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Clifford Analysis and Commutators on the Besov Spaces

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We study commutators between multiplication by a function, called the symbol, and Riesz transformations on the Besov spaces. We characterize symbols, by potential capacity, for which the associated commutators are bounded. Clifford analysis plays a key role in our approach. © 1999 Academic Press

1. INTRODUCTION

The study of commutators and their various generalizations plays an important role in harmonic analysis, PDE and other related areas. Much important work has been done in the past. We refer the reader to [CRW, JP, CLMS, LMWZ] and the references therein. Recently, it was revealed in [W1] that Clifford analysis provides a natural approach in the study of commutators and other related operators on function spaces of $\mathbb{R}^n$. For example, the relation between Hardy space H^1 and its dual BMO, and Div–Curl theorem (see [CLMS]) in the theory of compensated compactness can be associated with different parts of a bilinear form in certain Clifford valued function spaces. The power of Clifford analysis is that, by extending a function of n real variables monogenically (conju-analytically if $n=1$) to a function of $n+1$ variables with values in a complex Clifford algebra, one may use the "analytic" tools and associated function theory effectively. Works of this kind include the generalization of the Plemelj formula, the generalization of the three line theorems ([PS]), application of Clifford analysis on singular integral, Hardy spaces, and harmonic functions on Lipschitz domains and wavelets ([M1 and M2], [Mi]). One can also find some fundamental results and tools of Clifford analysis in [S] and [GM].

In this paper, we characterize bounded commutators on the Besov spaces on $\mathbb{R}^n$. Potential capacity is used to measure the "sizes" of the symbols of the commutators. Clifford analysis plays an important role in our approach.

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Let $C\infty_0(\mathbb{R}^n\backslash\{0\})$ be the set of all infinitely differentiable functions on $\mathbb{R}^n\backslash\{0\}$ with compact support. For $\alpha \in \mathbb{R}$, the space B_α is the Banach space of functions on $\mathbb{R}^n$ obtained as the completion of $C_0^\infty(\mathbb{R}^n\backslash\{0\})$ under the norm

$$\|f\|_\alpha = \left\{ \int_{\mathbb{R}^n} |\xi|^{2\alpha} \, |\hat{f}(\xi)|^2 \, d\xi \right\}^{1/2}.$$

Here $\hat{f}$ is the Fourier transformation of f. The space B_α is called a Besov space if $\alpha > 0$. Clearly $B_0 = L^2(\mathbb{R}^n)$.

For $j = 1, 2, ..., n$, the jth Riesz transformation $\mathscr{R}_j$ is defined on $C_0^\infty(\mathbb{R}^n)$ by

$$\mathscr{R}_j(f)(x) = \text{p.v.} \frac{c_n}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{x_j - u_j}{|x-u|^{n+1}} f(u) \, du.$$

For a function b on $\mathbb{R}^n$ and $j = 1, 2, ..., n$, the commutator $[b, \mathscr{R}_j]$ is defined by

$$[b, \mathscr{R}_j](f)(x) = b(x) \, \mathscr{R}_j(f)(x) - \mathscr{R}_j(bf)(x), \qquad \forall f \in C_0^\infty(\mathbb{R}^n).$$

For $0 < \alpha < n$, let $I_\alpha(x) = |x|^{\alpha - n}$, $x \in \mathbb{R}^n$. The α-Riesz capacity of a bounded op set $O \subset \mathbb{R}^n$ is defined by

$$\mathrm{Cap}_\alpha(O) = \inf\{ \|f\|_0^2 \colon I_\alpha * f \geqslant 1 \text{ on } O, \ f \geqslant 0\}.$$

We note that $\mathrm{Cap}_0(O) = |O|$ (obtained by letting α go to zero) is the Lebesgue measure of O in $\mathbb{R}^n$.

Denote by $T(O)$ the tent of O in $\mathbb{R}^{n+1}_+$, which is

$$T(O) = \{(x, y) \in \mathbb{R}^{n+1}_+ : \text{the ball centered at } x \text{ with radius } y \text{ contains in } O\}.$$

For a multi-index $\tau = (\tau_0, \tau_1 ..., \tau_n)$, let $|\tau| = \tau_0 + \tau_1 + \cdots + \tau_n$ and

$$D^\tau = \frac{\partial^{\tau_0}}{\partial y^{\tau_0}} \frac{\partial^{\tau_1}}{\partial x_1^{\tau_1}} \cdots \frac{\partial^{\tau_n}}{\partial x_n^{\tau_n}}.$$

The main result of this paper is

THEOREM. *Let $b \in L^1(\mathbb{R}^n, (1+|x|)^{-(n+1)} \, dx)$ and k be a nonnegative integer. For $\alpha = 1/2, 2/2, 3/2, ..., n/2$, the commutators $\{[b, \mathscr{R}_j]\}_1^n$ are bounded on the Besov space B_α if and only if there is a constant C_k such that*

$$\int_{T(O)} \sum_{|\tau| = 2\alpha + k} |D^\tau b(x, y)|^2 \, y^{2\alpha + 2k - 1} \, dx \, dy \leqslant C_k \, \mathrm{Cap}_\alpha(O) \tag{1.1}$$

holds for any bounded open set $O \subset \mathbb{R}^n$. Here $b(x, y)$ is the harmonic extension of b.

It is proved implicitly in this paper that the "if" part of the theorem is true for all $\alpha \geqslant 0$. We conjecture that the "only if" part is also true for all nonnegative α (other than half integers). For $\alpha = 0$ and $k = 1$, condition (1.1) is equivalent to the measure $|\nabla b(x, y)|^2 \, y \, dx \, dy$ being a Carleson measure, or b being in *BMO*. Therefore the main theorem in [CRW] corresponds to the case $\alpha = 0$ (not covered here). However the method in [CRW] cannot be used effectively to prove our theorem, because capacity is not a localized quantity. We note also that for $n = 1$, $k = 0$ and $0 < \alpha \leqslant 1/2$, the theorem was proved in [CM] (see also related work on the unit disk in [W2]).

Leaving more notations and terminologies later, let us outline roughly our approach of the main result. By using Clifford analysis and some technical results in Section 3, we prove in Section 5 that the commutators $\{[b, \mathcal{R}_j]\}_1^n$ are bounded on the Besov B_α if and only if the bilinear form

$$\int_{\mathbb{R}_+^{n+1}} \mathbf{F}(\mathcal{D}b) \, \mathbf{G} \, dx \, dy$$

is bounded by $C \, \|\mathbf{F}\|_\alpha \, \|\mathbf{G}\|_{-\alpha}$. Here $\mathcal{D}$ is the Cauchy–Riemann operator, $\mathbf{F}$ is a right monogenic function and $\mathbf{G}$ is a left monogenic function. Using integration by parts, this result can be further refined as that the measure

$$\sum_{|\tau| = m} |D^\tau \mathcal{D} b(x, y)|^2 \, y^m \, dx \, dy$$

is a $(m+1)/2$-Carleson measure and the following estimate holds:

$$\left| \int_{\mathbb{R}_+^{n+1}} \mathbf{F}(\mathcal{D}^{m+1} b) \, \mathbf{G} \, y^m \, dx \, dy \right| \leqslant C \, \|\mathbf{F}\|_\alpha \, \|\mathbf{G}\|_{-\alpha}.$$

Here m is the greatest integer of $2\alpha - 1$. Again using some technical results in Sections 3 and 5, we conclude that the above refinement is equivalent to that the measure

$$\sum_{|\tau| = 2\alpha + k} |D^\tau b(x, y)|^2 \, y^{2\alpha + 2k - 1} \, dx \, dy$$

is a α-Carleson measure. To obtain the main theorem, we then apply the result in Section 4, which characterizes α-Carleson measures in terms of potential capacity.

Preliminaries are in Section 2. Some technical results about Clifford analysis are gathered in Section 3. Properties of the α-Carleson measures and their characterization in terms of potential capacity are studied in Section 4. Section 5 contains the key approach of using Clifford analysis to obtain the main theorem. Throughout this paper, the letter "C" denotes a positive constant which may vary at each occurrence but is independent of the essential variables or quantities. We always use bold case letters or symbols for Clifford valued functions and function spaces.

2. PRELIMINARIES

In this section, we provide a minimum background of Clifford analysis needed in this paper. For a rich resource, we refer the reader to [Mi, M1, M2, GM, S and R].

The Clifford algebra $\mathbb{C}_{(n)}$ is the 2^n-dimensional algebra over $\mathbb{C}$ with the standard basis

$$\mathbf{e}_{\varnothing} \text{ and } \mathbf{e}_S = \mathbf{e}_{j_1}\mathbf{e}_{j_2}\cdots\mathbf{e}_{j_s}, \qquad 1 \leqslant j_1 < j_2 < \cdots < j_s \leqslant n;$$
$$\text{where} \quad S = \{j_1, ..., j_s\} \text{ is any ordered subset of } \{1, 2, ..., n\},$$

and the rules

$$\mathbf{e}_0 = \mathbf{e}_{\varnothing} = 1, \qquad \mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = -2\delta_{jk}, \qquad \overline{\mathbf{e}_j} = -\mathbf{e}_j, \qquad \overline{\mathbf{e}_R\mathbf{e}_S} = \overline{\mathbf{e}_S}\,\overline{\mathbf{e}_R}.$$

Here $1 \leqslant j, k \leqslant n$, δ_{jk} equals one if $j = k$ and zero otherwise; the overline is for conjugation; and R and S are ordered subsets of $\{1, 2, ..., n\}$. Elements in $\mathbb{C}_{(n)}$, called Clifford numbers, can be written as

$$\boldsymbol{\lambda} = \sum_S \lambda_S \mathbf{e}_S \qquad \text{with} \quad \{\lambda_S\} \subset \mathbb{C}.$$

Here $\lambda_0 = \lambda_{\{\varnothing\}}$, denoted by Scalar($\boldsymbol{\lambda}$), is called the scalar part of the Clifford number $\boldsymbol{\lambda}$. The product of two Clifford numbers $\boldsymbol{\lambda} = \sum_S \lambda_S \mathbf{e}_S$ and $\boldsymbol{\mu} = \sum_S \mu_S \mathbf{e}_S$ is defined by $\boldsymbol{\lambda\mu} = \sum_{S,R} \lambda_S \mu_R \mathbf{e}_S \mathbf{e}_R$. The magnitude of $\boldsymbol{\lambda}$ is $|\boldsymbol{\lambda}| = \sqrt{\text{Scalar}(\boldsymbol{\lambda}\bar{\boldsymbol{\lambda}})}$. We note that $\text{Scalar}(\boldsymbol{\lambda\mu}) = \text{Scalar}(\boldsymbol{\mu\lambda})$ (although $\boldsymbol{\lambda\mu} \neq \boldsymbol{\mu\lambda}$ in general). We observe also that $|\text{Scalar}(\boldsymbol{\lambda\mu})| \leqslant |\boldsymbol{\lambda}|\,|\boldsymbol{\mu}|$ and $|\boldsymbol{\lambda\mu}| \leqslant C\,|\boldsymbol{\lambda}|\,|\boldsymbol{\mu}|$, where C is a positive constant depending only on n.

