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Flows, View Obstructions, and the Lonely Runner

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We prove the following result: Let G be an undirected graph. If G has a nowhere zero flow with at most k different values, then it also has one with values from the set $\{1, ..., k\}$. When $k \geqslant 5$, this is a trivial consequence of Seymour's "six-flow theorem". When $k \leqslant 4$ our proof is based on a lovely number theoretic problem which we call the "Lonely Runner Conjecture:" Suppose k runners having nonzero constant speeds run laps on a unit-length circular track. Then there is a time at which all runners are at least $1/(k+1)$ from their common starting point. This conjecture appears to have been formulated by J. Wills (*Monatsch. Math.* **71**, 1967) and independently by T. Cusick (*Aequationes Math.* **9**, 1973). This conjecture has been verified for $k \leqslant 4$ by Cusick and Pomerance (*J. Number Theory* **19**, 1984) in a complicated argument involving exponential sums and electronic case checking. A major part of this paper is an elementary selfcontained proof of the case $k = 4$ of the Lonely Runner Conjecture. © 1998 Academic Press

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1. INTRODUCTION

Let $G=(V, E)$ be an undirected graph. A *nowhere zero flow* of G is an orientation of G supplied with a vector $f=(f_e)$ of positive integers indexed by $E(G)$, such that for every $v \in V(G)$ the sum of f_e on edges entering v is the same as that on edges leaving v. The number f_e is called the *value* of the edge e. The theory of nowhere zero flows is a major topic in combinatorics related to graph coloring and the cycle double cover conjecture; see [9, 14, 16]. The main result of this paper is the following.

THEOREM 1.1. *Let G be an undirected graph. If G has a nowhere zero flow with at most k distinct values, then it also has one with all values from the set* $\{1, ..., k\}$.

In view of the matroid duality [16, 15, 9, 11, 14] between vertex colorings and nowhere zero flows there is a cographic analogue to Theorem 1.1. A *coloring* of G is a function $c\colon V(G) \to \mathbb{R}$, so that for all $xy \in E$, $c(x) \neq c(y)$.

THEOREM 1.2. *If G has a coloring with real numbers so that the set* $\{|c(x)-c(y)| : xy \in E\}$ *has at most k distinct values, then G has a* $(k+1)$*-coloring (and thus one where* $|c(x)-c(y)| \in \{1, ..., k\}$ *for all* $xy \in E$.)

Theorem 1.2 is easy to prove. By orienting each edge toward the endpoint with the larger color and identifying the color classes, one obtains an acyclic digraph having maximum out-degree k. An easy greedy algorithm results in a $(k+1)$-coloring of G.

Theorem 1.1 is more difficult. Our proof relies on Seymour's *six-flow theorem* [13] and a number theoretic result of Cusick and Pomerance [6] to which we give a short proof. We state here the six-flow theorem. A graph is called *bridgeless*, if it has no bridge, where $e \in E$ is a *bridge* if $G-e$ has more components than G.

THEOREM 1.3. *Every bridgeless graph has a nowhere zero flow with values from the set* $\{1, ..., 5\}$.

There is a common generalization of Theorems 1.1 and 1.2 regarding flows in regular matroids (see [11, 15]) which is strongly suggested by Seymour's *regular matroid decomposition theorem* [12]. A matrix is *totally unimodular* if every subdeterminant belongs to $\{0, \pm 1\}$.

Conjecture 1.4. Let A be a totally unimodular matrix and suppose that $Af=0$ has a real solution $f=(f_e)$, where each f_e is nonzero and where $|\{|f_e| : e \in E(G)\}| \leqslant k$. Then there exists a solution $f'=(f'_e)$ with each $|f'_e| \in \{1, 2, ..., k\}$.

The analogous statement concerning *group-valued flows* [16, 9] is false. For example, the graph with two vertices and three parallel edges has a flow with range $\{1\}$ in Z_3, but not in the integers.

The paper is organized as follows. In Section 2, Conjecture 1.4 is reduced to the "Lonely Runner Problem"; in particular, Theorem 1.1 is reduced to the special case $k \leqslant 4$. A general proof technique for this problem is introduced in Section 3 and is applied to the case $k = 4$ in Section 4.

