is a λ-model. Then, by Fact 6.3, **E*(WP, RWP)** has enough points at V w.r.t. V. Because all retraction pairs in $P \cup Q$ defined in Lemma 8.8 are upper injective, **RWP** is also upper injective. By Proposition 6.5, **WP** has enough points at (a, a') w.r.t. (b, b') for every pair of (a, a'), $(b, b') \in V$. Especially we set $(a, a') = (b, b') = (p_0, p'_0)$. For every pair of arrows f, g: $(p_0, p'_0) \to (p_0, p'_0)$, if $f \neq g$, then there exists an arrow h: $(1, 1) \to (p_0, p'_0)$ in **WP** such that $f \circ h \neq g \circ h$. That is, there exists $x \in p_0$ such that $f(x) \neq g(x)$. This contradicts condition (3) in Definition 8.7. Hence $\mathfrak{M}$ is not a λ-model. ∎

9. Reconstruction of P_ω

In this section we will examine the relation between P_ω and ε-categories. We will reconstruct P_ω under the ε-category method. First we will define a complete order-enriched ccc **PFN**. Roughly speaking, **PFN** consists of $\{x_1, ..., x_n\}$ as objects and continuous functions on them, where $x_1, ..., x_n$ are finite subsets of ω. Next we will define a retraction map category **RFN** using Theorem 7.5. Finally we will show that **EFN** = **E(PFN, RFN)** has a reflexive object and that the generated λ-model from **EFN** with the reflexive object is isomorphic to P_ω.

9.1. Definition. We define some notations about the set ω of all natural numbers:

(i) We define *PFN* as the set of all $a = \{x_1, ..., x_n\}$, where $0 \leqslant n$, each x_i is a finite subset of ω, and a satisfies the condition: $\bigcup b \in a$ for any subset $b \subset a$. Note that each element of *PFN* is a cpo under the set inclusion $\subset$ as the partial order relation.

(ii) For $m, n \in \omega$,

$$\langle m, n \rangle = \tfrac{1}{2}(m+n)(m+n+1) + n.$$

(iii) For $n \in \omega$, $e_n = \{k_0, k_1, \ldots, k_{m-1}\}$ with

$$k_0 < k_1 < \cdots < k_{m-1} \quad \text{iff} \quad n = \sum_{i=0}^{m-1} 2^{k_i}.$$

(iv) For finite subsets $x, y \subset \omega$,

$$\langle x, y \rangle = \{\langle m, n \rangle \mid m \in x \text{ and } n \in y\}.$$

9.2. Definition. Let $p, q \in PFN$:

(i) For continuous function $f \colon p \to q$,

$$\text{graph}^{p,q}(f) = \{\langle m, n \rangle \mid e_n \in p \text{ and } m \in f(e_n)\}.$$

(ii) $FS(p, q) = \{z \mid (\exists f)$ (f is a continuous function from p to q and $z \subset \text{graph}^{p,q}(f))\}$.

(iii) For each $z \in FS(p, q)$, $\text{fun}^{p,q}(z)$ is the continuous function from p to q defined by

$$\text{fun}^{p,q}(z)(x) = \{m \mid (\exists e_n \subset x)(\langle m, n \rangle \in z)\}$$

for $x \in p$.

(iv) $PS(p, q) = \{\langle x, y \rangle \mid x \in p \text{ and } y \in q\}$.

Note that

$$FS(p, q) = \{z \mid z \subset \{\langle m, n \rangle \mid m \in \bigcup q \text{ and } e_n \in p\}\} \in PFN.$$

The above $\text{graph}^{p,q}$ and $\text{fun}^{p,q}$ come from graph and fun which appear in the P_ω-construction. As well as graph and fun in P_ω, $\text{graph}^{p,q}$ and $\text{fun}^{p,q}$ have the following properties:

(1) $\text{fun}^{p,q}(\text{graph}^{p,q}(f)) = f$, for every continuous function $f \colon p \to q$; and

(2) $\text{graph}^{p,q}(\text{fun}^{p,q}(z)) \supset z$, for every $z \in FS(p, q)$.