Suppose $\mathbf{f} = \sum_S f_S \mathbf{e}_S$ is a Clifford valued function, where $\{f_S\}$, the components of $\mathbf{f}$, are complex valued functions. We say $\mathbf{f}$ is in $\mathbf{L}^p$ if $|\mathbf{f}|$ is in L^p, or equivalently if its components are in L^p. For $\mathbf{f} \in \mathbf{L}^p$, the norm of $\mathbf{f}$ is defined by $\|\mathbf{f}\|_{\mathbf{L}^p} = \|\,|\mathbf{f}|\,\|_{L^p}$.

For $\mathbf{f}$, $\mathbf{g}$ in $\mathbf{L}^2(\Omega)$, define the pairing

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathbf{L}^2(\Omega)} = \int_\Omega \mathbf{f}\bar{\mathbf{g}}\, dv.$$

Here dv is the Lebesgue measure on Ω. Note that $\langle \cdot, \cdot \rangle_{\mathbf{L}^2(\Omega)}$ is not an inner product for Clifford valued $\mathbf{L}^2(\Omega)$ space. However

$$(\mathbf{f}, \mathbf{g})_{\mathbf{L}^2(\Omega)} = \text{Scalar}\langle \mathbf{f}, \mathbf{g} \rangle_{\mathbf{L}^2(\Omega)}$$

is the inner product on $\mathbf{L}^2(\Omega)$ associated to the norm $\|\cdot\|_{\mathbf{L}^2(\Omega)}$, and therefore we have

$$\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} \leqslant |\langle \mathbf{f}, \mathbf{f} \rangle_{\mathbf{L}^2(\Omega)}| \qquad \text{and} \qquad |\langle \mathbf{f}, \mathbf{g} \rangle_{\mathbf{L}^2(\Omega)}| \leqslant C\, \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)}\, \|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}.$$

We remark that under the pairing $(\cdot, \cdot)_{\mathbf{L}^2(\mathbb{R}^n)}$, the dual space of $\mathbf{B}_\alpha$ is $\mathbf{B}_{-\alpha}$.

The Fourier transformation, convolution and harmonic extension can be defined for Clifford valued functions. More explicitly, the Fourier transformation of a function $\mathbf{f}$ on $\mathbb{R}^n$ is

$$\hat{\mathbf{f}}(\chi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathbf{f}(x)\, e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^n;$$

the convolution of two functions $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ on $\mathbb{R}^n$ is

$$\boldsymbol{\varphi} * \boldsymbol{\psi}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \boldsymbol{\varphi}(x-u)\, \boldsymbol{\psi}(u)\, du;$$

and the harmonic extension of $\mathbf{f}(x)$ onto $\mathbb{R}^{n+1}_+$ is

$$\mathbf{f}(x, y) = \mathbf{f}_y(x) = \mathbf{f} * p_y(x),$$

where $p_y(x) = c_n(y/(|x|^2 + y^2)^{(n+1)/2})$ is the Poisson kernel and the constant c_n is chosen so that $\widehat{p_y}(\xi) = e^{-y|\xi|}$. Therefore $\widehat{\mathbf{f}_y}(\xi) = e^{-y|\xi|}\hat{\mathbf{f}}(\xi)$.

The Cauchy–Riemann operator is defined by

$$\mathscr{D} = \sum_{j=0}^{n} \frac{\partial}{\partial x_j} \mathbf{e}_j, \qquad (x_0 = y).$$

Here $(x_1, ..., x_n, y) = (x, y) \in \mathbb{R}^{n+1}_+$. Suppose $\mathbf{f} = \sum_S f_S \mathbf{e}_S$ is a Clifford valued function defined on an open set $\Omega \subseteq \mathbb{R}^{n+1}_+$. The left and right actions of $\mathscr{D}$ on $\mathbf{f}$ are defined, respectively, by

$$\mathscr{D}\mathbf{f} = \sum_0^n \sum_S \frac{\partial f_S}{\partial x_j} \mathbf{e}_j \mathbf{e}_S \qquad \text{and} \qquad \mathscr{D}_r \mathbf{f} = \sum_0^n \sum_S \frac{\partial f_S}{\partial x_j} \mathbf{e}_S \mathbf{e}_j.$$

A function $\mathbf{f}$ on $\Omega \subseteq \mathbb{R}^{n+1}_{+}$ is said to be left (or right) monogenic if $\mathscr{D}\mathbf{f} \equiv 0$ on Ω (or $\mathscr{D}_r \mathbf{f} \equiv 0$ On Ω). It is easy to see that $\bar{\mathscr{D}}\mathscr{D}$ is the Laplacian. Therefore left (or right) monogenic functions are harmonic functions.

For $\xi \in \mathbb{R}^n$, write $\boldsymbol{\xi} = \sum_{j=1}^{n} \xi_j \mathbf{e}_j$. Define the characteristic functions $\boldsymbol{\chi}_{\pm}(\xi)$ by

$$\boldsymbol{\chi}_{\pm}(\xi) = \frac{|\xi| \pm i\boldsymbol{\xi}}{2\,|\xi|}, \qquad \xi \neq 0. \tag{2.1}$$

Suppose $\mathbf{f} \in \mathbf{B}_0$ $(= \mathbf{L}^2(\mathbb{R}^n))$. It is well-known that $\mathbf{f}(x, y)$, the harmonic extension of $\mathbf{f}$ onto $\mathbb{R}^{n+1}_{+}$, is left or right monogenic if and only if $\boldsymbol{\chi}_{-}(\xi)\,\hat{\mathbf{f}}(\xi) = 0$ or $\hat{\mathbf{f}}(\xi)\,\boldsymbol{\chi}_{-}(\xi) = 0$, respectively. Some basic facts about $\boldsymbol{\chi}_{\pm}(\xi)$ are gathered in the following equalities:

$$\boldsymbol{\chi}_{+}(\xi) + \boldsymbol{\chi}_{-}(\xi) = 1, \qquad \boldsymbol{\chi}_{+}(\xi)\,\boldsymbol{\chi}_{-}(\xi) = 0, \qquad \overline{\boldsymbol{\chi}_{\pm}(\xi)} = \boldsymbol{\chi}_{\pm}(\xi). \tag{2.2}$$

Using the characteristic functions $\boldsymbol{\chi}_{\pm}(\xi)$, we construct the following two sets of right and left monogenic functions on $\mathbb{R}^{n+1}_{+}$. They play an important role in this paper.

$$\mathbf{R} = \{\mathbf{f}(x, y) = \mathbf{f} * p_y(x) = \hat{\mathbf{f}}(\xi) \in \mathbf{C}_0^{\infty}(\mathbb{R}^n \backslash \{0\}) \text{ and } \hat{\mathbf{f}}(\xi)\,\boldsymbol{\chi}_{-}(\xi) = 0\};$$

$$\mathbf{L} = \{\mathbf{g}(x, y) = \mathbf{g} * p_y(x) \colon \hat{\mathbf{g}}(\xi) \in \mathbf{C}_0^{\infty}(\mathbb{R}^n \backslash \{0\}) \text{ and } \boldsymbol{\chi}_{-}(\xi)\,\hat{\mathbf{g}}(\xi) = 0\}.$$

We end this section with the following standard results. For self contain we provide also proofs.

Lemma A. *Suppose* $\mathbf{F}$ *and* $\mathbf{G}$ *are right and left monogenic on* $\mathbb{R}^{n+1}_{+}$, *respectively. If the boundary functions* $\mathbf{F}_0(x) = F(x, 0)$ *and* $\mathbf{G}_0(x) = G(x, 0)$ *are in* $\mathbf{L}^2(\mathbb{R}^n)$, *then*

$$\mathbf{F}(x, y) + \sum_{j=1}^{n} \mathscr{R}_j(\mathbf{F}_0) * p_y(x)\,\mathbf{e}_j = 2\mathbf{F}(x, y);$$

$$\mathbf{G}(x, y) + \sum_{j=1}^{n} \mathbf{e}_j \mathscr{R}_j(\mathbf{G}_0) * p_y(x) = 2\mathbf{G}(x, y).$$

Proof. These identities are consequences of the Fourier transformation. Indeed it is easy to verify that $\mathbf{F}(x, y) = \mathbf{F}_0 * p_Y(x)$, and using the fact that

$$\widehat{\mathbf{F}_y}(\xi) = e^{-y|\xi|}\,\widehat{\mathbf{F}_0}(\xi), \quad \widehat{\mathscr{R}_j(\mathbf{F}_0)}(\xi) = i\,\frac{\xi_j}{|\xi|}\,\widehat{\mathbf{F}_0}(\xi) \quad \text{and} \quad \widehat{\mathbf{F}_0}(\xi)\,\boldsymbol{\chi}_{-}(\xi) = 0,$$

we have

$$\widehat{\mathbf{F}_y}(\xi) + \sum_{j=1}^{n} e^{-y|\xi|} \widehat{\mathscr{R}_j(\mathbf{F}_0)}(\xi)\, \mathbf{e}_j = 2e^{-y|\xi|} \widehat{\mathbf{F}_0}(\xi)\, \chi_+(\xi)$$
$$= 2e^{-y|\xi|} \widehat{\mathbf{F}_0}(\xi) = 2\widehat{\mathbf{F}_y}(\xi).$$

This yields the first formula. The second one can be established similarly. ▌

LEMMA B. *Suppose* $\alpha \in \mathbb{R}$ *and* τ *is a multi-index and* $\mathbf{f} \in \mathbf{R}$ (*or* $\mathbf{L}$). *Then*

$$\|D^{\tau}\mathbf{f}\|_{\alpha-|\tau|} \leqslant C\, \|\mathbf{f}\|_{\alpha}.$$

Moreover if $\alpha > 0$, $k \geqslant [\alpha]$ *and* $j \geqslant 0$, *then*

$$\left\|\frac{\partial^{k+1}\mathbf{f}}{\partial y^{k+1}}\right\|^2_{\mathbf{L}^2_{2k+1-2\alpha}} = C\, \|\mathbf{f}\|^2_{\alpha} \qquad \textit{and} \qquad \left\|\frac{\partial^{j}\mathbf{f}}{\partial y^{j}}\right\|^2_{\mathbf{L}^2_{2\alpha+2j-1}} = C\, \|\mathbf{f}\|^2_{-\alpha}.$$

Proof. These results are consequences of Plancherel's formula and the identity

$$\int_0^{\infty} e^{-2y|\xi|} y^{\beta}\, dy = \frac{C_{\beta}}{(2\,|\xi|)^{\beta+1}}, \qquad \beta > -1.$$

For example

$$\left\|\frac{\partial^{k+1}\mathbf{f}}{\partial y^{k+1}}\right\|^2_{\mathbf{L}^2_{2k+1-2\alpha}} = \int_{\mathbb{R}^{n+1}_+} \left|\widehat{\frac{\partial^{k+1}\mathbf{f}_y}{\partial y^{k+1}}}(\xi)\right|^2 y^{2k+1-2\alpha}\, d\xi\, dy$$
$$= \int_{\mathbb{R}^{n+1}_+} |\hat{\mathbf{f}}(\xi)|^2\, e^{-2y|\xi|}\, |\xi|^{2k+2}\, y^{2k+1-2\alpha}\, d\xi\, dy$$
$$= C\, \|\mathbf{f}\|^2_{\alpha}.$$

The proofs for others are similar. ▌

3. CALCULUS ON CLIFFORD VALUED FUNCTION AND FUNCTION SPACES

We start with the following lemma which shows how to transfer the derivatives in an integration which has a monogenic function as a factor of the integrand.

