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On Chevalley-Groups Acting on Projective Planes

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In [10] Hering introduced the concept of a strongly irreducible collineation group. Let π be a projective plane, K a collineation group of π; K is said to be strongly irreducible iff K fixes no points, lines triangles, or subplanes of π. If K is a finite group acting strongly irreducible on π generated by perspectivities, then Hering shows in [10] that K is either an extension of a 3-group with a subgroup of the automorphismgroup or there is a normal subgroup G in K, G a nonabelian finite simple group, such that $K \leqslant \mathrm{Aut}(G)$. The aim of this paper is to prove the following theorem.

THEOREM. *Let G be a finite Chevalley-group of normal or twisted type over a field with q elements and of rank m. Let $G \leqslant K \leqslant \mathrm{Aut}(G)$ and suppose that K acts strongly irreducible on a finite projective plane π. If K is generated by perspectivities, then $m \leqslant 2$.*

The case of a Chevalley-group of rank 1 has been treated by Hering and Walker in [11]. The occurring groups are $G \cong PSL(2, q)$ and $G \cong PSU(3, q)$. If $m = 2$, then Walker informed me that he has proved $G \cong PSL(3, q)$. If G is an alternating group, then by an unpublished paper of Hering and Walker, $G \cong A_n$, $n \leqslant 7$. If G is a sporadic simple group known at the time of writing, then $G \cong J_2$ [15]. Thus the problem of determining the structure of G is solved for all known simple groups G, and so it is quite probable that it is solved at all.

The proof of the theorem depends mainly on the following lemma due to Hering [10, (5.1)]: Suppose that a, b are perspectivities of π. Then ab is a generalized perspectivity or trivial. In particular ab is not planar. Thus if ab is an involution, then ab is a perspectivity with all consequences. We use this lemma in the following way. Let G be a counterexample to the theorem. Then $m \geqslant 3$ and so G is one of the following groups $L_n(q)$, $n \geqslant 4$, $\Omega_n^{\pm}(q)$, $n \geqslant 6$, $Psp_{2n}(q)$, $n \geqslant 2$, $PSU_n(q)$, $n \geqslant 5$, $F_4(q)$, $E_6(q)$, $E_7(q)$, $E_8(q)$ or ${}^2E_6(q)$. Furthermore G is generated by a class of subgroups V of order q corresponding to the long root of the underlying root system. We get

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$\langle V, V^g \rangle = Y$ is a p-group, $q = p^f$, or $Y \cong SL_2(q)$, for $g \in K$. For $x \in V^*$ Hering's lemma yields that $x^{-1}x^{\alpha}$ is a generalized perspectivity contained in $Y = \langle V, V^{\alpha} \rangle$, where α is a perspectivity in K. If Y is a p-group, then an easy play with parabolic subgroups of G yields a contradiction to the strongly irreducible action of K. Thus we may assume that $Y = SL_2(q)$. Now in almost all cases $N_G(Y)$ is a maximal subgroup of G. The action of G now forces $\alpha \in N_K(Y)$, as α and $N_K(Y)$ fixes the center and the axis of the generalized perspectivity $x^{-1}x^{\alpha}$. With the exception of some small cases we can reduce the situation to the case that $\alpha \in Y$. Then application of [21] yields that there is an involution in K acting as a perspectivity on π, provided q is odd. If q is even, it needs some further considerations to get the same result. Now there are lots of theorems available and we get a contradiction.

The given proof needs that π is finite in those cases where $x^{-1}x^{\alpha}$ is triangular. But it is likely that all arguments with some changes go over to the infinite case.

I hope all the notation are standard. All the notations concerning with Chevalley-groups can be found in Carter's book [6]. The remaining group-theoretical notation follows [9]; the geometric notation follows [10].

1. Properties of Chevalley-Groups

Let Δ be a root system in an Eucliean space E_n, and let k be a finite field of characteristic p such that $|k| = q$. A Chevalley-group associated with Δ, and defined over k, is a finite group generated by certain p-groups U_α, $\alpha \in \Delta$, called root subgroups, defined as in [19] for Chevalley-groups of normal type, and in [6, 18, 19] for a Chevalley group of twisted type. If Δ_0 is a root system generated by some subset of a fundamental system of roots in Δ, then $G_0 = \langle U_\alpha,\ \alpha \in \Delta_0 \rangle$ is a Chevalley group associated with the root system Δ_0.

The groups under consideration are assumed to have indecomposable root systems. Unless otherwise stated, G will denote throughout the paper a Chevalley-group with root system Δ, such that $Z(G) = 1$. Let B be the Borel-subgroup of G, U the Sylow p-subgroup of B, and H a p-complement of B. Then $U \trianglelefteq B$, $B = UH$ and H is abelian. There exists a subgroup $N \trianglerighteq H$ such that $W = N/H$ can be identified with a group generated by the reflections $w_1, \ldots, w_n$, corresponding to a fundamental set of roots $\alpha_1, \ldots, \alpha_n$ in the root system Δ. Setting $R = \{w_1, \ldots, w_n\}$ the pair (W, R) is a Coxeter-system [5] and the subgroups B, N define a Tits-system in G, with Weyl-group W. We shall view the elements of W as belonging to G when this causes no confusion. We shall use the notation $U_{\alpha_i} = U_i$ and $U_{-\alpha_i} = U_{-i}$, $1 \leqslant i \leqslant n$. We assume $w_i \in \langle U_i, U_{-i} \rangle$.

The Dynkin-diagram of an indecomposable root system of rank at least 3 will be given in Table I. The classical group notation is given in Table II. All the root systems are given explicitly at the end of [5]. The root system BC_n is not reduced and consists of the union of vectors on pp. 252 and 254 of [5]. In this system, roots have length 1, $\sqrt{2}$ or 2. A root α has length 2 if and only if $\alpha/2$ is a root, and in this case $U_\alpha = U_{\alpha/2}$ in the corresponding Chevalley-group.

For each subset $I \leqslant \{1,..., n\}$ set $W_I = \langle w_j \mid j \notin I \rangle$,

$$G_I = \langle B, U_{-j} \mid j \notin I \rangle = \langle B, W_I \rangle = BW_IB$$

$$L_I = \langle U_j, U_{-j} \mid j \notin I \rangle, \text{ the so-called Levi-factor.}$$

$$Q_I = \langle U_\alpha \mid \alpha > 0, \alpha = \sum m_j \alpha_j, m_j > 0 \text{ for some } j \in I \rangle.$$

(1.1) Lemma. *Let $I \subseteq \{1,..., n\}$. Then*

(i) *$Q_I \trianglelefteq G_I$, $Q_I L_I \trianglelefteq G_I$ and $G_I = Q_I L_I H$.*

(ii) *$Q_I = O_p(G_I)$.*

(iii) *L_I is a product of pairwise commuting covering groups Chevalley groups, and its structure can be found by deleting the vertices in I from the Dynkin diagram.*

Proof. [7, (2.2)].

(1.2) Lemma (Tits). *If L is a proper subgroup of G such that $U \leqslant L$, then $L \leqslant G_i$ for some i.*

Proof. [16, (1.6)].

(1.3) Proposition. *Let $G = PSO^{\pm}(l, q)'$, $l \geqslant 7$, q odd if l is odd. Then*

(i) *Q_1 is elementary abelian of order q^{l-2}.*

(ii) *$L_1 \cong SO^{\pm}(l-2, q)$ and acts on Q_1 as a group of F_q-linear transformations preserving a nondegenerated quadratic form.*

(iii) *Let r be the positive root in Δ of maximal height. Then $G_2 = N_G(U_r) = C_G(U_r)H$, $|U_r| = q$.*

(iv) *If q is odd, then U_r is an isotropic 1-space in Q_1. If q is even, U_r is a singular 1-space in Q_1.*

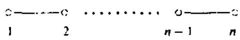
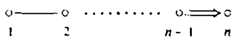
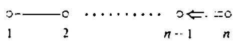
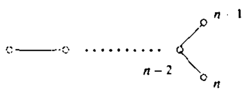
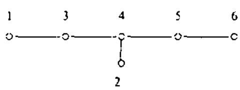
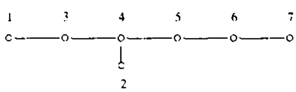
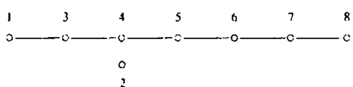
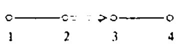
Proof. [7, (3.1)].

(1.4) Proposition. *Let $G = PSp(2n, q)$, $n \geqslant 2$. Then*

(i) *$|Q_1| = q^{2n-1}$. If q is odd, then Q_1 is special with center of order q. If q is even, Q_1 is elementary abelian.*

TABLE I

Dynkin-Diagrams[a]

	$A_n, n \geqslant 1$
	$B_n, n \geqslant 2$
	$C_n, n \geqslant 2$
	$D_n, n \geqslant 4$
	E_6
	E_7
	E_8
	F_4

[a] Arrows denote the short roots, if there are roots of different length.

TABLE II

Classical group notation	(B, N)-notation	Type
$PSO(2n+1, q)'$	$B_n(q)$	B_n
$PSp(2n, q)$	$C_n(q)$	C_n
$PSO^+(2n, q)'$	$D_n(q)$	D_n
$PSO^-(2n, q)'$	${}^2D_n(q)$	B_{n-1}
$PSU(2n, q)$	${}^2A_{2n-1}(q)$	C_n
$PSU(2n+1, q)$	${}^2A_{2n}(q)$	BC_n

(ii) *Let r be the root of maximal height. Then $Z(Q_1L_1) = U_r$ has order q and $G_1 = N_G(U_r) = C_G(U_r)H$. If q is odd, then $U_r = Z(Q_1)$. All elements of each nontrivial coset of U_r in Q_1 are conjugate in Q_1L_1.*

(iii) *$L_1 \cong Sp(2n-2, q)$ and acts on Q_1/U_r as a group of F_q-transformations preserving a nondegenerated alternating form. If q is odd, such a form is induced by the commutator function. If q is even and $q > 2$ for $n = 2$, then L_1 is indecomposable on Q_1.*

(iv) *There is a positive root s such that U_sU_r/U_r is central in U/U_r and is an isotropic 1-space in Q_1/U_r, $|U_s| = q$.*

(v) $Q_{12} = Q_2U_1$, $Q_2 \trianglelefteq G_{12}$, $Q_2L_{12} \trianglelefteq G_{12}$, $Q_2L_{12} \leqslant C_G(U_s)$, $G_{12} = (Q_2L_{12})\,U_1H = C_{G_{12}}(U_s)\,U_1H$.