9.3. Definition. We define the canonical complete order-enriched ccc **PFN**:

(a) The objects of **PFN** are syntactically defined as follows:

(1) for every $p \in PFN$, c_p is an object of **PFN**, and

(2) if a and b are objects of **PFN** then $(a \times b)$ and (b^a) are also objects of **PFN**.

(b) For each object a of **PFN**, we define $v(a) \in PFN$ as follows:

(1) $v(c_p) = p$,

(2) $v(a \times b) = PS(v(a), v(b))$ $(= \{\langle x, y\rangle \mid x \in v(a) \text{ and } y \in v(b)\})$,

and

(3) $v(b^a) = \{\text{graph}^{v(a),v(b)}(f) \mid f$ is a continuous function from $v(a)$ to $v(b)\}$.

For each pair of objects a and b of **PFN**, the arrows from a to b in **PFN** are all continuous functions from $v(a)$ to $v(b)$.

(c) We define the structure of order-enriched ccc in **PFN**. Let a, b, and c be objects of **PFN**:

(1) For each pair of arrows f, $g \in \mathbf{PFN}(a, b)$,

$$f \leqslant g \qquad \text{iff} \qquad f(x) \subset g(x) \text{ for all } x \in v(a).$$

(2) $1 = c_{\{\varnothing\}}$ and $!_a \in \mathbf{PFN}(a, 1)$ is defined by

$$!_a(x) = \varnothing \qquad \text{for} \quad x \in v(a).$$

(3) $\pi_1^{a,b} \in \mathbf{PFN}(a \times b, a)$ and $\pi_2^{a,b} \in \mathbf{PFN}(a \times b, b)$ are defined by

$$\pi_1^{a,b}(\langle x, y\rangle) = x \qquad \text{and} \qquad \pi_2^{a,b}(\langle x, y\rangle) = y$$
$$\text{for} \quad \langle x, y\rangle \in v(a \times b).$$

For each pair of $f \in \mathbf{PFN}(c, a)$ and $g \in \mathbf{PFN}(c, b)$, $\langle f, g\rangle \in \mathbf{PFN}(c, a \times b)$ is defined by

$$\langle f, g\rangle(z) = \langle f(z), g(z)\rangle \qquad \text{for} \quad x \in v(c).$$

(4) $\text{ev}^{a,b} \in \mathbf{PFN}(b^a \times a, b)$ is defined by

$$\langle f, g\rangle(z) = \langle f(z), g(z)\rangle \qquad \text{for} \quad x \in v(c).$$

(4) $\text{ev}^{a,b} \in \mathbf{PFN}(b^a \times a, b)$ is defined by

$$\text{ev}^{a,b}(\langle \text{graph}^{v(a),v(b)}(f), x\rangle) = f(x)$$
$$\text{for} \quad \langle \text{graph}^{v(a),v(b)}(f), x\rangle \in v(b^a \times a).$$

(This is well defined, because $\text{graph}^{p,q}$ is one-to-one by the remark (1) below Definition 9.2.) For each $f \in \mathbf{PFN}(c \times a, b)$, $\Lambda(f) \in \mathbf{PFN}(c, b^a)$ is defined by

$$\Lambda(f)(z) = \text{graph}^{v(a),v(b)}(k_z) \qquad \text{for} \quad z \in v(c),$$

where k_z is the continuous function from $v(a)$ to $v(b)$ defined by

$$k_z(x) = f(\langle z, x \rangle) \qquad \text{for} \quad x \in v(a).$$

9.4. LEMMA. **PFN** *is really a canonical complete order-enriched ccc.*

Proof. Clear. Here note that

$$f \leqslant g \qquad \text{iff} \qquad \text{graph}^{v(a),v(b)}(f) \subset \text{graph}^{v(a),v(b)}(g). \quad \blacksquare$$

9.5. DEFINITION. (i) We define

$$P = \{(i, j)\colon c_p \to c_q \mid p, q \in PFN \text{ and } p \subset q\},$$

where $(i, j)\colon c_p \to c_q$ is the retraction pair in **PFN** defined by

$$i(x) = x \qquad \text{for} \quad x \in p,$$

$$j(y) = \bigcup \{z \in p \mid z \subset y\} \qquad \text{for} \quad y \in q.$$

(ii) We define

$$Q_1 = \{(i, j)\colon c_{PS(p,q)} \to c_p \times c_q \mid p, q \in PFN\},$$

where $(i, j)\colon c_{PS(p,q)} \to c_p \times c_q$ *is the retraction pair in* **PFN** *defined by*

$$i(\langle x, y \rangle) = \langle x, y \rangle = j(\langle x, y \rangle)$$

for each $\langle x, y \rangle \in PS(p, q)(= v(c_p \times c_q))$.

