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Counting Objects Which Behave Like Irreducible Brauer Characters of Finite Groups

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1. INTRODUCTION

Let $N \lhd G$, where G is a finite group, so that G acts on $\mathrm{Irr}(N)$, the set of irreducible ordinary (complex) characters of N. Suppose $\theta \in \mathrm{Irr}(N)$ is G-invariant and write $\mathrm{Irr}(G \mid \theta)$ to denote the set of those $\chi \in \mathrm{Irr}(G)$ such that χ_N involves (and hence is a multiple of) θ. How large is $\mathrm{Irr}(G \mid \theta)$?

The answer to this question is well known, and not very hard to prove. Analogously to the fact that $|\mathrm{Irr}(G)|$ is equal to the number of conjugacy classes of G, it turns out that $|\mathrm{Irr}(G \mid \theta)|$ is equal to the number of certain "good" classes in the group G/N. There are two equivalent descriptions of which classes of G/N these are. One characterization depends on a certain complex twisted group algebra of G/N, and in this formulation, the result which counts $\mathrm{Irr}(G \mid \theta)$ is due to I. Schur [5]. The other formulation of this result, due to P. X. Gallagher [2], defines "goodness" internally to G; it depends on the character theory of G and its subgroups. Specifically, Gallagher defines an element of $\bar{g} \in G/N$ to be *θ-good* if some extension of θ to $\langle N, g \rangle$ is invariant in C, where $C/N = \mathbf{C}_{G/N}(\bar{g})$. The property of being θ-good is preserved by conjugacy in G/N and Gallagher shows that $|\mathrm{Irr}(G \mid \theta)|$ is equal to the number of classes of G/N which consist of θ-good elements.

Somewhat less well known is the corresponding result for irreducible Brauer characters with respect to some prime p. If $\theta \in \mathrm{IBr}(N)$ is G-invariant, we intend to count the elements of $\mathrm{IBr}(G \mid \theta)$, i.e., the irreducible Brauer characters of G whose restrictions to N are multiples of θ. In view of the fact that $|\mathrm{IBr}(G)|$ equals the number of p-regular classes of G, it is reasonable to guess that $|\mathrm{IBr}(G \mid \theta)|$ is the number of θ-good p-regular

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classes in G/N, where "θ-good" is defined as it was by Gallagher for ordinary characters. This guess is correct.

The modular version of Schur's count of $\mathrm{Irr}(G\,|\,\theta)$, working with twisted group algebras, was proved by K. Asano, M. Osima, and M. Takahasi [1] (see Chap. 3.6 of [4] for an exposition). It is not hard to translate this result into the character theoretic form mentioned above.

We are interested in generalizing further. Suppose π is any set of primes and write $\mathrm{cf}^0(G)$ to denote the vector space of complex valued functions defined on the set of π-elements of G and constant on conjugacy classes. In the cases where π is all primes or $\pi = p'$ (all primes but p), the functions Irr and IBr, respectively, pick out particular bases for $\mathrm{cf}^0(G)$.

Suppose for some arbitrary π we have a "basis-selection function" I, defined on the collection of all subgroups of G, such that $I(H)$ is a basis for $\mathrm{cf}^0(H)$ for all $H \subseteq G$. Assuming that I behaves more or less like Irr or IBr, we intend to show that $|I(G\,|\,\theta)|$ is equal to the number of "θ-good" π-classes in G/N, where $\theta \in I(N)$ is G-invariant. (We shall, of course, make precise the phrase "more or less" in the previous sentence.)

The point here is that there are interesting cases of well behaved basis-selection functions for all π. For instance, for π-separable and in particular solvable groups, there always exists such a function, called I_π. (We shall define this in Chapter 4.) The real purpose of this paper is to prove the analog of Gallagher's theorem for I_π for π-separable groups. By axiomatizing well behaved π-class-function basis-selection functions (which we call "character selectors") our proof also will work for Irr, reproving Gallagher's theorem, and for IBr, proving the character theoretic version of the Asano–Osima–Takahasi theorem.

Of course, we cannot hope to work with modular twisted group algebras in this generality, and so we have had to devise a new approach which is rather more in the spirit of Gallagher's paper than those of Schur and Asano *et al.*

2. Orthogonality

In this section we prove a very general sort of "character" orthogonality which includes for Brauer characters, the standard orthogonality between $\mathrm{IBr}(G)$ and the characters that Brauer denoted Φ_φ for $\varphi \in \mathrm{IBr}(G)$.

Let G be a finite group and fix an arbitrary nonempty normal subset $S \subseteq G$. As usual, $\mathrm{cf}(G)$ will denote the vector space of $\mathbb{C}$-valued class functions on G and we write $\mathrm{cf}^0(G)$ for the space of complex functions on S, constant on G-classes. Also, for $\alpha \in \mathrm{cf}(G)$, we write $\alpha^0 \in \mathrm{cf}^0(G)$ to denote the restriction of α to S.

Let $\mathscr{B}$ be an arbitrary $\mathbb{C}$-basis for $\mathrm{cf}^0(G)$. For each $\chi \in \mathrm{Irr}(G)$, therefore, there is a unique expression of the form

$$\chi^0 = \sum_{\varphi \in \mathscr{B}} d_{\chi\varphi} \varphi,$$

where the coefficients $d_{\chi\varphi} \in \mathbb{C}$ are the *decomposition numbers*. In this generality, of course, they need not be integers.

