Entropy and Følner function in algebras

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We introduce the notion of entropic Følner function for algebras and we study its relation with the isoperimetric profile and the lower transcendence degree. Under the assumption of a technical conjecture we use the Shannon inequality to derive a theorem on the lower transcendence degree of domains and division algebras. We finally discuss its relation with some old conjectures of M. Artin, L. Small and J. Zhang.

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Introduction

The present work was inspired by [8], where Gromov introduced the notion of entropic Følner function, and it should be thought as a continuation of [5].

We observed in [5] that the isoperimetric profile is a finer invariant than the lower transcendence degree introduced by J. Zhang in [11] (we will give the definitions later), and we have already seen how this geometric approach can give more insight in the understanding of this important invariant.

In this paper we introduce the related notion of entropic Følner function of an algebra (following [8]), and we study the relation of this invariant with the isoperimetric profile, the Følner function (already considered in [5]) and its entropic counterpart.

In doing this, as a first byproduct, we find a theorem on the lower transcendence degree of tensor product of algebras that extends some results in [11].

After introducing the notion of entropy, we find another result on algebras that are free left modules over a subalgebra, which extends another theorem in [11] about the lower transcendence degree.

Then we develop our entropic methods, and, under the assumption of a technical conjecture (Conjecture 1, which will be stated later), we use techniques like the Shannon inequality to prove an inequality relating the entropic Følner functions of a domain $A$ and of a division subalgebra of $A$. This will lead us to the following theorem (cf. [3]):
Theorem 0.1. Let $A$ be a domain over a field $K$ of characteristic $0$ and $D \supset K$ a division subalgebra of $A$, and assume Conjecture 1. If $A$ is not left algebraic over $D$, then $\text{Ld}(A) \geq \text{Ld}(D) + 1$, where $\text{Ld}$ denotes the lower transcendence degree.

After that, we check that Conjecture 1 is true when we have an element $x \in A$ left transcendent over $D$ such that $xD = Dx$. More generally we find a result on Ore extensions that extends another theorem in [11].

Finally we discuss some consequences of Theorem 0.1 and how these results are related to some old conjectures of M. Artin and J. Zhang, and of L. Small (recently proved by J. Bell [3]).

We think of all these results as another indication that the lower transcendence degree can be the “right” notion of transcendence degree for division algebras.

1. Preliminaries: isoperimetric profile

In this section we recall the definition of isoperimetric profile and some of its properties. For all that we say in this section and for an extensive study of this asymptotic invariant of algebras we refer to [5].

We start with some basic definitions and notation.

**Definition.** Given two functions $f_1, f_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ we say that $f_1$ is asymptotically faster than $f_2$, and we write $f_1 \succ f_2$, if there exist positive constants $\alpha$ and $\beta$ such that $\alpha f_1(\beta x) \geq f_2(x)$ for all $x \in \mathbb{R}_{\geq 0}$. We say that $f_1$ is asymptotically equivalent to $f_2$, and we write $f_1 \sim f_2$, if $f_1 \succ f_2$ and $f_2 \succ f_1$.

We will always consider associative algebras over a fixed field $K$ of characteristic $0$. Unless otherwise stated, they will all be infinite-dimensional over $K$.

**Definition.** We define a subframe of an algebra $A$ to be a finite-dimensional subspace containing the identity and a frame to be a subframe which generates the algebra (cf. [11]).

We will denote the dimension over $K$ of a vector space $V$ by $|V|$. We recall the definition of isoperimetric profile of an algebra (cf. [5]).

**Definition.** Let $A$ be an algebra over a field $K$ of characteristic zero. Given two subspaces $V$ and $W$ of $A$ we define the boundary of $W$ with respect to $V$ by

$$\partial_V(W) := VW/(VW \cap W).$$

Since we are interested in the dimension of the boundary, we can always assume that $1$ (the identity of $A$) is in $V$, i.e. $V$ is a subframe.

If $V$ is a subframe of $A$, we define the isoperimetric profile of $A$ with respect to $V$ to be the maximal function $I_*$ such that all finite-dimensional subspaces $W \subset A$ satisfy the following isoperimetric inequality

$$I_*(|W|; A, V) \leq |\partial_V(W)|.$$

Notice that for any $n \in \mathbb{N}$

$$I_*(n; A, V) = \inf |\partial_V(W)|,$$

where the infimum is taken over all subspaces $W$ of $A$ of dimension $n$. 
We say that an algebra $A$ has an isoperimetric profile if there exists a subframe $V$ of $A$ such that for any other subframe $V_1$ of $A$ we have

$$I_*(n; A, V_1) \ll I_*(n; A, V).$$

Otherwise we say that $A$ has no isoperimetric profile.

In case $A$ has an isoperimetric profile, we will refer to this function, or its asymptotic behavior, as the isoperimetric profile of $A$, and we will denote it also by $I_*(A)$. If the subframe $V$ of $A$ is such that $I_*(n; A, V) = I_*(A, V)$ is the isoperimetric profile of $A$ we will say that $V$ measures the profile of $A$.

**Warning.** For technical reasons, it is sometimes useful to think of a function $f(n)$ as defined on $\mathbb{N} \cup \{0\}$ as a real-valued function defined on $\mathbb{R}_{\geq 0}$, just by setting $f(n + t) := (1 - t)f(n) + tf(n + 1)$ for all $t \in (0, 1)$ and $n \in \mathbb{N} \cup \{0\}$. Also, if $f(n)$ is nonconstant and weakly monotone, we can always uniformly approximate it with a strictly monotone function.

Whenever we are interested in its asymptotic behavior, it is not a restriction to assume that our function has these properties. We will do this implicitly with our asymptotic invariants whenever it will be convenient.

**Example 1.1.** The algebra $K[x_1, x_2, \ldots]$ of polynomials in infinitely many variables has no isoperimetric profile.

On the other hand, any finitely generated algebra $A$ has an isoperimetric profile, which is measured by any frame $V \subseteq A$.

We have shown in [5, Theorem 0.0.5] that the following algebras have isoperimetric profile $I_*(n) \sim n^{1 - \frac{d}{2}}$, where $d$ is the Gelfand–Kirillov dimension (see [9] for definitions) of the algebra:

- finitely generated algebras of $GK$-dimension 1,
- finitely generated commutative domains,
- finitely generated prime $PI$ algebras,
- universal enveloping algebras of finite-dimensional Lie algebras,
- Weyl algebras,
- various quantum algebras.

Another basic example is the free algebra $A := K \langle x, y \rangle$ for which $I_*(n; A) \sim n$.

The isoperimetric profile of an algebra $A$ with respect to a subframe $V$ is always asymptotically sublinear, i.e. $I_*(n; A, V) \ll n$.

**Definition.** We say that an algebra $A$ is nonamenable if there exists a subframe $V \subseteq A$ such that $I_*(n; A, V) \sim n$. Otherwise we say that $A$ is amenable.

In this sense the isoperimetric profile can be thought of as a measure of the amenability of an algebra. The amenability of algebras has been studied by various authors (see e.g. [2,4–8]).

In the following proposition we recall two important properties of the isoperimetric profile (cf. [5, Corollary 2.2.2 and Proposition 2.4.4]).