Proof. [7, (3.2)].

(1.5) PROPOSITION. *Let $G = PSU(l, q)$, $l \geqslant 4$. Then*

(i) *Q_1 is special of order q^{2l-1} with center of order q.*

(ii) *There exists a uniquely determined root r such that $Z(Q_1) = Z(U_r)$ has order q. If l is odd, U_r is special of order q^3, while if l is even, U_r is elementary abelian. All elements of each nontrivial coset of $Z(Q_1)$ in Q_1 are conjugate in Q_1. Moreover $G_1 = N_G(Z(Q_1)) = C_G(Z(Q_1))H$.*

(iii) *$L_1 \cong SU(l-2, q)$ and acts on $Q_1/Z(Q_1)$ as a group of F_{q^2}-linear transformations preserving a nondegenerated hermitian form.*

(iv) *There is a positive root s such that $U_sZ(Q_1)/Z(Q_1)$ is central in $U/Z(Q_1)$ and is an isotropic 1-space of the unitary space $Q_1/Z(Q_1)$. Here $|U_s| = q^2$.*

(v) $Q_{12} = Q_2U_1$, $Q_2 \trianglelefteq G_{12}$, $Q_2L_{12} \trianglelefteq G_2$, $Q_2L_{12} \leqslant C_G(U_s)$, $G_{12} = C_{G_{12}}(U_s)\,U_1H = (Q_2L_{12})\,U_1H$.

Proof. [7, (3.3)].

Now we give some properties of the exceptional groups. Let r be the positive root of maximal height. By [5, pp. 260, 265, 269, 272] r is as follows:

$$F_4\colon r = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4,$$

$$E_6\colon r = \alpha_1 + 2\alpha_2 + \alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6,$$

$$E_7\colon r = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7,$$

$$E_8\colon r = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8.$$

The centralizer of r in W is W_i where $i = 1$ for $G = F_4(q)$, ${}^2E_6(q)$, $E_7(q)$, $i = 2$ for $G = E_6(q)$ and $i = 8$ for $G = E_8(q)$. Also $G_i = N_G(U_r)$.

(1.6) PROPOSITION. *Let $G = E_6(q)$, $E_7(q)$ or $E_8(q)$ and r and i as above. Then*

(i) *Q_i is special with center U_r and has order q^{21}, q^{33} or q^{57}, respectively.*

(ii) *$G_i = N_G(U_r) = C_G(U_r)H$, $L_i \leqslant C_G(U_r)$, $L_i/Z(L_i) \cong A_5(q)$, $D_6(q)$ or $E_7(q)$.*

(iii) *Q_i/U_r can be turned into an F_q-space such that the commutator function induces a nondegenerating form on Q_i/U_r. Moreover L_i acts on Q_0/U_r as a group of F_q-transformations preserving this form.*

Proof. [7, (4.4)].

(1.7) PROPOSITION. *Let $G = F_4(q)$. Then*

(i) *$|Q_1| = q^{15}$ and $L_1/Z(L_1) \cong PSp_6(q)$. If q is odd, then Q_1 is special with center U_r of order q, G_1 acts irreducibly on Q_1/U_r. If q is even, then $Q_1 = LS$ with $[L, S] = 1$, $L \cap S = U_r$, L special with center U_r and S elementary abelian of order q^7. Moreover G_1 acts irreducibly on S/U_r and Q_1/S.*

(ii) *$|Q_4| = q^{15}$ and $L_4 \cong SO(7, q)'$, G_4 has a normal elementary abelian subgroup R_4 of order q^7 such that $U_s < R_4 < Q_4$, $s = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$. G_4 acts irreducibly on Q_4/R_4.*

(iii) *If q is odd, then L_4 acts on R_4 as a group of F_q-transformations preserving a nondegenerated symmetric form. The isotropic 1-spaces of R_4 are conjugates of root-groups U_α with α a long root.*

(iv) *If q is even, then $U_s \lhd G_4$, L_4 acts on R_4 as a group of F_q-transformations preserving a quadratic form for which the radical of R_4 is U_s. The singular 1-spaces of R_4 are conjugates of groups U_α with α a long root.*

(v) *$G_1 = N_G(U_r) = C_G(U_r)H$. If q is even, $G_4 = N_G(U_s) = C_G(U_s)H$.*

Proof. [7, (4.5)].

(1.8) PROPOSITION. *Let $G = {}^2E_6(q)$. Then Q_1 is special of order q^{21} with center U_r of order q. G_1 acts* irr. *on Q_1/U_r. Moreover $G_1 = N_G(U_r) = C_G(U_r)H$.*

Proof. [7, (4.6)].

(1.9) PROPOSITION. *Let G be a Chevalley-group of rank at least 3. Let r be a long root of maximal height, $V_r = Z(U_r)$, and $J = \langle V_r, V_{-r}\rangle$. Then*

(i) *$J \cong SL_2(q)$. If q is odd, set $\langle t\rangle = Z(J)$. Then $t \in H$.*

(ii) *$N_G(J) = LJH$ where $[L, J] = 1$ and L is the Levi-factor of the parabolic subgroup $P = N_G(V_r)$.*

(iii) *Suppose q to be odd and G not isomorphic to $\Omega^\varepsilon_n(q)$, then $N_G(J) = C_G(t)$.*

(iv) *If $G = \Omega^\varepsilon_n(q) \neq \Omega^+_8(q)$, then $L = XJ^w$ for some $w \in W$. If q is odd, then $C_G(t) = XJJ^w H\langle w\rangle$.*

(v) *If $G = \Omega^+_8(q)$, then there exists a 4-group W_1 in W such that LJ is a central product of four conjugates of J under W_1. If q is odd, then $C_G(t) = LJHW_1$.*

(vi) *The isomorphism class of L and the weak closure of V in LJ are given in Table III.*

Proof. For q odd this is [2, (4.2)]. Suppose q to be even. Then (i) is well-known. Furthermore (iv)–(vi) is contained in [2, (4.2)].

Clearly $N_G(J) = JN_{N_G(V)}(J)$. Let $G = \bigcup_w UHwU^-_w$ the Bruhat-decomposition of G. Let w_0 be a word of greatest length in the generators $w_1, \ldots, w_n$. Then $J^{w_0} = J$, $(\Delta^+)^{w_0} = \Delta^-$ and $P^{w_0} = Q^{w_0} = Q^{w_0}LH$, where $Q = O_p(P)$. This yields $O_p(N_G(J)) = 1$ and so $N_Q(J) = V_r$. Thus $N_{N_G(V_r)}(J) = V_rLH$ and so (ii) is proved.

(1.10) LEMMA. *Let $g \in \mathrm{Aut}(G)$, $o(g)$ odd if $G = F_4(q)$, q even. Let $V = V_r$ as in (1.9). Then $\langle V^g, V\rangle$ is a p-group or $\langle V^g, V\rangle$ is conjugate to $J = \langle V_r, V_{-r}\rangle$. If $\langle V^g, V\rangle$ is a p-group, then $\langle V^g, V\rangle$ is conjugate to $\langle V_r, V_{w(r)}\rangle$ for some w contained in the Weyl-group.*

Proof. Let $G = BNB$ be the BN-decomposition of G and $g = xy$ with $x \in G$. Then $x = bhw\tilde{b}$, with $b, \tilde{b} \in B$, $h \in H$ and $w \in W$. We may assume $BH \leqslant N_G(V_r)$. Furthermore $(V_r)^w = V_{w(r)}$. Thus $V^g = (V_{w(r)})^{\tilde{b}y}$. Without loss

TABLE III

$G(q)$	L	$\langle V^G \cap LJ\rangle$
$L_n(q)$	$SL_{n-2}(q)$	LJ
$PSp_n(q)$	$Sp_{n-2}(q)$	LJ
$U_n(q)$	$SU_{n-2}(q)$	LJ
$\Omega^\varepsilon_n(q)$	$SL_2(q)\,SO^\varepsilon_{n-4}(q)$	LJ unless $n = 7$ or 8, $\varepsilon = -1$, where JJ^w
$F_4(q)$	$Sp_6(q)$	LJ
${}^2E_6(q)$	$SU_6(q)$	LJ
$E_6(q)$	$SL_6(q)/Z_{(q-1,3)}$	LJ
$E_7(q)$	$SO^-_{12}(q)$	LJ
$E_8(q)$	$E_7(q)$	LJ

$y \in N_G(U)$. As $o(y)$ is odd for $G = F_4(q)$ we get $(V_r)^y = V_r$. Now $\langle V^g, V\rangle$ is conjugate to $\langle V_{w(r)}, (V_r)^{y^{-1}\tilde{g}^{-1}}\rangle = \langle V_{w(r)}, V_r\rangle$. Now the assertion follows with [6].