For $\varphi \in \mathscr{B}$, we define $\tilde{\varphi} \in \mathrm{cf}(G)$ by

$$\tilde{\varphi} = \sum_{\chi \in \mathrm{Irr}(G)} \bar{d}_{\chi\varphi} \chi.$$

(2.1) LEMMA. *With the above notation, let $\varphi, \theta \in \mathscr{B}$. Then*

(a) $\tilde{\varphi}(g) = 0$ *if* $g \in G - S$, *and*

(b) $\displaystyle \frac{1}{|G|} \sum_{x \in S} \tilde{\varphi}(x) \overline{\theta(x)} = \begin{cases} 1 & \textit{if } \varphi = \theta \\ 0 & \textit{otherwise.} \end{cases}$

Proof. By the second orthogonality relation for ordinary characters, we have for $x \in S$ and $g \in G - S$ that

$$\begin{aligned} 0 &= \sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi(x)}\, \chi(g) = \sum_{\chi \in \mathrm{Irr}(G)} \bar{d}_{\chi\varphi} \overline{\varphi(x)}\, \chi(g) \\ &= \sum_{\varphi \in \mathscr{B}} \tilde{\varphi}(g)\, \overline{\varphi(x)}. \end{aligned}$$

By the linear independence of the functions $\varphi \in \mathscr{B}$, this shows that $\tilde{\varphi}(g) = 0$ for all φ, proving (a).

For (b), define $f_\varphi \colon \mathrm{cf}^0(G) \to \mathbb{C}$ by

$$f_\varphi(\alpha) = \frac{1}{|G|} \sum_{x \in S} \tilde{\varphi}(x)\, \overline{\alpha(x)}$$

and note that for $\chi \in \mathrm{Irr}(G)$, we have

$$f_\varphi(\chi^0) = \frac{1}{|G|} \sum_{g \in G} \tilde{\varphi}(g)\, \overline{\chi(g)} = \bar{d}_{\chi\varphi}$$

using part (a), the first orthogonality relation and the definition of $\tilde{\varphi}$.

Now $\bar{f}_\varphi$ is a $\mathbb{C}$-linear functional on $\mathrm{cf}^0(G)$. On vectors of the form χ^0, this functional agrees with the functional g_φ which picks out the coefficient of φ

in the expansion of a vector in terms of the basis $\mathscr{B}$. Since the χ^0 span $\mathrm{cf}^0(G)$, it follows that $\bar{f}_\varphi = g_\varphi$ and so

$$f_\varphi(\theta) = \bar{g}_\varphi(\theta) = \delta_{\varphi\theta},$$

the Kronecker symbol, as required. ∎

3. Character Selectors

We fix the following notation: Γ is an arbitrary group, π is a set of prime numbers, and Σ is the set of π-elements of Γ. (In other words, $x \in \Sigma$ iff $o(x)$ is finite and involves only primes in π.) For each finite subgroup $G \subseteq \Gamma$ we have the vector space $\mathrm{cf}^0(G)$ of G-class functions defined on $S = G \cap \Sigma$.

Suppose that for each finite $G \subseteq \Gamma$ we have fixed a particular $\mathbb{C}$-basis $I(G)$ for $\mathrm{cf}^0(G)$. We refer to I as a *basis-selection function* for (Γ, π). In this situation, if $\alpha \in \mathrm{cf}^0(G)$, we say that $\varphi \in I(G)$ is an *I-constituent* of α if φ occurs with nonzero coefficient in the expansion of α with respect to the basis $I(G)$.

We write $C(G)$ to denote the set of all sums of elements of $I(G)$. In other words, $\alpha \in C(G)$ iff α is a non-negative integer linear combination of $I(G)$, and $\alpha \neq 0$. (Thus $C(G)$ is related to $I(G)$ as $\mathrm{Char}(G)$ is to $\mathrm{Irr}(G)$.) Finally, if $t \in \Gamma$ and $\alpha \in \mathrm{cf}^0(G)$, we write $\alpha^t \in \mathrm{cf}^0(G^t)$ to denote the function given by $\alpha^t(x^t) = \alpha(x)$ for $x \in G \cap \Sigma$.

With all of this notation established, we are now ready for our principal definition.

(3.1) Definition. Let I be a basis-selection function for (Γ, π). Then I is a *character selector* provided that the following hold for all finite $G \subseteq \Gamma$ and all $\varphi \in I(G)$.

(a) If $H \subseteq G$, then $\varphi_H \in C(H)$.

(b) If G is a π-group, then $\varphi \in \mathrm{Irr}(G)$.

(c) If $t \in \Gamma$, then $\varphi^t \in I(G^t)$.

(d) If $N \lhd G$, then all I-constituents of φ_N are G-conjugate.

(e) If $\lambda \in \mathrm{Irr}(G)$ with $\lambda(1) = 1$, then $\lambda^0\varphi \in I(G)$.

(f) If $\chi \in \mathrm{Irr}(G)$, then $\chi^0 \in C(G)$.

Observe that if π is the set of all primes, then Irr is a character selector and that if $\pi = p'$, then IBr is a character selector provided that we are consistent in lifting modular roots of unity to complex roots of unity when constructing Brauer characters.

There is some redundancy in conditions (a)–(f) since we have not attempted to construct a "minimal" set of axioms. For example, in (a) we could drop the implicit assumption (included within the definition of "$C(G)$"), that φ_H is nonzero. To see this, observe that the weakened form of (a) together with (b) implies that if $K \subseteq G$ is a π-subgroup, then φ_K is either zero or a character of K. Therefore, if $\varphi(1)=0$, we have $\varphi_K=0$ for every π-subgroup K and thus $\varphi(x)=0$ for every π-element x. In other words, $\varphi=0$, which is a contradiction. Thus $\varphi(1)\neq 0$ and hence $\varphi_H \neq 0$ and we recover the full strength of (a).