**Proposition 1.1.**

1. Let $A$ be an algebra and let $\Omega$ be a right Ore set of regular elements. Then $A$ has an isoperimetric profile if and only if $A\Omega^{-1}$ does, and in this case $I_*(A) \sim I_*(A\Omega^{-1})$.
2. If $D$ is a nonamenable division subalgebra of $A$, then $A$ is nonamenable. If $D$ is an amenable division subalgebra of $A$, then $I_*(D, V) \ll I_*(A, V)$ for any subframe $V$ of $D$. 
Remark 1.1. Notice that in the second part of the proposition the assumption on $D$ cannot be simply removed: in fact there are even examples of amenable domains that contain nonamenable subalgebras (cf. [5]).

2. Følner function and its entropic counterpart

The following definitions follow quite closely Gromov’s [8].

**Definition.** Given a nonnegative asymptotically sublinear function $I(n) \preceq n$, we define its Følner function by

$$ F_I(n) := \min \{ N \in \mathbb{N} \mid I(N) \leq N/n \}.$$

We define its entropic Følner function $F_{\bullet}(n) = F_{I,\bullet}(n)$ as the inverse function of $1/J_I(n)$, where $J_I(n)$ is defined to be the maximal monotone decreasing minorant of $I(n)/n$ with the property that $J_I \circ \exp$ is convex. More precisely we want

$$ J_I(n) \leq I(n)/n \quad \text{for all } n,$$

$$ J_I(n) > J_I(n + 1) \quad \text{for all } n,$$

and

$$ J_I(\exp(tn + (1 - t)m)) \leq tJ_I(\exp(n)) + (1 - t)J_I(\exp(m))$$

for all $t \in [0, 1]$ and all $n$ and $m$; moreover, if there exists another function $J'$ with all these properties, then

$$ J_I(n) \geq J'(n) \quad \text{for all } n.$$

We say that $J_I(n)$ (or by abuse of speech $F_{\bullet}(n)$) is •-convex (cf. [8]).

**Example 2.1.** Let $I(n) = n^{1 - \frac{1}{d}}$ with $d \in \mathbb{R}, \ d \geq 1$. Hence of course

$$ \frac{I(n)}{n} = n^{-\frac{1}{d}}.$$

**Lemma 2.1.** If we set $g(x) := 1/x^{1/d}$, where $d \geq 1$, then $g \circ \exp$ is a convex function.

**Proof.** We have

$$ \frac{d}{dx} g(\exp(x)) = \frac{d}{dx} e^{-\frac{x}{d}} = -\frac{1}{d} e^{-\frac{x}{d}},$$

and

$$ \frac{d^2}{dx^2} g(\exp(x)) = \frac{1}{d^2} e^{-\frac{2x}{d}} > 0. \quad \square$$

Hence clearly in this case $J_I(n) = n^{-\frac{1}{d}}$, therefore $F_{I,\bullet}(n) = (1/J_I)^{-1}(n) = n^d$.

Another example is $I(n) = n/\log(n)^\alpha$ with $\alpha > 0$. 
**Lemma 2.2.** If we set \( g(x) := 1/\log(x)^\alpha \) where \( \alpha > 0 \), then \( g \circ \exp \) is a convex function.

**Proof.** We have
\[
\frac{d}{dx} g(\exp(x)) = \frac{d}{dx} x^{-\alpha} = -\alpha x^{-\alpha - 1},
\]
and
\[
\frac{d^2}{dx^2} g(\exp(x)) = (\alpha^2 + \alpha)x^{-\alpha - 2} > 0 \quad \text{for} \quad x > 0. \quad \Box
\]

Hence clearly in this case \( J_1(n) = 1/\log(n)^\alpha \), therefore \( F_{I,\ast}(n) = \exp(n^{\frac{1}{\alpha}}) \).

We define the Følner function and its entropic counterpart for algebras:

**Definition.** Given an amenable algebra \( A \) and a subframe \( V \) of \( A \), we define the **Følner function** \( F_\ast(n; A, V) \) with respect to \( V \) to be the Følner function of the isoperimetric profile \( I_\ast(n; A, V) \). Notice that, since \( |\partial V(W)| \geq I_\ast(|W|; A, V) \), this can be interpreted (cf. [5]) as the minimal dimension of a subspace \( W \) of \( A \) such that
\[
|\partial V(W)| \leq \left\lfloor \frac{|W|}{n} \right\rfloor.
\]

We define the **entropic Følner function** \( F_\ast(n; A, V) \) with respect to \( V \) to be the entropic Følner function of the isoperimetric profile \( I_\ast(n; A, V) \).

We say that the algebra \( A \) has a Følner function if there exists a subframe \( V \subseteq A \) such that
\[
F_\ast(n; A, V) \ll F_\ast(n; A, V_1)
\]
for any subframe \( V_1 \subseteq A \). We denote this function and its asymptotic equivalence class by \( F_\ast(A) \), and we say that \( V \) measures \( F_\ast(A) \) if \( F_\ast(A) \sim F_\ast(n; A, V) \).

We give similar definitions and we use similar notations for the entropic Følner function.

**Remark 2.1.** Notice that these functions are not defined for nonamenable algebras.

Also, by definition, the Følner function \( F_1(n) \) is a weakly monotone increasing function, and if \( I(n) \) is positive, then \( F_1(n) \geq n \) for all \( n \). It helps to keep in mind that the relation with the isoperimetric profile is **conjecturally** the following:
\[
I(n) \sim n/F_1^{-1}(n),
\]
where \( F_1^{-1}(n) \) denotes the inverse function of the Følner function.

To be able to give some example of entropic Følner functions of algebras we need to discuss its relation with the isoperimetric profile.

**3. Relation with the isoperimetric profile**

**Lemma 3.1.** Let \( I_1(n) \) and \( I_2(n) \) be two positive asymptotically sublinear functions. Then
\[
I_1(n) \succ I_2(n)
\]
implies
\[ F_{I_1,\bullet}(n) \succ F_{I_2,\bullet}(n). \]

In particular, \( I_1(n) \sim I_2(n) \) implies \( F_{I_1,\bullet}(n) \sim F_{I_2,\bullet}(n) \).

**Proof.** By assumption there are positive constants \( \alpha, \beta > 0 \) (not depending on \( n \)) such that
\[ \alpha I_1(\beta n) \geqslant I_2(n) \quad \text{for all } n. \]

By definition of \( F \), we have
\[ \frac{1}{F_{I_1,\bullet}(n)} \leqslant \frac{I_2(n)}{n} \leqslant \frac{\alpha' I_1(\beta n)}{\beta n}, \]

where \( \alpha' := \alpha \cdot \beta \). Hence, again by definition of \( F \),
\[ \frac{1}{F_{I_2,\bullet}(n)} \leqslant \frac{\alpha'}{F_{I_1,\bullet}(\beta n)}, \]
or
\[ \gamma F_{I_1,\bullet}(\beta n) \leqslant F_{I_2,\bullet}(n), \]

where \( \gamma := 1/\alpha' \). Since both \( \gamma F_{I_1,\bullet}^{-1}(\beta n) \) and \( F_{I_2,\bullet}^{-1}(n) \) are monotone increasing functions, this implies
\[ (1/\beta)F_{I_1,\bullet}(1/\gamma n) \geqslant F_{I_2,\bullet}(n). \]

Keeping in mind the computations of isoperimetric profiles in Example 1.1 and Lemma 3.1, Example 2.1 gives the computation of the entropic Følner function for most of the algebras with known isoperimetric profile. In fact, for the list of amenable algebras in Example 1.1, the entropic Følner function will be \( F_\bullet(n; A) \sim \gamma^d \), where \( d \) is the Gelfand–Kirillov dimension of \( A \).