(1.11) LEMMA. *Let $G \neq PSU_m(q)$ or $PSp_{2n}(q)$ and $E = \langle V_{w(r)}, V_r\rangle$ elementary abelian. Then there is a $g \in G$ such that $E^g \leqslant O_p(N_G(V_r))$.*

Proof. Application of (1.3)–(1.9) yield that the centralizer of r in the Weyl-group W acts on $\Delta_0 = \{s \mid s \in \Delta, s$ long and $V_s \leqslant O_p(N_G(V_r))\}$. Suppose now $G \neq PSO^\varepsilon(n, q)$. Then the Weyl-group acts transitively on $\Delta_1 = \{s \mid s \in \Delta,\ s$ long and $V_s \leqslant L$, the Levifactor of $N_G(V_r)\}$. As $G \neq PSU_m(q)$ or $PSp_{2n}(q)$ we get $\Delta_0 \neq \emptyset$. Thus there is an element $\tilde{w} \in W$ with $(V_r)^{\tilde{w}} \neq V_r$ and $V_r^{\tilde{w}} \leqslant O_p(N_G(V_r))$. Now $(\Delta_0)^{\tilde{w}} \neq \Delta_0$. Thus we may assume that there is a root $s \in \Delta_0$ such that $\tilde{w}^{-1}(s) = w(r)$. Now

$$\langle V_r, V_{w(r)}\rangle^{\tilde{w}} = \langle V_{w(r)}, V_s\rangle \leqslant O_p(N_G(V_r)).$$

Let now $G = PSO^\varepsilon(n, q)$. Then look at G_1. Set $\Delta_0 = \{s \in \Delta,\ s$ long, $V_s \leqslant L_1\}$. Then W_1 is transitive on Δ_0. Thus we may assume $\langle V_{w(r)}, V_r\rangle$ is contained in $Q_1 O_p(N_G(V_r)) = O_p(N_G(V_r))U_1$. As $V_{w_2(r)}$ and $U_{\alpha_1+\alpha_2} = U_{w_2(\alpha_1)}$ are contained in $O_p(N_G(V_r))$ we get the assertion.

(1.12) LEMMA. *Let $\langle V_r, V_{w(r)}\rangle$ be a nonabelian p-group. Then $\langle V_r, V_{w(r)}\rangle$ is special with center $V_{r+w(r)}$ and $r + w(r)$ is long. Furthermore $\langle V_r, V_{w(r)}\rangle \leqslant O_p(N_G(V_{r+w(r)}))$.*

Proof. If $w(r)$ is positive, then $\langle V_r, V_{w(r)}\rangle$ is special with center $V_{r+w(r)}$ by [7, (4.8)]. Thus assume $-w(r)$ is positive. As r is of maximal height, $[V_r, V_{-w(r)}] = 1$ by [7, (4.8)]. Set $\tilde{w} = w_{w(r)}$. Then $(\langle V_r, V_{w(r)}\rangle)^{\tilde{w}} = \langle V_{r+w(r)}, V_{-w(r)}\rangle$. Now [7, (4.8)] yields that $\langle V_r, V_{w(r)}\rangle$ is special with center $V_{r+w(r)}$ and $r + w(r)$ is long.

As $V_{w(r)}$ and $V_{-(r+w(r))}$ are not contained in $C_G(V_r)$ we get $\langle V_{r+w(r)}, V_{-w(r)}\rangle \leqslant O_p(N_G(V_r))$ and by conjugation we get the conclusion.

(1.13) LEMMA. *Let $G = PSU(n, q)$ or $PSp(2n, q)$ and $\langle V_r, V_{w(r)}\rangle$ a p-group, then $[V_r, V_{w(r)}] = 1$.*

Proof. By (1.4) and (1.5) we get that V_r is weakly closed in $O_p(N_G(V_r))$ with respect to G. Now (1.12) yields the conclusion.

(1.14) PROPOSITION. *Let J be as in (1.9) and $K \nleqslant G$ such that $JL \leqslant K$. If q is odd, then $K \leqslant C_C(t)$ or $G = L_4(q)$ and $K \leqslant \mathrm{Aut}(PSp_4(q))$ or $G = L_n(q)$ and K is a subgroup of the stabilizer of a 2-space or a $(n-2)$-space in the natural representation of $SL_n(q)$, or $G = \Omega_m^{\pm}(3)$, $m \leqslant 8$.*

Proof. Suppose first $O_p(K) \neq 1$. Then $O_p(K) \cap O_p(N_G(V)) \neq 1$. As $V \nleqslant O_p(K)$ the action of LH on $O_p(N_G(V))/V$ is not irreducible. Thus $G = L_n(q)$ by (1.3)–(1.8). Let M be the natural module for G. Then $M = M_1 \oplus M_2$ where M_1 is a natural module for J and M_2 is a trivial J-module. Thus we may assume that $O_p(K)$ stabilizes M_1 or M_2 and the structure of $L_n(q)$ yields now $K = O_p(K) JLH$ is a subgroup of the stabilizer of a 2-space or a $(n-2)$-space.

So we may assume $O_p(K) = 1$. Suppose now $O_2(K) \neq 1$. Clearly $[V, O_2(K)] \leqslant J \cap O_2(K)$. Furthermore the action of L on $O_p(N_G(V))$, (1.3) – (1.8), yields

$$C_{O_2(K)}(V) \leqslant \langle t \rangle \qquad \text{or} \qquad C_{C_{O_2(K)}(V)}(O_p(N_G(V))) \neq 1.$$

But as $N_G(V)$ is p-constraint we get $C_{O_2(K)}(V) \leqslant \langle t \rangle$. This yields $\Omega_1(O_2(K)) = \langle t \rangle$ and so $K \leqslant C_G(t)$.

So we may assume that $F(K)$ is a $\{2, p\}'$-group. Take $a \in F(K)$. Then $\langle V^a, V \rangle \leqslant VF(K)$. By (1.10), $\langle V, V^a \rangle$ is a p-group. Thus $[F(K), V] = 1$ and so $F(K) \leqslant C_G(t)$. So we may assume $F(K) = 1$. Let E be a component of $E(K)$. Then $V \leqslant N_G(E)$. Suppose $[V, E] \neq 1$. By Baer's theorem [9, (3.8.2)] we get $V \leqslant E$ and so $J \leqslant E$ or $q = 3$ and $t \in E$. Suppose now $E = E(K)$. Then the conclusion follows with [1] and (1.3) – (1.9). Thus $E(K) = EF$, $F \neq 1$. Then $F \leqslant C_G(t)$ and so $F \leqslant C_G(V)$. This yields $JL \nleqslant EV$. Suppose $G \neq \Omega_n^{\varepsilon}(q)$. Then there is a $V^g \leqslant JL$ such that $V^g \nleqslant EV$. Thus $[V^g, E] = 1$ and so $[L, EV] = 1$. As $E \leqslant {}_G(V^g)$ we get now the contradiction $EV \cong SL_2(q)$.

Let $G = \Omega_n^{\varepsilon}(q)$ and $n \neq 7$ and $n \neq 8$, $\varepsilon = -1$. Then $L = J^w X$. If $X \leqslant E$, then $[J^w, E] = 1$, contradicting $t \in J^w$. Thus $J^w \leqslant EV$. Now we may assume $[X, E] = 1$, yielding the same contradiction as above. Thus $G = \Omega_7(q)$ or $\Omega_8^-(q)$. Furthermore $JJ^w \leqslant E$. Thus $JL \cap E = JJ^w$. Then a Sylow p-subgroup of $C_E(V)$ is of order q^2. Application of [1] yields now a contradiction.

(1.15) Proposition. *Let J be as in* (1.9) *and $K \nleqslant G$ such that $JL \leqslant K$. Suppose q to be even. Then $K \leqslant N_G(JL)$ or $G = F_4(q)$ and $K = PSp_8(q)$, $G = L_4(q)$ and $K = PSp_4(q)$, $G = L_n(q)$ and K is contained in the stabilizer of a 2-space or a $(n-2)$-space in the natural representation of G, or $G = L_4(2)$, $SU_4(2)$ or $\Omega_8^+(2)$.*

Proof. Suppose $q > 2$. Then $O(K) = \langle C_{O(K)}(v) \mid v \in V^* \rangle = C_{O(K)}(V)$. Thus by (1.9), $O(K) \leqslant N_G(JL)$. So we may assume $O(K) = 1$. Set $R = O_2(K)$. Suppose $R \neq 1$. Then $C_R(V) \neq 1$. Thus $C_R(V) \leqslant O_2(N_G(V))$ and so LH cannot act irreducible on $O_2(N_G(V))/V$. This yields $G = L_n(q)$ or $F_4(q)$. If $G = F_4(q)$, $Z(O_2(N_G(V))) \leqslant R$ as LH acts indecomposable on $Z(O_2(N_G(V)))$. Thus $V \leqslant R$, a contradiction. Hence $G = L_n(q)$ and an easy argument shows that K is contained in the stabilizer of a 2-space or a $(n-2)$-space.

So we may assume $O_2(K)=1$. Set $K_1=\langle J^g \mid J^g \leqslant K\rangle$. Then by [23], K_1 is a direct product $X_1 \times X_2 \times \cdots \times X_r$ with Chevalley-groups X_i. We may assume $J \leqslant X_1$. If $O_2(N_G(V)) \leqslant K$, then $K=G$ by (1.2). Thus $O_2(N_G(V)) \nleqslant K$. Suppose $V=O_2(N_K(V))$. Then $X_1 \cong L_2(q)$ and so $X_1=J$. Furthermore $K \leqslant N_G(JL)$. Thus $V \neq O_2(N_K(V))$ and so $G=L_n(q)$ or $G=F_4(q)$. As $O_2(N_{X_1}(V)) \neq V$ we get by inspection $X_1=Sp_4(q)$ or $X_1=Sp_8(q)$ and $JL \leqslant X_1$. Then $K_1=X_1=K$. The proposition is proved.

Suppose now $q=2$. Suppose $O_2(C_K(V)) \neq V$. If $O_2(C_G(V)) \leqslant K$, we get the assumption with (1.2). Thus $G=L_n(q)$ and $O_2(C_K(V))$ is elementary abelian of order 2^{n-1} or $G=F_4(q)$ and $O_2(C_K(V))$ is elementary abelian of order 2^7.