Other examples of redundancy in (a)–(f) are that in (b), it would be sufficient to assume that $\varphi \in \mathrm{Char}(H)$; the irreducibility would then follow using (f). Similarly, in (c), it would suffice to have $\varphi^t \in C(G^t)$ and in (e), the assumption that $\lambda^0\varphi \in C(G)$ would be enough, since the operations "conjugate by t" and "multiply by λ^0" are invertible.

We mention that by (a) applied with $H=1$ and (b) with $G=1$, we see that $\varphi(1)$ is a positive integer for all $\varphi \in I(G)$.

We close this section with two easy general lemmas about character selectors.

(3.2) LEMMA. *Let I be a character selector for (Γ, π) and let $\varphi \in I(G)$ where $G \subseteq \Gamma$ is finite. Define $\hat{\varphi} \in \mathrm{cf}(G)$ by*

$$\hat{\varphi}(g) = \varphi(g_\pi),$$

where $g = g_\pi g_{\pi'}$ is the usual decomposition of $g \in G$ into commuting π- and π'-elements. Then $\hat{\varphi}$ is a generalized character.

Proof. Certainly φ is a class function and so by Brauer's characterization of characters, it is enough to show that $\hat{\varphi}_E$ is a generalized character of E whenever $E \subseteq G$ is Brauer elementary.

Write $E = P \times Q$ where P is a π-group and Q is a π'-group. If $x \in P$ and $y \in Q$, then $\hat{\varphi}(xy) = \varphi(x)$ and so $\hat{\varphi}_E = \varphi_P \times 1_Q$. However, by 3.1(a) and (b), $\varphi_P \in \mathrm{Char}(P)$ and thus $\hat{\varphi}_E \in \mathrm{Char}(E)$. ∎

If $H \subseteq G \subseteq \Gamma$ and $\theta \in I(H)$, we write $I(G \mid \theta)$ to denote the set of $\varphi \in I(G)$ such that θ is an I-constituent of φ_H. (There is actually a slight notational problem here which we will ignore. The difficulty is that it may not be possible to recover H from a knowledge of θ, which is defined as a function with domain $H \cap \Sigma$.)

(3.4) LEMMA. *Let $\theta \in I(H)$ where $H \subseteq G \subseteq \Gamma$ and I is a character selector for (Γ, π). Then $I(G \mid \theta) \neq \varnothing$.*

Proof. Since $\mathrm{Irr}(H)$ spans $\mathrm{cf}(H)$, the functions ψ^0 span $\mathrm{cf}^0(H)$ as ψ runs over $\mathrm{Irr}(H)$. We can therefore choose $\psi \in \mathrm{Irr}(H)$ such that θ is an

I-constituent of ψ^0. Now choose $\chi \in \mathrm{Irr}(G|\psi)$ and observe that θ is an I-constituent of $(\chi^0)_H$. (This uses 3.1(f) applied in H.) It follows that θ is an I-constituent of φ_H for some I-constituent φ of χ^0. ∎

4. π-Separable Groups

In this section we discuss $I_\pi(G)$ when G is π-separable and we observe that I_π is a character selector for (Γ, π) if Γ is π-separable. We use the notation established previously so that, for instance, α^0 is the restriction of $\alpha \in \mathrm{cf}(G)$ to the set S of π-elements of G.

(4.1) Theorem. *Let G be finite and π-separable. Then there exists a unique basis $I_\pi(G)$ for* $\mathrm{cf}^0(G)$ *which satisfies*:

(DP) *If $\chi \in \mathrm{Irr}(G)$, then χ^0 is a non-negative integer linear combination of $I_\pi(G)$.*

(FS) *If $\varphi \in I_\pi(G)$, then $\varphi = \chi^0$ for some $\chi \in \mathrm{Irr}(G)$.*

In the case where $\pi = p'$ (and so G is p-solvable) we have $I_\pi(G) = \mathrm{IBr}(G)$. That $\mathrm{IBr}(G)$ satisfies (DP) is just the fact that the Brauer decomposition numbers are non-negative integers. (The initials "DP" stand for "decomposition property".) That $\mathrm{IBr}(G)$ satisfies (FS) for p-solvable groups is exactly the Fong–Swan theorem.

Proof of 4.1. That a basis satisfying (DP) and (FS) actually exists is most of the content of [3]. The uniqueness, on the other hand, is easy to see. In fact, conditions (FS) and (DP) yield that

$$I_\pi(G) = \{\chi^0 \mid \chi \in \mathrm{Irr}(G) \text{ and } \chi^0 \neq \alpha^0 + \beta^0 \text{ for any } \alpha, \beta \in \mathrm{Char}(G)\}.$$

This gives the uniqueness. ∎

Note that the above proof gives a relatively simple formula for $I_\pi(G)$ and it is a triviality that the subset of $\mathrm{cf}^0(G)$ defined by that formula spans $\mathrm{cf}^0(G)$. What seems far from easy to prove is the fact that this subset is linearly independent.

(4.2) Proposition. *Let Γ be π-separable. Then $I = I_\pi$ is a character selector for (Γ, π).*

Proof. Let $G \subseteq \Gamma$ be finite and let $\varphi \in I_\pi(G)$. By (FS), choose $\psi \in \mathrm{Irr}(G)$ with $\psi^0 = \varphi$. If $H \subseteq G$, then $\varphi_H = (\psi_H)^0$ and this lies in $C(H)$ by (DP) applied in H and the fact that $\varphi_H \neq 0$ since $\varphi(1) = \psi(1) \neq 0$. This proves 3.1(a).

If G is a π-group, then $\varphi = \psi$, proving 3.1(b). For 3.1(c), observe that $\varphi' = (\psi^0)' = (\psi')^0$ and this lies in $C(G')$ by (DP) and the fact that $\varphi' \neq 0$. We remarked in Chapter 3 that it suffices to show $\varphi' \in C(G')$ in order to prove (c).