It is important to notice that by Proposition 1.1 and again by Lemma 3.1, also their localizations have the same entropic Følner function, while their Gelfand–Kirillov dimension is in many cases infinite.

We record here a lemma which will be useful later.

**Lemma 3.2.** For any three positive functions \( f_1(n), f_2(n) \) and \( g(n) \) such that \( f_1 \) and \( g \) are increasing
\[ f_1(n) \succ f_2(n) \]

implies
\[ f_1(n)g(n) \succ f_2(n)g(n). \]

**Proof.** By assumption, there exist \( \alpha, \beta > 0 \) such that for all \( n \in \mathbb{N} \)
\[ \alpha f_1(\beta n) \geqslant f_2(n). \]

Since \( f_1 \) is increasing, we may assume that \( \beta \geqslant 1 \). Since \( g \) is increasing, we have
\[ \alpha f_1(\beta n)g(\beta n) \geqslant f_2(n)g(\beta n) \geqslant f_2(n)g(n). \]

\[ \square \]
4. Relations with the lower transcendence degree

**Definition.** Given a nonnegative asymptotically sublinear function \( I(n) \preceq n \), we define its **lower transcendence degree** by \( \text{Ld}(I) = 0 \) if \( I(m) = 0 \) for some \( m \); otherwise

\[
\text{Ld}(I) := \sup\{d \in \mathbb{R}_{\geq 0} \mid \exists c > 0: I(n) \geq cn^{1-\frac{1}{d}} \text{ for all } n \in \mathbb{N}\}.
\]

It follows immediately that, if \( \text{Ld}(I) \neq 0 \), then \( \text{Ld}(I) \geq 1 \).

The previous definition was inspired by the following definition, which was introduced by Zhang in [11].

**Definition.** Given an algebra \( A \), we define its **lower transcendence degree** by

\[
\text{Ld}(A) := \sup_{V} \text{Ld}(I_{*}(n; A, V))
\]

where the supremum is taken over all subframes \( V \) of \( A \).

We first show the relation between the lower transcendence degree and the Følner function.

**Lemma 4.1.** Let \( I(n) \) be a positive asymptotically sublinear function, and let \( d \in \mathbb{R}, d \geq 1 \). There exists \( b > 0 \) such that

\[
I(n) \geq bn^{1-\frac{1}{d}} \text{ for all } n \in \mathbb{N}
\]

if and only if there exists \( c > 0 \) such that

\[
F_{I}(n) \geq cn^{d} \text{ for all } n \in \mathbb{N}.
\]

**Proof.** Assume there exists \( b > 0 \) such that for all \( N \in \mathbb{N} \)

\[
I(N) \geq b \cdot N^{1-\frac{1}{d}}.
\]

If \( I(N) \leq N/n \), then

\[
b \cdot N^{1-\frac{1}{d}} \leq I(N) \leq N/n,
\]

hence

\[
N \geq b^{d} \cdot n^{d},
\]

i.e. \( F_{I}(n) \geq c \cdot n^{d} \) for \( c := b^{d} \).

Suppose now that we have a \( c > 0 \) such that \( F_{I}(n) \geq cn^{d} \) for all \( n \in \mathbb{N} \). By contradiction, if \( I(n) \preceq n^{1-\frac{1}{d}} \), then for every \( r \in \mathbb{N} \) we can find an \( N_{r} \) such that

\[
I(N_{r}) \preceq \frac{1}{r} \cdot (N_{r})^{1-\frac{1}{d}} = \frac{N_{r}}{r \cdot (N_{r})^{1/d}}.
\]

Hence, by the definition of Følner function,

\[
F_{I}(r \cdot (N_{r})^{1/d}) \leq N_{r}.
\]
This implies
\[ N_r \geq c (r \cdot (N_r r)^{1/d})^d = cr^d N_r. \]

For \( r \) big enough this gives a contradiction. \( \square \)

The following corollaries are immediate consequences of the lemma.

**Corollary 4.2.** For a positive asymptotically sublinear function \( I(n) \)
\[ Ld(I) = \sup \{ d \in \mathbb{R} | \exists c > 0: F_I(n) \geq c n^d \text{ for all } n \in \mathbb{N} \}. \]

**Corollary 4.3.** Given an amenable algebra \( A \) with \( Ld(A) \neq 0 \),
\[ Ld(A) = \sup \sup_{V} \{ d \in \mathbb{R} | \exists c > 0: F_*(n; A, V) \geq c n^d \text{ for all } n \in \mathbb{N} \}, \]

where the supremum is taken over all subframes \( V \) of \( A \).

The following corollary extends a result in [11].

**Corollary 4.4.** Given two finitely generated algebras \( A \) and \( B \),
\[ Ld(A \otimes B) \leq Ld(A) + Ld(B). \]

**Proof.** This follows immediately from Corollary 4.3 and Corollary 4.1.2 of [5]. \( \square \)

We now investigate the relation between the lower transcendence degree and the entropic Følner function.

**Lemma 4.5.** Let \( I(n) \) be a positive asymptotically sublinear function, and let \( d \in \mathbb{R}, d \geq 1 \). If there exists a constant \( c > 0 \) such that
\[ F_I(n) > c n^d \text{ for all } n, \]
then
\[ \frac{1}{n^{1/d}} < \frac{I(c n)}{c n} \text{ for all } n. \]

**Proof.** If not, then for some \( n \in \mathbb{N} \)
\[ \frac{1}{n^{1/d}} \geq \frac{I(c n)}{c n}, \]
hence by definition of \( F_I \) we have
\[ c n \geq F_I(n^{1/d}). \]

Setting \( m := n^{1/d} \) we get
\[ c m^d \geq F_I(m), \]
which gives a contradiction. \( \square \)
**Corollary 4.6.** Let $I(n)$ be a positive asymptotically sublinear function, and let $d \in \mathbb{R}$, $d \geq 1$. If there exists a constant $c > 0$ such that

$$F_I(n) > c n^d \quad \text{for all } n,$$

then

$$F_\bullet(n) = F_{I,\bullet}(n) \geq c n^d \quad \text{for all } n.$$

**Proof.** Using the first of the previous lemmas

$$\frac{1}{n^{1/d}} < \frac{I(cn)}{c n} \quad \text{for all } n.$$

Using the second of the previous lemmas and the definition of $F_\bullet$

$$\frac{1}{n^{1/d}} \leq \frac{1}{F_{-1}(cn)},$$

hence

$$F_{-1}(cn) \leq n^{1/d}.$$

Since $F_{-1}(cn)$ and $n^{1/d}$ are both monotone increasing functions, this implies

$$\frac{1}{c} F_\bullet(n) \geq n^d,$$

as we wanted. □

Using now Lemma 4.1 we get the following corollaries.

**Corollary 4.7.** Let $I(n)$ be a positive asymptotically sublinear function. Then

$$Ld(I) \leq \sup \{d \in \mathbb{R}, \ d \geq 1 \mid \exists c > 0: F_{I,\bullet}(n) \geq c n^d\}.$$

**Corollary 4.8.** For an amenable algebra $A$ with $Ld(A) \neq 0$,

$$Ld(A) \leq \sup \sup_{V} \{d \in \mathbb{R}, \ d \geq 1 \mid \exists c > 0: F_\bullet(n; A, V) \geq c n^d\},$$

where the supremum is taken over all subframes $V$ of $A$.