Set $\tilde{K}=K/O(K)$. Suppose $O_2(\tilde{K}) \neq 1$. Then $O_2(C_K(V)) \neq V$ or $\tilde{V}=O_2(\tilde{K})$. If $\tilde{V}=O_2(\tilde{K})$, then $K=O(K)\,C_K(V)$. The structure of $C_G(V)$ yields $O(K)=O_3(K)$. The action of L on $O_2(C_G(V))$ yields that V cannot centralize a subgroup of order 9 in $O_3(K)$. Furthermore V inverts no subgroup of order 9. By [24, (3.14)] $|O_3(K)| \leqslant 27$. If $K \nleqslant N_G(JL)$, then $O_3(K)$ has to be extraspecial of order 27. Thus $O_3(C_G(V)/O_2(C_G(V))) \neq 1$. Now (1.9) yields $G=U_n(2)$ or $\Omega_n^{\pm}(2)$. The action of L on $O_3(K)$ shows $G=U_n(2)$ and $[L, O_3(K)]=1$. But the action of G on the natural module shows that G cannot contain such a subgroup. Thus $O_2(K) \neq V$ and so $G=L_n(2)$ or $F_4(2)$. If $V \leqslant O_2(K)$, we get a contradiction as above. Thus $V \nleqslant O_2(K)$ and so $G=L_n(2)$. Now it is easy to see that a Sylow 2-subgroup of G is contained in K. By (1.2) we get that K is the stabilizer of a 2-space or a $(n-2)$-space in the natural representation of G. Hence we have shown $O_2(\tilde{K})=1$. Then $J \cap O(K)=1$ and so $[V, O(K)]=1$. So we may assume $O(K) \leqslant Z(K)$.

Set $E_1=E(K)$. Let E be a component of E_1 with $1 \neq [E, V] \leqslant E$. Suppose $V \nleqslant E$. Then by [23], $E \cong \Omega_n^{\pm}(2^m)$, $\Omega_n(3)$, $\Omega_n(5)$ or A_n. As V inverts no subgroup of order 9, $E \trianglelefteq ELH$ and so $C_E(V)=O_2(C_E(V))\,N_E(J)$. Let D be $O_3(J)$. Then $D \leqslant E$. If $E=\Omega_n^{\pm}(2^m)$, then $C_E(V) \cong Sp(n-2, 2^m)$. But an easy argument shows that $\Omega_n^{\pm}(2^m)$ contains no subgroup $Z_3 \times Sp(n-2, 2^m)$. If $E \cong A_n$, then $C_E(V) \cong A_{n-2}$. As A_n contains no subgroup $Z_3 \times A_{n-2}$, we get $n=6$. Now $O_2(C_E(V)) \neq 1$, and so $|O_2(C_K(V))|=4$, a contradiction. Suppose $E \cong \Omega_n^{\pm}(p)$, $p=3, 5$. As $N_E(J) \lhd N_G(J)$ and there are only two isomorphisms between $\Omega_m^{\pm}(p)$ and a Chevalley-group in characteristic 2, namely, $\Omega_5(3) \cong U_4(2) \cong \Omega_6^-(2)$ and $\Omega_4^-(3) \cong Sp_4(2)'$, we get $G=U_6(2)$, $\Omega_8^-(2)$ or $Sp_6(2)$. As $N_E(J)$ has to normalize an element of order 3 in E, we get a contradiction to the structure of E.

Thus we have $V \leqslant E$. Clearly JL normalizes E. The structure of the groups in [23] yields $E \cong M(22)$ or $M(23)$ of $O_2(C_E(V))=V$. Now $C_E(V)$ is contained in $N_E(J)$ a contradiction as $V \leqslant C_E(V)'$ [8, Chaps. 17, 18]. Thus $O_2(C_E(V))>V$. Thus $GL_n(2)$ or $F_4(2)$. Furthermore $O_2(C_K(V)) \leqslant E$. As E is a Chevalley-group by [22], we get by inspection $JK \leqslant E$ and $E \cong Sp(8, 2)$. $G=F_4(2)$.

(1.16) LEMMA. *Let* $G = L_2(q)$, *q even. Let* $g \in G$ *with* $o(g) \neq 3$ *if* $3 \mid q+1$. *Then there is an element* $h \in G$ *such that* $o(g(g^i)^h) = 2$ *for some i.*

Proof. Let T be a Sylow 2-subgroup of G, $r \in N_G(T)$ with $o(r) = q-1$ and $x \in N_G(\langle r \rangle)$, $o(x) = 2$. Then each element of G is conjugate to an element tx with $t \in T$. Suppose $g = tx$. If there is some j such that $g \not\sim g^j$ in G set $i = -j$. Now g^j is conjugate to $t_1 x$ for some t_1 in T. Choose h with $(g^j)^h = t_1 x$. Then $g(g^i)^h = txxt_1 = tt_1$. As $g \not\sim g^j$ we have $t \neq t_1$. Thus $o(g(g^i)^h) = 2$. Suppose now that $g \sim g^j$ for all j. Then $o(g) = 3$. Thus $3 \mid q-1$. But then g normalizes a Sylow 2-subgroup S of G and so $o(g(g^{-1})^s) = 2$ for some $s \in S$.

(1.17) LEMMA. *Let* $G = \Omega_6^-(q)$ *or* $\Omega_8^+(q)$. *For* $g \in G$, $o(g) = 3 \mid q+1$, *there is a nontrivial 2-group T such that* $g \in C_G(T)$.

Proof. Let M be the natural module for G. M is a direct sum of 2-dim totally anisotiopic G-spaces. If g acts trivially on two of them, then there is an element of order 2 in G interchanging these spaces and acting trivially on the remaining spaces. Hence g centralizes this element.

Thus we may suppose that there are two spaces where g acts nontrivially on. Then there is an involution i interchanging these spaces and acting trivially on the remaining spaces. Now the structure of $SO_4^+(q)$ shows that i may be chosen in the centralizer of g.

2. PERSPECTIVITIES

(2.1) LEMMA. *Let T be a special 2-group acting on a projective plane* π. *Let* α *be a perspectivity of odd order such that* $[T, \alpha] = T$, *and* $[Z(T), \alpha] = 1$. *Then* $Z(T)$ *contains nontrivial perspectivities.*

Proof. Choose $t \in T - Z(T)$. Set $u = t^{-1}t^{\alpha}$. Then u is a generated perspectivity. If $o(u) = 2$, then u is a persepectivity and there is an nontrivial element $z \in Z(T)$ such that $u \sim uz$ in T. Thus z is a perspectivity. So we may assume $o(u) = 4$. Set $z = u^2$ and $\pi_1 = \mathrm{Fix}(z)$. Suppose π_1 to be a plane. Then u and α act on π_1 and α induces a perspectivity and u induces a generalized perspectivity of order 2 on π_1. Thus u induces a perspectivity on π_1. As the axis of u is ZZ^t, Z the center of α, we get that Z is fixed by u. But then $u^{-1}u^{\alpha} = w$ is a perspectivity contained in T. If $o(w) = 2$, we get the assumption as above. If $o(w) = 4$, then w^2 is a perspectivity, $w^2 \in Z(T)$.

For the remainder of this chapter let $G = G(q)$ be a Chevalley-group different from $\Omega_n^{\pm}(2)$, $\Omega_n^{\pm}(3)$, $n \leqslant 8$, of rank at least 3, and K a subgroup of $\mathrm{Aut}(G)$ acting strongly irreducible on a finite projective plane π and containing perspectivities $\alpha \neq 1$. Let the notation be as in (1.9).

(2.2) LEMMA. *Let $V_1 \leqslant V$. Then V_1 is planar or regular.*

Proof. Suppose to be false. Let V_1 be a minimal counterexample. Then each subgroup of V_1 different from V_1 is planar. Let $V_1 = V_2\langle v\rangle$. Set $\pi_1 = \mathrm{Fix}(V_2)$. Then v induces a generalized perspectivity or a perspectivity on π_1. Furthermore $\mathrm{Fix}(V_1) = \mathrm{Fix}_{\pi_1}(v)$. Let L be the Levifactor of $N_G(V)$. Suppose that L stabilizes te center of v.

According to (1.9) choose $g \in G$ with $(V_1)^g \leqslant L$. If $(V_1)^g$ has the same center as V_1, then JL stabilizes this center. If J does not stabilize the center of V_1, then all V_1^g contained in L have the same axis. As J is contained in the Levifactor of $N_G((V_1)^g)$ we get that J fixes the axis of V_1.

By duality we may assume that JL fixes the center of V_1. But the stabilizer of the center of V_1 in $O_p(N_G(V))$ is of index at most three in $O_p(C_G(V))$. Now the action of L on $O_p(C_G(V))$ yields that U stabilizes the center of V_1. But now G stabilizes the center of V_1 by (1.2).

Thus we have that L does not stabilize the center of V_1. Then V_1 has to be triangular and so L has to contain a subgroup of index at most three. This yields $q = 2$ or 3 and $G = \Omega_n^{\pm}(q)$. Let $L = J^w X$. As $n > 8$, there is a $g \in G$ such that $(V_1)^g \leqslant X$. Thus $\mathrm{Fix}(V_1) = \mathrm{Fix}((V_i)^g)$ and so LJ normalizes $\mathrm{Fix}(V_1)$. As above we get, that U normalizes $\mathrm{Fix}(V_1)$. Application of (1.2) implies that G normalizes $\mathrm{Fix}(V_1)$, a contradiction.