Putting (d) aside for the moment, let $\lambda \in \text{Irr}(G)$ with $\lambda(1) = 1$. Then $\lambda^0\varphi = (\lambda\psi)^0 \in C(G)$ by (DP). This suffices to prove 3.1(e). Note that 3.1(f) is essentially just property (DP).

Finally, to prove 3.1(d), let $N \lhd G$. Since $\varphi_N \in C(N)$, we can use (FS) in N to choose $\theta \in \text{Char}(N)$ with $\theta^0 = \varphi_N$ and such that $\mu^0 \in I(N)$ for each irreducible constituent μ of θ. Let ρ be the regular character of G/N (viewed in Char(G)) and observe that $(\theta^G)^0 = (\psi\rho)^0$. (This is because both functions vanish on $(G - N) \cap \Sigma$ and both equal $|G:N|\, \theta^0 = |G:N|\, \varphi_N$ on $N \cap \Sigma$.)

Now $(\psi\rho)^0 = \psi^0 + ((\rho - 1_G)\psi)^0$ and since the second term lies in $C(G)$ (or is zero) by (DP), it follows that $\varphi = \psi^0$ is an I-constituent of $(\theta^G)^0$. We conclude that φ is an I-constituent of ξ^0 for some irreducible constituent ξ of θ^G. Therefore, all I-constituents of φ_N are in fact I-constituents of ξ_N^0. (This uses 3.1(a).) However, ξ_N has some irreducible constituent μ which is a constituent of θ and so $\mu^0 \in I(N)$. It follows that all I-constituents of φ_N are of the form $(\mu^0)^g \in I(N)$ (by 3.1(c)) and we are done. ∎

5. Abelian and Cyclic Factors

We now fix the character selector I for (Γ, π). Also, fix $N \lhd G \subseteq \Gamma$ and $\theta \in I(N)$ and assume that θ is G-invariant. We intend to count the elements of $I(G \mid \theta)$, the "characters" $\varphi \in I(G)$ such that θ is an I-constituent of φ_N. (By 3.1(d), this means that $\varphi_N = a\theta$ for some positive integer a.)

(5.1) Lemma. *Suppose G/N is abelian and let $\varphi \in I(G \mid \theta)$. Then*

$$I(G \mid \theta) = \{\lambda^0\varphi \mid \lambda \in \text{Irr}(G/N)\}.$$

Proof. Let ρ be the regular character of G/N, viewed as a character of G. If $\eta \in I(G \mid \theta)$, then $\eta(1)\, \rho^0\varphi = \varphi(1)\, \rho^0\eta$ and η is an I-constituent of the right side. Since ρ is a sum of linear characters and using 3.1(e), the fact that η is an I-constituent of $\rho^0\varphi$ yields that $\eta = \lambda^0\varphi$ for some irreducible constituent λ of ρ. Conversely, for any such λ, we have $(\lambda^0\varphi)_N = \varphi_N$ and so $\lambda^0\varphi \in I(G \mid \theta)$, again using 3.1(e). ∎

Next, we need an easy lemma about ordinary characters.

(5.2) Lemma. *Suppose that G/N is cyclic. If $\Xi \in \text{Char}(N)$ is G-invariant, then Ξ extends to a character of G.*

Proof. We may assume without loss that Ξ is an orbit sum for the action of G on $\mathrm{Irr}(N)$. Let ψ be an irreducible constituent of Ξ and let T be the inertia group of ψ in G. Since T/N is cyclic, we know that the invariant (in T) irreducible character ψ extends to some $\eta \in \mathrm{Irr}(T)$. Also, $\chi = \eta^G$ is irreducible and $[\chi_N, \psi] = [\eta_N, \psi] = 1$. It follows that $\chi_N = \Xi$. ∎

We now return to our study of the situation where $\theta \in I(N)$ and θ is invariant in G.

(5.3) LEMMA. *Suppose G/N is cyclic. Then every element of $I(G \mid \theta)$ is an extension of θ. Furthermore, if G/N is a π-group, and we fix $\varphi \in I(G \mid \theta)$, then all $\lambda^0\varphi$ are distinct for $\lambda \in \mathrm{Irr}(G/N)$ and hence $|I(G \mid \theta)| = |G:N|$.*

Proof. By Lemma 3.2, θ uniquely determines some generalized character $\hat{\theta}$ of N which is necessarily G-invariant and so is a $\mathbb{Z}$-linear combination of G-orbit sums on $\mathrm{Irr}(N)$, each of which extends to B by 5.2. Therefore, $\hat{\theta} = \Lambda_N$ for some generalized character Λ of G.

We also know that $\hat{\theta}^0 = \theta$ by 3.2 and thus $(\Lambda^0)_N = \theta$. Now write $\Lambda^0 = \alpha + \beta$ where α is a $\mathbb{Z}$-linear combination of elements of $I(G \mid \theta)$ and where no I-constituent of β lies in this set. Since α_N is a multiple of θ and θ is not an I-constituent of β_N, we conclude that $\theta = \alpha_N + \beta_N = \alpha_N$.

Now let $\varphi \in I(G \mid \theta)$. By 5.1, all elements of $I(G \mid \theta)$ have equal restrictions to N, and therefore, $\theta = \alpha_N$ is an integer multiple of φ_N. On the other hand, $\varphi_N \in C(N)$, and this forces $\varphi_N = \theta$ and proves the first assertion.