Before moving on with other results, we need to talk about entropy.
5. Entropy

We discuss the discrete setting first.

Consider \( a_1, \ldots, a_r \in \mathbb{N} \setminus \{0\} \) with \( \sum_{i=1}^r a_i = n \). We have

\[
\log n = -\sum_{k=1}^n \frac{1}{n} \log \frac{1}{n} = -\sum_{i=1}^r \frac{a_i}{n} \log \frac{1}{a_i} \\
= -\sum_{i=1}^r \frac{a_i}{n} \log \frac{1}{a_i} - \sum_{i=1}^r \frac{a_i}{n} \log \frac{a_i}{n} \\
= \sum_{i=1}^r \frac{a_i}{n} \log a_i + \left( -\sum_{i=1}^r \frac{a_i}{n} \log \frac{a_i}{n} \right).
\]

Suppose we have \( a_{ij} \in \mathbb{N} \) (not necessarily all non-zero), \( 1 \leq i \leq r \) and \( 1 \leq j \leq r_i \) such that \( \sum_{j=1}^{r_i} a_{ij} = a_i \), and define \( b_j := \sum_{i=1}^r a_{ij} \). We have

\[
-\sum_{i=1}^r \frac{a_i}{n} \log \frac{a_i}{n} = -\sum_{i=1}^r \left( \sum_{j=1}^{r_i} \frac{b_j a_{ij}}{n} \right) \log \left( \sum_{j=1}^{r_i} \frac{b_j a_{ij}}{b_j} \right) \\
\geq -\sum_{i=1}^r \sum_{j=1}^{r_i} \frac{b_j a_{ij}}{n} \left( \frac{a_{ij}}{b_j} \log \frac{a_{ij}}{b_j} \right) \\
= -\sum_{i,j} \frac{a_{ij}}{n} \log \frac{a_{ij}}{b_j}
\]

with the convention that the terms with \( a_{ij} = 0 \) do not appear, and where the inequality follows from the convexity of the function \( x \log x \) for \( x > 0 \) (whose second derivative is \( 1/x \)). If we assume also that \( 0 \leq a_{ij} \leq 1 \) for all \( i \) and \( j \), then the last term becomes

\[
-\sum_{i,j} \frac{a_{ij}}{n} \log \frac{a_{ij}}{b_j} = -\sum_{j} \frac{b_j}{n} \log \frac{1}{b_j} = \sum_{j} \frac{b_j}{n} \log b_j.
\]

Consider now a finite set \( X \) and a partition \( P \) of it. Given a subset \( Y \subset X \), we call \( P|Y := \{Y \cap \alpha \mid \alpha \in P\} \) the partition induced on \( Y \) by \( P \).

We define an entropy on \( X \): if \( Y \subset X \) is a subset of \( X \), the entropy of \( Y \) with respect to \( P \) is defined to be

\[
\text{ent}_P(Y) := -\sum_{s \in P|Y} \frac{|s|}{|Y|} \log \frac{|s|}{|Y|}.
\]

We define instead the entropy of the partition \( P|Y \) as

\[
\text{ent}(P|Y) := \sum_{s \in P|Y} \frac{|s|}{|Y|} \log |s|.
\]
Notice that if we consider the discrete partition \( P_X := \{ \{ x \} \mid x \in X \} \), then

\[
\text{ent}(Y) := \text{ent}_{P_X}(Y) = -\sum_{i=1}^{\lfloor |Y| \rfloor} \frac{1}{|Y|} \log \frac{1}{|Y|} = \log |Y|.
\]

At the beginning of this section we proved the following results. With the same notation as above, we have the decomposition

\[
\text{ent}(Y) = \text{ent}_{P}(Y) + \text{ent}(P|Y);
\]
given another partition \( P_1 \) with the property that \( |s \cap t| \leq 1 \) for all \( s \in P \) and \( t \in P_1 \), we have the so-called Shannon inequality

\[
\text{ent}(Y) \geq \text{ent}(P|Y) + \text{ent}(P_1|Y).
\]

The basic example is \( A \times B \) with the two obvious coordinate partitions.

We establish here the notation in the linear algebra setting. Given a finite-dimensional vector space \( W \) over \( K \), consider a decomposition \( W = W_1 \oplus \cdots \oplus W_r \), \( r \geq 1 \), and call it \( P \) (for “partition”). We define

\[
\text{ent}(W) := \log |W|,
\]

\[
\text{ent}(P|W) := \sum_{i=1}^{r} \frac{|W_i|}{|W|} \log |W_i|
\]

and

\[
\text{ent}_{P}(W) := \text{ent}(W) - \text{ent}(P|W) = -\sum_{i=1}^{r} \frac{|W_i|}{|W|} \log \frac{|W_i|}{|W|}.
\]

To apply these entropic methods (in particular the Shannon inequality) we need to discuss free left modules over subalgebras and decompositions of vector spaces in algebras.

6. Free left modules over subalgebras

The following discussion is close to Section 2.4 of [5]. Suppose that \( B \subset A \) is a subalgebra and \( A \) is a free left \( B \)-module. We don’t assume that \( A \) is a domain.

We have \( A = \bigoplus_i B a_i \) where \( a_i \in A \). Given any subspace \( W \) of \( A \) we can find \( a_1, \ldots, a_n \) such that \( W \subset \bigoplus_{i=1}^n B a_i \). We can choose a basis of \( W \) of the form

\[
\{ w_i^1 a_1 + y_i^1 \}_{i=1}^{p_1} \cup \{ w_i^2 a_2 + y_i^2 \}_{i=1}^{p_2} \cup \cdots \cup \{ w_i^n a_n + y_i^n \}_{i=1}^{p_n}
\]

where \( w_i^j \in B \) and \( y_i^j \in \bigoplus_{k \neq j} B a_k \), such that for each \( j \), \( \{ w_i^j \}_{i=1}^{p_j} \) are linearly independent. Notice that \( \{ w_i^j a_j + y_i^j \}_{i=1}^{p_j} \) corresponds to a basis of \( (W \cup \bigoplus_{k \neq j} B a_k)/(W \cup \bigoplus_{k \neq j} B a_k) \). Let \( W_j \) denote the subspace generated by \( \{ w_i^j \}_{i=1}^{p_j} \) and let \( W_j \) denote the subspace generated by \( \{ w_i^j a_j + y_i^j \}_{i=1}^{p_j} \). Then

\[
W = W_1 \oplus W_2 \oplus \cdots \oplus W_n
\]
and hence

$$|W| = \sum_j |W_j| = \sum_j |W'_j|. $$

Let $V$ be a subframe of $B$. We have

$$V W_1 = \left\{ xa_1 + y \mid x \in V W'_1 \text{ and } y \in \bigoplus_{i=2}^n Ba_i \right\}. $$