(2.3) LEMMA. *Let q be even, $o(\alpha) = r$ an odd prime, then $[V, \alpha] \nsubseteq V$.*

Proof. Suppose $[V, \alpha] \subseteq V$. By (2.2), $[V, \alpha] = 1$. Now α acts on $O_2(C_G(V))$. By (2.1) we get that $Z(O_2(C_G(V)))$ contains a nontrivial perspectivity. Now (2.2) yields $G = Sp_{2n}(q)$ or $F_4(q)$. As $C_G(V)$ acts indecomposable on $Z(O_2(C_G(V)))$ we get that all involutions in $Z(O_2(C_G(V)))$ are perspectivities contradicting (2.2).

(2.4) PROPOSITION. *Suppose $o(\alpha) = r$, prime. Then $\langle V, V^\alpha\rangle \cong SL_2(q)$ or $[V, \alpha] = 1$.*

Proof. Suppose $o(\alpha) = 2$ and $G = F_4(q)$, q even. By [15], there is a nontrivial perspectivity $\beta \in Z(U)$. By (2.2), $\beta \notin V$. Set $Z = Z(O_2(N_G(V)))$. By (1.7), $Z = \langle \beta^a \mid a \in N_G(V)\rangle$. Thus by [15, (2.7)] all involutions in Z are perspectivities. As $V \leqslant Z$, we get a contradiction.

Suppose now $\langle V, V^\alpha\rangle \ncong SL_2(q)$. By (1.10), $\langle V, V^\alpha\rangle$ is a p-group. Assume $G \neq PSU_m(q)$ or $PSp_{2n}(q)$. By (1.11) and (1.12), $\langle V, V^\alpha\rangle$ is contained in $O_p(N_G(Y)) = R$ for some $Y \sim V$ in G. Set $h = v^{-1}v^\alpha$ for some $v \in V^*$. By [10, (5.1)] h is a generalized perspectivity. Thus $\mathrm{Fix}(R) \subseteq \mathrm{Fix}(h)$. Let L be the Levi factor of $N_G(Y)$. By (2.2), Y is planar. Set $\mathrm{Fix}(Y) = \pi_1$. Then h induces a generalized perspectivity on π_1. If h is a perspectivity, then α fixes the center or the axis of h. If h is ot a perspectivity, α fixes both the axis and the center. Thus we may assume that α fixes the center of h. If L fixes the center of h, then the action of L on $O_p(N_G(Y))$ yields that R fixes the center too. If $G \neq L_n(q)$, then (1.2) implies that α normalizes Y.

Suppose that α normalizes Y. By (2.2), α centralizes Y. If $r = p$, then [15, (2.5)] implies that $Z(U)$ contains perspectivities. Now (2.2) and (1.3)–(1.8) yields $G = F_4(q)$, q even. But then $r = 2$, a contradiction. Thus $r \neq p$. Now we may assume that α normalizes L [6, Chap. 12]. By (1.14) and (1.15) we get that α normalizes $\tilde{J} = J^x$, $V^x = Y$. As α centralizes Y, α centralizes $\tilde{J}$. Suppose $[L, \alpha] = 1$. Then α acts fix-point-free on R/Y. Thus α normalizes a conjugate Y_1 of Y but $[Y_1, \alpha] \neq 1$, a contradiction to (2.2). As no element of order p is a perspectivity we get that there is an element $l \in L$ such that $\alpha^l \neq \alpha$ and α^l and α have different axis. Thus $\mathrm{Fix}(L\langle\alpha\rangle)$ is contained in a triangle. Furthermore $\tilde{J}$ fixes $\mathrm{Fix}(L\langle\alpha\rangle)$. Thus $\tilde{J}L$ fixes the center of h, contradicting (1.2).

Thus α cannot normalize Y. Thus $G = L_n(q)$ and a subgroup isomorphic to EM, $M \cong SL_{n-1}(q)$ and E elementary abelian of order q^{n-1} stabilizes the center of h. Now α normalizes E. As all elements of $E^\#$ are conjugate to v in G we get $[\alpha, E] = 1$ by (2.2). But this contradicts the structure of $\mathrm{Aut}(L_n(q))$.

So we have shown that L cannot fix the center of h. Thus $\mathrm{Fix}(R)$ is empty or a triangle. Suppose that $\mathrm{Fix}(R)$ is a triangle. Then L has a subgroup of index at most 3. Thus $G = \Omega_n^{\pm}(q)$, $q = 2$ or 3.

Suppose $q = 2$. By (2.2), $h^2 \neq 1$ and h^2 is planar. Let $\pi_1 = \mathrm{Fix}(h^2)$. By [15, (2.7)] all elements in $O_2(N_G(Y))/Y$ induce perspecitivities on π_1, with the same center or the same axis. As α fixes the center and the axis of h, we may assume that all perspectivities of R/Y have the same center. But then L fixes the center of h, a contradiction.

Let $q = 3$. Suppose that α normalizes R. Then α centralizes Y by (2.2). As above we get $r \neq p$. Thus we may assume that α normalizes L. We have $L = J^w X$. Thus α normalizes X. Suppose that α centralizes X, then we get a contradiction as above. As X contains conjugates of Y we get that there is a conjugate, α^x, with $x \in X$, such that α and α^x have different centers and axes. Hence $\mathrm{Fix}(X)$ is contained in a triangle. This yields $\mathrm{Fix}(X) = \mathrm{Fix}(R)$. Thus $P = \langle JL, N_G(Y)\rangle$ fixes $\mathrm{Fix}(R)$. By (1.2) $P = G$, a contradiction.

Thus $[\alpha, Y] \neq Y$. Set $P = \langle C_{N_G(Y)}(\mathrm{Fix}(R)), \alpha\rangle \cap G$. Let M be the natural module for G. Then the action of P on M is irreducible. Thus $O_3(P) = 1$. By Baer's theorem [9, (3.8.2)] we get a conjugate Y_1 of Y such that $\langle Y, Y_1\rangle \cong SL_2(3)$. As all groups $\langle Y, Y^g\rangle \cong SL_2(3)$ are conjugate under $N_G(Y)$ we get $\tilde{J} \subseteq P$. Thus the whole Weyl-group is contained in P. But then all U_r are contained in P. This yields $P = G$, a contradiction.

Thus $\mathrm{Fix}(R) = \varnothing$. Then h is triangular on $\mathrm{Fix}(Y)$. But now application of [10, (3.13)] yields a contradiction.

It remain the cases $G = PSU_m(q)$ and $G = PSp_{2n}(q)$. Let M be the natural module of G and M_1 a maximal isotropic subspace of M. Let P be the stabilizer of M_1 in G. Then $P/O_p(P) \cong SL_n(q)$ for $G = PSp_{2n}(q)$ and $P/O_p(P) \cong SL_t(q^2)$, $t = [m/2]$, for $G = PSU_m(q)$. Furthermore we may assume that $h = v^{-1}v^{\alpha} \in Z(O_p(P))$. Suppose $\mathrm{Fix}(Z(O_p(P))) \neq \varnothing$. Then the center of h is contained in $\mathrm{Fix}(Z(O_p(P)))$. Thus P fixes the center of h. Now the action of P on $O_p(P)$ yields that there is a conjugate Y of V such that $h \in O_p(N_G(Y))$. By (2.2), (1.4) and (1.5) we get that Y is planar with $\mathrm{Fix}(Y) = \pi_1$. Let $\mathrm{Fix}(O_p(N_G(Y)) = \varnothing$. Then h is triangular on π_1. Now application of [10, (3.13)] yields a contradiction, as $N_G(Y)$ contains no normal subgroup $K \leqslant O_p(N_G(Y))$ such that $|O_p(N_G(Y))/K| = 9$. Thus $\mathrm{Fix}(O_p(N_G(Y))$ contains the center of h. Let be the Levifactor of $N_G(Y)$; then L fixes the center of h. But now $\langle P, L\rangle$ fixes the center of h. By (1.2) this group is G, a contradiction. Thus $\mathrm{Fix}(Z(O_p(P)) = \varnothing$. Then $p = 3$ and h is triangular. Again application of [10, (3.13)] yields a contradiction.

(2.5) PROPOSITION. *Let q be odd, α a perspectivity with $o(\alpha) = r$, r a prime, then the involution $t \in J$ is a perspectivity.*

Proof. By (2.4) we may assume $J = \langle V, V^{\alpha}\rangle$. Let $v \in V^{\#}$. Set $h = v^{-1}v^{\alpha}$. Then h is a generalized perspectivity. If h is a perspectivity the proposition follows with [21]. Thus h is not a perspectivity. Let L be the Levifactor of $N_G(V)$. Then $L \leqslant C_G(h)$. Let Z be the center of h. By (1.9) there is an element $g \in G$ such that $h^g \in L$. Let T be the center of h^g. (If h is triangular, then the center of h is one point in $\mathrm{Fix}(h)$ stabilied by α.) Suppose $Z = T$. Then J fixes Z. Suppose that L does not fix Z. Then $q = 3$ and $G = \Omega_n^{\pm}(3)$. Furthermore h is triangular. Let $L = J^w X$. Then we may assume $h^g \in X$. Thus $\mathrm{Fix}(h) = \mathrm{Fix}(h^g)$. Now $\langle LJ, g\rangle$ normalizes $\mathrm{Fix}(h)$. By (1.14), $\langle LJ, g\rangle = G$, a contradiction. So we have proved that LJ fixes Z if $Z = T$.

Suppose $Z \neq T$. Let a be the axis of h. Then $T \in a$. Suppose that all conjugates of h contained in L have the same center. Then as there is and $g \in G$ such that $J^g \leqslant L$, all conjugates of h contained in J have the same center Z. Thus J fixes Z and so LJ fixes Z too.

Suppose now that there are two conjugates h_1, h_2 of h contained in L with different centers. Then the axis a of h is fix under LJ.

As α fixes the axis of h we may assume by duality that LJ fixes the center of h. Then $\langle LJ, \alpha\rangle$ fixes the center of h. Suppose that $\langle LJ, \alpha\rangle \cap G \not\leqslant C_G(t)$. Then by (1.14) we get $G = L_4(q)$ and $\langle LJ, \alpha\rangle \cap G \cong PSp_4(q)$ or $G = L_n(q)$ and $\langle JL, \alpha\rangle \cap G$ is contained in the stabilizer of a 2-space in the natural representation of G.

