J. Math. Anal. Appl. 348 (2008) 404–410

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On weak type inequalities for dyadic maximal functions

Antonios D. Melas *, Eleftherios Nikolidakis

Department of Mathematics, University of Athens, Panepistimiopolis 15784, Athens, Greece

ARTICLE INFO

Article history:
Received 6 March 2008
Available online 9 July 2008
Submitted by L. Grafakos

Keywords:
Dyadic maximal
Distribution function

ABSTRACT

We obtain sharp estimates for the localized distribution function of the dyadic maximal function $M_d\phi$, given the local L^1 norms of ϕ and of $G \circ \phi$ where G is a convex increasing function such that $G(x)/x \to +\infty$ as $x \to +\infty$. Using this we obtain sharp refined weak type estimates for the dyadic maximal operator.

1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is defined by

$$M_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|\,du:\ x \in Q,\ \ Q \subseteq \mathbb{R}^n \text{ is a dyadic cube}\right\} \tag{1.1}$$

for every $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ where the dyadic cubes are the cubes formed by the grids $2^{-N}\mathbb{Z}^n$ for $N = 0, 1, 2, \ldots.$

As it is well known it satisfies the following weak type $(1,1)$ inequality

$$\left|\left\{x \in \mathbb{R}^n:\ M_d\phi(x) > \lambda\right\}\right| \leqslant \frac{1}{\lambda}\int_{\{M_d\phi>\lambda\}} |\phi(u)|\,du \tag{1.2}$$

for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$ from which it is easy to get the following L^p inequality

$$\|M_d\phi\|_p \leqslant \frac{p}{p-1}\|\phi\|_p \tag{1.3}$$

for every $p > 1$ and every $\phi \in L^p(\mathbb{R}^n)$ which is best possible (see [1,2] for the general martingales and [13] for dyadic ones).

An approach for studying such maximal operators is the introduction of the so-called Bellman functions (see [5]) related to them which reflect certain deeper properties of them by localizing. Such functions related to the L^p inequality (1.3) have been precisely evaluated in [3]. Actually defining for any $p > 1$,

$$\mathcal{B}_p(F, f, L) = \sup\left\{\frac{1}{|Q|}\int_Q (M_d\phi)^p:\ \mathrm{Av}_Q\left(\phi^p\right) = F,\ \mathrm{Av}_Q(\phi) = f,\ \sup_{R:Q\subseteq R}\mathrm{Av}_R(\phi) = L\right\} \tag{1.4}$$

* Corresponding author.
E-mail address: amelas@math.uoa.gr (A.D. Melas).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.07.001

where Q is a fixed dyadic cube, R runs over all dyadic cubes containing Q, ϕ is nonnegative in $L^p(Q)$ and the variables F, f, L satisfy $0 \leqslant f \leqslant L$, $f^p \leqslant F$ which is independent of the choice of Q (so we may take $Q = [0,1]^n$) it has been shown in [3] that

$$\mathcal{B}_p(F, f, L) = \begin{cases} F\omega_p\left(\frac{pL^{p-1}f-(p-1)L^p}{F}\right)^p & \text{if } L < \frac{p}{p-1}f, \\ L^p + \left(\frac{p}{p-1}\right)^p(F - f^p) & \text{if } L \geqslant \frac{p}{p-1}f, \end{cases} \tag{1.5}$$

where $\omega_p : [0,1] \to [1, \frac{p}{p-1}]$ is the inverse function of $H_p(z) = -(p-1)z^p + pz^{p-1}$. Actually this has been shown in a much more general setting of tree like maximal operators on probability spaces and the corresponding Bellman function is always the same. Also in [4] Bellman functions related to local L^p, L^q inequalities have been determined, which turned out to be considerably more complicated than those in (1.5).

For more information and results on the Bellman approach we refer to [5–7] and for exact determinations of various Bellman functions (which usually is a difficult task) see [1–3,8–12].

One may look at (1.4) as an extremum problem which reflects the deeper structure of the dyadic maximal function since it encodes information not only about the size of the function but also a measure of its variance. In this spirit we will study here a corresponding extremum problem for the standard weak-L^p quasi-norms. Therefore we define

$$\mathcal{B}_{p,\infty}(F, f, L) = \sup\left\{ \frac{1}{|Q|}\|M_d\phi\|^p_{L^{p,\infty}(Q)}\colon\ \mathrm{Av}_Q(\phi^p) = F,\ \mathrm{Av}_Q(\phi) = f,\ \sup_{R:Q\subseteq R} \mathrm{Av}_R(\phi) = L \right\} \tag{1.6}$$

where $\|M_d\phi\|_{L^{p,\infty}(Q)} = \sup\{\lambda|\{M_d\phi \geqslant \lambda\} \cap Q|^{1/p}\colon\ \lambda > 0\}$ is the corresponding local weak-L^p quasi-norm. In this note we will among other things explicitly compute the above function.