Since

$$\sum_{i=2}^n V W_i \subset \bigoplus_{i=2}^n Ba_i \quad \text{and} \quad \left( \bigoplus_{i=2}^n Ba_i \right) \cap Ba_1 = 0,$$

we have

$$\left| \sum_{i=1}^n V W_i \right| \geq |V W'_1| + \sum_{i=2}^n V W_i |.$$

By induction on $n$ we have

$$\left| \sum_{i=1}^n V W_i \right| \geq \sum_{i=1}^n |V W'_i|, $$

and hence

$$|\partial_V(W)| = |V W| - |W| = \left| \sum_{i=1}^n V W_i \right| - \sum_{i=1}^n |W_i|$$

$$\geq \sum_{i=1}^n |V W'_i| - \sum_{i=1}^n |W'_i| = \sum_{i=1}^n |\partial_V(W'_i)|.$$

**Remark 6.1.** Notice that in general the spaces $\{W'_i\}_{i=1}^n$ are not subspaces of $W$, hence strictly speaking they don’t provide a decomposition of $W$. Also, as subspaces of $B$ they could intersect nontrivially. But since we are only interested in dimensions, by abuse of speech we will call $\{W'_i\}_{i=1}^n$ a decomposition of $W$, and we will denote it by $P_B$. Accordingly, we define the entropies

$$\text{ent}_{P_B}(W) := -\sum_{i=1}^n \frac{|W'_i|}{|W|} \log \frac{|W'_i|}{|W|} \quad \text{and} \quad \text{ent}(P_B|W) := \sum_{i=1}^n \frac{|W'_i|}{|W|} \log |W'_i|. $$

**Theorem 6.1.** Suppose that $B \subset A$ are amenable algebras and $A$ is a free left $B$-module (we don’t assume that $A$ or $B$ are domains). If $V \leq B$ is a subframe of $B$, then

$$F_\bullet(n; B, V) \leq F_\bullet(n; A, V).$$
Proof. Let $W \subseteq A$ be a subspace with $|W| = F_\pi(n; A, V)$. Therefore

$$|\partial_V(W)| \leq \frac{|W|}{n}.$$ 

By the previous discussion, we can find a decomposition $P_B$ of $W$, $\{W'_i\}_{i=1}^r$, where $W'_i \subseteq B$ for all $i$, $\sum_{i=1}^r |W'_i| = |W|$ and such that

$$|\partial_V(W)| \geq \sum_{i=1}^r |\partial_V(W'_i)|.$$ 

We have

$$\frac{1}{n} \geq \frac{|\partial_V(W)|}{|W|} \geq \sum_{i} \frac{|\partial_V(W'_i)|}{|W|} \geq \sum_{i} \frac{I_\pi(|W'_i|; B, V)}{|W|} \geq \sum_{i} \frac{|W'_i|}{|W|} J(|W'_i|; B, V)$$

where $J(|W'_i|; B, V) := 1/F_{\pi}^{-1}(|W'_i|; B, V)$. Using the convexity of $J \circ \exp$ we get

$$\sum_{i} \frac{|W'_i|}{|W|} J(|W'_i|; B, V) \geq J \left( \exp \sum_{i=1}^r \frac{|W'_i|}{|W|} \log |W'_i|; B, V \right)$$

$$= J(\exp(\text{ent}(P_B|W)); B, V).$$

From this it follows that

$$\text{ent}(P_B|W) \geq \log(F_\pi(n; B, V)).$$

From the general equality

$$\log |W| = \text{ent}(|W|) = \text{ent}(P_B|W) + \text{ent}_{P_B}(W)$$

we get

$$|W| = F_\pi(n; A, V) \geq F_\pi(n; B, V) \cdot \exp(\text{ent}_{P_B}(W)) \geq F_\pi(n; B, V).$$

By Corollaries 4.3 and 4.8, this theorem provides an extension of the result of J. Zhang [11] that if $B \subseteq A$ are two algebras, and $A$ is a free left $B$-module, then

$$\text{Ld}(A) \geq \text{Ld}(B).$$

Remark 6.2. Notice that the analogous of Theorem 6.1 for isoperimetric profiles (see [5, Proposition 2.4.1]) required the technical assumption of $I_\pi(n; B, V)$ being subadditive. Here, instead of the subadditivity of the isoperimetric profile we have used the intrinsic $\bullet$-convexity of the entropic Følner function.
7. Decompositions in algebras

We need to discuss decompositions in algebras. This is the most technical part of the paper.

Let \( A \supseteq K \) be a domain, \( W \subseteq A \) a finite-dimensional subspace, and \( v \in A \) an element in \( A \) which is not algebraic over \( K \). We want to prove the following lemma.

Lemma 7.1. Given \( v \) not algebraic over \( K \) and a finite-dimensional subspace \( W \subseteq A \), we can find a decomposition

\[
W = \bigoplus_i W_i
\]

such that

1. each \( W_i \) contains an element \( w_i \) such that \( W_i = \bigoplus_{j=0}^s K v^j w_i \) for some \( s \geq 0 \);
2. \( vW \cap W = \bigoplus_i (vW_i \cap W_i) \).

As a consequence of the lemma we will have

\[
\left| \partial_v(W) \right| = |vW| - |vW \cap W| = |W| - |vW \cap W|
\]

\[
= \left| \bigoplus_i W_i \right| - \left| \bigoplus_i (vW_i \cap W_i) \right|
\]

\[
= \sum_i |W_i| - \sum_i |vW_i \cap W_i| = \sum_i \left| \partial_v(W_i) \right|. 
\]

Proof of Lemma 7.1. We will use the induction on the dimension of the space \( W \).

Consider the chain of subspaces

\[
W = W^{(0)} \supseteq W^{(1)} \supseteq \cdots,
\]

where

\[
W^{(i)} := \{ w \in W \mid v^i w \in W \}.
\]

Observe that \( W^{(1)} \) is a subspace of \( W \) and \( |W^{(1)}| = |vW \cap W| \).

Claim. For all \( i \geq 1 \), if \( W^{(i-1)} \neq 0 \), then

\[
W^{(i)} \leq W^{(i-1)}
\]

Proof. If not, for some \( i \)

\[
vW^{(i-1)} = W^{(i-1)}. 
\]

Since \( A \) is a domain, it follows that the element \( v \) is algebraic, a contradiction. \( \square \)

Hence we have a finite chain of subspaces

\[
W = W^{(0)} \supseteq W^{(1)} \supseteq \cdots \supseteq W^{(r)} \supseteq W^{(r+1)} = (0).
\]
Since $vW^{(i)} \subseteq W^{(i-1)}$, it follows that $v(W^{(i)}/W^{(i+1)}) \subseteq W^{(i-1)}/W^{(i)}$.

(⋆) This mapping is an injection.

**Proof.** If $w \in W^{(i)}$ and $vw \in W^{(i)}$, then in fact $w \in W^{(i+1)}$, which implies $w + W^{(i+1)} = 0$ in $W^{(i)}/W^{(i+1)}$. □

Thus we have

$$|W^{(0)}/W^{(1)}| \geq |W^{(1)}/W^{(2)}| \geq \cdots \geq |W^{(r)}.|

**Case 1.** $|W^{(0)}/W^{(1)}| = |W^{(1)}/W^{(2)}| = \cdots = |W^{(r)}|$. Let $w_1, w_2, \ldots, w_d$ be a basis of $W_r$. Then $\{v^j w_i \mid 1 \leq j \leq r, 1 \leq i \leq d\}$ is a basis of $W$ and, if we set $W_i := \text{span}_K \{v^j w_i \mid 1 \leq j \leq r\}$, then $W = \sum_1^d W_i$ is the decomposition we have been looking for.