2. RUNNERS AND FLOWS

Let us informally state the *Lonely Runner Problem*: At time zero, k participants depart from the origin of a unit length circular track to run repeated laps. Each runner maintains a constant nonzero speed. Is it true that regardless of what the speeds are, there exists a time at which the k runners are simultaneously at least $1/(k+1)$ units from the starting point? The term "lonely runner" reflects an equivalent formulation in which there are $k+1$ runners with distinct speeds. Is there a time at which a given runner is "lonely," that is, at distance at least $1/(k+1)$ from the others? This poetic title (given by the second author) made its way through an internet inquiry (of the second and last authors) up to the cover page of a public relations booklet for the Weizmann Institute in Israel [22].

We introduce some notation. The sets of real numbers and positive integers are denoted $\mathbb{R}$ and $\mathbb{N}$, respectively. The residue class of $a \in \mathbb{R}$ modulo 1 (called the *fractional part* of a) is denoted by $\langle a \rangle$. We view the unit-length circle C as the set $\{\langle a \rangle : a \in \mathbb{R}\}$, which we frequently identify with the real interval $[0, 1)$. An instance of the Lonely Runner Problem consists of a set of *runners* $R := \{1, 2, ..., k\}$ and a *speed vector* $v := (v_1, ..., v_k)$ having nonzero real entries. At time $t = 0$, each $r \in R$ begins running on C from the point 0 maintaining the constant speed v_r. The *position* of runner r on C at time t is $\langle tv_r \rangle$. The position of R at time t is the vector $\langle tv \rangle := (\langle tv_1 \rangle, ..., \langle tv_k \rangle) \in [0, 1)^k$. A vector $x = (x_1, ..., x_k) \in [0, 1)^k$ is a *position* (for the speed vector v) if there exists $t \in \mathbb{R}$ with $x = \langle tv \rangle$. The set of all positions is denoted $X = X(v) \subseteq [0, 1)^k$. The distance between two points on C is the length of the shorter of the two (arc) intervals between them. We say that $r \in R$ is *distant* (from 0) in $x \in X$ or *at time t* if $x_r = \langle tv_r \rangle \in [1/(k+1), k/(k+1)]$. A subset $R' \subseteq R$ is *distant* (in some position x) if each $r \in R'$ is distant in x. (Here, k is understood to equal $|R|$, not $|R'|$).

The aforementioned internet inquiry led us to the following assertion, which we call the *Lonely Runner Conjecture*. This conjecture appears to have been introduced by Wills [17] and again, independently by Cusick [3].

Conjecture 2.1. For all $k \in \mathbb{N}$ and $v \in (\mathbb{R} - \{0\})^k$, there exists a position where R is distant.

This problem appears in two different contexts. Cusick [3–6] was motivated by a beautiful application in n dimensional geometry—*view obstruction problems.* Our statement of the problem is closer to the diophantine approximation approach of Wills [1, 17–21]. A more general conjecture appears in [2]. The cases $k = 2, 3, 4$ were first proved in [17, 1, 6], respectively.

THEOREM 2.2. *If $k \leqslant 4$, then for any $v \in (\mathbb{R} - \{0\})^k$ there exists a time at which R is distant.*

The proof by Cusick and Pomerance [6] of the case $k = 4$ is not easy and requires a computer check. In Sections 3 and 4 we provide a simple self-contained proof. Section 3 also contains a very short proof for the case $k = 3$. We now prove Theorem 1.1 using Theorems 2.2 and 1.3.

Proof of Theorem 1.1. Let f be a nowhere zero flow with k different values. If $k \geqslant 5$, then the result is a trivial consequence of Theorem 1.3 since any graph having a nowhere zero flow must be bridgeless. If $k \leqslant 4$, then by Theorem 2.2 there exists $t \in \mathbb{R}$ such that the fractional part of each entry of tf is in the interval $[1/(k+1), k/(k+1)]$. The flow tf is a feasible flow in the edge-capacitated network (G, l, u), where $l = \lfloor tf \rfloor$ and $u = \lceil tf \rceil$ (we take floors and ceilings componentwise). But then there also exists a feasible integer-valued flow for (G, l, u), (Ford and Fulkerson, [7]), in which each edge e has value either $\lfloor tf_e \rfloor$ or $\lceil tf_e \rceil$. Let us denote this flow by $\lfloor tf \rceil$. Thus $tf - \lfloor tf \rceil$ is a flow with all entries in $[-k/(k+1), -1/(k+1)] \cup [1/(k+1), k/(k+1)]$. Multiplying this flow by $k+1$ and reorienting the edges corresponding to negative entries yields a flow with values in $[1, k]$. Again, there also exists then an integer flow with values in $[1, k]$. ∎

Note. We may loosely denote the final flow in the proof of Theorem 1.1 as $\lfloor (k+1)(f - \lfloor tf \rceil) \rceil$.