Actually as in [3] we will take the more general approach of defining Bellman functions with respect to the maximal operator on a nonatomic probability space (X, μ) equipped with a tree $\mathcal{T}$ (see Definition 1.1). Then we can define the maximal operator associated to $\mathcal{T}$ as follows

$$M_{\mathcal{T}}\phi(x) = \sup\left\{ \frac{1}{\mu(I)} \int_I |\phi|\, d\mu\colon\ x \in I \in \mathcal{T} \right\} \tag{1.7}$$

for every $\phi \in L^1(X, \mu)$. The above maximal operator is related to the theory of martingales and satisfies essentially the same inequalities as M_d.

Next we let $G : [0, +\infty) \to [0, +\infty)$ be a strictly convex and increasing function and such that $\lim_{x\to+\infty} \frac{G(x)}{x} = +\infty$ and for any f, F, λ such that $0 < f < \lambda$, $G(f) < F$ we define

$$\mathcal{D}_G(\lambda, f, F) = \sup\left\{ \mu\big(\{M_{\mathcal{T}}\phi \geqslant \lambda\}\big)\colon\ \phi \geqslant 0,\ \phi \in L^1(X, \mu),\ \int_X \phi\, d\mu = f,\ \int_X G\circ\phi\, d\mu = F \right\}. \tag{1.8}$$

Then we will in Theorem 1 find the exact form of the above function. This gives the best possible behavior of the distribution function of the maximal operator and can be thought of as a sharp refinement of the classical weak type inequality (1.2).

Using this we will then solve corresponding to (1.6) local extremum problems but with the more general functional $\sup\{H(\lambda)\mu(\{M_d\phi \geqslant \lambda\})\colon\ \lambda > 0\}$ where H is another convex function (in a sense at most as strong as G) and then we will use this to find the solution of extremal problems like (1.6) but with mixed norms.

A common feature in all those computations is that the corresponding functions are independent from the particular tree $\mathcal{T}$ used, and therefore we have suppressed the $\mathcal{T}$ from them.

2. The main result

As in [3] we will let (X, μ) be a nonatomic probability space (i.e. $\mu(X) = 1$). Two measurable subsets A, B of X will be called almost disjoint if $\mu(A \cap B) = 0$. Then we give the following.

Definition 1. A set $\mathcal{T}$ of measurable subsets of X will be called a tree if the following conditions are satisfied:

(i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have $\mu(I) > 0$.
(ii) For every $I \in \mathcal{T}$ there corresponds a finite subset $\mathcal{C}(I) \subseteq \mathcal{T}$ containing at least two elements such that:
  (a) the elements of $\mathcal{C}(I)$ are pairwise almost disjoint subsets of I,
  (b) $I = \bigcup \mathcal{C}(I)$.
(iii) $\mathcal{T} = \bigcup_{m\geqslant 0} \mathcal{T}_{(m)}$ where $\mathcal{T}_{(0)} = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup_{I\in\mathcal{T}_{(m)}} \mathcal{C}(I)$.
(iv) We have $\lim_{m\to\infty} \sup_{I\in\mathcal{T}_{(m)}} \mu(I) = 0$.

The elements of such a tree $\mathcal{T}$ behave in a similar to the dyadic cubes manner, in particular if the intersection of two elements of $\mathcal{T}$ has positive measure then one is contained in the other. For more details as well as for a proof of the following lemma we refer to [3].

Lemma 1. *For every $I \in \mathcal{T}$ and every α such that $0 < \alpha < 1$ there exists a subfamily $\mathcal{F}(I) \subseteq \mathcal{T}$ consisting of pairwise almost disjoint subsets of I such that*

$$\mu\Bigl(\bigcup_{J\in\mathcal{F}(I)} J\Bigr) = \sum_{J\in\mathcal{F}(I)} \mu(J) = (1-\alpha)\mu(I). \tag{2.1}$$

Also we will need the following.