**Case 2.** Let $|W^{(0)}/W^{(1)}| = |W^{(1)}/W^{(2)}| = \cdots = |W^{(i-1)}/W^{(i)}| \geq |W^{(i)}/W^{(i+1)}|$. This means that $vW^{(i)} + W^{(i)}$ is a proper subspace of $W^{(i-1)}$.

Choose an element $w \in W^{(i-1)} \setminus (vW^{(i)} + W^{(i)})$. Let $W' := \text{span}_K \{v^j w, 0 \leq j \leq i - 1\}$, and choose a subspace $W^{(i-1)}$ such that

$$vW^{(i)} + W^{(i)} \leq W^{(i-1)} \leq W^{(i-1)}$$

and

$$(Kw + W^{(i)}/W^{(i)}) \oplus \widehat{W}^{(i-1)/W^{(i)}} = W^{(i-1)}/W^{(i)}$$

is a direct sum. Let $W'' := \sum_{j=0}^{i-1} v^j \widehat{W}^{(i-1)}$.

**Claim.** We claim that $W = W' \oplus W''$ is a direct sum.

**Proof.** From $|W^{(0)}/W^{(1)}| = |W^{(1)}/W^{(2)}| = \cdots = |W^{(i-1)}/W^{(i)}|$ and (⋆) it follows that $W = \sum_{j=0}^{i-1} v^j W^{(i-1)}/W^{(i)}$. This implies $W = W' + W''$. Now let $\sum_{j=0}^{i-1} \alpha_j v^j w = \sum_{j=0}^{i-1} v^j \tilde{w}_{i-1,j}$ where $\tilde{w}_{i-1,j} \in \tilde{W}^{(i-1)}$. Again from (⋆) it follows that for any $0 \leq j \leq i - 1$, $v^j w \in W^{(i-j)}$ and $v^j \tilde{w}_{i-1,j} \subseteq W^{(i-j)}$ are linearly independent modulo $W^{(i-j)}$. If $j$ is maximal with the property that $\alpha_j \neq 0$ or $\tilde{w}_{i-1,j} \neq 0$ then $\alpha_j v^j w = v^j \tilde{w}_{i-1,j}$ modulo $W^{(i-j)}$, a contradiction. Hence $W = W' \oplus W''$. □

Let us check that $vW \cap W \subseteq (vW' \cap W') + (vW'' \cap W'')$.

Let $a \in W$, $a = \sum_{j=0}^{i-1} \alpha_j v^j w + \sum_{j=0}^{i-1} v^j \tilde{w}_{i-1,j} + va \in W$. This implies that $v^i (\alpha_i w + \tilde{w}_{i-1,i-1}) \in W$ and hence $\alpha_i w + \tilde{w}_{i-1,i-1} \in W^{(0)}$. Hence $\alpha_i = 0$ and $\tilde{w}_{i-1,i-1} = 0$. Therefore $va = \sum_{j=0}^{i-2} \alpha_j v^j w + \sum_{j=0}^{i-2} v^j \tilde{w}_{i-1,j} \in (vW' \cap W') + (vW'' \cap W'')$.

This shows that $vW \cap W \subseteq (vW' \cap W') + (vW'' \cap W'')$ and hence $vW \cap W = (vW' \cap W') \oplus (vW'' \cap W'')$ is a direct sum.

Now we can apply the induction assumption on the dimension to $W''$. □

For future reference, we will denote such a decomposition of $W$ provided by the lemma as $P_v$.

**8. Technical conjecture**

To be able to use the Shannon inequality we need two decompositions with certain properties. These decompositions come from the discussions in the two previous sections.

We are interested in the situation where $A$ is a domain, and $D \subset A$ is a division algebra with $K \subset D$. In this case $A$ has a natural structure of (free) left $D$-module. We assume also that there is
an element $x \in A$ which is transcendent on the left over $D$. This of course implies the transcendence of $x$ over $K$, hence we can use the previous lemma to get our decomposition $P_x$ given by $W = \bigoplus_{i=1}^{r} W_i$ with the stated properties. On the other hand, given a subframe $V_D$ of $D$ we can use the discussion preceding Theorem 6.1 to get a decomposition $P_D$ of subspaces of $D \{W_j\}_{j=1}^{s}$ with the stated properties.

Notice that none of the mentioned decompositions is unique. In order to apply the Shannon inequality we need to assume some combinatorial properties of the dimensions of these decompositions.

**Conjecture 1.** There exist two such decompositions $P_x$ and $P_D$ given by $\{W_i\}_{i=1}^{r}$ and $\{W'_j\}_{j=1}^{s}$ respectively with the property that there exists an $r \times s$ matrix $|a_{i,j}|$ with $a_{i,j} \in \{0, 1\}$, $\sum_{j} a_{i,j} = |W_i|$ and $\sum_{i} a_{i,j} = |W'_j|$ for all $i$ and $j$.

We summarize our discussion in the following proposition.

**Proposition 8.1.** In the setting $K \subset D \subset A$, where $D$ is a division algebra and $A$ is an amenable domain, given a subframe $V_D \subset D$ of $D$ and an element $x \in A$ which is not left algebraic over $D$, for any finite-dimensional subspace $W \leq A$, Conjecture 1 implies that we can find two decompositions $P_x$ and $P_D$ of $W$ given by $\{W_i\}_{i=1}^{r}$ and $\{W'_j\}_{j=1}^{s}$ respectively with the following properties:

- $|\partial_{V_D}(W)| \geq \sum_{j} |\partial_{V_D}(W'_j)|$;
- $|\partial_x(W)| = \sum_{i} |\partial_x(W_i)|$, where $W_i = \bigoplus_{j=0}^{s} Kx_i w_i$ for some $w_i \in W$. In particular $|\partial_x(W_i)| = 1$ for all $i$;
- there exist $a_{i,j} \in \{0, 1\}$ such that $\sum_{j} a_{i,j} = |W_i|$ and $\sum_{i} a_{i,j} = |W'_j|$ for all $i$ and $j$.

9. Main consequence of Conjecture 1

We state here the main consequence of Conjecture 1. Theorem 0.1 will be an immediate corollary.

**Theorem 9.1.** Let $A$ be an amenable domain, and $D \subset A$ be a division subalgebra of $A$, where $K \subset D$. Let $x \in A \setminus D$ be algebraic independent on the left over $D$. Let $V_D \subset D$ be a subframe of $D$, and let $V := V_D + Kx$. Then Conjecture 1 implies

$$F_\bullet(n; A, V) \geq F_\bullet(n; D, V_D) \cdot n.$$  

Before proving this theorem, let’s see how does it imply Theorem 0.1.

**Proof of Theorem 0.1.** Proposition 1.1 implies that $D$ is also amenable. Then the result follows immediately from Theorem 9.1 and Corollaries 4.3 and 4.8.

In order to prove Theorem 9.1 we need the following proposition.

**Proposition 9.2.** Let $K \subset D \subset A$, where $D$ is a division algebra and $A$ is a domain; let $V_D \subset D$ be a subframe of $D$, $x \in A$ an element which is not left algebraic over $D$, and set $V := V_D + Kx$. We assume Conjecture 1, and we call $P_x$ and $P_D$ the decompositions given by Proposition 8.1.