Lemma 3.1. *Suppose k is a positive integer, $\mathbf{G}$ is left monogenic on $\mathbb{R}^{n+1}_+$ and the Clifford valued function $\boldsymbol{\varphi} \in \mathbf{C}^k(\mathbb{R}^{n+1}_+)$ satisfies*

$$(D^\tau \boldsymbol{\varphi})\, \mathbf{G}(x, y) \to 0 \qquad \text{as} \quad |x| \to \infty,$$

for all multi-indices τ with $|\tau| \leqslant k$. Then

$$\int_{\mathbb{R}^{n+1}_+} \frac{\partial^k (\boldsymbol{\varphi}\mathbf{G})}{\partial y^k}\, y^\beta \, dx\, dy = \int_{\mathbb{R}^{n+1}_+} (\mathscr{D}_r^k \boldsymbol{\varphi})\, \mathbf{G}\, y^\beta \, dx\, dy. \tag{3.1}$$

Proof. Since $\mathbf{G}$ is left monogenic, we have $\partial \mathbf{G}/\partial y = -\sum_{j=1}^n \mathbf{e}_j (\partial \mathbf{G}/\partial x_j)$. Therefore

$$\frac{\partial^k(\boldsymbol{\varphi}\mathbf{G})}{\partial y^k} = \frac{\partial^{k-1}}{\partial y^{k-1}} \left(\frac{\partial \boldsymbol{\varphi}}{\partial y}\, \mathbf{G} - \boldsymbol{\varphi} \sum_{j=1}^n \mathbf{e}_j \frac{\partial \mathbf{G}}{\partial x_j} \right).$$

Using integration by parts on x_j, together with the assumption that $(\partial \boldsymbol{\varphi}/\partial x_j)\, \mathbf{G} \to 0$ as $|x| \to \infty$, we get

$$-\int_{\mathbb{R}^{n+1}_+} \boldsymbol{\varphi} \mathbf{e}_j \frac{\partial \mathbf{G}}{\partial x_j}\, dx = \int_{\mathbb{R}^{n+1}_+} \frac{\partial \boldsymbol{\varphi}}{\partial x_j}\, \mathbf{e}_j \mathbf{G}\, dx.$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} \frac{\partial^k(\boldsymbol{\varphi}\mathbf{G})}{\partial y^k}\, y^\beta \, dx\, dy \\
&\quad = \int_{\mathbb{R}^{n+1}_+} \frac{\partial^{k-1}}{\partial y^{k-1}} \left(\frac{\partial \boldsymbol{\varphi}}{\partial y}\, \mathbf{G} \right) y^\beta \, dx\, dy + \int_{\mathbb{R}^{n+1}_+} \frac{\partial^{k-1}}{\partial y^{k-1}} \left(\sum_{j=1}^n \frac{\partial \boldsymbol{\varphi}}{\partial x_j}\, \mathbf{e}_j \mathbf{G} \right) y^\beta \, dx\, dy \\
&\quad = \int_{\mathbb{R}^{n+1}_+} \frac{\partial^{k-1}((\mathscr{D}_r \boldsymbol{\varphi})\, \mathbf{G})}{\partial y^{k-1}}\, y^\beta \, dx\, dy.
\end{aligned}$$

By this iterative formula and the assumption, we can easily obtain formula (3.1). ∎

As an application of Lemma 3.1, we prove the following theorem of integration by parts, which is fundamental in Section 5.

Theorem 3.2. *Suppose m is a nonnegative integer and $\mathbf{B}$ is left monogenic on $\mathbb{R}^{n+1}_+$ with $y\mathbf{B}(x, y) \in \mathbf{L}\infty(\mathbb{R}^{n+1}_+)$. Then there is a bilinear form $II_m(\cdot, \cdot)$ on $\mathbf{R} \times \mathbf{L}$ such that for any $\mathbf{F} \in \mathbf{R}$ and $\mathbf{G} \in \mathbf{L}$*

$$\int_{\mathbb{R}^{n+1}_+} \mathbf{F}\bar{\mathbf{B}}\mathbf{G}\, dx\, dy = \frac{(-1)^m}{m!} \int_{\mathbb{R}^{n+1}_+} \mathbf{F}(\mathscr{D}^m \bar{\mathbf{B}})\, \mathbf{G}\, y^m \, dx\, dy + II_m(\mathbf{F}, \mathbf{G}). \tag{3.2}$$

Moreover the bilinear form $II_m(\mathbf{F}, \mathbf{G})$ *can be estimated by*

$$|II_m(\mathbf{F}, \mathbf{G})| \leqslant C \sum_{|\tau| + |\tau'| = m,\ |\tau'| \geqslant 1} \int_{\mathbb{R}^{n+1}_+} |D^{\tau'} \mathbf{F}|\ |D^{\tau} \overline{\mathbf{B}}|\ |\mathbf{G}|\ y^m\, dx\, dy. \tag{3.3}$$

To prove Theorem 3.2 we need the following result, which is also needed later.

For $\beta > -1$ and $p \geqslant 1$, denote by $\mathbf{L}^p_\beta = \mathbf{L}^p_\beta(\mathbb{R}^{n+1}_+)$ the space of all Clifford valued functions $\mathbf{f}$ defined on $\mathbb{R}^{n+1}_+$ with the norm

$$\|\mathbf{f}\|_{\mathbf{L}^p_\beta} = \left\{ \int_{\mathbb{R}^{n+1}_+} |\mathbf{f}(x, y)|^p\, y^\beta\, dx\, dy \right\}^{1/p}.$$

LEMMA 3.3. *Suppose* $\beta > -1$, $p \geqslant 1$ *and the Clifford valued function* $\mathbf{h} \in \mathbf{L}^p_\beta$ *is harmonic. Let* τ *be a multi-index and* $\delta > 0$. *Then*

(1) $\int_{\mathbb{R}^{n+1}_+} |y^{|\tau|} D^\tau \mathbf{h}(x, y)|^p\, y^\beta\, dx\, dy \leqslant C\, \|\mathbf{h}\|^p_{\mathbf{L}^p_\beta}$;

(2) $y^{|\tau|} D^\tau \mathbf{h}(x, y) \to 0$ *uniformly as* $|x| \to \infty$ *or* $y \to \infty$ *for* $y > \delta$;

(3) $\mathbf{h}(x, y + \delta)$ *is in* $\mathbf{L}^\infty(\mathbb{R}^{n+1}_+)$;

For complex valued $\mathbf{h}$ and $\beta = 0$, Lemma 3.3 was proved in [W3]. For $\beta \neq 0$, the proof is similar.

Proof of Theorem 3.2. Assume $k \geqslant 1$ is an integer. It is a standard result (see for example [W3] for $\mathbf{B}$ is harmonic)

$$\left\| y^{k+1} \frac{\partial^k \mathbf{B}}{\partial y^k} \right\|_{\mathbf{L}^\infty(\mathbb{R}^{n+1}_+)} \leqslant C\, \|y \mathbf{B}\|_{\mathbf{L}^\infty(\mathbb{R}^{n+1}_+)}$$

Therefore by Lemmas B and 3.3, we have

$$y^{k+1} \frac{\partial^k}{\partial y^k} \{\mathbf{F} \overline{\mathbf{B}} \mathbf{G}\} \to 0 \qquad \text{as} \quad y \to \infty \quad \text{or} \quad |x| \to \infty.$$

Using integration by parts, it is hence easy to establish the formula

$$\int_0^\infty \mathbf{F} \overline{\mathbf{B}} \mathbf{G}\, dy = \frac{(-1)^m}{m!} \int_0^\infty \frac{\partial^m}{\partial y^m} \{\mathbf{F} \overline{\mathbf{B}} \mathbf{G}\}\ y^m\, dy, \qquad \forall x \in \mathbb{R}^n.$$

By Lemma 3.1, we have therefore

$$\int_{\mathbb{R}^{n+1}_+} \mathbf{F} \overline{\mathbf{B}} \mathbf{G}\, dx\, dy = \frac{(-1)^m}{m!} \int_{\mathbb{R}^{n+1}_+} (\mathscr{D}^m_r(\mathbf{F} \overline{\mathbf{B}}))\, \mathbf{G}\, y^m\, dx\, dy. \tag{3.4}$$

For $0 \leqslant j_1, ..., j_m \leqslant n$, denote by

$$\vec{\kappa} = (j_1, ..., j_m), \qquad D_{\vec{\kappa}} = \frac{\partial^m}{\partial x_{j_1} \cdots \partial x_{j_m}}, \qquad \mathbf{e}_{\vec{\kappa}} = \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_m}.$$

Let $\kappa = (\kappa_0, \kappa_1, ..., \kappa_n)$ with κ_l be the number of times of the integer l appears in $\vec{\kappa}$. Then $D_{\vec{\kappa}} = D^{\kappa}$. Since

$$\mathscr{D}^m = \sum_{\vec{\kappa}} D_{\vec{\kappa}} \mathbf{e}_{\vec{\kappa}} = \sum_{\vec{\kappa}} D^{\kappa} \mathbf{e}_{\vec{\kappa}},$$

using Leibniz's rule we have

$$\mathscr{D}_r^m(\mathbf{F}\overline{\mathbf{B}}) = \sum_{\vec{\kappa}} D^{\kappa}(\mathbf{F}\overline{\mathbf{B}})\, \mathbf{e}_{\vec{\kappa}} = \sum_{\vec{\kappa}} \sum_{\tau \leqslant \kappa} \binom{\kappa}{\tau} (D^{\kappa - \tau}\mathbf{F})(D^{\tau}\overline{\mathbf{B}})\, \mathbf{e}_{\vec{\kappa}}.$$

Note that $\sum_{\vec{\kappa}} D^{\kappa} \overline{\mathbf{B}} \mathbf{e}_{\vec{\kappa}} = \mathscr{D}^m \overline{\mathbf{B}}$. Therefore (3.4) can be expanded further as

$$\int_{\mathbb{R}^{n+1}_+} \mathbf{F}\overline{\mathbf{B}}\mathbf{G}\, dx\, dy = \frac{(-1)^m}{m!} \int_{\mathbb{R}^{n+1}_+} \mathbf{F}(\mathscr{D}_r^m \overline{\mathbf{B}})\, \mathbf{G}\, y^m\, dx\, dy$$
$$+ \frac{(-1)^m}{m!} \sum_{\vec{\kappa}} \sum_{\tau < \kappa} \binom{\kappa}{\tau} \int_{\mathbb{R}^{n+1}_+} (D^{\kappa - \tau}\mathbf{F})(D^{\tau}\overline{\mathbf{B}})\, \mathbf{e}_{\vec{\kappa}}\, \mathbf{G}\, y^m\, dx\, dy.$$

This is enough to derive the desired result. ∎

The following estimate is needed in Section 5.