Lemma 2. *Let G be a convex increasing function on $[0,+\infty)$ such that $\lim_{x\to+\infty}\frac{G(x)}{x} = +\infty$ and let (Y,μ) be a nonatomic measure space with $\delta = \mu(Y) < +\infty$. Then given $\alpha, \beta > 0$ there exists a nonnegative measurable function ψ on Y such that $\int_Y \psi\, d\mu = \alpha$ and $\int_Y G\circ\psi\, d\mu = \beta$ if and only if $\delta G(\frac{\alpha}{\delta}) \leqslant \beta$.*

Proof. One direction is just Jensen's inequality. For the other if $\delta G(\frac{\alpha}{\delta}) \leqslant \beta$ for any t such that $0 < t \leqslant \delta$ we choose a measurable subset $C(t)$ of Y such that $\mu(C(t)) = t$ (this is possible since μ is nonatomic) and define $\psi_t = \frac{\alpha}{t}\chi_{C(t)}$. Clearly $\int_Y \psi_t\, d\mu = \alpha$ and $\int_Y G\circ\psi_t\, d\mu = tG(\frac{\alpha}{t})$. But now the assumptions on G, α, β easily imply that there exists t as above with $tG(\frac{\alpha}{t}) = \beta$. $\square$

Now we state the main result of this note.

Theorem 1. *Let G be a C^1 strictly convex increasing function on $[0,+\infty)$ such that $\lim_{x\to+\infty}\frac{G(x)}{x} = +\infty$ and let $0 < f < \lambda$, $G(f) < F$ be given. Then $\mathcal{D}_G(\lambda, f, F)$ is equal to $\frac{f}{\lambda}$ when $f\frac{G(\lambda)}{\lambda} \leqslant F$ and it is equal to the unique solution k in $(0, \frac{f}{\lambda})$ of the equation*

$$(1-k)G\left(\frac{f-\lambda k}{1-k}\right) + kG(\lambda) = F \tag{2.2}$$

when $f\frac{G(\lambda)}{\lambda} > F$.

Proof. Let $\phi \geqslant 0$ be measurable and such that $\int_Y \phi\, d\mu = f$ and $\int_Y G\circ\phi\, d\mu = F$ and consider the set $E = \{M_{\mathcal{T}}\phi \geqslant \lambda\}$. It is easy to see as in the dyadic case that E is the union of a family $\{I_i\}$ (finite or countable) of pairwise almost disjoint elements of $\mathcal{T}$ such that $\int_{I_i}\phi\, d\mu \geqslant \lambda\mu(I_i)$. Let

$$k = \mu(E), \qquad x_i = \int_{I_i}\phi\, d\mu, \qquad a_i = \mu(I_i), \qquad y_i = \int_{I_i} G\circ\phi\, d\mu,$$
$$\bar{x} = \int_{X\setminus E}\phi\, d\mu \quad \text{and} \quad \bar{y} = \int_{X\setminus E} G\circ\phi\, d\mu. \tag{2.3}$$

We have

$$x_i \geqslant \lambda a_i, \qquad \sum_i a_i = k, \qquad \bar{x} + \sum_i x_i = f \quad \text{and} \quad \bar{y} + \sum_i y_i = F. \tag{2.4}$$

Upon setting $A = \sum_i x_i$ the convexity of G implies

$$\sum_i a_i G\left(\frac{x_i}{a_i}\right) \geqslant \sum_i a_i G\left(\frac{\sum_i x_i}{\sum_i a_i}\right) = kG\left(\frac{A}{k}\right),$$
$$G\left(\frac{x_i}{a_i}\right) = G\left(\frac{1}{\mu(I_i)}\int_{I_i}\phi\right) \leqslant \frac{1}{\mu(I_i)}\int_{I_i} G\circ\phi = \frac{y_i}{a_i} \quad \text{and}$$
$$G\left(\frac{\bar{x}}{1-k}\right) = G\left(\frac{1}{\mu(X\setminus E)}\int_{X|E}\phi\right) \leqslant \frac{1}{\mu(X\setminus E)}\int_{X|E} G\circ\phi = \frac{\bar{y}}{1-k}. \tag{2.5}$$

Hence

$$G\left(\frac{f-A}{1-k}\right) = G\left(\frac{\bar{x}}{1-k}\right) \leqslant \frac{\bar{y}}{1-k} = \frac{F-\sum_i y_i}{1-k} \leqslant \frac{F-\sum_i a_i G(\frac{x_i}{a_i})}{1-k} \leqslant \frac{F-kG(\frac{A}{k})}{1-k} \tag{2.6}$$

which gives

$$C_k(A) = (1-k)G\left(\frac{f-A}{1-k}\right) + kG\left(\frac{A}{k}\right) \leqslant F \tag{2.7}$$

where $\lambda k = \sum_i \lambda a_i \leqslant \sum_i x_i = A \leqslant f$.