(iii) We define

$$Q_2 = \{(i, j)\colon c_{FS(p,q)} \to (c_q)^{c_p} \mid p, q \in PFN\},$$

where $(i, j)\colon c_{FS(p,q)} \to (c_q)^{c_p}$ is the retraction pair in **PFN** defined by

$$i(z) = \text{graph}^{p,q}(\text{fun}^{p,q}(z)) \qquad \text{for} \quad z \in FS(p, q),$$

$$j(z') = z' \qquad \text{for} \quad z' \in v((c_q)^{c_p}).$$

(Note that $j \circ i \geqslant \text{id}_{c_{FS(p,q)}}$ and $i \circ j = \text{id}_{(c_q)^{c_p}}$ by the remarks (1) and (2) below Definition 9.2.)

9.6. PROPOSITION. *Let $Q = Q_1 \cup Q_2$. Then P and Q satisfy the conditions in Theorem 7.5. So we can define the generated retraction map category* **RFN** *of* **PFN** *from $P \cup Q$ by Theorem 7.5.*

Proof. We will show that condition (2) in Theorem 7.5 is satisfied. The rest is clear.

Let p, q, p', $q' \in PFN$ be given, and suppose $p \subset p'$ and $q \subset q'$. We define the objects a, a', b, b', c, and c' of **PFN** by $a = c_p$, $a' = c_{p'}$, $b = c_q$, $b' = c_{q'}$, $c = c_{FS(p,q)}$, and $c' = c_{FS(p',q')}$. Then clearly $FS(p, q) \subset FS(p', q')$. There are three retraction pairs uniquely defined in P:

$$(i_a, j_a)\colon a \to a',$$
$$(i_b, j_b)\colon b \to b',$$
$$(i_c, j_c)\colon c \to c',$$

and two retraction pairs uniquely defined in Q:

$$(i, j)\colon c \to b^a,$$
$$(i', j')\colon c' \to b'^{a'}.$$

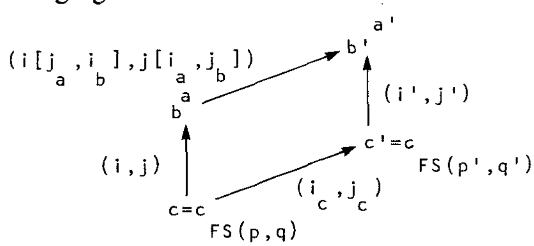
We define

$$(i[j_a, i_b], j[i_a, j_b]) = (i_b, j_b)^{(i_a, j_a)}$$
$$(= (\Lambda(i_b \circ \mathrm{ev}^{a,b} \circ (\mathrm{id}_{b^a} \times j_a)), \Lambda(j_b \circ \mathrm{ev}^{a',b'} \circ (\mathrm{id}_{b'^{a'}} \times i_a))).$$

Then we must show that

$$i[j_a, i_b] \circ i = i' \circ i_c$$
$$j \circ j[i_a, j_b] = j_c \circ j',$$

which the following figure illustrates:



We will show the first equation $i[j_a, i_b] \circ i = i' \circ i_c$. Let $z \in FS(p, q) \subset FS(p', q')$. It can be proved that

$$(i[j_a, i_b] \circ i)(z) = \mathrm{graph}^{p',q'}(i_b \circ \mathrm{fun}^{p,q}(z) \circ j_a)$$
$$(i' \circ i_c)(z) = \mathrm{graph}^{p',q'}(\mathrm{fun}^{p',q'}(z)).$$

For every $x' \in p'$,

$$(i_b \circ \mathrm{fun}^{p,q}(z) \circ j_a)(x') = \{m \mid (\exists e_n \subset j_a(x'))(\langle m, n \rangle \in z)\}$$

and

$$\mathrm{fun}^{p',q'}(z)(x') = \{m \mid (\exists e_n \subset x')(\langle m, n \rangle \in z)\}.$$