Let $G = L_4(q)$, $\langle LJ, \alpha\rangle \cap G \cong PSp_4(q)$. As $\alpha \notin C(t)$ we have $o(\alpha) \neq 2$. Thus α induces an automorphism on $PSp_4(q)$. Suppose $\langle LJ, \alpha\rangle \leqslant G$. If $o(\alpha)$ divides q^2+1, then α is conjugate in a subgroup S of $PSp_4(q)$, $S \cong L_2(q^2)$.

Application of [21] yields now that there are involutions in G which are perspectivities. By [15] there is an involution contained in the center of a Sylow 2-subgroup of G, which is a perspectivity. By conjugation we get now that t is a perspectivity.

Let $o(\alpha)$ divide q^2-1. Then α is conjugate in a subgroup of $PSp_4(q)$ isomorphic to $SL_2(q) * SL_2(q)$. As in [21, Satz 1] we get that the involution in the center of this group has to be a perspectivity.

Let $o(\alpha)=p$. As $q>3$ we may assume by [15, (2.5)] that $\alpha \in Z(U)=V$. But this contradicts (2.2). Thus we have $\alpha \notin G$. Then $q=t^r$. If $r \nmid |PSp(4,q)|$, then by conjugation we may assume that α acts on V, contradicting (2.2). Thus $r \mid |PSp_4(q)|$. Furthermore we may assume that $PSp_4(q)$ contains no perspectivities of order r. If $r=p$, we get a contradiction with [15, (2.5)]. If $r \mid q^2+1$, then a Sylow r-subgroup of $\langle LJ, \alpha\rangle$ has to be abelian. Thus $t^2 \equiv -1(r)$. Then r divides $(q^2+1)/(t^2+1)$ and so a field automorphism of $PSp_4(q)$ cannot centralize a Sylow r-subgroup. Thus $r \mid q^2-1$. Let r^a the exact divisor of q^2-1. Then a Sylow r-subgroup of $PSp_4(q)$ is abelian of type (r^a, r^a). Thus a Sylow r-subgroup of $\langle LJ, \alpha\rangle$ has to be abelian, or $r=3$ by [15, (2.5)]. In the former case we get a contradiction as above. Thus suppose $r=3$ and Sylow r-subgroups of $\langle LJ, \alpha\rangle$ not to be abelian. The action of a field automorphism yields now $a=1$. But always 9 divides t^6-1, a contradiction.

Let now $\langle JL, \alpha\rangle \cap G$ be contained in the stabilizer of a 2-space. Set $N=O_p(\langle JL, \alpha\rangle \cap G)$. Then α acts on N and so, by (2.2), $[\alpha, N]=1$. As $[\alpha, t] \neq 1$, we get $r=p$. As a field automorphism cannot centralize N, we get $\alpha \in N$. But then $\langle V, V^\alpha\rangle$ is a p-group, contradicting the choice of V and α. Thus we have shown that $[\alpha, t]=1$.

By (1.9) and $J=\langle V, V^\alpha\rangle$ we get $\alpha \in N_G(J)$. Set $\alpha=\beta\delta$ with $\beta \in \mathrm{Aut}(J)$ and $\delta \in C_G(J)$. Obviously $r \neq 2$. Thus $\beta=j\gamma$ with $j \in J$ and γ a field automorphism of J. Suppose now $\beta \notin J$. By (2.2) we have $r \neq p$. If $r=3$, then a Sylow 3-subgroup of $J\langle\alpha\rangle$ has exactly one elementary abelian subgroup of order 9. Let $\langle x, \alpha\rangle$ be this subgroup. Then $\alpha \sim \alpha x \sim \alpha x^2$. Thus there is a subgroup $F\langle\alpha\rangle$ of $J\langle\alpha\rangle$ such that $F\langle\alpha\rangle \cong SL_2(3)$. By [10, (2.5)] we get that t is a perspectivity. Thus we may assume $r \neq 3$. If $r \nmid |J|$ then α acts on a Sylow p-subgroup of J by the Frattini argument. But this contradicts (2.2). Thus $r \mid |J|$. As a Sylow r-subgroup of $J\langle\alpha\rangle$ is nonabelian, we get by [15, (2.5)] a perspectivity η inside of J. Application of [21, Satz 1] yields now the assertion.

Thus we may assume $\beta=j \in J$. By [21, Satz 1] we may assume $\delta \neq 1$ and $j \neq 1$. If $r \neq p$, then there is a conjugate j^g of j in J such that $o(jj^g)=4$ [21, Lemma 3]. As $r \neq p$ we have $j^{-1} \sim j$ in J. Thus there is an element $k \in J$ such that $o(j^k j^{-1})=4=o(\alpha^k \alpha^{-1})$. As in [21, Satz 1] we het that t is a perspectivity.

So we may assume $r=p$. Let $Y \in \mathrm{Syl}_p(J)$, $j \in Y$. Choose

$k \in N_J(Y) - C_J(Y)$. Then $(\alpha^{-1})^k\alpha$ is a generalized perspectivity contained in Y. But this contradicts (2.2). Thus the proposition is proved.

(2.6) PROPOSITION. *Let α be a perspectivity, $o(\alpha) = r$, r a prime. Then q is odd.*

Proof. Let q be even. By (2.4) we may assume $J = \langle V, V^\alpha \rangle$. Let L be the Levifactor of $N_G(V)$ and $h = v^{-1}v^\alpha$ for some $v \in V^\#$. Then h is a generalized perspectivity. If h is a perspectivity we may choose α as h. So assume h not to be a perspectivity. Then α fixes the center and the axis of h. By (1.9) there is an element $h^g \in L$. Let Z the center of h and T the center of h^g. If $T = Z$, then J fixes Z or $q = 2$ and h is triangular.

Suppose that J fixes Z, then L fixes Z or $q = 2$ and h is triangular. Thus assume h to be triangular. Then LJ normalizes Fix(h). Furthermore we may assume Fix(h^g) = Fix(h) and so g normalizes Fix(h). As $\langle JL, g \rangle \neq G$ we get, by (1.15), $G = F_4(2)$ and $\langle JL, g \rangle \cong PSp_8(2)$. But then JL fixes the center of h. Thus we have that JL fixes the center of h if there is a conjugate of h contained in L, X respectively, with the same center.

Suppose now $T \neq Z$. Then $T \in a$, the axis of h. If all the conjugates of h contained in L, X, respectively, have the same center T, then we get that all the cnjugates of h contained in J have the same center. Thus Z is fixed under JL. Thus there are conjugates h_1, h_2 with different centers. Then a is fixed under LJ. As a summary we get by duality that Z is fixed under $\langle LJ, \alpha \rangle$.

Suppose $\alpha \notin N(JL)$. Set $N = \langle LJ, \alpha \rangle \cap G$. By (1.15) we get $G = F_4(q)$ and $N \cong PSp_8(q)$, $G = L_4(q)$ and $N \cong PSp_4(q)$, or $G = L_n(q)$ and N is contained in the stabilizer of a 2-space in the natural representation of G.

Suppose $G = L_4(q)$ and $N \cong PSp_4(q)$. As $\alpha \notin N_G(J)$ we get $o(\alpha) \neq 2$. Then α induces an automorphism on N. Suppose $\alpha \in N$. If $r \mid q^2 + 1$, then α is conjugate in a subgroup of $PSp_4(q)$ isomorphic to $L_2(q^2)$. Application of (1.16) yields now that there are involutory perspectivities in N. Then [15, (2.5)] yields that there are perspectivities in V, contradicting (2.2). Thus $r \mid q^2 - 1$. Then α is conjugate into JL in N. But JL is contained in the stabilizer of a 2-space. Thus α normalizes an elementary abelian 2-subgroup in G. By (2.2), α centralizes this 2-subgroup. As $r \neq 2$ this contradicts the structure of $L_4(q)$. Thus we have that α induces an outer automorphism on N. As in (2.5) we get a contradiction.

Let N be contained in the stabilizer of a 2-space. Set $P = O_2(N)$. Then α acts on P. By (2.2), $[\alpha, P] = 1$. This yields $o(\alpha) = 2$. Now application of [15, (2.5)] yields that V contains nontrivial perspectivities, contradicting (2.2).

Assume finally $G = F_4(q)$, $N \cong PSp_8(q)$. Let U_s be a rootsubgroup of G, s a short root, and $U_s \leqslant N$. Let $u \in U_s^\#$. By [23], uu^α is of order 1, 2, 4 or odd. If $o(uu^\alpha) = 4$, then $(uu^\alpha)^2$ is conjugate to u in G. By (2.2), u cannot be a perspectivity. Thus uu^α is a generalized perspectivity and the center and the axis of uu^α is fixed under α. As all axes of N-conjugates of α intersect in one point we get that the center of $t = uu^\alpha$ is a fix point of N. Furthermore the axis is fixed under N. As all involution of G are planar we get $o(t) = 4$ or odd. Suppose $o(t)$ to be odd. Then an argument like in (1.10) yields that $\langle U_s, (U_s)^\alpha\rangle$ is conjugate to $\langle U_s, U_{-s}\rangle$ in G. Thus t is centralized by a subgroup $T \cong PSp_6(q)$ of G. Obviously this subgroup is not contained in N and so the center of t is fixed under $T_1 = \langle T, N\rangle$. By (1.15) we have $T_1 = G$, a contradiction. Thus $o(t) = 4$. Set $z = t^2$. Let $\pi_2 = \mathrm{Fix}(z)$. Then t acts as a perspectivity on π_2. Let t be in $O_2(C_G(z))$. Then $T = \langle t^{C_G(z)}\rangle/\langle z\rangle$ is generated by perspectivities. Now by [15, (2.7)] we may assume that all these perspectivities have the same center Z. Thus Z is fixed under $C_G(z)$ and N. Application of (1.2) yields that Z is fixed under G, a contradiction. Thus $t \notin O_2(C_G(z))$. Then $|[t, O_2(C_G(z))/\langle z\rangle]| \geqslant 4$. By [15] all elements in $\langle t\langle z\rangle, [t, O_2(C_G(z))/\langle z\rangle]\rangle$ are perspectivities on π_2 with the same center Z. Then $O_2(C_G(z))$ fixes Z and so $\langle N, O_2(C_G(z))\rangle$ fixes Z. It is easy to see that $O_2(C_G(z)) \nleq N$. Thus by (1.15) we have Z is fixed under G, a contradiction. So we have shown that $\alpha \in N(JL)$.