Conversely given $0 < k < 1$ and $\lambda k \leqslant A < f$ satisfying (2.7) we use Lemma 1 to choose a pairwise almost disjoint family $\{I_i\}$ of elements of $\mathcal{T}$ such that $k = \sum_i \mu(I_i)$ and then use Lemma 2 on $Y = X \setminus \bigcup_i I_i$ we choose a nonnegative measurable function ψ on Y satisfying $\int_Y \psi\, d\mu = f - A$ and $\int_Y G \circ \psi\, d\mu = F - kG(\frac{A}{k})$ the condition in Lemma 2 being here just (2.7). Thus by defining

$$\phi = \frac{A}{k}\sum_i \chi_{I_i} + \psi\chi_Y \tag{2.8}$$

it is easy to see that $\int_Y \phi\, d\mu = f$ and $\int_Y G \circ \phi\, d\mu = F$ and moreover since clearly $M_{\mathcal{T}}\phi \geqslant \lambda$ on $\bigcup_i I_i$ we get $\mathcal{D}_G(\lambda, f, F) \geqslant k$.

Thus $\mathcal{D}_G(\lambda, f, F)$ is the supremum of all k in $(0, \frac{f}{\lambda})$ for which there exists at least one A in $[\lambda k, f)$ such that (2.7) holds.

Now observing that

$$C_k'(A) = -G'\left(\frac{f-A}{1-k}\right) + G'\left(\frac{A}{k}\right) > 0 \quad \text{if } \frac{A}{k} > \frac{f-A}{1-k} \tag{2.9}$$

that is when $A \geqslant fk$ and since $\lambda > f$ we conclude that (2.7) holds for some A as above if and only if it holds at $A = \lambda k$.

But defining now

$$R(k) = (1-k)G\left(\frac{f-\lambda k}{1-k}\right) + kG(\lambda) \tag{2.10}$$

the convexity of G implies that

$$R'(k) = G(\lambda) - G\left(\frac{f-\lambda k}{1-k}\right) - \left(\lambda - \frac{f-\lambda k}{1-k}\right)G'\left(\frac{f-\lambda k}{1-k}\right) > 0 \tag{2.11}$$

for any k in $(0, \frac{f}{\lambda})$. Moreover $R(0) = G(f) < F$ and $R(\frac{f}{\lambda}) = f\frac{G(\lambda)}{\lambda}$ and these easily complete the proof of the theorem. □

It is obvious that when $\lambda \leqslant f$ the expression $\mathcal{D}_G(\lambda, f, F)$ is equal to 1. Specializing now the above theorem to the case $G(x) = x^p$ where $p > 1$ we get the following.

Corollary 1. *For any $p > 1$, $\mathcal{D}_p(\lambda, f, F)$ is equal to $\frac{f}{\lambda}$ when $f^{p-1} < \lambda^{p-1} \leqslant \frac{F}{f}$ and it is equal to the unique solution k in $(0, \frac{f}{\lambda})$ of the equation*

$$\frac{(f-k\lambda)^p}{(1-k)^{p-1}} + k\lambda^p = F \tag{2.12}$$

when $\lambda^{p-1} f > F$.

In particular

$$\mathcal{D}_2(\lambda, f, F) = \begin{cases} \frac{f}{\lambda} & \text{if } f < \lambda \leqslant \frac{F}{f}, \\ \frac{F-f^2}{F-2\lambda f+\lambda^2} & \text{if } \frac{F}{f} < \lambda. \end{cases} \tag{2.13}$$

Next Theorem 1 implies that with f, F fixed the function $k(\lambda) = \mathcal{D}_G(\lambda, f, F)$ satisfies $k(\lambda)G(\lambda) < F$ as $\lambda \to +\infty$ and so since $\frac{G(x)}{x} \to +\infty$ that $k(\lambda)\lambda \to 0$ as $\lambda \to +\infty$. Using this into (2.2) and letting $\lambda \to +\infty$ we get the following.

