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Chebyshev polynomials of the second, third and fourth kinds in approximation, indefinite integration, and integral transforms *

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Abstract

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Chebyshev polynomials of the third and fourth kinds, orthogonal with respect to $(1+x)^{1/2}(1-x)^{-1/2}$ and $(1-x)^{1/2}(1+x)^{-1/2}$, respectively, on $[-1, 1]$, are less well known than traditional first- and second-kind polynomials. We therefore summarise basic properties of all four polynomials, and then show how some well-known properties of first-kind polynomials extend to cover second-, third- and fourth-kind polynomials. Specifically, we summarise a recent set of first-, second-, third- and fourth-kind results for near-minimax constrained approximation by series and interpolation criteria, then we give new uniform convergence results for the indefinite integration of functions weighted by $(1+x)^{-1/2}$ or $(1-x)^{-1/2}$ using third- or fourth-kind polynomial expansions, and finally we establish a set of logarithmically singular integral transforms for which weighted first-, second-, third- and fourth-kind polynomials are eigenfunctions.

Keywords: Chebyshev polynomials; Jacobi polynomials; orthogonality; minimax approximation; near-minimax; constrained; expansion; interpolation; indefinite integration; integral transforms; singular; hypersingular

1. Definitions and basic properties

The Chebyshev polynomials $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$ of the first, second, third and fourth kinds are defined, respectively, on $[-1, 1]$ according to the following trigonometric

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formulae:

$$T_n(x) = \cos n\theta, \qquad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta},$$
$$V_n(x) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}, \qquad W_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \tag{1}$$

where $x = \cos\theta$, $0 \leqslant \theta \leqslant \pi$.

The nomenclature of "third- and fourth-kind Chebyshev polynomials" appears to have been first used by Gautschi (e.g., [2]). Since $\sin\theta = (1-x^2)^{1/2}$, $\cos\frac{1}{2}\theta = [\frac{1}{2}(1+x)]^{1/2}$, $\sin\frac{1}{2}\theta = [\frac{1}{2}(1-x)]^{1/2}$, it follows that $T_n(x)$, $(1-x^2)^{1/2}U_n(x)$, $(1+x)^{1/2}V_n(x)$, $(1-x)^{1/2}W_n(x)$ are proportional to cosine or sine functions in θ, namely $\cos n\theta$, $\sin(n+1)\theta$, $\cos(n+\frac{1}{2})\theta$, $\sin(n+\frac{1}{2})\theta$, each of which oscillates between precisely $n+1$ extrema of equal magnitude. We may therefore deduce the following minimax property.

Property 1.1 (minimax property). The polynomials $2^{1-n}T_n(x)$, $2^{-n}U_n(x)$, $2^{-n}V_n(x)$ and $2^{-n}W_n(x)$ have the smallest Chebyshev norm (i.e., maximum magnitude) on $[-1, 1]$ amongst all monic polynomials weighted by 1, $(1-x^2)^{1/2}$, $(1+x)^{1/2}$ and $(1-x)^{1/2}$, respectively.

The 4 polynomials are in fact Jacobi polynomials, orthogonal with respect to $(1-x)^{\alpha}(1+x)^{\beta}$ for α, $\beta = \pm\frac{1}{2}$ according to the following property.

Property 1.2 (orthogonality property). $\{T_n(x)\}$, $\{U_n(x)\}$, $\{V_k(x)\}$, $\{W_n(x)\}$ are orthogonal on $[-1, 1]$ with respect to $(1-x^2)^{-1/2}$, $(1-x^2)^{1/2}$, $(1+x)^{1/2}(1-x)^{-1/2}$, $(1-x)^{1/2}(1+x)^{-1/2}$, respectively.

We only have space to give a few of the formulae that hold for these polynomials. In particular, all four polynomials share the same recurrence relation

$$p_n = 2xp_{n-1} - p_{n-2}, \quad p_0 = 1,$$

but with different starting polynomials p_1, namely $p_1 = x$, $2x$, $2x-1$, $2x+1$ for first, second, third and fourth kinds. It is also clear that the third- and fourth-kind polynomials are essentially the same polynomial, but viewed from different ends of the interval, and specifically it is readily seen that

$$W_n(x) = (-1)^n V_n(-x). \tag{2}$$

Hence, it is normally sufficient to establish properties for third-kind polynomials, and then deduce analogous properties for fourth kind (by replacing x by $-x$).

A key pair of formulae, for the third and fourth polynomials, establishes a strong link with first and second kinds:

$$V_n(x) = u^{-1}T_{2n+1}(u), \qquad W_n(x) = U_{2n}(u), \tag{3}$$

where $u = [\frac{1}{2}(1+x)]^{1/2} = \cos\frac{1}{2}\theta$ for $x = \cos\theta$. A further pair of formulae may be added to (3), namely

$$T_n(x) = T_{2n}(u), \qquad U_n(x) = \tfrac{1}{2}u^{-1}U_{2n+1}(u).$$

It is clear from these formulae and (3) that T_n, U_n, V_n and W_n together form all first- and second-kind polynomials in the new variable u (weighted by u^{-1} in two cases).