LEMMA 3.4. *Suppose $\sigma > -1$, k is a positive integer and $\boldsymbol{\varphi} \in \mathbf{C}_0^{\infty}(\mathbb{R}^{n+1}_+)$. Then*

$$\int_{\mathbb{R}^{n+1}_+} \frac{|\boldsymbol{\varphi}(x, y)|^2}{y^{2k}}\, y^{-\sigma}\, dx\, dy \leqslant C \int_{\mathbb{R}^{n+1}_+} |\mathscr{D}_r^k \boldsymbol{\varphi}(x, y)|^2\, y^{-\sigma}\, dx\, dy.$$

Proof. We first show that

$$\int_{\mathbb{R}^{n+1}_+} \frac{|\boldsymbol{\varphi}(x, y)|^2}{y^{2k}}\, y^{-\sigma}\, dx\, dy \leqslant C \int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial^k \boldsymbol{\varphi}(x, y)}{\partial y^k} \right|^2 y^{-\sigma}\, dx\, dy. \tag{3.5}$$

In fact integrating both sides of the following trivial inequality

$$\frac{\partial}{\partial y} |\boldsymbol{\varphi}(x, y)|^2 \leqslant 2\, |\boldsymbol{\varphi}(x, y)| \left| \frac{\partial \boldsymbol{\varphi}}{\partial y}(x, y) \right|$$

over the interval $[0, y]$, we obtain

$$|\varphi(x, y)|^2 = \int_0^y \frac{\partial}{\partial x} |\varphi(x, s)|^2 \, ds \leqslant 2 \int_0^y |\varphi(x, s)| \left| \frac{\partial \varphi}{\partial s}(x, s) \right| ds.$$

Integrating again the left and right sides of above inequality over the interval $[0, \infty)$ with respect to the measure $y^{-\sigma-2}\,dy$, then using Fubini's theorem, we get

$$\begin{aligned}\int_0^\infty \frac{|\varphi(x, y)|^2}{y^2} y^{-\sigma}\, dy &\leqslant 2 \int_0^\infty \int_0^y |\varphi(x, s)| \left| \frac{\partial \varphi}{\partial s}(x, s) \right| ds \frac{1}{y^{2+\sigma}}\, dy \\ &= 2 \int_0^\infty \left(\int_s^\infty \frac{dy}{y^{2+\sigma}} \right) |\varphi(x, s)| \left| \frac{\partial \varphi}{\partial s}(x, s) \right| ds \\ &= \frac{2}{1+\sigma} \int_0^\infty |\varphi(x, s)| \left| \frac{\partial \varphi}{\partial s}(x, s) \right| \frac{ds}{s^{1+\sigma}}.\end{aligned}$$

Applying Schwarz's inequality to the last integral above, we continue the estimate by

$$\leqslant \frac{2}{1+\sigma} \left(\int_0^\infty \left| \frac{\partial \varphi}{\partial s}(x, s) \right|^2 s^{-\sigma}\, ds \right)^{1/2} \left(\int_0^\infty \frac{|\varphi(x, s)|^2}{s^2} s^{-\sigma}\, ds \right)^{1/2}.$$

Therefore

$$\int_{\mathbb{R}_+^{n+1}} \frac{|\phi(x, y)|^2}{y^2} y^{-\sigma}\, dx\, dy \leqslant \frac{4}{(1+\sigma)^2} \int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial \phi}{\partial y}(x, y) \right|^2 y^{-\sigma}\, dx\, dy.$$

Replacing σ by $\sigma + 2k - 2$ in above inequality, we have

$$\begin{aligned}&\int_{\mathbb{R}_+^{n+1}} \frac{|\varphi(x, y)|^2}{y^{2k}} y^{-\sigma}\, dx\, dy \\ &\qquad \leqslant \frac{4}{(2k+\sigma-1)^2} \int_{\mathbb{R}_+^{n+1}} \frac{|(\partial \varphi/\partial y)(x, y)|^2}{y^{2k-2}} y^{-\sigma}\, dx\, dy.\end{aligned}$$

This iterative estimate yields (3.5).

We show next that

$$\int_{\mathbb{R}_+^{n+1}} |\mathscr{D}_r \varphi(x, y)|^2\, y^{-\sigma}\, dx\, dy = \int_{\mathbb{R}_+^{n+1}} |\bar{\mathscr{D}}_r \varphi(x, y)|^2\, y^{-\sigma}\, dx\, dy. \tag{3.6}$$

In fact, using the identity $\mathscr{D}\bar{\mathscr{D}} = \bar{\mathscr{D}}\mathscr{D}$ and integration by parts, we have

$$\int_{\mathbb{R}^{n+1}_+} (\mathscr{D}_r\varphi)(\overline{\mathscr{D}_r\varphi})\, y^{-\sigma}\, dx\, dy = \int_{\mathbb{R}^{n+1}_+} (\mathscr{D}_r\varphi)(\bar{\mathscr{D}}\bar{\varphi})\, y^{-\sigma}\, dx\, dy$$

$$= -\int_{\mathbb{R}^{n+1}_+} \varphi(\mathscr{D}\bar{\mathscr{D}}\bar{\varphi} - \sigma y^{-\sigma-1})\, dx\, dy$$

$$= -\int_{\mathbb{R}^{n+1}_+} \varphi(\bar{\mathscr{D}}\mathscr{D}\bar{\varphi} - \sigma y^{-\sigma-1})\, dx\, dy$$

$$= \int_{\mathbb{R}^{n+1}_+} (\bar{\mathscr{D}}_r\varphi)(\overline{\bar{\mathscr{D}}_r\varphi})\, y^{-\sigma}\, dx\, dy.$$

The scalar part of above identity is (3.6).

Finally the desired result follows from estimate (3.5), the fact that $\partial/\partial y = \frac{1}{2}(\mathscr{D}_r + \bar{\mathscr{D}}_r)$ and identity (3.6). ∎

We now turn to the Clifford valued function space $\mathbf{L}^2_\beta$. The following lemma constructs a dense set in $\mathbf{L}^2_\beta$, which consists of left monogenic functions and the image of the Cauchy–Riemann operator. This result is needed in Section 3.

LEMMA 3.5. *For $\beta > -1$, the set*

$$\mathbf{S} = \{\overline{\mathbf{\Phi}(x, y)} + y^{-\beta}\mathscr{D}_r\varphi(x, y) \colon \mathbf{\Phi} \in \mathbf{L}, \varphi \in \mathbf{C}^\infty_0(\mathbb{R}^{n+1}_+)\}$$

is dense in $\mathbf{L}^2_\beta$. Moreover if $\mathbf{\Phi} \in \mathbf{L}$ and $\varphi \in \mathbf{C}^\infty_0(\mathbb{R}^{n+1}_+)$, then

$$\|\bar{\mathbf{\Phi}} + y^{-\beta}\mathscr{D}_r\varphi\|^2_{\mathbf{L}^2_\beta} = \|\bar{\mathbf{\Phi}}\|^2_{\mathbf{L}^2_\beta} + \|y^{-\beta}\mathscr{D}_r\varphi\|^2_{\mathbf{L}^2_\beta}.$$

Proof. The norm identity is trivial. For denseness, we only need to show that for $\mathbf{\Psi} \in \mathbf{L}^2_\beta \cap \mathbf{C}^1(\mathbb{R}^{n+1}_+)$ the equation

$$(\mathbf{\Psi}, \mathscr{D}_r\varphi)_{\mathbf{L}^2_0} = (\mathbf{\Psi}, y^{-\beta}\mathscr{D}_r\varphi)_{\mathbf{L}^2_\beta} = 0, \qquad \forall \varphi \in \mathbf{C}^\infty_0(\mathbb{R}^{n+1}_+)$$

implies that $\bar{\mathbf{\Psi}}$ is left monogenic.

In fact, integration by parts yields

$$\langle \overline{\mathscr{D}\bar{\mathbf{\Psi}}}, \varphi \rangle_{\mathbf{L}^2_0} = -\langle \mathbf{\Psi}, \mathscr{D}_r\varphi \rangle_{\mathbf{L}^2_0} = 0.$$

This is enough to conclude that $\mathscr{D}\bar{\mathbf{\Psi}} \equiv 0$. ∎

4. α-CARLESON MEASURES

In this section, we study properties of α-Carleson measures and characterize them in terms of capacity. Results here have little to do with Clifford analysis.

A non-negative measure μ on $\mathbb{R}^{n+1}_+$ is called an α-Carleson measure if the following inequality holds:

$$\int_{\mathbb{R}^{n+1}_+} |f * p_y(x)|^2 \, d\mu(x, y) \leqslant C \, \|f\|^2_\alpha, \qquad \forall f \in C_0^\infty(\mathbb{R}^n \backslash \{0\}).$$

It is clear that in the above definition f can be replaced by a Clifford valued $\mathbf{f}$.

0-Carleson measures are just the classical Carleson measures. For $\alpha \leqslant 0$, α-Carleson measures can be characterized by the requirement that the inequality $\mu(T(\mathbb{B})) \leqslant C \, |\mathbb{B}|^{1-(2\alpha/n)}$ holds for any open ball $\mathbb{B} \subset \mathbb{R}^n$ (see for example [T, p. 372] for $\alpha = 0$; [L] for $\alpha < 0$ and $n = 1$). For $\alpha > 0$ and $n = 1$, characterizations of α-Carleson measures can be found in [St] and [KS].

We need some definitions. Assume $s > 0$. Let $\delta_s = (2\pi)^{-n/2} \int_{|x| < s} p_1(x) \, dx$. For $(x, y) \in \mathbb{R}^{n+1}$ denote by $\mathbb{B}_s(x, y)$ the ball centered at (x, y) with radius sy. For $x \in \mathbb{R}^n$, denote by $\Gamma_s(x) = \{(u, t) \in \mathbb{R}^{n+1}_+ : |u - x| \leqslant st\}$ the cone with vertex at x and opening s. For an open set $O \subset \mathbb{R}^n$ denote the tent of O in $\mathbb{R}^{n+1}_+$ with opening s by

$$T_s(O) = \{(x, y) \in \mathbb{R}^{n+1}_+ : \text{the ball centered at } x \text{ with radius } sy \text{ contains in } O\}.$$

Given a function $\mathbf{F}$ defined in $\mathbb{R}^{n+1}_+$, the nontangential maximal function (of opening s) of $\mathbf{F}$ is defined by

$$N_s(\mathbf{F})(x) = \sup_{(u,\, t) \in \Gamma_s(x)} |\mathbf{F}(u, t)|.$$

For $\sigma \geqslant 0$, define $\widehat{p_y^{(\sigma)}}(\xi) = |\xi|^\sigma \, e^{-y\,|\xi|}$. For a function g on $\mathbb{R}^n$, let

$$A_s^{(0)}(g)(x) = \sup_{t > 0} \left(\frac{1}{|\{u\colon |u - x| < t\}|} \int_{\{u\colon |u - x| < st\}} |g(u)|^2 \, du \right)^{1/2} \qquad \text{and}$$

$$A_s^{(\sigma)}(g)(x) = \left(\int_{\Gamma_s(x)} |g * p_t^{(\sigma)}(u)|^2 \, t^{2\sigma - n - 1} \, du \, dt \right)^{1/2}, \qquad \text{if} \quad \sigma > 0.$$