Given $n \in \mathbb{N}$, for any finite-dimensional subspace $W \leq A$ such that

$$|\partial_V(W)| \leq |W|/n$$

we have

$$\text{ent}(P_x|W) \geq \log(F_\bullet(n; A, Kx))$$
and

\[ \text{ent}(P_D|W) \geq \log (F_\star(n; A, V_D)). \]

**Proof.** We have

\[ \frac{1}{n} \geq \frac{|\partial V(W)|}{|W|} \geq \frac{|\partial x(W)|}{|W|} = \sum_i \frac{|\partial x(W_i)|}{|W|} \]

\[ \geq \sum_i \frac{I_\star(|W_i|; A, Kx)}{|W|} \geq \sum_i \frac{|W_i|}{|W|} J(|W_i|; A, Kx) \]

where \( J(|W_i|; A, Kx) := 1/F_\star^{-1}(|W_i|; A, Kx). \) Using the convexity of \( J \circ \exp \) we get

\[ \sum_i \frac{|W_i|}{|W|} J(|W_i|; A, Kx) \geq J \left( \exp \sum_i \frac{|W_i|}{|W|} \log |W_i|; A, Kx \right) \]

\[ = J(\exp(\text{ent}_{P_D}(W)); A, Kx). \]

From this we get

\[ \text{ent}(P_x|W) \geq \log (F_\star(n; A, Kx)), \]

as we wanted.

For the second inequality we can use a similar argument:

\[ \frac{1}{n} \geq \frac{|\partial V(W)|}{|W|} \geq \frac{|\partial V_D(W)|}{|W|} \geq \sum_j \frac{|\partial V_D(W_j')|}{|W|} \]

\[ \geq \sum_j \frac{I_\star(|W_j'|; A, V_D)}{|W|} \geq \sum_j \frac{|W_j'|}{|W|} J(|W_j'|; A, V_D) \]

where \( J(|W_j'|; A, V_D) := 1/F_\star^{-1}(|W_j'|; A, V_D). \) Using the convexity of \( J \circ \exp \) we get

\[ \sum_j \frac{|W_j'|}{|W|} J(|W_j'|; A, V_D) \geq J \left( \exp \sum_j \frac{|W_j'|}{|W|} \log |W_j'|; A, V_D \right) \]

\[ = J(\exp(\text{ent}_{P_D}(W)); A, V_D). \]

From this we get

\[ \text{ent}(P_D|W) \geq \log (F_\star(n; A, V_D)), \]

as we wanted. \( \square \)

We are now ready to finish the proof of Theorem 9.1.
Proof of Theorem 9.1. Given \( n \in \mathbb{N} \), let \( W \subseteq A \) be a finite-dimensional subspace such that \( |W| = F_*(n; A, V) \). We can apply Proposition 8.1 to get the decompositions \( P_x \) and \( P_D \) of \( W \) with the stated properties. Notice that with those properties we can apply the Shannon inequality to get

\[
\log |W| = \text{ent}(W) \geq \text{ent}(P_x|W) + \text{ent}(P_D|W).
\]

By our choice of \( W \) we have

\[
\frac{1}{n} \geq \frac{\partial_V(W)}{|W|},
\]

hence we can apply Proposition 9.2 to get

\[
\log F_*(n; A, V) = \log |W| \geq \text{ent}(P_x|W) + \text{ent}(P_D|W)
\]

\[
\geq \log \left( F_*(n; A, Kx) \right) + \log \left( F_*(n; A, V_D) \right).
\]

Therefore

\[
F_*(n; A, V) \geq F_*(n; A, Kx) F_*(n; A, V_D).
\]

By Proposition 1.1,

\[
I_*(n; A, V_D) \succ I_*(n; D, V_D),
\]

hence applying Lemmas 3.1 and 3.2 we get

\[
F_*(n; A, V) \geq F_*(n; A, Kx) F_*(n; A, V_D) \succ F_*(n; A, Kx) F_*(n; D, V_D).
\]

It only remains to observe that \( I_*(n; A, Kx) = 1 \) for all \( n \); this can be seen by looking at the subspaces \( Z_n := \text{span}_K \{1, x, x^2, x^3, \ldots, x^{n-1} \} \), for which clearly \( |Z_n| = n \) and \( |\partial_k(Z_n)| = 1 \). Also, if for some subspace \( W \) we had \( |\partial_k(W)| = 0 \), then \( xW = W \), contradicting the transcendence of \( x \) over \( K \).

Now it's clear from the definition of \( F_\bullet \) that \( F_\bullet(n; A, Kx) = n \), completing the proof. \( \square \)

Remark 9.1. Notice that in the proof of this theorem, to get the inequality we only used the properties of the decompositions stated in Proposition 8.1.

10. Application to Ore extensions

In this section we apply our entropic methods to the situation of an Ore extension. For the definition of an Ore extension we refer to [9].

The following theorem extends results in [11] (cf. also [5, Section 2.8]).

Theorem 10.1. Let \( A \) be a domain over \( K \), \( \sigma \) a linear automorphism of \( A \) and \( \delta \) a \( \sigma \)-derivation. If \( A[x, \sigma, \delta] \) is a domain, then for a subframe \( V \subseteq A \) we have

\[
n F_\bullet(n; A, V) \preceq F_*(n; A[x, \sigma, \delta], V + Kx).
\]

Proof. There is a natural filtration of \( A[x, \sigma, \delta] \) determined by the degree of \( x \), such that the associated graded algebra is isomorphic to \( A[x, \sigma] \). Hence there is a valuation \( \nu \) from \( A[x, \sigma, \delta] \) to \( A[x, \sigma] \) (cf. [5, Section 2.8]), which by Theorem 2.7.2 in [5] gives

\[
I_*(A[x, \sigma, \delta], W) \succ I_*(A[x, \sigma], W).
\]
for any graded subframe \( W = \bigoplus_{i=0}^{m} W_i x^i \). Hence by Lemma 3.1 we have also

\[
F_\bullet \left( A[x, \sigma, \delta], W \right) \succcurlyeq F_\bullet \left( A[x, \sigma], W \right).
\]

Hence, it's enough to show that \( F_\bullet(n; A[x, \sigma], V + Kx) \succcurlyeq n F_\bullet(n; A, V) \), where \( V \) is a subframe of \( A \). First observe that the leading-term map of \( A[x, \sigma] \) is a valuation from \( A[x, \sigma] \) to itself. Again by Theorem 2.7.2 in [5] and Lemma 3.1, it's enough to consider only the graded subspaces of \( A[x, \sigma] \).

Let \( V \) be a subframe of \( A \). Given a graded subspace \( Z \subset A[x, \sigma] \), we have \( Z = \bigoplus_{i=0}^{n} Z_i x^i \), where \( Z_i \subset A \) for all \( i \). Notice that this decomposition corresponds to the one in the discussion preceding Theorem 6.1. We call this decomposition \( P_A \).

First of all observe that

\[
\left| \partial_V(Z) \right| = \sum_{i} |V Z_i| - \sum_{i} |Z_i| = \sum_{i} \left| \partial_V(Z_i) \right|.
\]

Now consider the decomposition \( P_x \) given by \( Z = \bigoplus_{j} Z_j = \bigoplus_{j} Kx^j z_j \), provided by Lemma 7.1. Since \( Z \) is graded with respect to \( x \) and \( xA = Ax \), from the construction of \( P_x \) it is clear that we can choose the \( z_j \)'s homogeneous in \( x \), i.e. \( z_j = z_j x^{m_j} \) with \( m_j \in \mathbb{N} \) and \( z_j \in A \) for all \( j \).