We remark that this proof can be directly generalized to flows in regular matroids by applying Hoffman's theorem [8] in order to define $f' = \lfloor (k+1)(f - \lfloor tf \rceil) \rceil$. Thus, Conjecture 1.4 is a weak form of the Lonely Runner Conjecture.

THEOREM 2.3. *For any $k \in \mathbb{N}$, if the Lonely Runner Conjecture holds true for k runners, then the statement of Conjecture* 1.4 *holds true for that particular value of k.*

The remainder of this paper is devoted to the Lonely Runner Conjecture. Wills [17] reduced the Lonely Runner Conjecture from the case of irrational speeds to the rational case. So when proving any case $k \geqslant 1$, one can assume without loss of generality that $v \in \mathbb{N}^k$, whence the speeds express the number of laps the runners make in unit time. One can further assume that $t \in [0, 1)$, although there is usually no advantage in doing so.

Proof of Theorem 2.2 *when* $k \leqslant 2$. The case $k = 1$ is trivial. In case $k = 2$ we prove a stronger statement:

> *Suppose* $v_1, v_2 \in \mathbb{N}$ *are relatively prime speeds. At any time* t, *the nearer runner has distance at most* $\lfloor (v_1 + v_2)/2 \rfloor / (v_1 + v_2)$. *Moreover, this bound is achieved at time* $t = \tau/(v_1 + v_2)$ *for some* $\tau \in \mathbb{N}$.

Whenever the distance from 0 to the nearer runner is maximum, we have $\langle tv_1 \rangle = 1 - \langle tv_2 \rangle$. This equality holds if and only if t is an integer multiple of $1/(v_1 + v_2)$. For such t, both runners are at distance $a/(v_1 + v_2)$ for some integer $a \leqslant \lfloor (v_1 + v_2)/2 \rfloor$. Since $\gcd(v_1, v_1 + v_2) = 1$ we can solve the congruence air $v_1 \tau \equiv \lfloor (v_1 + v_2)/2 \rfloor \bmod v_1 + v_2$, to obtain a time at which the bound on a is achieved, proving the statement. ∎

3. PREJUMPS

We state the fact that *the set* X *of positions is closed under addition modulo* 1 in a particular form suggesting a technique used by all the proofs hereafter.

(1) If $x_1, x_2 \in X$ and $\alpha \in \mathbb{Z}$, then the vector $x = \langle x_1 + \alpha x_2 \rangle \in [0, 1)^k$ is also in X. If moreover, $x_1 = \langle t_1 v \rangle$, $x_2 = \langle t_2 v \rangle$, and $t \equiv t_1 + \alpha t_2 \bmod 1$, then $x = \langle tv \rangle$.

Our use of (1) is as follows. We first note the existence of certain "key" positions in X which we call *prejumps*. In the proof of our main result, it sometimes becomes convenient to add one of these prejumps to a position that has already been constructed, thereby obtaining a position in which all runners are distant. Our first example of prejumps will be used in a short proof of the case $k = 3$. (Compare with the proofs in [1, 3].)

(2) Let $v \in \mathbb{N}^k$, $k \geqslant 3$. If $\gcd(v_1, ..., v_{k-1})$ does not divide v_k, then there exists a time when R is distant if and only if there exists a time when $R \backslash \{k\}$ is distant.

Proof. Let $d \geqslant 2$ be the greatest common divisor defined in the statement, and suppose without loss of generality that $\gcd(d, v_k) = 1$. Then

$$\left\langle \frac{0}{d} v_r \right\rangle = \left\langle \frac{1}{d} v_r \right\rangle = \cdots = \left\langle \frac{d-1}{d} v_r \right\rangle = 0 \qquad \text{for} \quad r = 1, ..., k-1,$$

whereas

$$\left\{\left\langle \frac{0}{k} v_k \right\rangle, \left\langle \frac{1}{d} v_k \right\rangle, ..., \left\langle \frac{d-1}{d} v_k \right\rangle\right\} = \left\{\frac{0}{d}, \frac{1}{d}, ..., \frac{d-1}{d}\right\}.$$

Let now $x = \langle tv \rangle$ be a position where $R\backslash\{k\}$ is distant. Since $R\backslash\{k\}$ is also distant in each of the d positions $\langle x + (j/d)\, v \rangle$ $(1 = 0, 1, ..., d-1)$, it suffices to show that k is distant in one of these positions. However, this follows from the fact that $1/d$ is at most the length $1 - 2/(k+1)$ of the interval of distant positions since $k \geqslant 3$ and $d \geqslant 2$.