If $\langle m, n\rangle \in z$, then $e_n \in p$ $(\subset p')$. For every $e_n \in p \subset p'$,

$$e_n \subset j_a(x') \left(= \bigcup \{x \in p \mid x \subset x'\}\right) \quad \text{iff} \quad e_n \subset x'$$

Thus, for all $x' \in p'$,

$$(i_b \circ \mathrm{fun}^{p,q}(z) \circ j_a)(x') = \mathrm{fun}^{p',q'}(z)(x').$$

Therefore $i[j_a, i_b] \circ i = i' \circ i_c$.

Next we will show that $j \circ j[i_a, j_b] = j_c \circ j'$. Let $\mathrm{graph}^{p',q'}(f') \in v(b'^{a'})$ Then, we can prove the following:

$$\begin{aligned}&(i \circ j[i_a, j_b])(\mathrm{graph}^{p',q'}(f'))\\ &\quad = \mathrm{graph}^{p,q}(j_b \circ f' \circ i_a)\\ &\quad = \left\{\langle m, n\rangle \,\middle|\, e_n \in p \text{ and } m \in \bigcup \{y \in q \mid y \subset f'(e_n)\}\right\}\end{aligned}$$

and

$$\begin{aligned}&(j_c \circ j')(\mathrm{graph}^{p',q'}(f'))\\ &\quad = j_c(\mathrm{graph}^{p',q'}(f))\\ &\quad = \bigcup \{z \in FS(p, q) \mid z \subset \mathrm{graph}^{p',q'}(f')\}\\ &\quad = \bigcup \{\mathrm{graph}^{p,q}(f) \mid f \text{ is a continuous function from } p \text{ to } q\\ &\qquad\qquad \text{and } \mathrm{graph}^{p,q}(f) \subset \mathrm{graph}^{p',q'}(f')\}.\end{aligned}$$

Because $(j_{b'} \circ f' \circ i_a)(x) \subset f'(x)$ for every $x \in p$,

$$\mathrm{graph}^{p,q}(j_b \circ f' \circ i_a) \subset \mathrm{graph}^{p',q'}(f').$$

So $\mathrm{graph}^{p,q}(j_b \circ f' \circ i_a) \subset (j_c \circ j')(\mathrm{graph}^{p',q'}(f'))$.

Conversely let $\langle m, n\rangle \in (j_c \circ j')(\mathrm{graph}^{p',q'}(f'))$ be given. Then there is a continuous function f from p to q such that

$$\langle m, n\rangle \in \mathrm{graph}^{p,q}(f) \subset \mathrm{graph}^{p',q'}(f').$$

Because $e_n \in p$, $m \in f(e_n) \in q$ and $f(e_n) \subset f'(e_n)$,

$$\langle m, n\rangle \in \mathrm{graph}^{p,q}(j_b \circ f' \circ i_b).$$

Hence $\mathrm{graph}^{p,q}(j_b \circ f' \circ i_a) \subset (j_c \circ j')(\mathrm{graph}^{p',q'}(f'))$. Since $\mathrm{graph}^{p',q'}(f')$ is arbitrary, we conclude that

$$i \circ j[i_a, j_b] = j_c \circ j'. \quad \blacksquare$$

9.7. PROPOSITION. *Let* **EFN** = **E(PFN, RFN)**. *We define U as the set of all* $c_p \in |\mathbf{PFN}|$, *where* $p \in PFN$. *Then U is an object of* **EFN**, $U \subset U^U$ *and* $I[U, U^U] \circ J[U, U^U] = \mathrm{ID}_{U^U}$. *Moreover* P_ω *and* $\mathfrak{M} = \mathfrak{M}(\mathbf{EFN}, U, I[U, U^U], J[U, U^U])$ *are isomorphic. That is, there is a bijective function from* P_ω *onto* $\mathfrak{M}$ *that preserves the application* $\cdot$, *and the elements k and s. (The definition of isomorphism between two λ-algebras appears in Barendregt* (1984).)