Corollary 2. *We have for any G as in Theorem 1 that $\lim_{\lambda\to+\infty} G(\lambda)\mathcal{D}_G(\lambda, f, F) = F - G(f)$. In particular if $p > 1$ we have $\lim_{\lambda\to+\infty} \lambda^p \mathcal{D}_p(\lambda, f, F) = F - f^p$.*

Remark 1. If Q is C^1 strictly concave and increasing function on $[0, +\infty)$ satisfying $\lim_{x\to+\infty} \frac{Q(x)}{x} = 0$ then the proof of Theorem 1 can be carried out with minor modifications and by reversing the inequalities to give that whenever $0 < f < \lambda$ and $0 < F < Q(f)$ the corresponding function $\mathcal{D}_Q(\lambda, f, F)$ is equal to $\frac{f}{\lambda}$ when $f\frac{Q(\lambda)}{\lambda} \geqslant F$ and to the unique solution k in $(0, \frac{f}{\lambda})$ of Eq. (2.2) with Q replacing G otherwise. Thus the function $\mathcal{D}_p(\lambda, f, F)$ can be computed and for $0 < p < 1$. In particular for $p = 1/2$ we get

$$\mathcal{D}_{1/2}(\lambda, f, F) = \begin{cases} \frac{f}{\lambda} & \text{if } f < \lambda \leqslant (\frac{f}{F})^2, \\ \frac{f-F^2}{f-2F\sqrt{\lambda}+\lambda} & \text{if } (\frac{f}{F})^2 < \lambda. \end{cases} \tag{2.14}$$

Remark 2. One can generalize the above result as follows. Instead of fixing the L^1 norm and the higher "quasi-norm" defined by $G\circ\phi$ one could fix two quasi-norms a lower one and a higher one. To describe the result, given G is as in Theorem 1 and g another strictly convex and increasing function on $[0,+\infty)$ we define for any f, F, λ such that $0<\lambda$, $G(f)<F$

$$\mathcal{D}_{G,g}(\lambda, f, F) = \sup\left\{\mu\big(\{M_{\mathcal{T}}\phi \geqslant \lambda\}\big):\ \phi \geqslant 0 \text{ measurable},\ \int\limits_X g\circ\phi\, d\mu = f,\ \int\limits_X G\circ g\circ\phi\, d\mu = F\right\}. \tag{2.15}$$

Then the following holds

$$\mathcal{D}_{G,g}(\lambda, f, F) = \mathcal{D}_G\big(g(\lambda), f, F\big). \tag{2.16}$$

The proof in the case $g(\lambda) > f$ is similar to that of Theorem 1 by taking $x_i = \int_{I_i} g\circ\phi\, d\mu$, $y_i = \int_{I_i} G\circ g\circ\phi\, d\mu$, $\bar{x} = \int_{X\setminus E} g\circ\phi\, d\mu$ and $\bar{y} = \int_{X\setminus E} G\circ g\circ\phi\, d\mu$ in (2.3) instead and noting that $x_i \geqslant g(\lambda)a_i$ by Jensen's inequality and so $A \geqslant g(\lambda)k$, and, for the lower bound, taking $\phi = g^{-1}(\frac{A}{k})\sum_i \chi_{I_i} + g^{-1}\circ\psi\chi_Y$ in (2.8) instead. If $g(\lambda) \leqslant f$ then to show that $\mathcal{D}_{G,g}(\lambda, f, F) = 1$ we take $A = fk$ for any $k<1$ then take ϕ as before and let $k\to 1$. Of course for the upper bound one could just use Theorem 1 for the function $g\circ\phi$ since by Jensen's inequality $g(M_{\mathcal{T}}\phi(x)) \leqslant M_{\mathcal{T}}(g\circ\phi)(x)$ for all $x\in X$.

Now we let $H: [0,+\infty) \to [0,+\infty)$ be a strictly convex and increasing function and such that $H(0)=0$ and given any ψ measurable in X we define

$$|\psi|_{H,\infty} = \sup\big\{H(\lambda)\mu\big(\{|\psi| \geqslant \lambda\}\big):\ \lambda > 0\big\}. \tag{2.17}$$

Then we consider the following

$$\mathcal{B}_{G,H,\infty}(F, f) = \sup\left\{|M_d\phi|_{H,\infty}:\ \phi \geqslant 0,\ \phi\in L^1(\mu),\ \int\limits_X \phi\, d\mu = f,\ \int\limits_X G\circ\phi\, d\mu = F\right\}. \tag{2.18}$$

In view of the above corollary it is clear that this will be $+\infty$ if H is stronger than G in the sense $H(x)/G(x) \to +\infty$ as $x\to+\infty$. Next we prove the following.