Then

$$\|A_s^{(\sigma)}(g)\|_0 = C s^{(n+1)/2} \, \|g\|_0. \tag{4.1}$$

In fact, for $\sigma = 0$ the above identity is standard, for $\sigma > 0$ straightforward computation yields

$$\begin{aligned}\|A_s^{(\sigma)}(g)\|_0^2 &= \int_{\mathbb{R}^n} \int_{\Gamma_s(x)} |g * p_t^{(\sigma)}(u)|^2 \, t^{2\sigma - n - 1} \, du \, dt \, dx \\ &= Cs^{n+1} \int_{\mathbb{R}_+^{n+1}} |g * p_t^{(\sigma)}(u)|^2 \, t^{2\sigma - 1} \, du \, dt \\ &= Cs^{n+1} \int_{\mathbb{R}_+^{n+1}} |\hat{g}(\xi)|^2 \, |\xi|^{2\sigma} \, e^{-2t|\xi|} t^{2\sigma - 1} \, du \, dt \\ &= Cs^{n+1} \, \|g\|_0^2.\end{aligned}$$

LEMMA 4.1. *Suppose p, $s > 0$ and μ is a non-negative measure on $\mathbb{R}_+^{n+1}$. Then*

$$\int_{\mathbb{R}_+^{n+1}} |f(x, y)|^p \, d\mu(x, y) \leqslant C \int_{\mathbb{R}_+^{n+1}} |N_s(F) * p_y(x)|^p \, d\mu(x, y).$$

Proof. For $t > 0$ let $\Omega_t = \{(x, y) \in \mathbb{R}_+^{n+1} : |F(x, y)| > t\}$. It is standard that

$$\int_{\mathbb{R}_+^{n+1}} |F(x, y)|^p \, d\mu(x, y) = \int_0^\infty \mu(\Omega_t) \, dt^p.$$

To estimate $\mu(\Omega_t)$, consider any compact subset K of Ω_t. Associate each point $(x, y) \in K$ with $\mathbb{B}(x; sy)$, which is the ball in $\mathbb{R}^n$ centered at x with radius sy. Clearly the point (x, y) is in the tent $T_{s/2}(\mathbb{B}(x; sy))$. Therefore the set of tents $\{T_{s/2}(\mathbb{B}(x, sy)) : (x, y) \in K\}$ covers K. By compactness, there are finitely many tents $\{T_{s/2}(\mathbb{B}_j)\}_1^m$ cover K, i.e. $K \subset \bigcup_1^m T_{s/2}(\mathbb{B}_j)$. Since $u \in \mathbb{B}(x; sy)$ if and only if $(x, y) \in \Gamma_s(u)$, it is easy to see that

$$\mathbb{B}_j \subset O_t^* = \{x \in \mathbb{R}^n : N_s(F)(x) > t\}$$

holds for each $j = 1, 2, ..., m$. Then $K \subset \bigcup_1^m T_{s/2}(\mathbb{B}_j) \subseteq T_{s/2}(O_t^*)$, and therefore

$$\mu(\Omega_t) \leqslant \mu(T_{s/2}(O_t^*)).$$

Let $O_t^* = \bigcup O_j$ be the decomposition of the open set O_t^* in $\mathbb{R}^n$, where each O_j is a connected open set and $O_j \cap O_k = \varnothing$ if $j \neq k$. For each j, it is clear that for any point, $(x, y) \in T_{s/2}(O_j)$,

$$\frac{1}{t} N_s(F) * p_y(x) \geqslant \chi_{O_j} * p_y(x) \geqslant (2\pi)^{-n/2} \int_{|u| < sy/2} p_y(u) \, du = \delta_{s/2}.$$

Thus $N_s(F) * p_y(x) \geqslant \delta_{s/2} t$ on $\bigcup T_{s/2}(O_j) = T_{s/2}(O_t^*)$. Therefore we have

$$\mu(T_{s/2}(O_t^*)) \leqslant \mu(\{(x, y) \in \mathbb{R}_+^{n+1} : N_s(F) * p_y(x) > \delta_{s/2} t\}).$$

This implies

$$\int_0^\infty \mu(\Omega_t)\, dt^p \leqslant \delta_{s/2}^{-p} \int_{\mathbb{R}_+^{n+1}} |N_s(F) * p_y(x)|^p \, d\mu(x, y).$$

The proof is complete. ∎

THEOREM 4.2. *Suppose $\beta < \alpha < n$ and the non-negative measure μ is an α-Carleson measure. Then the measure $y^{2(\alpha-\beta)}\, d\mu(x, y)$ is a β-Carleson measure.*

Proof. For $\alpha \leqslant 0$, the result is trivial. For $\alpha > 0$ and the ball $\mathbb{B}(u; t) \subset \mathbb{R}^n$, let $f(x) = p_t(x-u)$. It is easy to verify that $f * p_y(x) \geqslant Ct^{-n}$ for all $(x, y) \in T_1(\mathbb{B}(u; t))$ and $\|f\|_\alpha^2 = Ct^{-n-2\alpha}$. We have therefore

$$\mu(T_1(\mathbb{B}(u; t))) \leqslant Ct^{2n} \int_{\mathbb{R}_+^{n+1}} |f * p_y(x)|^2 \, d\mu(x, y) \leqslant Ct^{2n} \|f\|_\alpha^2 \leqslant Ct^{n-2\alpha}.$$

Thus, if $\beta < 0$, then

$$\int_{T_1(\mathbb{B}(u; t))} y^{2(\alpha-\beta)}\, d\mu(x, y) \leqslant Ct^{2(\alpha-\beta)} \mu(T_1(\mathbb{B}(u; t))) \leqslant Ct^{n-2\beta}.$$

This implies that $y^{2(\alpha-\beta)}\, d\mu(x, y)$ is a β-Carleson measure.

Assume now $\beta \geqslant 0$. First we note that, for $0 < \sigma < n$, f is in B_σ if and only if f can be represent as $f = I_\sigma * g$ with $\|f\|_\sigma = \|g\|_0$. Let $\sigma = \alpha_\beta$ and write $F_g(x, y) = y^\sigma I_\beta * g * p_y(x)$. It suffices to show that

$$\int_{\mathbb{R}_+^{n+1}} |F_g(x, y)| \, d\mu(x, y) \leqslant C \|g\|_0^2, \qquad \forall g \in B_0.$$

Since μ is an α-Carleson measure, by Lemma 4.1 and the identity (4.1) it suffices to show that

$$N_s(F_g)(x) \leqslant CI_\alpha * A_{2s}^{(\sigma)}(g)(x). \tag{4.2}$$

Let $\rho = s/(1+2s)$. For $(x', y) \in \mathbb{R}^{n+1}_+$, the volume of the ball $\mathbb{B}_\rho(x', y)$ satisfies $|\mathbb{B}_\rho(x', y)| = Cy^{n+1}$. And for any $(u, t) \in \mathbb{B}_\rho(x', y)$ we have $1/C \leqslant t/y \leqslant C$. Applying the mean value inequality to the subharmonic function $|I_\beta * g * p_t(u)|^2$ on the ball $\mathbb{B}_\rho(x', y)$, we have

$$|F_g(x', y)|^2 \leqslant \frac{y^{2\sigma}}{|\mathbb{B}_\rho(x', y)|} \int_{\mathbb{B}_\rho(x', y)} |I_\beta * g * p_t(u)|^2 \, du \, dt$$

$$\leqslant C \int_{\mathbb{B}_\rho(x', y)} \frac{|F_g(u, t)|^2}{t^{n+1}} \, du \, dt.$$

Since $\mathbb{B}_\rho(x', y) \subset \Gamma_{2s}(x)$ for any point $(x', y) \in \Gamma_s(x)$, we have

$$N_s(F_g)(x) \leqslant C \left\{ \int_{\Gamma_{2s}(x)} |F_g(u, t)|^2 \, t^{-n-1} \, du \, dt \right\}^{1/2}.$$

Finally, we notice that F_g can be represented as $F_g(u, t) = t^\sigma I_\alpha * g * p_t^{(\sigma)}(u)$. Hence the estimate (4.2) can be obtained by applying Minkowski's inequality to the right hand side of the above estimate. ▮

THEOREM 4.3. *Let $0 < \rho < 1$ and $\alpha < n$. Suppose μ is a nonnegative measure on $\mathbb{R}^{n+1}_+$. Then μ is an α-Carleson measure if and only if the average of μ defined by*

$$d\nu(x, y) = \left(|\mathbb{B}_\rho(x, y)|^{-1} \int_{\mathbb{B}_\rho(x, y)} d\mu \right) dx \, dy$$

is an α-Carleson measure.

Proof. Assume first that ν is an α-Carleson measure. Let $r = \rho/(1+\rho)$. It is not hard to check that for any $(x, y) \in \mathbb{R}^{n+1}_+$.

$$\{(u, t) \in \mathbb{R}^{n+1}_+ : \mathbb{B}_r(u, t) \ni (x, y)\} \subset \mathbb{B}_\rho(x, y)$$

and

$$\frac{|\mathbb{B}_\rho(x, y)|}{(1+2\rho)^{n+1}} \leqslant |\mathbb{B}_r(u, t)|.$$

Applying the mean value inequality to the subharmonic function $|F(u, t)|^2 = |f * p_t(u)|^2$ on the ball $\mathbb{B}_r(u, t)$, together with Fubini's theorem, we have

$$\int_{\mathbb{R}_+^{n+1}} |F(u,t)|^2\, d\mu(u,t) \leqslant \int_{\mathbb{R}_+^{n+1}} \frac{1}{|\mathbb{B}_r(u,t)|} \int_{\mathbb{B}_r(u,t)} |F(x,y)|^2\, dx\, dy\, d\mu(u,t)$$

$$\leqslant (1+2\rho)^{n+1} \int_{\mathbb{R}_+^{n+1}} |F(x,y)|^2\, dv(x,y)$$

$$\leqslant C\, \|f\|_\alpha^2.$$

Assume now μ is an α-Carleson measure. Since the result is trivially true for $\alpha \leqslant 0$, we assume $\alpha > 0$. Consider $g \in B_0$. By Fubini's theorem, we have

$$\int_{\mathbb{R}_+^{n+1}} |I_\alpha * g * p_t(u)|^2\, dv(u,t) = \int_{\mathbb{R}_+^{n+1}} |F(x,y)|^2\, d\mu(x,y), \tag{4.3}$$

where

$$F(x,y) = \left(\int_{\mathbb{R}_+^{n+1}} \frac{|I_\alpha * g * p_t(u)|^2}{|\mathbb{B}_\rho(u,t)|} \chi_{\mathbb{B}_\rho(u,t)}(x,y)\, du\, dt \right)^{1/2}.$$

Let $s = \rho/(1-\rho)$. As in the proof of Theorem 4.2, we can get $N_s(F)(x) \leqslant CI_\alpha * A_{2s}^{(0)}(g)(x)$. Applying this estimate, the identity (4.1) and Lemma 4.1 to the right hand side of (4.3), we can easily conclude that v is an α-Carleson measure. ∎

THEOREM 4.4. *Suppose $s > 0$ and $0 < \alpha \leqslant n/2$. A non-negative measure μ on $\mathbb{R}_+^{n+1}$ is an α-Carleson measure if and only if*

$$\mu(T_s(O)) = \int_{T_s(O)} d\mu(x,y) \leqslant C\, \mathrm{Cap}_\alpha(O). \tag{4.4}$$

holds for every bounded open set $O \subset \mathbb{R}^n$.