As $V^\alpha \leqslant J$, we have $\alpha \in N(J)$. Thus $\alpha = \beta\delta$, $\delta \in C_G(J)$, $\beta \in \mathrm{Aut}(J)$. Further $\beta = j\gamma$, $j \in J$. Suppose $\gamma \neq 1$. Then γ induces a field automorphism of order r on J. If $r \nmid |J|$, than α normalizes a Sylow 2-subgroup of J. But this contradicts (2.2). Thus $r \mid |J|$. If $r \neq 3$, then we may assume by [15, (2.5)] that α centralizes a Sylow r-subgroup of J. But this contradicts the structure of $\mathrm{Aut}(L_2(q))$. Thus $r = 3$ and for a Sylow 3-subgroup R of J we have $|R : C_R(\alpha)| = 3$. But then all elements of order three in $J\langle\alpha\rangle - J$ are conjugate under J. Thus α acts nontrivial on a Sylow 2-subgroup of J, a contradiction to (2.2) as this Sylow 2-subgroup is conjugate to V in J.

Thus we may assume $\alpha = j\delta$, $\delta \in C(J)$. Furthermore $j \neq 1$. By (2.3), α centralizes no conjugate of V. According to (1.9) let Y be a conjugate of V contained in L. Then $\langle Y, Y^\alpha\rangle \cong L_2(q)$ by (2.4). Furthermore α normalizes $\langle Y, Y^\alpha\rangle$; otherwise we get a contradiction as above. Let $y \in Y^\#$ an element fixing the axis or the center of α. Then $y^{-1}y^\alpha$ is a perspectivity contained in $\langle Y, Y^\alpha\rangle$, contradicting (2.2) and (2.3). Thus $y^{-1}y^\alpha$ is a generalized perspectively with center $a \cap b$ where a is the axis of α and b the axis of α^y. Hence the center of $y^{-1}y^\alpha$ is the center Z of h. As J conjugate to $\langle Y, Y^\alpha\rangle$ we get that the Levifactor L_1 of $C_G(Y)$ stabilizes Z. Thus Z is stabilized by $\langle JL, \alpha, L_1\rangle$. If L_1 is not contained in LJ, then by (1.15), $G = F_4(q)$ and $\langle JL, \alpha, L_1\rangle \cap G \cong Sp_8(q)$. But then we get a contradiction as above. Thus we may assume $L_1 \leqslant JL$, yielding $G = \Omega_6^{\pm}(q)$ or $\Omega_8^{\pm}(q)$. If $G \neq \Omega_8^{\pm}(q)$, then $JL = L_2(q) \times L_2(q)$. Now as above we get $\alpha = j_1 j_2 \varepsilon$ with $j_1 \in J$, $j_2 \in \langle Y, T^\alpha\rangle$ and $\varepsilon \in C_G(LJ)$. But $C_G(LJ) = 1$. Now (1.16) yields that an involution in LJ has to be a perspectivity or $o(\alpha) = 3 \mid q + 1$. In the former case we get that all the involutions in V are perspectivities contradicting

(2.2). Let $G = \Omega_8^+(q)$, then $LJ \cong L_2(q) \times L_2(q) \times L_2(q) \times L_2(q)$. With the same argument as above we get $o(\alpha) = 3$ and $3 \mid q+1$.

Suppose $G \neq \Omega_8^-(q)$, $o(\alpha) = 3 \mid q+1$. If $G = \Omega_6^+(q) \cong L_4(q)$, then LJ acts on a 2-group. Thus there are involutory perspectivities in G. Now [15, (2.5)] yields that there are perspectivities in V, a contradiction to (2.2). If $G = \Omega_6^-(q)$ or $\Omega_8^+(q)$, then (1.17) yields that there is a 2-group T in G such that α acts on $O_2(C_G(T))$. As α cannot centralize this group an easy argument and (2.1) yield that there are involutory involution inside of G. Now [15, (2.5)] yields that there are perspectivities in $V^\#$, contradicting (2.2).

Thus $G = \Omega_8^-(q)$ and $LJ = J \times J^w \times X$, $X \cong L_2(q^2)$. Furthermore we have $\alpha = j_1 j_2 x$, with $j_1 \in J$, $j_2 \in J^w$ and $x \in X$. Application of (1.16) yields that there are involutory perspectivities in G or $o(\alpha) = 3$. In the former case we get that there are perspectivities in $V^\#$ with [15, (2.5)] and so we have a contradiction. In the latter case we have as $3 \mid q^2 - 1$, that α normalizes a Sylow 2-subgroup T of X. Thus there are involutory perspecitivities in G again. This final contradiction proves the proposition.

3. The Proof of the Main Theorem

In this section we assume the hypothesis of the theorem with $m > 3$. Then we have shown in Section 2 that $G = \Omega_n^{\pm}(2)$ or $\Omega_n^{\pm}(3)$, $n \leqslant 8$, or $G = G(q)$ is a Chevalley-group of odd characteristic, and the involution t in J is a perspectivity.

(3.1) Lemma. *Let $G \neq \Omega_n^{\pm}(q)$, $q = 2, 3$, $n \leqslant 8$. Then $t \in J$ is planar.*

Proof. By (1.9), $t \in H$ and so t acts on U. We have $U = O_p(C_G(V))U_1$, with $[U_1, t] = 1$ and $[O_p(C_G(V))/V, t] = O_p(C_G(V))/V$. According to (1.9) choose $Y \sim V$ in G, $Y \leqslant U_1$. Then $[Y, t] = 1$. As $[Y, O_p(C_G(V))] \neq 1$, we have a conjugate Y_1 with $O_p(C_G(V))Y = O_p(C_G(V))Y_1$ and $[Y_1, t] \neq 1$. But this contradicts (2.4).

(3.2) Lemma. *We have $G \neq \Omega_n^{\pm}(2)$, $n \leqslant 8$.*

Proof. Suppose $G = \Omega_n^{\pm}(2)$, $n \leqslant 8$. If $G \cong L_4(2)$ and no involution in K is a perspectivity, we get a perspectivity α of order 3, 5, or 7. As $L_2(5)$ and $L_2(7)$ are subgroups of G we get with [21, Satz 2] that $o(\alpha) = 3$. But a Sylow 3-subgroup of G acts nontrivially on a 2-group, as can be seen from $A_4 \times A_4 \subseteq GL_4(2) \cong A_8$. But then there are involutory perspectivities in G. Thus we may assume that K contains involutory perspectivities. Application of [15, (2.5)] yields now that the involution in the center of a Sylow 2-

subgroup of G is a perspectivity. But then all involutions of G are perspectivities. Application of [12] yields a contradiction.

Assume now $G \cong U_4(2)$. Suppose that there is no involutory perspectivity in K. Then $o(\alpha)=3$ or 5. As $L_2(5)$ is contained in G we get with [21, Satz 2] that $o(\alpha)=3$. As $G=\Omega_6^-(2)$ we get that α centralizes a 2-group T by application of (1.17). But then (2.1) yields a contradiction. Thus there are involutory perspectivities in K and so by [15] there are involutory perspectivities in $Z(S)$, S a Sylow 2-subgroup of G. Now the stabilizer N of a maximal isotropic subspace of the natural module of G is an extension of an elementary abelian subgroup of order 16 by $L_2(4)$. Thus by [15, (2.7)] all involutions in the elementary abelian subgroup E are perspectivities with the same center Z. Then $\langle\langle C_G(e) \mid e \in E^\#\rangle, N\rangle$ stabilizes Z. But then application of (1.2) yields that G stabilizes Z, a contradiction.

Suppose now $G=\Omega_8^+(2)$. If there are no involutory perspectivities in K, then $o(\alpha)=3$, 5 or 7. As $L_2(7)$ is contained in G we get by [21, Satz 2] that $o(\alpha)\neq 7$. As $L_2(4)\times L_2(4)\cong \Omega_4^-(2)\times\Omega_4^-(2)$ is contained in G we get with (1.16) that $o(\alpha)=3$. Application of (2.1) and (1.17) yields $\alpha \notin G$. Let S be a Sylow 3-subgroup of K containing α. Set $S_1=S\cap G$. Then $S_1=Z_3\times Z_3 \wr Z_3$. Let E be the elementary abelian subgroup of order 81 in S_1. Then $S\leqslant N_K(E)$. Furthermore $N_G(E)/E\cong E_8\Sigma_4$ as can be easy seen by inspection of the action of $N_G(E)$ on the natural module of G. By [15, (2.5)] we may assume that $|E:C_E(\alpha)|\leqslant 3$. Suppose $[\alpha,E]=1$. Then $[\alpha,N_G(E)]\leqslant E$. Thus there is an involution i in $N_G(E)$ such that $[\alpha,i]=1$. Now α acts on $O_2(C_G(i))$. As $O_2(C_G(i))$ is extraspecial or elementary abelian, (2.1) yields a contradiction. Thus $C_K(E)=E$. Now $N_G(E)$ induces on the subgroups of order 3 of E orbits of length 4, 8, 12 and 16. Thus α leaves all these orbits invariant. Thus there is an orbit $\langle e_1\rangle\sim\langle e_2\rangle\sim\langle e_3\rangle\sim\langle e_4\rangle$ of $N_G(E)$ invariant under α such that $E=\langle e_1,e_2,e_3,e_4\rangle$. But as $[E,\alpha]\neq 1$ we get $|E:C_E(\alpha)|\geqslant 9$, contradicting [15, (2.5)]. Thus we have proved that there are involutory perspectivities in K, and so by [15] there are involutory perspectivities i in the center of a Sylow 2-subgroup of G. Let N be the stabilizer of a singular point in the natural representation of G. Then $N/O_2(N)\cong\Omega_6^+(2)$ and $O_2(N)$ is elementary abelian of order 2^6. Application of [15, (2.7)] yields that all involutions of $O_2(N)$ are perspectivities with the same center Z. Now $\langle N, C_G(i) \mid i\in O_2(N)^\#\rangle$ fixes Z. By (1.2) this group is equal to G, a contradiction.