Theorem 2. *Assume G is a C^2 increasing function on $[0,+\infty)$ satisfying $G''>0$ on $(0,+\infty)$, $G(0)=G'(0)=0$ and $\lim_{x\to+\infty}\frac{G(x)}{x} = +\infty$ and let H be a strictly convex and increasing function on $[0,+\infty)$, such that $H(0)=0$. Moreover we assume that the function $\frac{G}{H}$ is increasing on $(0,+\infty)$. Then we have*

$$\mathcal{B}_{G,H,\infty}(F, f) = \frac{H(\tau(\frac{F}{f}))}{\tau(\frac{F}{f})} f \tag{2.19}$$

where τ is the inverse of the function $\tilde{G}(x) = \frac{G(x)}{x}$ on $(0,+\infty)$.

Proof. Fix f, F as in Theorem 1. Given $\phi\geqslant 0$ be measurable and such that $\int_Y \phi\, d\mu = f$ and $\int_Y G\circ\phi\, d\mu = F$, Theorem 1 and the convexity of H implies that

$$H(\lambda)\mu\big(\{M_{\mathcal{T}}\phi \geqslant \lambda\}\big) \leqslant H(\lambda)\mathcal{D}_G(\lambda, f, F) = \frac{H(\lambda)}{\lambda} f \leqslant \frac{H(\tau(\frac{F}{f}))}{\tau(\frac{F}{f})} f \tag{2.20}$$

when $\lambda \leqslant \tau(\frac{F}{f})$. On the other hand the same theorem implies that there is ϕ as above with $H(\lambda_0)\mu(\{M_{\mathcal{T}}\phi \geqslant \lambda_0\}) \geqslant \frac{H(\lambda_0)}{\lambda_0} f$ where $\lambda_0 = \tau(\frac{F}{f})$. In particular this implies that $\mathcal{B}_{G,H}(f, F) \geqslant \frac{H(\lambda_0)}{\lambda_0} f$. Hence in view of the other part of Theorem 1 upon setting $k(\lambda)$ to denote the unique solution of (2.2) when $\lambda > \lambda_0$ the proof will be complete once we have shown that $H(\lambda)k(\lambda)$ is strictly decreasing on $\lambda > \lambda_0$. Differentiating (2.2) with k replaced by $k(\lambda)$ which is legitimate in view of the implicit function theorem we easily get setting $x(\lambda) = \frac{f - k(\lambda)\lambda}{1-k(\lambda)}$ that

$$\frac{d}{d\lambda}\log H(\lambda)k(\lambda) = \frac{H'(\lambda)}{H(\lambda)} - \frac{k'(\lambda)}{k(\lambda)} = \frac{H'(\lambda)}{H(\lambda)} - \frac{G'(\lambda) - G'(x(\lambda))}{G(\lambda) - G(x(\lambda)) - (\lambda - x(\lambda))G'(x(\lambda))} \tag{2.21}$$

(the implicit function theorem can be applied since $G(\lambda) - G(x(\lambda)) - (\lambda - x(\lambda))G'(x(\lambda)) > 0$) and this expression has the same sign as

$$W(x) = H'(\lambda)\big[G(\lambda) - G(x) - (\lambda - x)G'(x)\big] - H(\lambda)\big[G'(\lambda) - G'(x)\big] \tag{2.22}$$

evaluated at $x = x(\lambda)$. Moreover since $f<\lambda$ we have $0 < x(\lambda) < \lambda$ and so it suffices to prove that $W(x) \leqslant 0$ on $[0,\lambda]$. But $W(0) = H'(\lambda)G(\lambda) - H(\lambda)G'(\lambda) \leqslant 0$ since $\frac{G}{H}$ is increasing and $W(\lambda) = 0$. Also

$$W'(x) = G''(x)\big[-(\lambda - x)H'(\lambda) + H(\lambda)\big] \tag{2.23}$$

and so $W'(\lambda) > 0$ and W' has at most one zero in $(0, \lambda)$. These easily imply that $W(x) \leqslant 0$ on $[0, \lambda]$ and thus complete the proof of the theorem. □

Specializing the above theorem to the case where $G(x) = x^p$, $H(x) = x^q$ where $p > 1$, $1 \leqslant q \leqslant p$ we easily get the following.