This theorem generalizes the characterization of classical Carleson measures (the case of $\alpha = 0$). Our approach is similar to the proof in [St], which pertains to the case $n = 1$ on the unit disk of complex plane.

Proof of Theorem 4.4. We prove the "only if" part first. Let $O = \bigcup O_j$ be an open set in $\mathbb{R}^n$, where O_j is connected open set and $O \cap O_k = \varnothing$ if $j \neq k$. By the definition of capacity, there is a test function $g \in B_0$ with $g \geqslant 0$, $I_\alpha * g(x) \geqslant \chi_O(x)$ and $\|g\|_0^2 \leqslant 2\, \mathrm{Cap}_\alpha(O)$. Let $f = I_\alpha * g$. Then $\|f\|_\alpha = \|g\|_0$. As in the proof of Lemma 4.1, we have

$$f * p_y(x) \geqslant \delta_s, \qquad \forall (x,y) \in T_s(O).$$

Therefore

$$\int_{T_s(O)} d\mu \leqslant \delta_s^{-2} \int_{\mathbb{R}^{n+1}_+} |f * p_y(x)|^2 \, d\mu(x, y) \leqslant C \, \|f\|_\alpha^2 \leqslant C \operatorname{Cap}_\alpha(O).$$

We now prove the "if" part. Let F_g be the function defined in the proof of Theorem 4.2 with $\sigma = 0$. Then as in the proof of Lemma 4.1, we have

$$\mu(\{(x, y) \in \mathbb{R}^{n+1}_+ : |F_g(x, y)| > t\}) \leqslant \mu(T_s(\{x \in \mathbb{R}^n : N_{2s}(F_g)(x) > t\})),$$

and $N_{2s}(F_g)(x) \leqslant CI_\alpha * A_{4s}^{(0)}(g)(x)$. These, together with the assumption, imply that

$$\mu(T_s(\{x \in \mathbb{R}^n : N_{2s}(F_g)(x) > t\})) \leqslant \operatorname{Cap}_\alpha\left(\left\{x \in \mathbb{R}^n : I_\alpha * A_{4s}^{(0)}(g)(x) > \frac{t}{C}\right\}\right).$$

Therefore

$$\int_{\mathbb{R}^{n+1}_+} |I_\alpha * g * p_y(x)|^2 \, d\mu(x, y) \leqslant C \int_0^\infty \operatorname{Cap}_\alpha(\{x \in \mathbb{R}^n : I_\alpha * A_{4s}^{(0)}(x) > t\}) \, dt^2$$

It was proved in [A] that the integral on the right hand side of the above estimate is bounded by $C \, \|A_{4s}^{(0)}(g)\|_0^2$, and therefore bounded by $C \, \|g\|_0^2$. This completes our proof. ∎

5. COMMUTATORS AND BILINEAR FORMS OF MONOGENIC FUNCTIONS

In this section, we always assume that b is a complex valued function and the commutators $\{[b, \mathcal{R}_j]\}_1^n$ are bounded on $B_0 = L^2(\mathbb{R}^n)$, or equivalently (see [CRW]) the symbol function b is in BMO. Therefore $b \in L^1(\mathbb{R}^n, (1 + |x|)^{-(n+1)} \, dx)$ and $\sup_{(x, y) \in \mathbb{R}^n} |(\partial b / \partial x_j)(x, y)| \, y \leqslant C \, \|b\|_{BMO}$. In fact, the following result indicates that the assumption is necessary.

Lemma 5.1. *Suppose $\alpha > \beta \geqslant 0$ and $1 \leqslant j \leqslant n$. If the commutator $[b, \mathcal{R}_j]$ is bounded on B_α then it is bounded on B_β.*

Proof. In fact the boundedness of $[b, \mathcal{R}_j]$ on B_α or on $B_{-\alpha}$ (the dual of B_α) are equivalent. Therefore interpolation theory yields the desired result. ∎

The following theorem ties the boundedness of the commutators with the boundedness of certain bilinear form defined on the monogenic function spaces.

THEOREM 5.2. *The commutators* $\{[b, \mathcal{R}_j]\}_1^n$ *are bounded on* B_α *if and only if the bilinear from defined on* $\mathbf{R} \times \mathbf{L}$ *by*

$$\int_{\mathbb{R}_+^{n+1}} \mathbf{F}(x, y)(\mathcal{D}b(x, y))\,\mathbf{G}(x, y)\,dx\,dy, \qquad \forall \mathbf{F} \in \mathbf{R}, \quad \mathbf{G} \in \mathbf{L} \tag{5.1}$$

is bounded by $C\,\|\mathbf{F}\|_\alpha\,\|\mathbf{G}\|_{-\alpha}$.

Proof. Let f and g in $C_0^\infty(\mathbb{R}^n\backslash\{0\})$ be complex valued functions and $1 \leqslant j \leqslant n$. Then

$$\langle [b, \mathcal{R}_j](f), \bar{g}\rangle = \langle f\mathcal{R}_j(g) + \mathcal{R}_j(f)\,g, \bar{b}\rangle.$$

It is a straightforward computation that

$$fg - \sum_{j=1}^n \mathcal{R}_j(f)\,\mathcal{R}_j(g) = -\sum_{j=1}^n \{f\mathcal{R}_j(\mathcal{R}_j(g)) + \mathcal{R}_j(f)\,\mathcal{R}_j(g)\}.$$

$$\mathcal{R}_j(f)\,\mathcal{R}_k(g) - \mathcal{R}_k(f)\,\mathcal{R}_j(g) = f\mathcal{R}_j(\mathcal{R}_k(g)) + \mathcal{R}_j(f)\,\mathcal{R}_k(g) - \{f\mathcal{R}_k(\mathcal{R}_j(g)) + \mathcal{R}_k(f)\,\mathcal{R}_j(g)\}.$$

Since Riesz transformations $\{\mathcal{R}_j\}_1^n$ are bounded on B_α, we conclude that for $1 \leqslant j, k \leqslant n$ the bilinear forms

$$\langle f\mathcal{R}_j(g) + \mathcal{R}_j(f)\,g, \bar{b}\rangle_{L^2(\mathbb{R}^n)}, \quad \left\langle fg - \sum_{j=1}^n \mathcal{R}_j(f)\,\mathcal{R}_j(g), \bar{b}\right\rangle_{L^2(\mathbb{R}^n)} \quad \text{and}$$

$$\langle \mathcal{R}_j(f)\,\mathcal{R}_k(g) - \mathcal{R}_k(f)\,\mathcal{R}_j(g), \bar{b}\rangle_{L^2(\mathbb{R}^n)}$$

are bounded (by $C\,\|f\|_\alpha\,\|g\|_{-\alpha}$) on $B_\alpha \times B_{-\alpha}$ if and only if the commutators $\{[b, \mathcal{R}_j]\}_1^n$ are bounded on B_α. Note that this statement is still true if f and g are replaced by Clifford valued functions $\mathbf{f}$ and $\mathbf{g}$, respectively.

For any $\mathbf{f}$ and $\mathbf{g}$ in $\mathbf{C}_0^\infty(\mathbb{R}^n\backslash\{0\})$, let

$$\mathbf{F}(x, y) = \mathbf{f} * p_y(x) + \sum_{j=1}^n \mathcal{R}_j(\mathbf{f}) * p_y(x)\,\mathbf{e}_j, \tag{5.2a}$$

$$\mathbf{G}(x, y) = \mathbf{g} * p_y(x) + \sum_{j=1}^n \mathbf{e}_j\mathcal{R}_j(\mathbf{g}) * p_y(x). \tag{5.2b}$$

It is easy to check that $\mathbf{F}$ and $\mathbf{G}$ are right and left monogenic functions on $\mathbb{R}_+^{n+1}$, respectively. By Lemma A, we know also that functions

$\mathbf{F} \in \mathbf{R}$ and $\mathbf{G} \in \mathbf{L}$ can always be written as (5.2a) and (5.2b), respectively. Since

$$\mathbf{FG} = \mathbf{fg} - \sum_{j=1}^{n} \mathcal{R}_j(\mathbf{f})\, \mathcal{R}_j(\mathbf{g}) + \sum_{j=1}^{n} \{\mathbf{f}\mathbf{e}_j \mathcal{R}_j(\mathbf{g}) + \mathcal{R}_j(\mathbf{f})\, \mathbf{e}_j \mathbf{g}\}$$
$$+ \sum_{j \neq k} \{\mathcal{R}_j(\mathbf{f})\, \mathbf{e}_j \mathbf{e}_k \mathcal{R}_k(\mathbf{g}) - \mathcal{R}_k(\mathbf{f})\, \mathbf{e}_j \mathbf{e}_k \mathcal{R}_j(\mathbf{g})\},$$

we conclude that the bilinear form

$$\langle \mathbf{FG}, \bar{b} \rangle_{\mathbf{L}^2(\mathbb{R}^n)}, \qquad \forall \mathbf{F} \in \mathbf{R}, \quad \mathbf{G} \in \mathbf{L}$$

is bounded by $C\, \|\mathbf{F}\|_\alpha\, \|\mathbf{G}\|_{-\alpha}$ if and only if the commutators $\{[b, \mathcal{R}_j]\}_1^n$ are bounded on B_α. It is easy to see that $b(x, y)\, \mathbf{F}(x, y)\, \mathbf{G}(x, y) \to 0$ as $y \to \infty$. Therefore we can establish the formula

$$\langle \mathbf{FG}, \bar{b} \rangle_{\mathbf{L}^2(\mathbb{R}^n)} = -\int_{\mathbb{R}^{n+1}_+} \frac{\partial}{\partial y} \{b\mathbf{FG}\}\, dx\, dy$$

Applying Lemma 3.1 to the integral on the right hand side above, we get the desired result. ∎

When we apply Theorem 3.2 to the bilinear form (5.1), we get two bilinear forms as displaced in (3.2) with $\mathbf{B}\quad(x, y) = \overline{\mathcal{D}b(x, y)}$. We need to show these two bilinear forms are bounded under appropriate conditions. The following three theorems serve the purposes in a more general way.

Theorem 5.3. *Suppose $\sigma > -1$, $\alpha < n$ and $\mathbf{B}(x, y)$ is a harmonic function on $\mathbb{R}^{n+1}_+$. Then the measure $|\mathbf{B}(x, y)|^2\, y^\sigma\, dx\, dy$ is an α-Carleson measure if and only if the measure*

$$\sum_{j=0}^{n} \left| \frac{\partial \mathbf{B}}{\partial x_j}(x, y) \right|^2 y^{\sigma+2}\, dx\, dy$$

is an α-Carleson measure.