Now assume G/N is a π-group. If we can show that the $\lambda^0\varphi$ are distinct for $\lambda \in \mathrm{Irr}(G/N)$, it will follow by 5.1 that $|I(G \mid \theta)| = |G:N|$. To complete the proof, therefore, it will suffice to assume that $\lambda^0\varphi = \varphi$ and prove $\lambda = 1_G$.

Let $K = \ker \lambda$. We have $N \subseteq K \subseteq G$ and we need to show that $K = G$. Let $\varphi_K = \mu$ and note that $\mu \in I(K)$ since $\mu_N = \theta \in I(N)$. Let $\tilde{\mu}$ be as in the notation of Chapter 2 and note that $\tilde{\mu} \in \mathrm{Char}(K)$ since the decomposition numbers $d_{\chi\mu}$ are non-negative integers by 3.1(f). Since μ is G-invariant and uniquely determines $\tilde{\mu}$, it follows that $\tilde{\mu}$ is G-invariant and thus extends to $\Delta \in \mathrm{Char}(G)$ by 5.2.

Now let $\hat{\varphi}$ be the generalized character of G corresponding to φ as in 3.2. Then $[\Delta, \hat{\varphi}]$ is an integer which we proceed to compute.

If $g \in G - K$, we write $g = g_\pi g_{\pi'}$ where g_π and $g_{\pi'}$ are respectively the π and π'-parts of g. Since G/K is a π-group, we have $g_{\pi'} \in K$ and thus $g_\pi \notin K$ and $\lambda(g_\pi) \neq 1$. We have, however, $\varphi(g_\pi) = \lambda(g_\pi)\,\varphi(g_\pi)$ since $\varphi = \lambda^0\varphi$, and this implies that $\varphi(g_\pi) = 0$. Thus $\hat{\varphi}(g) = 0$ for all $g \in G - K$ and we have

$$[\Delta, \hat{\varphi}] = \frac{1}{|G:K|}\,[\Delta_K, \hat{\varphi}_K] = \frac{1}{|G:K|}\,[\tilde{\mu}, \hat{\varphi}_K].$$

By 2.1(a), $\tilde{\mu}$ vanishes on all non-π-elements of K. If $x \in K$ is a π-element, then $\hat{\varphi}(x) = \varphi(x) = \mu(x)$ and so

$$[\tilde{\mu}, \hat{\varphi}_K] = \frac{1}{|K|} \sum_{x \in K \cap \Sigma} \tilde{\mu}(x) \overline{\mu(x)} = 1$$

by 2.1(b). This yields that $[\Delta, \hat{\varphi}] = 1/|G:K|$ and since this must be an integer, we conclude that $K = G$ as required. ∎

We investigate further the situation where G/N is a cyclic π-group.

(5.4) LEMMA. *Assume $G = \langle N, g \rangle$ where g is a π-element. Let $\varphi \in I(G \mid \theta)$ and $\psi \in I(G)$. Then $\psi \in I(G \mid \theta)$ iff*

$$0 \neq \sum_{x \in Ng \cap \Sigma} \tilde{\varphi}(x) \overline{\psi(x)}.$$

where $\tilde{\varphi}$ is as in 2.1.

Proof. Define the function α: $G/N \to \mathbb{C}$ by

$$\alpha(Nu) = \sum_{x \in Nu \cap \Sigma} \tilde{\varphi}(x) \overline{\psi(x)}.$$

Since G/N is an abelian group, α is certainly a class function and we can write

$$\alpha = \sum_{\lambda \in \operatorname{Irr}(G/N)} a_\lambda \lambda$$

for some choice of complex numbers a_λ. Then

$$\begin{aligned} a_\lambda = [\alpha, \lambda] &= \frac{1}{|G/N|} \sum_{\bar{u} \in G/N} \alpha(\bar{u}) \overline{\lambda(\bar{u})} \\ &= \frac{1}{|G/N|} \sum_{\bar{u}} \sum_{x \in Nu \cap \Sigma} \tilde{\varphi}(x) \overline{\psi(x)}\, \overline{\lambda(x)} \\ &= \frac{1}{|G:N|} \sum_{x \in G \cap \Sigma} \tilde{\varphi}(x) \overline{(\lambda^0 \psi)(x)} \end{aligned}$$

and by 2.1(b), this is nonzero iff $\lambda^0 \psi = \varphi$. By 5.1, therefore, all $a_\lambda = 0$ if $\psi \notin I(G \mid \theta)$ and thus $\alpha = 0$ in this case. If, on the other hand, $\psi \in I(G \mid \theta)$, then $\varphi = \lambda^0 \psi$ for some unique λ by 5.3 and thus $\alpha = a_\lambda \lambda$ for that λ and α never takes on the value zero. The result now follows. ∎

(5.5) COROLLARY. *Let $\varphi \in I(G \mid \theta)$ and suppose φ extends θ. If $\bar{g} \in G/N$ is a π-element, then we can choose a π-element $x \in Ng$ such that $\varphi(x) \neq 0$.*

Proof. Without loss, we may assume $G = \langle N, g \rangle$ and apply 5.4 to conclude that

$$0 \neq \sum_{x \in Ng \cap \Sigma} \tilde{\varphi}(x)\, \overline{\varphi(x)}.$$

The result follows. ∎

6. Good Elements

We continue to hold fixed Γ and π, the character selector I, the subgroups $N \lhd G \subseteq \Gamma$, and a G-invariant "character" $\theta \in I(N)$.

(6.1) Definition. A π-element $\bar{g} \in G/N$ is *θ-good* if every element of $I(\langle N, g \rangle \mid \theta)$ is invariant in C, where $C/N = \mathbf{C}_{G/N}(\bar{g})$.