Proof of Theorem 2.2 *when* $k \leqslant 3$. We assume that the speeds v_1, v_2, v_3 are distinct positive integers having no common factor. If all three speeds are odd, then $\langle \frac{1}{2} v \rangle = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, so we may assume that v_2 is even. By (2) we may further assume that v_1 and v_3 are odd. So $\langle \frac{1}{2} v \rangle = (\frac{1}{2}, 0, \frac{1}{2})$, and this will provide our prejump $x_1 = \langle t_1 v \rangle$, $t_1 := \frac{1}{2}$.

Consider the time interval $T := [1/4v_2, 3/4v_2]$, during which runner 2 is for the first time in the distant region $[\frac{1}{4}, \frac{3}{4}]$. For $r = 1, 3$, let $T_r = \{t \in [0, 1)$: $\langle tv_r \rangle \in [\frac{1}{4}, \frac{3}{4}]\}$.

If $T\backslash(T_1 \cup T_3) = \emptyset$, then use (1) with the defined prejump x_1, an arbitrary $t_2 \in T\backslash(T_1 \cup T_3)$, and $\alpha = 1$: $\langle (t_1 + t_2) v \rangle = (\frac{1}{2}, 0, \frac{1}{2}) + \langle t_2 v \rangle$. Since 2 is the only distant runner at time t_2, $\{1, 2, 3\}$ is distant at time $t_1 + t_2$.

We may now assume $T \subseteq T_1 \cup T_3$. Suppose that $T \subseteq T_i$ for some $i \in \{1, 3\}$. Then T is contained in one of the closed intervals comprising T_i, which implies $v_2 \geqslant v_i$. Furthermore, i first becomes distant no later than 2 does, so $v_2 \leqslant v_i$ which contradicts $v_2 \neq v_i$.

Thus $T \subseteq T_1 \cup T_3$, $T \cap T_i \neq \emptyset$ $(i = 1, 3)$. Both $T \cap T_1$ and $T \cap T_3$ consist of disjoint closed intervals and their union is T. Hence, $\emptyset \neq (T \cap T_1) \cap (T \cap T_3) = T \cap T_1 \cap T_3$, and we are done. ∎

4. THE CASE $k = 4$

Before completing the proof of Theorem 2.2, we set some notation and present two more prejump facts which hold true whenever $k+1$ is prime. The notation $a \mid b$ means that a divides b. For fixed $k \geqslant 2$ we partition the circle $C = [0, 1)$ as $\{0\} \cup C_1 \cup C_2$, where

$$C_1 := \left(0, \frac{1}{k+1}\right) \cup \left(\frac{k}{k+1}, 1\right) \cup \left\{\frac{1}{k+1}, \frac{2}{k+1}, ..., \frac{k}{k+1}\right\},$$

$$C_2 := \left(\frac{1}{k+1}, \frac{2}{k+1}\right) \cup \left(\frac{2}{k+1}, \frac{3}{k+1}\right) \cup \cdots \cup \left(\frac{k-1}{k+1}, \frac{k}{k+1}\right).$$

Given a speed vector $v \in \mathbb{N}^k$ and a position $x \in X = X(v)$, we define $D := \{r \in R\colon (k+1) \mid v_r\}$ and partition the runners R as $R_0(x) \cup R_1(x) \cup R_2(x)$, where

$$R_0(x) := D \cup \{r \in R\colon x_r = 0\},$$
$$R_1(x) := \{r \in R \backslash D\colon x_r \in C_1\},$$
$$R_2(x) := \{r \in R \backslash D\colon x_r \in C_2\}.$$

(3) Let $k+1$ be prime, and suppose there exists $x \in X$ in which D is distant and $|R_2(x)| < |R_0(x)|$. Then there exists a time when R is distant.

Proof. We consider the list of k positions $\langle x + j/(k+1)\, v \rangle$ $(1 = 1, 2, ..., k)$. Since $k+1$ is prime, we have

$$\left\langle \frac{1}{k+1} v_r \right\rangle = \cdots = \left\langle \frac{k}{k+1} v_r \right\rangle = 0 \qquad \text{if} \quad r \in D,$$

$$\left\{ \left\langle \frac{1}{k+1} v_r \right\rangle, ..., \left\langle \frac{k}{k+1} v_r \right\rangle \right\} = \left\{ \frac{1}{k+1}, ..., \frac{k}{k+1} \right\} \qquad \text{if} \quad r \in R \backslash D.$$

Using this, it is straightforward to check that, for $m = 0, 1, 2$, each runner in $R_m(x)$ is distant in exactly $k - m$ of the listed positions. Thus, there are at most $|R_1(x)| + 2\,|R_2(x)|$ positions in the list in which R is not distant. If $|R_2(x)| < |R_0(x)|$) then $|R_1(x)| + 2\,|R_2(x)| < k$, so R is distant in at least one of the k listed positions. ∎

Here is an easy corollary.