Proof. We start with the "only if" part. Using the reproducing formula for harmonic function on the ball ([ABR, p. 157]), we have for $0 \leqslant j \leqslant n$

$$y^2 \left| \frac{\partial \mathbf{B}}{\partial x_j}(x, y) \right|^2 \leqslant \frac{C}{|\mathbb{B}_{1/2}(x, y)|} \int_{\mathbb{B}_{1/2}(x, y)} |\mathbf{B}(u, t)|^2\, du\, dt, \quad \forall (x, y) \in \mathbb{R}^{n+1}_+.$$

Therefore

$$\sum_{j=0}^{n} \left| \frac{\partial \mathbf{B}}{\partial x_j} \right|^2 y^{2+\sigma} \leqslant \frac{C}{|\mathbb{B}_{1/2}(x, y)|} \int_{\mathbb{B}_{1/2}(x, y)} |\mathbf{B}(u, t)|^2 t^{\sigma} \, du \, dt.$$

The desired result follows from Theorem 4.3.

For the "if" part, it suffices to show that for a complex valued harmonic function $b = \mathbf{B}$ the result is true.

For $\mathbf{F} \in \mathbf{R}$, $\mathbf{G} \in \mathbf{L}$ and $\boldsymbol{\phi} \in \mathbf{C}_0^{\infty}(\mathbb{R}_+^{n+1})$, consider the following identities

$$\int_{\mathbb{R}_+^{n+1}} \mathbf{F}(x, y)\, b(x, y)\, \mathbf{G}(x, y)\, y^{\sigma} \, dx \, dy$$
$$= -\frac{1}{\sigma+1} \int_{\mathbb{R}_+^{n+1}} \mathbf{F}(x, y)(\mathscr{D} b(x, y))\, \mathbf{G}(x, y)\, y^{\sigma+1} \, dx \, dy,$$

$$\int_{\mathbb{R}_+^{n+1}} \mathbf{f}(x, y)(\mathscr{D} b(x, y))\, \overline{\boldsymbol{\phi}(x, y) \, dx \, dy}$$
$$= -\int_{\mathbb{R}_+^{n+1}} \mathbf{F}(x, y)\, b(x, y)\, \overline{\mathscr{D}_r \boldsymbol{\phi}(x, y)} \, dx \, dy$$

Since $|\mathscr{D} b|^2 = \sum_{j=0}^{n} |\partial b / \partial x_j|^2$, we know by the assumption that $|\mathscr{D} b(x, y)|^2 y^{\sigma+2} \, dx \, dy$ is an α-Carleson measure. Hence

$$\left| \int_{\mathbb{R}_+^{n+1}} \mathbf{F}(x, y)\, b(x, y)\, \mathbf{G}(x, y)\, y^{\sigma} \, dx \, dy \right| \leqslant C \, \|\mathbf{F}\|_{\alpha} \, \|\mathbf{G}\|_{\mathbf{L}_{\sigma}^2},$$

$$\left| \int_{\mathbb{R}_+^{n+1}} \mathbf{F}(x, y)\, b(x, y)\, \overline{\mathscr{D}_r \boldsymbol{\phi}(x, y)} \, dx \, dy \right| \leqslant C \, \|\mathbf{F}\|_{\alpha} \left\| \frac{\boldsymbol{\phi}}{y^{1+\sigma}} \right\|_{\mathbf{L}_{\sigma}^2}$$
$$\leqslant C \, \|\mathbf{F}\|_{\alpha} \, \|y^{-\sigma} \mathscr{D}_r \boldsymbol{\phi}\|_{\mathbf{L}_{\sigma}^2}.$$

The last estimate above is obtained by using Lemma 3.4.

Combine above two estimates together, we have

$$\left| \int_{\mathbb{R}_+^{n+1}} \mathbf{F}(x, y)\, b(x, y)(\mathbf{G}(x, y) + y^{-\sigma} \overline{\mathscr{D}_r \boldsymbol{\phi}(x, y)})\, y^{\sigma} \, dx \, dy \right|$$
$$\leqslant C(\|\mathbf{F}\|_{\alpha} + \|y^{-\sigma} \mathscr{D}_r \boldsymbol{\phi}\|_{\mathbf{L}_{\sigma}^2}).$$

By Lemmas 3.5, we have therefore

$$\int_{\mathbb{R}_+^{n+1}} |\mathbf{F}(x, y)\, b(x, y)|^2 \, y^{\sigma} \, dx \, dy \leqslant C \, \|\mathbf{F}\|_{\alpha}^2.$$

The proof is complete. ∎

Suppose $\beta > -1$ and $\mathbf{B}$ is a Clifford valued function on $\mathbb{R}^{n+1}_+$. Define the bilinear form $T(\cdot,\ \cdot)$ on $\mathbf{R}\times\mathbf{L}$ by

$$T(\mathbf{F}, \mathbf{G}) = \int_{\mathbb{R}^{n+1}_+} \mathbf{F}(x, y)\, \overline{\mathbf{B}(x, y)}\, \mathbf{G}(x, y)\, y^{\beta}\, dx\, dy, \qquad \mathbf{F}\in\mathbf{R}, \quad \mathbf{G}\in\mathbf{L}. \tag{5.3}$$

THEOREM 5.4. *Suppose $\alpha < n$ and $\mathbf{B}$ is left monogenic on $\mathbb{R}^{n+1}_+$. Then the bilinear form $T(\mathbf{F}, \mathbf{G})$, defined by (5.3), is bounded by $C\,\|\mathbf{F}\|_{\alpha}\,\|\mathbf{G}\|_{\mathbf{L}^2_\beta}$ and $|\mathbf{B}(x, y)|^2\, y^{\beta+2}\, dx\, dy$ is an $(\alpha-1)$-Carleson measure if and only if the measure.*

$$|\mathbf{B}(x, y)|^2\, y^{\beta}\, dx\, dy$$

is an α-Carleson measure.

Proof. The "If" part follows from Theorem 4.2 and the following estimate:

$$\left|\int_{\mathbb{R}^{n+1}_+} \mathbf{F}\overline{\mathbf{B}}\mathbf{G}\, y^{\beta}\, dx\, dy\right| \leqslant C\,\|\mathbf{F}\overline{\mathbf{B}}\|_{\mathbf{L}^2_\beta}\,\|\mathbf{G}\|_{\mathbf{L}^2_\beta} \leqslant C\,\|\mathbf{F}\|_{\alpha}\,\|\mathbf{G}\|_{\mathbf{L}^2_\beta}.$$

For the "only if" part, we prove first that

$$\int_{\mathbb{R}^{n+1}_+} |\mathbf{F}\overline{\mathbf{B}}|^2\, y^{\beta}\, dx\, dy \leqslant C\,\|\mathbf{F}\|^2_{\alpha}, \qquad \forall \mathbf{F}\in\mathbf{R}. \tag{5.4}$$

Because of Lemma 3.5 and the assumption that the bilinear form T is bounded, we only need to prove that

$$|\langle \mathbf{F}\overline{\mathbf{B}}, \mathscr{D}_r\boldsymbol{\varphi}\rangle_{\mathbf{L}^2_0}| \leqslant C\,\|\mathbf{F}\|_{\alpha}\,\|y^{-\beta}\mathscr{D}_r\boldsymbol{\varphi}\|_{\mathbf{L}^2_\beta}, \qquad \forall\boldsymbol{\varphi}\in\mathbf{C}^{\infty}_0(\mathbb{R}^{n+1}_+).$$

In fact, using integration by parts, we have

$$\langle \mathbf{F}\overline{\mathbf{B}}, \mathscr{D}_r\boldsymbol{\varphi}\rangle_{\mathbf{L}^2_0} = \int_{\mathbb{R}^{n+1}_+} \mathbf{F}\overline{\mathbf{B}}\overline{\mathscr{D}}(\bar{\boldsymbol{\varphi}})\, dx\, dy = -\int_{\mathbb{R}^{n+1}_+} (\overline{\mathscr{D}}_r(\mathbf{F}\overline{\mathbf{B}}))\, \bar{\boldsymbol{\varphi}}\, dx\, dy$$

Since $\mathscr{D}\mathbf{B} \equiv 0$, we have

$$\overline{\mathscr{D}_r(\mathbf{F}\overline{\mathbf{B}})} = \sum_{j=0}^{n} \frac{\partial \mathbf{F}}{\partial x_j}\, \overline{\mathbf{B}}\,\overline{\mathbf{e}_j}.$$

By the assumption that the measure $|\mathbf{B}(x, y)|^2\, y^{\beta+2}\, dx\, dy$ is an $(\alpha-1)$-Carleson measure, we have therefore

$$|(\mathbf{F}\overline{\mathbf{B}}, \mathscr{D}_r \boldsymbol{\varphi})_{\mathbf{L}^2(\mathbb{R}^{n+1}_+)}| \leqslant C \left(\sum_{j=0}^{n} \left\| \frac{\partial \mathbf{F}}{\partial x_j} \right\|_{\alpha-1} \right) \|y^{-\beta-1} \boldsymbol{\varphi}\|_{\mathbf{L}^2_\beta}$$
$$\leqslant C\, \|\mathbf{F}\|_\alpha\, \|y^{-\beta} \mathscr{D}_r \boldsymbol{\varphi}\|_{\mathbf{L}^2_\beta}.$$

This last inequality is obtained by Lemmas B and 3.4. Hence the estimate (5.4) is proved.

Finally if we let $\mathbf{F} = f + \sum_{j=1}^n \mathscr{R}_j(f)\, \mathbf{e}_j$, then $|\mathbf{F}|^2 = \mathbf{F}\overline{\mathbf{F}}$ and therefore

$$|\mathbf{F}\overline{\mathbf{B}}|^2 = \text{Scalar}(\mathbf{B}\overline{\mathbf{F}}\mathbf{F}\overline{\mathbf{B}}) = |\mathbf{F}|^2\, \text{Scalar}(\mathbf{B}\overline{\mathbf{B}}) = |\mathbf{F}|^2\, |\mathbf{B}|^2.$$

Estimate (5.4) yields therefore $\int_{\mathbb{R}^{n+1}_+} |\mathbf{F}|^2\, |\mathbf{B}|^2\, dx\, dy \leqslant C\, \|\mathbf{F}\|^2_\alpha$. This is enough. ∎

THEOREM 5.5. *Suppose* $\sigma > -1$, $\alpha < n$ *and* $\mathbf{B}$ *is left monogenic on* $\mathbb{R}^{n+1}_+$. *Then the measure*

$$\sum_{|\tau| = m} |D^\tau \overline{\mathbf{B}}(x, y)|^2\, y^\sigma\, dx\, dy$$

is an α*-Carleson measure if and only if the measure* $|\mathscr{D}^{m-k} \overline{\mathbf{B}}|^2\, y^\sigma\, dx\, dy$ *is an* $(\alpha-k)$*-Carleson measure for every* $k = 0, 1, ..., m$.

Proof. We show the "only if" part first. Since $\mathscr{D}\mathbf{B} \equiv 0$, we have $\mathscr{D}\overline{\mathbf{B}} = 2(\partial \overline{\mathbf{B}}/\partial y)$. This implies that the measure

$$|\mathscr{D}^m \overline{\mathbf{B}}|^2\, y^\sigma\, dx\, dy = 2^{2m} \left| \frac{\partial^m \overline{\mathbf{B}}}{\partial y^m} \right|^2 y^\sigma\, dx\, dy$$

is an α-Carleson measure. By Theorem 4.2, we know that the measure

$$\sum_{|\tau'| = m-1} \sum_{j=0}^{n} \left| \frac{\partial D^{\tau'} \overline{\mathbf{B}}(x, y)}{\partial x_j} \right|^2 y^{\sigma+2}\, dx\, dy$$

is an $(\alpha-1)$-Carleson measure. Therefore by Theorem 5.3, the measure

$$\sum_{|\tau'| = m-1} |D^{\tau'} \overline{\mathbf{B}}(x, y)|^2\, y^\sigma\, dx\, dy$$

is an $(\alpha-1)$-Carleson measure. Repeating the process above, we get the desired result.