Corollary 3. *Given $p > 1$ and q with $1 \leqslant q \leqslant p$ we have for any nonnegative measurable ϕ that*

$$\|M_{\mathcal{T}}\phi\|_{q,\infty} \leqslant \|\phi\|_p^{\frac{p(q-1)}{q(p-1)}} \|\phi\|_1^{\frac{p-q}{q(p-1)}} \tag{2.24}$$

and this is sharp in the sense that the right-hand side is the supremum of the left-hand side over all ϕ's with fixed L^1 and L^p norms. In particular when $p = q$ we get the sharp inequality

$$\|M_{\mathcal{T}}\phi\|_{q,\infty} \leqslant \|\phi\|_q. \tag{2.25}$$

Note that inequality (2.24) follows from (2.25) via Hölder's inequality. The main point though is the sharpness of (2.24) when the L^1 and L^p norms of ϕ are fixed. We also remark that in the case $p = q$ the value of the L^1 norm of ϕ does not appear in the corresponding supremum which is in sharp contrast with the corresponding problem involving strong L^p norms mentioned in the Introduction.

Also specializing (2.16) to the case where $G(x) = x^{p_2/p_1}$, $g(x) = x^{p_1}$ where $1 \leqslant p_1 < p_2$ and then using the above theorem with $H(x) = x^q$ where $p_1 \leqslant q \leqslant p_2$ one easily obtains the following.

Corollary 4. *Given $p_2 > p_1 \geqslant 1$ and q with $p_1 \leqslant q \leqslant p_2$ we have for any nonnegative measurable ϕ that*

$$\|M_{\mathcal{T}}\phi\|_{q,\infty} \leqslant \|\phi\|_{p_2}^{\frac{p_2(q-p_1)}{q(p_2-p_1)}} \|\phi\|_{p_1}^{\frac{p_1(p_2-q)}{q(p_2-p_1)}} \tag{2.26}$$

and this is sharp in the sense that the right-hand side is the supremum of the left-hand side over all ϕ's with fixed L^{p_1} and L^{p_2} norms.

Similar remarks as with Corollary 3 apply here.

Now let G be as in Theorem 2 and let $q \geqslant 1$. For any $0 < f < L$ and $F > G(f)$ we define

$$\mathcal{B}_{G,q,\infty}(F, f, L) = \sup\left\{ \big\|\max(M_{\mathcal{T}}\phi, L)\big\|_{q,\infty}^q \colon \phi \geqslant 0,\ \phi \in L^1(\mu),\ \int_X \phi\, d\mu = f,\ \int_X G \circ \phi\, d\mu = F \right\}. \tag{2.27}$$

This is a generalized version of the function defined in (1.6). Then using Theorems 1 and 2 we get the following.

Theorem 3. *If G is as in Theorem 2 and $q \geqslant 1$ is such that $x^{-q}G(x)$ is increasing in $x > 0$ then we have*

$$\mathcal{B}_{G,q,\infty}(F, f, L) = \begin{cases} \tau(\frac{F}{f})^{q-1} f & \text{if } f < L < \tau(\frac{F}{f})^{1-\frac{1}{q}} f^{\frac{1}{q}}, \\ L^q & \text{if } \tau(\frac{F}{f})^{1-\frac{1}{q}} f^{\frac{1}{q}} \leqslant L. \end{cases} \tag{2.28}$$

In particular if $p > 1$ and $p \geqslant q$ then

$$\mathcal{B}_{p,q,\infty}(F, f, L) = \begin{cases} F^{\frac{q-1}{p-1}} f^{\frac{p-q}{p-1}} & \text{if } f < L < F^{\frac{q-1}{q(p-1)}} f^{\frac{p-q}{q(p-1)}}, \\ L^q & \text{if } F^{\frac{q-1}{q(p-1)}} f^{\frac{p-q}{q(p-1)}} \leqslant L, \end{cases} \tag{2.29}$$

and so for any $p > 1$,

$$\mathcal{B}_{p,\infty}(F, f, L) = \begin{cases} F & \text{if } f < L < F^{1/p}, \\ L^p & \text{if } F^{1/p} \leqslant L. \end{cases} \tag{2.30}$$

Proof. Given any ϕ as in (2.27) and any $\lambda > f$ we have $\lambda^q \mu(\{\max(M_d\phi, L) \geqslant \lambda\}) \leqslant L^q$ if $\lambda \leqslant L$ and it is $\leqslant \lambda^q \mathcal{D}_G(\lambda, f, F)$. On the other hand the proof of Theorem 2 implies that $\lambda^q \mathcal{D}_G(\lambda, f, F)$ is decreasing if $\lambda > \tau(\frac{F}{f})$. This combined with Theorem 1 implies that $\mathcal{B}_{G,q,\infty}(F, f, L)$ is equal to L^q if $L \geqslant \tau(\frac{F}{f})$ and to $\max(L^q, \tau(\frac{F}{f})^{q-1} f)$ otherwise and this easily completes the proof. □