(4) Suppose that $k+1$ is prime and the only speed which it divides is v_2. If there exists $d \in \mathbb{N}$ dividing at least $k/2$ different speeds, but not dividing v_2, then there exists a time when R is distant.

Proof. Let $R' := \{r \in R\colon d \mid v_r\}$. Since $d \geqslant 2$ and $2 \notin R'$, there exists $j \in \{0, ..., d-1\}$ such that runner 2 is distant in $x := \langle (j/d)\, v \rangle$. We have that $x_r = 0$ for each $r \in R'$, so $R_0(x) \supseteq \{2\} \cup R'$, and therefore $|R_0(x)| \geqslant 1 + |R'| > k/2 = |R|/2$, whence $|R_0(x)| > |R_2(x)|$. Since $D = \{2\}$ is distant, we are done by (3). ∎

Proof of Theorem 2.2. We assume $k = 4$, $R = \{1, 2, 3, 4\}$, all speeds are distinct and have no common prime factor. Consider the (proper) subset $D = \{r \in R\colon 5 \mid v_r\}$. If $|D| = 0$, then R is distant at time $\frac{1}{5}$. Suppose $2 \leqslant |D| \leqslant 3$. By induction on k there exists a position y where D is distant. Either we are done at y, or some runner in $R \backslash D$ is not distant, whence $|R_0(y)| + |R_1(y)| \geqslant |D| + 1 \geqslant 3$, so $|R_2(y)| \leqslant 1$, whereas $|R_0(y)| \geqslant |D| \geqslant 2 > 1 \geqslant |R_2(y)|$ and we are done by (3). We henceforth assume $D = \{2\}$, whence $2 \in R_0(x)$ for every position x.

If no runner is faster than 2, then at time $1/5v_2$, 2 is the only distant runner, whence $|R_2(v/5v_2)| = 0$, $|R_0(v/5v_2)| = 1$, and we are again done by (3). We thus assume $v_1 > v_2, v_3, v_4$.

At least one of v_3, v_4, say v_3, is not equal to $v_1 - v_2$. Since v_2, v_3 are distinct and less than v_1, the assumptions $v_3 \neq v_2$ and $v_3 \neq v_1 - v_2$ imply $v_3 \not\equiv \pm v_2 \bmod v_1$. If $d := \gcd(v_1, v_3) > 1$, then if d divides v_2, we are done by (2); if it does not, we are done by (4).

Thus we can assume $\gcd(v_1, v_3) = 1$. Then there exists $\alpha \in \mathbb{N}$, $\alpha v_3 \equiv 1 \bmod v_1$. Let x be the position at time α/v_1. We have $x_1 = 0$ and $x_3 = 1/v_1 < 1/v_2 \leqslant 1/5$, so $1, 2 \in R_0(x)$ and $3 \in R_1(x)$. If $D = \{2\}$ is distant in x, then we are done by (3) since $1, 2 \in R_0(x)$, whereas $3 \in R_1(x)$, so $|R_2(x)| \leqslant 1$. So we may assume 2 is not distant in x.

We notice two facts. First, the distance of x_2 from 0 is at least twice that of x_3 (this follows from $v_2 \not\equiv 0, \pm v_3 \bmod v_1$ and $\gcd(\alpha, v_1) = 1$, which implies $x_2 = \langle \alpha/v_1 \rangle \neq 0, \pm 1/v_1$, whence $x_2 \in [2/v_1, 1 - 2/v_1]$.) Second, if a runner has distance $\delta \leqslant 1/4$ from 0 in some position $z \in X$, then it has distance 2δ in position $\langle 2z \rangle$. Let x' be the first position in the sequence $\langle 2x \rangle, \langle 4x \rangle, \langle 8x \rangle, \ldots$ in which 2 is distant. As before, $1, 2 \in R_0(x')$, whereas, by the two facts and the minimality in the choice of x', $x'_3 \in (0, 1/5)$ so $3 \in R_1(x')$, and we are again done by (3). ∎

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