Proof. It is clear that U is really an object of **EFN**. We show that $U \subset U^U$. Let $d \in U$. Then $d = c_r$ for some $r \in PFN$. There are finite subsets x, $Y \subset \omega$ such that

$$\bigcup r = \{\langle m, n\rangle \mid m \in y \text{ and } n \in x\},$$

since $\langle -, - \rangle$ is a one-to-one map from $\omega \times \omega$ onto ω.

Define

$$p = \left\{ z \,\middle|\, z \subset \bigcup \{e_n \mid n \in x\} \right\}$$

$$q = \{z \mid z \subset y\}.$$

Then $p \in PFN$ and $q \in PFN$. For every $v \in r$,

$$v \subset \bigcup r \subset \left\{ \langle m, n\rangle \,\middle|\, m \in \bigcup q \text{ and } e_n \in p \right\}$$

and so

$$v \in \left\{ z \,\middle|\, z \subset \left\{ \langle m, n\rangle \mid m \in \bigcup q \text{ and } e_n \in p \right\} \right\} = FS(p, q).$$

Hence $r \subset FS(p, q)$. Therefore $c_r \leqslant c_{FS(p,q)} \leqslant (c_q)^{c_p} \in U^U$ and $c_r \in U^U$. Since $d = c_r$ is arbitrary, $U \subset U^U$.

Next we show that $I[U, U^U] \circ J[U, U^U] = \mathrm{ID}_{U^U}$. Because all the retraction pairs in $P \cup Q_1 \cup Q_2$ defined in Definition 9.5 are lower injective, the generated retraction map category **RFN** is lower injective, too. Here we use Lemma 2.3. By Lemma 5.2(vii), $(I[U, U^U], J[U, U^U])$ is lower injective, namely, $I[U, U^U] \circ J[U, U^U] = \mathrm{ID}_{U^U}$.

For the proof that P_ω and $\mathfrak{M}$ are isomorphic, we need the precise construction of $\mathfrak{M}$, the definition of P_ω as a λ-model, and the definition of isomorphisms between two λ-algebras, which appear in Barendregt (1984). We only go through the key points of the proof.

First we will show that $\mathbf{EFN}(1, U)$ and $\mathbf{EFN}(U, U)$ correspond to P_ω and $[P_\omega \to P_\omega]$, respectively, where $[P_\omega \to P_\omega]$ means the set of all continuous functions from P_ω to P_ω. We define the pair of functions $K\colon \mathbf{EFN}(1, U) \to P_\omega$ and $L\colon \mathbf{EFN}(U, U) \to [P_\omega \to P_\omega]$ by

$$K(H) = \bigcup \{h(\varnothing) \mid h \in H\} \qquad \text{for } H \in \mathbf{EFN}(1, U),$$

$$L(F)(x) = \bigcup \{f(y) \mid y \subset x,\, f \in F, \text{ and } (\exists p \in PFN)(\mathrm{dom}(f) = c_p \text{ and } y \in p)\} \quad \text{for } F \in \mathbf{EFN}(U, U) \text{ and } x \in P_\omega.$$

Clearly K and L are bijective. Indeed the following K^{-1} and L^{-1} are the inverse functions of K and L, respectively:

$$K^{-1}(x) = \{h \mid (\exists p \in PFN)(h \in \mathbf{PFN}(1, c_p) \text{ and } h(\varnothing) \subset x)\} \quad \text{for } x \in P_\omega,$$

$$L^{-1}(k) = \{f \mid (\exists p, q \in PFN)(f \in \mathbf{PFN}(c_p, c_q) \text{ and } (\forall x \in p)(f(x) \subset k(x))\} \qquad \text{for} \quad k \in [P_\omega \to P_\omega].$$

Moreover K and L preserve the order and satisfy the property: $K(F \circ H) = L(F)(K(H))$ for every pair of $F \in \mathbf{EFN}(U, U)$ and $H \in \mathbf{EFN}(1, U)$. These can be shown by simple calculation.