Also by using Corollary 4 and defining

$$\mathcal{B}_{p_1,p_2,q,\infty}(F,f,L)=\sup\left\{\left\|\max(M_{\mathcal{T}}\phi,L)\right\|_{q,\infty}^{q}:\ \phi\geqslant 0 \text{ measurable},\ \int_X \phi^{p_1}\,d\mu=f,\ \int_X \phi^{p_2}\,d\mu=F\right\}, \tag{2.31}$$

we get the following.

Corollary 5. *Given $p_2>p_1\geqslant 1$ and q with $p_1\leqslant q\leqslant p_2$ we have*

$$\mathcal{B}_{p_1,p_2,q,\infty}(F,f,L)=\begin{cases} F^{\frac{q-p_1}{p_2-p_1}}f^{\frac{p_2-q}{p_2-p_1}} & \text{if } f<L<F^{\frac{q-p_1}{q(p_2-p_1)}}f^{\frac{p_2-q}{q(p_2-p_1)}},\\ L^q & \text{if } F^{\frac{q-p_1}{q(p_2-p_1)}}f^{\frac{p_2-q}{q(p_2-p_1)}}\leqslant L. \end{cases} \tag{2.32}$$

Acknowledgment

The authors would like to thank the referee for suggesting Remark 2 and Corollaries 4 and 5.

References

[1] D.L. Burkholder, Martingales and Fourier analysis in Banach spaces, in: C.I.M.E. Lectures, Varenna (Como), Italy, 1985, in: Lecture Notes in Math., vol. 1206, 1986, pp. 61–108.
[2] D.L. Burkholder, Boundary value problems and sharp inequalities for martingale transforms, Ann. Probab. 12 (1984) 647–702.
[3] A.D. Melas, The Bellman functions of dyadic-like maximal operators and related inequalities, Adv. Math. 192 (2005) 310–340.
[4] A.D. Melas, Sharp general local estimates for Dyadic-like maximal operators and related Bellman functions, submitted for publication.
[5] F. Nazarov, S. Treil, The hunt for a Bellman function: Applications to estimates for singular integral operators and to other classical problems of harmonic analysis, Algebra i Analiz 8 (5) (1996) 32–162.
[6] F. Nazarov, S. Treil, A. Volberg, The Bellman functions and two-weight inequalities for Haar multipliers, J. Amer. Math. Soc. 12 (4) (1999) 909–928.
[7] F. Nazarov, S. Treil, A. Volberg, Bellman Function in Stochastic Optimal Control and Harmonic Analysis (How Our Bellman Function Got Its Name), Oper. Theory Adv. Appl., vol. 129, Birkhäuser Verlag, 2001, pp. 393–424.
[8] L. Slavin, V. Vasyunin, Sharp results in the integral-form John–Nirenberg inequality, submitted for publication.
[9] L. Slavin, A. Volberg, The explicit BF for a dyadic Chang–Wilson–Wolff theorem. The *s*-function and the exponential integral, Contemp. Math., in press.
[10] V. Vasyunin, The sharp constant in the reverse Holder inequality for Muckenhoupt weights, Algebra i Analiz 15 (1) (2003) 73–117.
[11] V. Vasyunin, A. Volberg, The Bellman functions for a certain two weight inequality: The case study, Algebra i Analiz 18 (2) (2006).
[12] V. Vasyunin, A. Volberg, Monge–Ampere equation and Bellman optimization of Carleson embedding theorem, preprint.
[13] G. Wang, Sharp maximal inequalities for conditionally symmetric martingales and Brownian motion, Proc. Amer. Math. Soc. 112 (1991) 579–586.

Further reading

[14] L. Grafakos, S. Montgomery-Smith, Best constants for uncentered maximal functions, Bull. London Math. Soc. 29 (1) (1997) 60–64.
[15] A.D. Melas, E. Nikolidakis, Dyadic-like maximal operators on integrable functions and Bellman functions related to Kolmogorov's inequality, submitted for publication.
[16] A. Volberg, Bellman approach to some problems in harmonic analysis, in: Seminaire des Equations aux derivées partielles, Ecole Polytéchnique, 2002, exposé XX, pp. 1–14.