The "if" part can be proved by induction on m. For $m = 1$, it suffices to show that for each $j = 0, 1, ..., n$ the measure $|\partial \overline{\mathbf{B}}/\partial x_j|^2\, y^\sigma\, dx\, dy$ is an α-Carleson measure. Since $|\overline{\mathbf{B}}|^2\, y^\sigma\, dx\, dy$ is an $(\alpha-1)$-Carleson measure, by

Theorem 5.3 we know $|\partial\bar{\mathbf{B}}/\partial x_j|^2\, y^{\sigma+2}\, dx\, dy$ is also an $(\alpha-1)$-Carleson measure. It is easy to verify that

$$(\sigma+1)\int_{\mathbb{R}^{n+1}_+} \mathbf{F}\,\frac{\partial\bar{\mathbf{B}}}{\partial x_j}\,\mathbf{G}\,y^{\sigma}\,dx\,dy = -\int_{\mathbb{R}^{n+1}_+} \mathbf{F}(\mathscr{D}\bar{\mathbf{B}})\,\frac{\partial \mathbf{G}}{\partial x_j}\,y^{\sigma+1}\,dx\,dy + \int_{\mathbb{R}^{n+1}_+} \sum_{k=0}^{n} \frac{\partial \mathbf{F}}{\partial x_k}\,\bar{\mathbf{B}}\mathbf{e}_k\,\frac{\partial \mathbf{G}}{\partial x_j}\,y^{\sigma+1}\,dx\,dy - (\sigma+1)\int_{\mathbb{R}^{n+1}_+} \frac{\partial \mathbf{F}}{\partial x_j}\,\bar{\mathbf{B}}\mathbf{G}\,y^{\sigma}\,dx\,dy.$$

Therefore we can show that the bilinear form

$$\int_{\mathbb{R}^{n+1}_+} \mathbf{F}\,\frac{\partial\bar{\mathbf{B}}}{\partial x_j}\,\mathbf{G}\,y^{\sigma}\,dx\,dy$$

is bounded by $\|\mathbf{F}\|_{\alpha}\,\|\mathbf{G}\|_{L^2_\sigma}$. By Theorem 5.4, we complete the proof for $m=1$. The argument for the final part of the induction is similar. ∎

The last result needed in this paper is

THEOREM 5.6. *Suppose* $\alpha \geqslant \frac{1}{2}$ *and* $m=[2\alpha-1]$. *The commutators* $\{[b, \mathscr{R}_j]\}_1^n$ *are bounded on* B_α *if and only if the estimate*

$$\left|\int_{\mathbb{R}^{n+1}_+} \mathbf{F}(\mathscr{D}^{m+1}b)\,\mathbf{G}\,y^{m}\,dx\,dy\right| \leqslant C\,\|\mathbf{F}\|_{\alpha}\,\|\mathbf{G}\|_{-\alpha}, \tag{5.5}$$

holds for $\mathbf{F}\in\mathbf{R}$ *and* $\mathbf{G}\in\mathbf{L}$, *and the measure* $\sum_{|\tau|=m} |D^{\tau}\mathscr{D}b(x,y)|^2\, y^m\, dx\, dy$ *is a* $(m+1)/2$-*Carleson measure.*

Proof. Let σ be the number such that $\alpha=\sigma+(m+1)/2$. Then $0\leqslant\sigma<\frac{1}{2}$. We prove the "only if" part by induction on m.

For $m=0$, estimate (5.5) is true because of Theorem 5.2. Since $1/2\leqslant\alpha$, we have by Theorems 5.1 and 5.2 and Lemma B that the bilinear form

$$\int_{\mathbb{R}^{n+1}_+} \mathbf{F}(\mathscr{D}b)\,\mathbf{G}\,dx\,dy$$

is bounded by $C\,\|\mathbf{F}\|_{1/2}\,\|\mathbf{G}\|_{\mathbf{L}^2_0}$. Since $y\mathscr{D}b(x,y)\in\mathbf{L}^{\infty}(\mathbb{R}^{n+1}_+)$ (i.e. $|\mathscr{D}b|^2\,y^2\,dx\,dy$ is a $-1/2$-Carleson measure), by Theorem 5.4 we have that $|\mathscr{D}b|^2\,dx\,dy$ is a 1/2-Carleson measure.

Assume the result is true for $m \leqslant k$. Consider $m = k+1$. For any $l = 0, 1, ..., k$, since $\sigma + (l+1)/2 < \alpha$, we have that $\{[b, \mathcal{R}_j]\}_1^n$ are bounded on $B_{(l+1)/2}$ by Theorem 5.1. Therefore by induction the measure

$$\sum_{|\kappa| = l} |D^\kappa \mathcal{D} b(x, y)|^2 \, y^l \, dx \, dy$$

is a $(l+1)/2$-Carleson measure.

In order to prove the estimate (5.5), by Theorem 3.2, we only need to prove that, with $\mathbf{F} \in \mathbf{R}$, $\mathbf{G} \in \mathbf{L}$ and $\overline{\mathbf{B}} = \mathcal{D} b$, the bilinear form $II_{k+1}(\mathbf{F}, \mathbf{G})$, defined by formula (3.2), is bounded by $C \|\mathbf{F}\|_\alpha \|\mathbf{G}\|_{-\alpha}$. Since the estimate (3.3) yields

$$|II_{k+1}| \leqslant C \sum_{|\tau| + |\tau'| = k+1, \, |\tau'| \geqslant 1} \|(D^{\tau'} \mathbf{F})(D^\tau \mathcal{D} b)\|_{\mathbf{L}^2_{k+1-2\sigma}} \|G\|_{\mathbf{L}^2_{k+1+2\sigma}},$$

and $\|G\|_{\mathbf{L}^2_{k+1+2\sigma}} = \|\mathbf{G}\|_{-\alpha}$ is clear, we only need to show that

$$\|(D^{\tau'} \mathbf{F})(D^\tau \mathcal{D} b)\|_{\mathbf{L}^2_{k+1-2\sigma}} \leqslant C \|\mathbf{F}\|_\alpha, \qquad \text{for} \quad |\tau| + |\tau'| = k+1 \quad \text{and} \quad |\tau'| \geqslant 1. \tag{5.6}$$

If $|\tau'| > \alpha$, then $D^{\tau'} \mathbf{F} \in \mathbf{B}_{\alpha - |\tau'|} = \mathbf{L}^2_{1 - 2|\tau'| + 2\alpha}$, by Lemma B. Since

$$\|y^{1+|\tau|} D^\tau \mathcal{D} b\|_{\mathbf{L}^\infty(\mathbb{R}^{n+1}_+)} \leqslant C \|y \mathcal{D} b\|_{\mathbf{L}^\infty(\mathbb{R}^{n+1}_+)} \leqslant C,$$

we have therefore

$$\begin{aligned} \|(D^{\tau'} \mathbf{F})(D^\tau \mathcal{D} b)\|_{\mathbf{L}^2_{k+1-2\sigma}} &\leqslant C \|D^{\tau'} \mathbf{F}\|_{\mathbf{L}^2_{k-1-2\sigma-2|\tau|}} \\ &= C \|D^{\tau'} \mathbf{F}\|_{\mathbf{L}^2_{2|\tau'|-2\alpha-1}} \leqslant C \|F\|_\alpha. \end{aligned}$$

If $|\tau'| < \alpha$, then $|\tau'| \leqslant (k+2)/2$. Let $l = k+2-2|\tau'|$. We have $0 \leqslant l \leqslant k$ because of $|\tau'| \geqslant 1$. Therefore $\sum_{|\kappa| = l} |D^\kappa \mathcal{D} b(x, y)|^2 \, y^l \, dx \, dy$ is a $(l+1)/2$-Carleson measure. Since $k+1-|\tau'| = l + |\tau'| - 1 \geqslant l$, by Theorem 5.3, $\sum_{|\tau| = k+1-|\tau'|} |D^\tau \mathcal{D} b(x, y)|^2 \, y^k \, dx \, dy$ is a $(l+1)/2$-Carleson measure. Note that $\sigma < 1/2$ and $l/2 + \sigma = \alpha - |\tau'|$. Hence by Theorem 4.2 $\sum_{|\tau| = k+1-|\tau'|} |D^\tau \mathcal{D} b(x, y)|^2 \, y^{k+1-2\sigma} \, dx \, dy$ is a $(\alpha - |\tau'|)$-Carleson measure. Thus

$$\|(D^{\tau'} \mathbf{F})(D^\tau \mathcal{D} b)\|_{\mathbf{L}^2_{k+1-2\sigma}} \leqslant C \|D^{\tau'} \mathbf{F}\|_{\alpha - |\tau'|} \leqslant C \|F\|_\alpha.$$

Estimate (5.6) is proved.

Since $\{[b, \mathcal{R}_j]\}_1^n$ are bounded on $B_{k+1/2}$, we have

$$\left| \int_{\mathbb{R}^{n+1}_+} \mathbf{F}(\mathcal{D}^{k+2} b) \, \mathbf{G} \, y^{k+1} \, dx \, dy \right| \leqslant C \|\mathbf{F}\|_{(k+2)/2} \|\mathbf{G}\|_{\mathbf{L}^2_{k+1}}.$$

Since $|\mathscr{D}^{k+1}b|^2\, y^k\, dx\, dy$ is a $(k+1)/2$-Carleson measure, we have by Theorems 4.2 and 4.3 that $|\mathscr{D}^{k+2}b|^2\, y^{k+3}\, dx\, dy$ is a $k/2$-Carleson measure. Therefore by Theorem 5.4 $|\mathscr{D}^{k+2}b|^2\, y^{k+1}\, dx\, dy$ is a $(k+2)/2$-Carleson measure. This implies, by Theorem 5.5, that $\sum_{|\tau|=k+1} |D^\tau \mathscr{D}b|^2\, y^{k+1}\, dx\, dy$ is a $k+2/2$-Carleson measure.

To prove the "if" part, we note that the case of $m=0$ follows from Theorem 5.2. Assume $m \geqslant 1$. As above, one can prove that for $\mathbf{F} \in \mathbf{R}$ and $\mathbf{G} \in \mathbf{L}$ the bilinear form $II_m(\mathbf{F}, \mathbf{G})$ is bounded by $C\,\|\mathbf{F}\|_\alpha\,\|\mathbf{G}\|_{-\alpha}$, if the measure $\sum_{|\tau|=m} |D^\tau \mathscr{D}b(x, y)|^2\, y^m\, dx\, dy$ is a $(m+1)/2$-Carleson measure. The desired result follows from Theorem 3.2 and Theorem 5.2. ∎

Proof of the main theorem. It is a consequence of Theorems 4.4, 5.3, 5.4, 5.5 and 5.6. ∎

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