Next we define the pair of functions $\Phi\colon \mathbf{EFN}(1, U) \to \mathbf{EFN}(U, U)$ and $\Psi\colon \mathbf{EFN}(U, U) \to \mathbf{EFN}(1, U)$ by

$$\Phi(H) = EV^{U,U} \circ \langle I[U, U^U] \circ H \circ !_U, ID_U \rangle \qquad \text{for} \quad H \in \mathbf{EFN}(1, U),$$

$$\Psi(F) = J[U, U^U] \circ \Lambda(F \circ \Pi_2^{1,U}) \qquad \text{for} \quad F \in \mathbf{EFN}(U, U).$$

Then Φ and Ψ correspond to graph and fun in P_ω, respectively. Here we repeat the definitions of fun and graph in P_ω:

$$\mathrm{fun}(z)(x) = \{m \mid (\exists e_n \subset x)(\langle m, n \rangle \in z)\} \qquad \text{for} \quad z, x \in P_\omega,$$

$$\mathrm{graph}(f) = \{\langle m, n \rangle \mid m \in f(e_n)\} \qquad \text{for} \quad f \in [P_\omega \to P_\omega].$$

We will show that

$$L(\Phi(H)) = \mathrm{fun}(K(H)) \qquad \text{for every } H \in \mathbf{EFN}(1, U),$$

$$K(\Psi(F)) = \mathrm{graph}(L(F)) \qquad \text{for every } F \in \mathbf{EFN}(U, U).$$

Let $H \in \mathbf{EFN}(1, U)$ and $x \in P_\omega$. If $y \in p \in PFN$, then we can prove that

$$\bigcup \{f(y) \mid f \in \Phi(H) \text{ and } \mathrm{dom}(f) = c_p\}$$

$$= \bigcup \{(\mathrm{ev}^{c_p, c_q} \circ \langle i[c_{FS(p,q)}, (c_q)^{c_p}] \circ h \circ !_{c_p}, \mathrm{id}_{c_p}\rangle)(y) \mid q \in PFN \text{ and } h \in H(1, c_{FS(p,q)})\}$$

$$= \bigcup \{\mathrm{fun}^{p,q}(h(\varnothing))(y) \mid q \in PFN \text{ and } h \in 1, c_{FS(p,q)})\}.$$

Thus

$$\begin{aligned}
&L(\Phi(H))(x)\\
&= \bigcup \{f(y) \mid y \subset x, f \in \Phi(H), \text{ and } (\exists p \in PFN)(\mathrm{dom}(f) = c_p \text{ and } y \in p)\}\\
&= \bigcup \{\mathrm{fun}^{p,q}(h(\varnothing))(y) \mid p, q \in PFN, y \in p, y \subset x, \text{ and } h \in H(1, c_{FS(p,q)})\}\\
&= \{m \mid (\exists p, q \in PFN)(\exists h \in H(1, c_{FS(p,q)}))(\exists e_n \in p)(e_n \subset x \text{ and } \langle m, n\rangle \in h(\varnothing))\}\\
&= \{m \mid (\exists e_n \subset x)(\exists h \in H)(\langle m, n\rangle \in h(\varnothing))\}\\
&= \{m \mid (\exists e_n \subset x)(\langle m, n\rangle \in K(H))\}\\
&= \mathrm{fun}(K(H)).
\end{aligned}$$

On the other hand, if $F \in \mathbf{EFN}(U, U)$, then

$$\begin{aligned}
K(\Psi(F)) &= \bigcup \{h(\varnothing) \mid h \in \Psi(F)\}\\
&= \bigcup \{(j[c_{FS(p,q)}, (c_q)^{c_p}] \circ \Lambda(f \circ \pi_2^{1,c_p}))(\varnothing) \mid p, q \in PFN \text{ and } f \in F(c_p, c_q)\}\\
&= \bigcup \{\mathrm{graph}^{p,q}(f) \mid p, q \in PFN \text{ and } f \in F(c_p, c_q)\}\\
&= \{\langle m, n\rangle \mid (\exists p \in PFN)(\exists f \in F)(\mathrm{dom}(f) = c_p, e_n \in p, \text{ and } m \in f(e_n))\}\\
&= \{\langle m, n\rangle \mid m \in L(F)(e_n)\}\\
&= \mathrm{graph}(L(F)).
\end{aligned}$$

From the above arguments we can conclude that P_ω and $\mathfrak{M}$ are isomorphic. ▮

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