Note that by 5.3, all of the elements of $I(\langle N, g \rangle \mid \theta)$ are in fact extensions of θ. Also, by 5.1, they can all be obtained from any one of them by multiplication by the various λ^0 for $\lambda \in \mathrm{Irr}(\langle N, g \rangle / N)$. Since these linear characters are certainly C-invariant, it follows that in order to show that $\bar{g}$ is θ-good, it suffices to check that some particular extension of θ to $\langle N, g \rangle$ is C-invariant.

It should be clear, and we will use the fact without proof, that θ-"goodness" is preserved by conjugacy in G/N. We can now formally state our main result.

(6.2) Theorem. *Let I be a character selector for (Γ, π) and let $N \lhd G \subseteq \Gamma$ with G finite. If $\theta \in I(N)$ is G-invariant, then $|I(G \mid \theta)|$ is equal to the number of conjugacy classes of θ-good π-elements in G/N.*

Perhaps it is worth observing that in the case that G/N is a cyclic π-group, we essentially know Theorem 6.2 already. By 5.3, $|I(G \mid \theta)| = |G:N|$ in this case, and so we need to check that every element $\bar{g} \in G/N$ is θ-good. Note that $C = G$. Choose $\varphi \in I(G \mid \theta)$and let $\mu = \varphi_{\langle N, g \rangle}$. By 5.3, we have $\varphi_N = \theta$ and thus μ extends θ and μ is G-invariant. As we remarked earlier, this suffices to establish that $\bar{g}$ is θ-good.

We now begin work toward a proof of Theorem 6.2.

(6.3) Lemma. *Let $\psi \in I(G \mid \theta)$ and suppose $x \in G$ is a π-element such that $\bar{x} = xN \in G/N$ is not θ-good. Then $\psi(x) = 0$.*

Proof. By 5.3, choose an extension φ of θ to $\langle N, x \rangle$. Every I-constituent of $\psi_{\langle N, x \rangle}$ lies over θ and hence is of the form $\lambda^0 \varphi$ for some $\lambda \in \mathrm{Irr}(\langle N, x\rangle/N)$. It follows that $\psi_{\langle N,x\rangle} = \Lambda^0\varphi$ for some $\Lambda \in \mathrm{Char}(\langle N, x\rangle/N)$.

Because $\bar{x}$ is not θ-good, there exists $c \in G$ with $x^c \in Nx$ such that $\varphi^c \neq \varphi$. Now $(\varphi^c)_N = \theta^c = \theta$ and thus $\varphi^c = \mu^0\varphi$ for some nontrivial $\mu \in \mathrm{Irr}(\langle N, x\rangle/N)$. Since c centralizes $\bar{x}$, we have $\Lambda^c = \Lambda$ and so it follows that $\psi^c_{\langle N,x\rangle} = \mu^0\psi_{\langle N,x\rangle}$ and we have

$$\psi(cxc^{-1}) = \psi^c(x) = \mu(x)\,\psi(x).$$

Since $\psi(cxc^{-1}) = \psi(x)$ because ψ is a class function, and since $\mu(x) \neq 1$, it follows that $\psi(x) = 0$. ∎

Our next step is to fix a set T of coset representatives for those cosets of N in G which are π-elements of G/N. We also want to fix for each $t \in T$ a "character" $\theta_t \in I(\langle N, t\rangle)$ extending θ. The extensions θ_t exist by 5.3 and by 5.5 we can arrange that each $t \in T$ is a π-element and that $\theta_t(t) \neq 0$. Of course, the choices of T and $\{\theta_t \mid t \in T\}$ are not unique and so objects we define in terms of them may not be canonical.

We define a function σ on the set $\Sigma \cap G$ of π-elements of G by $\sigma(x) = \theta_t(x)/\theta_t(t)$ where t is the unique element of $Nx \cap T$. Although σ need not be a class function, we can get some information about how $\sigma(x^g)$ is related to $\sigma(x)$.

(6.4) LEMMA. *Let x, $g \in G$ where x is a π-element. Then in the notation established above, we have*

$$\sigma(x^g) = \sigma(x)\,\sigma(t^g),$$

where $t \in Nx \cap T$.

Proof. We have $\theta_t \in I(\langle N, t\rangle)$ and we write $\varphi = (\theta_t)^g \in I(\langle N, t\rangle^g)$ so that $\varphi_N = \theta^g = \theta$. Let $s \in Nt^g \cap T$. Then θ_s and φ are both extensions of θ to $\langle N, s\rangle$ and we can write $\theta_s = \lambda^0\varphi$ for some $\lambda \in \mathrm{Irr}(\langle N, s\rangle/N)$.

Now

$$\sigma(x^g) = \frac{\theta_s(x^g)}{\theta_s(s)} = \frac{\lambda(x^g)\,\varphi(x^g)}{\theta_s(s)} = \frac{\lambda(x^g)\,\theta_t(x)}{\theta_s(s)}.$$

Similarly,

$$\sigma(t^g) = \frac{\lambda(t^g)\,\theta_t(t)}{\theta_s(s)}$$

and so

$$\sigma(x)\,\sigma(t^g) = \frac{\theta_t(x)}{\theta_t(t)} \cdot \frac{\lambda(t^g)\,\theta_t(t)}{\theta_s(s)} = \frac{\lambda(t^g)\,\theta_t(x)}{\theta_s(s)}.$$

Since $Nx = Nt$, we have $Nx^g = Nt^g$ and thus $\lambda(x^g) = \lambda(t^g)$ and the result follows. ∎

We can use the function σ to characterize the complex vector subspace of $\mathrm{cf}^0(G)$ spanned by $I(G \mid \theta)$. We shall denote this space by $\mathrm{cf}^0(G \mid \theta)$.