Suppose now $G=\Omega_8^-(2)$. Assume further that no involution in K is a perspectivity. Then $o(\alpha)=3$, 5, 7, 17. As $L_2(7)$ and $L_2(5)$ are contained in G we get, with [21], $o(\alpha)\neq 5$ and 7. By [4, Satz 5] $L_2(16)\cong\Omega_4^-(4)$ is contained in G. Thus (1.16) yields that $o(\alpha)\neq 17$. As a Sylow 3-subgroup of G is contained in a subgroup isomorphic to $\Omega_6^-(2)$, we get a contradiction with (1.17). Thus there are involutory perspectivities in G. Let N be the stabilizer of a singular point in the natural representation of G. Then $O_2(N)$

is elementary abelian and $N/O_2(N) \cong \Omega_6^-(2)$. Thus all involutions in $O_2(N)$ are perspectivities with the same center Z. Application of (1.2) yields now the contradiction that Z is a fixpoint of G.

(3.3) LEMMA. *There is no counterexample to the main theorem.*

Proof. By Section 2, (3.1) and (3.2), it is enough to show that $G \neq \Omega_n^{\epsilon}(3)$, $n \leqslant 8$.

Let $G = \Omega_6^+(3) \cong L_4(3)$. Suppose that there is no involutory perspectivity in K. Then $o(\alpha) = 3, 5, 13$. It is $\Omega_4^-(3) \cong L_2(9)$ contained in G. Thus by [21, Satz 2], $o(\alpha) \neq 5$. The stabilizer N of a point in the natural representation of G is an extension of an elementary abelian group E of order 27 by $SL_3(3)$. Thus we may assume $\alpha \in N$. Then $O_3(N)$ contains a generalized perspectivity β. Application of [10, (3.13)] yields that the center Z of β is a fixpoint of $O_3(N)$ and then of N. Let M be a complement of $O_3(N)$ in N. Then M contains perspectivities. As Z is a fix point of M, Z is contained in the axes of all perspectivities of M. Let N_1 be the stabilizer of a hyperplane in the natural representation of G. We may assume $M \leqslant N_1$. Then Z is a fix point of N_1. As $\langle N, N_1 \rangle = G$, we get a contradiction. So we have proved that K and then G contains involutory perspectivities.

Let $G = \Omega_6^-(3) \cong U_4(3)$. Suppose that K contains no involutory perspectivities. Then $o(\alpha) = 3, 5, 7$. As $U_3(3) \subseteq U_4(3)$ we get that $L_2(7)$ is a subgroup of G. Furthermore [14], $L_2(5)$ is a subgroup of G. Thus by [21], $o(\alpha) = 3$. There is a subgroup N of G such that $O_3(N)$ is elementary abelian of order 81 and $N/O_3(N) \cong A_6$. By [15, (2.5)] we may assume $\alpha \in O_3(N)$. Then all nontrivial elements of $O_3(N)$ are perspectivities with the same center. Application of (1.2) and [14] yields $G = \langle N, C_G(t) \mid t \in O_3(N)^{\#} \rangle$ fixes the center of α, a contradiction. Thus K and then G contains involutory perspectivities.

Let $G = \Omega_7(3)$. Suppose that G contains no involutory perspectivities. Then $o(\alpha) = 3$, 5, 7 or 13. As $L_2(7)$ and $L_2(5)$ are contained in G we get $o(\alpha) = 3$ or 13. By [8], G is generated by a class i^G of 3-transpositions. Thus we may assume $ii^{\alpha} = j$ is of order 3. Then [8, sect. 15] yields $N_G(\langle j \rangle) \cong \Sigma_3 \times E_{81} \Sigma_6$. Set $R = O_3(N_G(\langle j \rangle))$. Then $N_G(R)/R$ contains $\Omega_5(3) \cong PSp_4(3)$. As j is a generalized perspectivity we get with [10, (3.13)] that $N_G(R)$ fixes the center Z of j. As $N_G(R)$ is a maximal subgroup of G by (1.2) we get $\alpha \in N_G(R)$ and so $o(\alpha) = 3$. Let N be the stabilizer of a maximal isotropic subspace in the natural representation of G. Then $O_3(N)$ is of order 3^6 and $N/O_3(N) \cong SL_3(3)$. Thus we may assume $\alpha \in N$. By [15, (2.5)] we may assume $\alpha \in O_3(N)$. But then the center of α is a fix point of N. Thus a Sylow 3-subgroup P of G fixes the center of α. We may assume $P \leqslant N_G(R)$. Set $M = \langle \alpha^P \rangle$. Then $M \trianglelefteq P$ and so $Z(P) \cap M \neq 1$. Thus we may assume

$\alpha \in Z(P)$. But then $\alpha \in R$ and so Z is the center of α. Now $G = \langle N_G(R), N \rangle$ fixes Z, a contradiction. Thus there are involutory perspectivities in G.

Let $G = \Omega_8^+(3)$. Suppose that there are no involutory perspectivities in G. Then $o(\alpha) = 3$, 5, 7 or 13. As $L_2(7)$ and $\Omega_4^-(3) \times \Omega_4^-(3) \cong L_2(9) \times L_2(9)$ are contained in G we get $o(\alpha) = 3$ or 13. Let N be the stabilizer of an isotropic point in the natural representation of G. Then $O_3(N)$ is elementary abelian of order 3^6 and $N/O_3(N)$ is isomorphic to $SO_6^+(3)$. Thus we may assume $\alpha \in N$, if $\alpha \in G$. If $\alpha \notin G$, then $o(\alpha) = 3$. But then [15, (2.5)] yields that there are perspectivities in G. Thus we may assume $\alpha \in G$. Suppose $o(\alpha) = 3$. By [15, (2.5)] we may assume $\alpha \in O_3(N)$. Then N stabilizes the center of α. Thus $G = \langle N, C_G(\alpha) \rangle$ stabilizes the center of α, a contradiction. Thus $o(\alpha) = 13$. Then there is a generalized perspectivity $j = \beta^{-1}\beta^{\alpha} \in O_3(N)$. Application of [10, (3.13)] yields that N fixes the center Z of j. Thus Z is the intersection of the axes of perspectivities contained in N. Let $M \leqslant G$, $M \cong \Omega_7(3)$, and $\alpha \in M$. As above we get that M fixes a point Y. Thus Y is the intersection of the axes of the perspectivities in M. As $M \cap N \not\subseteq N_G(\langle \alpha \rangle)$ we get $Z = Y$. But then (1.2) yields that G fixes Z, a contradiction. Thus there are involutory perspectivities in G.

Let $G = \Omega_8^-(3)$. Suppose that there are no involutory perspectivities in G. Then $o(\alpha) = 3$, 5, 7, 13 or 41. As $L_2(7)$, $L_2(5)$ and $L_2(3^4) \cong \Omega_4^-(3^2)$ are involved in G we get, with [21], $o(\alpha) = 3$ or 13. Suppose $o(\alpha) = 3$. Let N be the stabilizer of an isotropic point in the natural representation of G. Then $N/O_3(N) \cong SO_6^-(3)$. We may assume $\alpha \in N$. By [15, (2.5)], we get $\alpha \in O_3(N)$. But then N stabilizes the center Z of α. Thus $G = \langle N, C_G(\alpha) \rangle$ fixes Z, a contradiction. Now $o(\alpha) = 13$. Let M be a subgroup of G isomorphic to $\Omega_7(3)$. As above we get that M fixes the intersection Z of the axes of the perspectivities contained in M. Let $g \in G - N_G(M)$. Then an easy counting argument shows that $M \cap M^g \not\leqslant N_G(\langle \alpha \rangle)$. Thus $\langle M, M^g \rangle$ fixes Z. Now a Sylow argument and application of [3, 20] yield $G = \langle M, M^g \rangle$, a contradiction. Thus there are involutory perspectivities in G.

So we have proved that in all cases there are involutory perspectivities in G. By [15] the involution in the center of a Sylow 2-subgroup of G is a perspectivity. Thus t, where t means the involution in (1.9), is a perspectivity. As $L_4(3)$ contains a subgroup $N \cong E_{16}\Sigma_5$, [13], and $U_4(3)$ a subgroup $N \cong E_{16}A_6$, [14], and $\Omega_7(3)$ a subgroup $N \cong E_{64}A_7$, [17], with $t \in O_2(N)$, we get that N fixes the center of t. As $\Omega_7(3)$ is a subgroup of $\Omega_8^-(3)$ we get in all cases that $\langle C_G(t), N \rangle$ fixes the center of t. Now application of [1, Corollary II] yields that $F^*(\langle C_G(t), N \rangle)$ is a Chevalley-group of odd characteristic. A careful checking of the possible orders yields $G = L_4(3)$ and $F^*(\langle C_G(t), N \rangle) \cong PSp_4(3)$. But then there are only two classes of involutions in G, both contained in $O_2(N)$. Thus all involutions of G are perspectivities. Application of [12] yields now the contradiction.

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