In order to be able to apply the proof of Theorem 9.1, we want to check that \( P_A \) and \( P_x \) satisfy the conditions in Proposition 8.1. It remains to check only the last property. We have

\[
Z'_i \cap Z_j x^j = \bigoplus_{k} (K x^k Z_i \cap Z_j x^j) = \bigoplus_{k} (K x^k z_i x^{m_j} \cap Z_j x^j)
\]

\[
= \bigoplus_{k} (K z_i^{m_j - k} x^{k + m_j} \cap Z_j x^j) = K z_i^{m_j - k} x^j \cap Z_j x^j,
\]

which is \( (0) \) or \( K z_i^{m_j - k} x^j \) depending on \( i \) and \( j \). But since \( Z_j x^j = \bigoplus_{i} K z_i^{m_j - k} x^j \), if we set \( a_{i,j} := |Z'_i \cap Z_j x^j| \) we get the matrix that we wanted.

Therefore now we can apply the proof of Theorem 9.1 to get the result. \( \Box \)

**Remark 10.1.** Notice that in the proof of the previous theorem we essentially checked Conjecture 1 in the case when \( x \) has the property that \( xD = Dx \).

**Remark 10.2.** In [5, Section 2.8], in the analogous situation for the isoperimetric profile, we were not able to find an inequality as strong as this one (cf. [5, Remark 3]). This is a further indication that the entropic Følner function can be a better tool than the isoperimetric profile in studying algebras.

### 11. Old conjectures

We list some old conjectures of M. Artin, L. Small and J. Zhang (cf. [11]).

Consider a domain \( A \) which contains a subring \( D \) which is a division ring. Hence \( A \) can be viewed as a right \( D \)-module \( A_D \).

**Given a subset \( S \subset A \) we will denote by \( |S|_D \) the right dimension over \( D \) of the right span of \( S \) over \( D \).**

Given a subframe \( V \) of \( A \), we can define the boundary

\[
|\partial_V(W)|_D := |V W|_D - |W|_D,
\]

and with this we can define the isoperimetric profile \( I_\delta(n; A_D, V) \) in the obvious way. With this we can define the Følner function of \( A_D \), denoted \( F_\bullet(n; A_D, V) \), its entropic Følner function, denoted \( F_\bullet(n; A_D, V) \), and its lower transcendence degree, denoted \( L_d A_D \).
Conjecture 2 (J. Zhang). Let $A$ be a domain, and let $D \subset A$ be a subalgebra of $A$ which is a division algebra. Then

$$LDA \geq LdD + LdAD.$$ 

This is the strongest conjecture, and it’s clearly related to Theorem 9.1.

It’s worthwhile to notice that Theorem 9.1 suggests that the lower transcendence degree measures somehow the transcendence of $A$ over $D$ on the left, while this conjecture guesses that the jump from $LdD$ to $LdA$ should be measured by $LdAD$, which is the lower transcendence degree of $A$ as right $D$-module.

Given a subset $S \subset A$ we will denote by $D|S|$ the left dimension over $D$ of the right span of $S$ over $D$.

We can give all the definitions of boundary, isoperimetric profile, Følner function, entropic Følner function and lower transcendence degree of $DA$ as we did for $AD$, and we denote by $LdDA$ the corresponding lower transcendence degree. We formulate the following conjecture.

Conjecture 3. Let $A$ be a domain, and let $D \subset A$ be a subalgebra of $A$ which is a division algebra. Then

$$LDA \geq LdD + LdDA.$$ 

Conjecture 4 (L. Small). Let $A$ be a finitely generated Ore domain which is not locally PI and $F$ a commutative subalgebra of the quotient division algebra of $A$. Then $GK \dim(F) \leq GK \dim(A) - 1$.

Notice that this last conjecture has been recently proved by Jason Bell in [3].

Conjecture 5 (J. Zhang). Suppose that there is a chain of finitely generated division algebras

$$K = Q_0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_n = Q$$

such that each $Q_i$ is an infinite-dimensional right $Q_{i-1}$-space. Then $n \leq Ld(Q)$.

Conjecture 6 (M. Artin). Let $D$ be a division algebra over an algebraically closed field $K$ with $GK \dim(D) > 1$. Then $Ld(D) \geq 2$.

We mention here that Conjecture 2 implies Conjectures 4 and 5, and Conjecture 5 implies Conjecture 6. Also, Conjecture 6 implies the famous Artin–Stafford gap theorem, conjectured by M. Artin and T. Stafford in [1] and proved by A. Smoktunowicz in [10], which states that there are no finitely generated connected graded domains with Gelfand–Kirillov dimension strictly between 2 and 3.

In [11, Section 8] there is a complete discussion of the relations of all these statements.

12. Further consequences of Conjecture 1

We discuss some further consequences of Conjecture 1, related to the previous conjectures. First we prove that Conjecture 1 implies a weaker version of Conjecture 4.

Theorem 12.1. Let $A$ be a finitely generated Ore domain and $F$ a commutative subalgebra of the quotient division algebra $Q(A)$ of $A$, and assume Conjecture 1. If $Q(A)$ is not left algebraic over $F$, then $GK \dim(F) \leq GK \dim(A) - 1$.

Proof. It follows from Theorem 0.1 and known properties of the lower transcendence degree (cf. [11]) that

$$GK \dim(F) = LdF \leq LdA - 1 \leq GK \dim(A) - 1.$$  □
The following theorem is related to Conjecture 5, and it follows immediately from Theorem 0.1.

**Theorem 12.2.** Consider a chain

\[ K = D_0 \subset D_1 \subset \cdots \subset D_n \subset A, \]

where for \( i \geq 1 \), \( D_i \) is not left algebraic over \( D_{i-1} \). Assuming Conjecture 1 we get

\[ \text{Ld}(A) \geq n. \]

We prove a theorem related to Conjecture 6.

**Theorem 12.3.** Let \( D \) be a division algebra. If there exists an element \( a \in D \) transcendental over \( K \) such that \( D \) is not left algebraic over the subfield \( F \subset D \) generated by \( a \), then Conjecture 1 implies that \( \text{Ld}(D) \geq 2 \).

**Proof.** We know that \( \text{Ld}F = \text{tr.deg}_K(F) = 1 \). Hence, applying Theorem 0.1 we get

\[ \text{Ld}D \geq \text{Ld}F + 1 = 2. \]

Hence this theorem reduces Conjecture 6 to Conjecture 1 and the following conjecture:

**Conjecture 7.** If \( D \) is a division algebra over an algebraically closed field \( K \) with \( \text{GK dim}(D) > 1 \), then there exists an element \( a \in D \) transcendental over \( K \) such that \( D \) is not left algebraic over the subfield \( F \subset D \) generated by \( a \).

Notice that, using the previous results, Conjectures 4, 5 and 6 would follow from Conjecture 1 and a positive solution to the following version of Kurosh’s problem:

**Problem.** If a finitely generated division algebra \( D \) is left algebraic over a (not necessarily central) subfield \( F \subset D \), then \( D \) is a finite left \( F \)-module.

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**References**