(6.5) LEMMA. *Let $\psi \in \mathrm{cf}^0(G)$. Then the following are equivalent*:

(i) $\psi \in \mathrm{cf}^0(G \mid \theta)$.

(ii) $\psi(g) = \sigma(g)\,\psi(t)$ *whenever* $g \in G \cap \Sigma$ *and* $t \in Ng \cap T$.

Proof. Suppose ψ satisfies (i) and let $g \in G \cap \Sigma$ and $t \in Ng \cap T$. Every I-constituent of $\psi_{\langle N, g\rangle}$ lies over θ and thus has the form $\lambda^0\theta_t$ for some $\lambda \in \mathrm{Irr}(\langle N, g\rangle/N)$. It follows that $\psi_{\langle N, g\rangle} = \Lambda^0\theta_t$ for some $\Lambda \in \mathrm{cf}(G/N)$. (Note that since $Nt = Ng$, we have $\Lambda(g) = \Lambda(t)$.) We have

$$\psi(g) = \Lambda(g)\,\theta_t(g) = \Lambda(t)\,\theta_t(t)\,\frac{\theta_t(g)}{\theta_t(t)} = \psi(t)\,\sigma(g)$$

and ψ satisfies (ii).

Conversely, assume ψ satisfies (ii). Write $\psi = \alpha + \beta$ where every I-constituent of α lies in $I(G \mid \theta)$ and no I-constituent of β lies in this set. Our object is to show that $\beta = 0$.

By the first part of the proof, α satisfies (ii) and it follows that $\beta = \psi - \alpha$ does too, since condition (ii) is linear in ψ.

For each $t \in T$, we have by 5.4 that

$$\begin{aligned} 0 &= \sum_{x \in Nt \cap \Sigma} \tilde{\theta}_t(x)\,\overline{\beta(x)} \\ &= \sum_x \tilde{\theta}_t(x)\,\overline{\sigma(x)}\,\overline{\beta(t)} \\ &= \overline{\beta(t)/\theta_t(t)} \sum_x \tilde{\theta}_t(x)\,\theta_t(x). \end{aligned}$$

The sum in the last expression is nonzero by 5.4 and we conclude that $\beta(t) = 0$ for all $t \in T$. Since $\beta(g) = \beta(t)\,\sigma(g) = 0$ for $g \in Nt \cap \Sigma$, we see that β is identically zero, as required. ∎

Now choose a subset $T_0 \subseteq T$ such that cosets $Nt \in G/N$ for $t \in T_0$ form a set of representatives for the conjugacy classes of θ-good π-elements of G/N.

(6.6) THEOREM. *Restriction defines a $\mathbb{C}$-vector space isomorphism from* $\mathrm{cf}^0(G \mid \theta)$ *onto the space of all complex functions on* T_0.

Since $I(G \mid \theta)$ is a basis for $\mathrm{cf}^0(G \mid \theta)$, Theorem 6.6 tells us that $|I(G \mid \theta)| = |T_0|$ and thereby proves Theorem 6.2.

Proof of Theorem 6.6. Suppose $\psi \in \mathrm{cf}^0(G|\theta)$ is in the kernel of the restriction map so that $\psi(t)=0$ for all $t \in T_0$. We want to show that $\psi=0$; in other words, we need to establish that $\psi(x)=0$ for all $x \in G \cap \Sigma$. By Lemma 6.3, $\psi(x)=0$ if $\bar{x}$ is not θ-good and so we may assume that $\bar{x}$ is θ-good and thus $\bar{x}^g=\bar{t}$ for some $g \in G$ and $t \in T_0$.

Now

$$\psi(x)=\psi(x^g)=\sigma(x^g)\,\psi(t)=0$$

where we have used that fact that (i) $\Rightarrow$ (ii) in Lemma 6.5. This proves that the restriction map is injective.

Now let α be any complex function defined on T_0. To complete the proof, we must find $\psi \in \mathrm{cf}^0(G|\theta)$ such that $\psi(t)=\alpha(t)$ for all $t \in T_0$.

Let $x \in G \cap \Sigma$. Define

$$\psi(x)=\begin{cases}0 & \text{if } \bar{x} \text{ is not } \theta\text{-good} \\ \sigma(x^g)\,\alpha(t) & \text{if } \bar{x} \text{ is } \theta\text{-good,}\end{cases}$$

where $g \in G$ and $t \in T_0$ are chosen so that $\bar{x}^g=\bar{t}$ in the latter case. Note that the θ-good element $\bar{x}$ uniquely determines t but does not uniquely determine g and so we need to show that if also $h \in G$ and $\bar{x}^h=\bar{t}$, then $\sigma(x^g)=\sigma(x^h)$. In other words, we need $\theta_t(x^g)=\theta_t(x^h)$.

Now $h^{-1}g \in C$ where $C/N=\mathbf{C}_{G/N}(\bar{t})$ and since $\bar{t}$ is θ-good, we have $\theta_t^{h^{-1}g}=\theta_t$ and hence $\theta_t^{h^{-1}}=\theta_t^{g^{-1}}$. Thus

$$\theta_t(x^g)=\theta_t^{g^{-1}}(x)=\theta_t^{h^{-1}}(x)=\theta_t(x^h)$$

and ψ is well defined.

If $t \in T_0$, then $\bar{t}$ is θ-good and so to evaluate $\psi(t)$ we can take $g=1$. This gives $\psi(t)=\sigma(t)\,\alpha(t)$. Since $\sigma(t)=1$ by the definition of σ, we have $\psi(t)=\alpha(t)$. To complete the proof, it suffices to show that $\psi \in \mathrm{cf}^0(G|\theta)$.

First, we show that ψ is a class function. Let $x, y \in G \cap \Sigma$ with $y=x^g$ for some $g \in G$. If neither $\bar{x}$ nor $\bar{y}$ is θ-good, then $\psi(x)=0=\psi(y)$ as required. The remaining possibility is that both $\bar{x}$ and $\bar{y}$ are θ-good. Choose $h \in G$ and $t \in T$ such that $\bar{y}^h=\bar{t}$. Then $\bar{x}^{gh}=\bar{t}$ and we have

$$\psi(x)=\sigma(x^{gh})\,\alpha(t) \qquad \text{and} \qquad \psi(y)=\sigma(y^h)\,\alpha(t)$$

and these, of course, are equal since $x^{gh}=y^h$.

To show that $\psi \in \mathrm{cf}^0(G|\theta)$, it suffices to show that ψ satisfies 6.5(ii). Let $x \in G \cap \Sigma$ and $s \in Nx \cap T$. We need to check that $\psi(x)=\sigma(x)\,\psi(s)$ and so we must compute $\psi(x)$ and $\psi(s)$. Now $\bar{x}=\bar{s}$ and if this element is not θ-good, then $\psi(x)=0=\psi(s)$ and the condition is satisfied.

Suppose $\bar{x} = \bar{s}$ is θ-good and choose $g \in G$ and $t \in T_0$ with $\bar{x}^g = \bar{t}$. Then

$$\psi(x) = \sigma(x^g)\,\alpha(t) = \sigma(x)\,\sigma(s^g)\,\alpha(t) = \sigma(x)\,\psi(s),$$

where the second equality follows by Lemma 6.4. This completes the proof. ▮

7. Further Remarks

We continue with our usual notation so that $N \lhd G \subseteq \Gamma$ and $\theta \in I(N)$ is G-invariant where I is a character selector for (Γ, π). The following result provides a slight shortcut to establishing that an element is θ-good.

(7.1) Lemma. *Let $\bar{g}$ be a π-element of G/N and let $C/N = \mathbf{C}_{G/N}(\bar{g})$. Let $\varphi \in I(\langle N, g\rangle \mid \theta)$. Then $\bar{g}$ is θ-good iff $\varphi^c = \varphi$ for every π-element of C.*

Proof. The "only if" part is trivial and so we assume that $\varphi^c = \varphi$ when $c \in C$ is a π-element. If $c \in C$ is arbitrary, we can write $c = c_\pi c_{\pi'}$, the $\pi - \pi'$-decomposition in C, and since $\varphi^{c_\pi} = \varphi$, it suffices to show that φ is invariant under all π'-elements of C.

Let $c \in C$ be a π'-element. Then $\varphi^c \in I(\langle N, g\rangle \mid \theta)$ and so $\varphi^c = \lambda^0 \varphi$ for some $\lambda \in \mathrm{Irr}(G/N)$. Now λ is C-invariant, and it follows that

$$\varphi^{c^m} = (\lambda^m)^0\,\varphi$$

for every integer m. Taking $m = o(c)$ and appealing to Lemma 5.3, we conclude that $\lambda^m = 1_G$. The multiplicative order of λ, however, is a π-number since $\langle N, g\rangle/N$ is a π-group. It follows that $\lambda = 1_G$ and $\varphi^c = \varphi$, as required. ▮

(7.2) Corollary. *The π-element $\bar{g} \in G/N$ is θ-good iff θ is extendible to every group H where $\langle N, g\rangle \subseteq H \subseteq \Gamma$ and H/N is abelian and $H/\langle N, g\rangle$ is a cyclic π-group.*

Proof. By 7.1, $\bar{g}$ is θ-good iff for every choice of H as in the statement of the corollary, some extension φ of θ to $\langle N, g\rangle$ is H-invariant. By 5.3, however, φ is H-invariant iff φ extends to H. Therefore, $\bar{g}$ is θ-good iff for each H, some extension of θ to $\langle N, g\rangle$ further extends to H. In other words, $\bar{g}$ is θ-good iff θ extends to H for each choice of H. ▮

Using 7.2, we can state a useful application of our main theorem.

(7.3) COROLLARY. *Let I_1 and I_2 be character selectors for (Γ_1, π) and (Γ_2, π). Let $N_i \lhd G_i \subseteq \Gamma_i$ with G_i finite and let $\theta_i \in I_i(N_i)$ be G_i-invariant. Let*

$$f: G_1/N_1 \to G_2/N_2$$

be an isomorphism. Assume for every two-generator abelian π-subgroup $H/N_1 \subseteq G_1/N_1$ that θ_1 extends to H iff θ_2 extends to K, where $K/N_2 = f(H/N_1)$. Then $|I_1(G_1 \mid \theta_1)| = |I_2(G_2 \mid \theta_2)|$.

Proof. By 7.2, we see that if $\bar{g}$ is a π-element of G_1/N_1, then $\bar{g}$ is θ_1-good iff $f(\bar{g})$ is θ_2-good. The result then follows by Theorem 6.2. ∎

REFERENCES

1. K. ASANO, M. OSIMA, AND M. TAKAHASI, Über die Darstellung von Gruppen durch Kollineationen in Körpern der Characteristic *p*, *Proc. Phys. Math. Soc. Japan* **19** (1937), 199–209.
2. P. X. GALLAGHER, The number of conjugacy classes in a finite group, *Math. Z.* **118** (1970), 175–179.
3. I. M. ISAACS, Characters of π-separable groups, *J. Algebra* **86** (1984), 98–128.
4. G. KARPILOVSKY, "Projective Representations of Finite Groups," Dekker, New York, 1985.
5. I. SCHUR, Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, *J. Reine Angew. Math.* **127** (1904), 20–50.