It is finally useful to give simple formulae for differentiation of suitably weighted polynomials, as follows:

$$T_n'(x) = nU_{n-1}(x), \qquad \left[(1-x^2)^{1/2}U_{n-1}(x)\right]' = -n(1-x^2)^{-1/2}T_n(x),$$
$$\left[(1+x)^{1/2}V_n(x)\right]' = (n+\tfrac{1}{2})(1+x)^{-1/2}W_n(x), \tag{4}$$
$$\left[(1-x)^{1/2}W_n(x)\right]' = -(n+\tfrac{1}{2})(1-x)^{-1/2}V_n(x).$$

2. Near-minimax constrained approximation

The common minimax property (Property 1.1) suggests that a partial sum of a series expansion in weighted Chebyshev polynomials (of first, second, third or fourth kind) should be close to a minimax weighted polynomial approximation, and a similar property should hold for interpolation at Chebyshev polynomial zeros. Indeed, in [6], a set of such results is obtained, which extend existing results for first-kind Chebyshev polynomials. Any projection P_n of a function f in a space F onto a polynomial of degree n satisfies

$$\| f - P_n f \|_\infty \leqslant (1 + \| P_n \|_\infty) \| f - B_n f \|_\infty,$$

where B_n is the (nonlinear) best minimax approximation operator, and $P_n f$ may therefore be described as near-minimax within a relative distance $\| P_n \|_\infty$. The latter constant is thus important in measuring a bound on the distance from $B_n f$.

Mason and Elliott [6] define series projections $S^{(1)}$, $S^{(2)}$, $S^{(3)}$ and $S^{(4)}$ from spaces $C[-1, 1]$, $C_{\pm 1}[-1, 1]$, $C_{-1}[-1, 1]$ and $C_1[-1, 1]$ to partial sums of degree n of expansions in $\{T_n(x)\}$, $\{(1-x^2)^{1/2}U_{n-1}(x)\}$, $\{(1+x)^{1/2}V_n(x)\}$ and $\{(1-x)^{1/2}W_n(x)\}$, respectively. Here $C_{a,b,\dots}[-1, 1]$ denotes continuous functions vanishing at $a, b, \dots$. They also define analogous projections $L^{(1)}$, $L^{(2)}$, $L^{3)}$ and $L^{(4)}$ by interpolation at zeros of $T_{n+1}(x)$, $U_{n+1}(x)$, $V_{n+1}(x)$ and $W_{n+1}(x)$, respectively. They then show that all eight projection norms are apparently asymptotically proportional to log n; in some cases the behaviour is only demonstrated numerically, but a formula for the projection norm is obtained in all cases. The numerical values of all projection norms are less than 5 for all $n \leqslant 500$, and so the corresponding approximations may justifiably be described as "near-minimax".

3. Indefinite integration by third- and fourth-kind polynomials

Consider the determination of the indefinite integral

$$h(x) = \int_{-1}^{x} (1+x)^{-1/2} f(x)\, \mathrm{d}x, \tag{5}$$

where f is a given function and the integrand is square integrable. Similar integrals were determined in [5] for weights 1 and $(1-x^2)^{-1/2}$ (in place of $(1+x)^{-1/2}$), using Chebyshev polynomials of the first and second kind, and we adopt an analogous approach for (5) using third- and fourth-kind polynomials. Precisely the same approach can also be adopted for the weight $(1-x)^{-1/2}$, but with x replaced by $-x$, and with third- and fourth-kind polynomials interchanged.

Suppose that f_n is the polynomial of degree n obtained as a partial sum of the expansion of f in fourth-kind Chebyshev polynomials

$$f_n = \sum_{k=0}^{n} C_k W_k(x), \tag{6}$$

where

$$C_k = \frac{1}{\pi}\int_{-1}^{1} (1-x)^{1/2}(1+x)^{-1/2} f(x) W_k(x)\,\mathrm{d}x,$$

and define

$$h_n(x) = \int_{-1}^{x} (1+x)^{-1/2} f_n(x)\,\mathrm{d}x. \tag{7}$$

Then, from (4), (6),

$$h_n(x) = \sum_{k=0}^{n} C_k\left(k+\tfrac{1}{2}\right)^{-1}(1+x)^{1/2} V_k(x). \tag{8}$$

Thus, an approximation h_n to h has been determined explicitly and simply. From (1), (5) and (8), setting $x = \cos\theta$, $0 \leqslant \theta \leqslant \pi$,

$$\begin{aligned} h(x) - h_n(x) &= \int_{-1}^{x}\left[(1+x)^{-1/2} f(x) - \sum_{k=0}^{n} C_k (1+x)^{-1/2} W_k(x)\right]\mathrm{d}x \\ &= \int_{\theta}^{\pi}\left[2^{1/2}\sin\tfrac{1}{2}\theta\, f(\cos\theta) - \sum_{k=0}^{n} 2^{1/2} C_k \sin\left(k+\tfrac{1}{2}\right)\theta\right]\mathrm{d}\theta. \end{aligned}$$

Hence, since the integral from θ to π of a positive function is bounded by the integral from 0 to π,

$$\begin{aligned} \| h - h_n \|_\infty &\leqslant \int_{0}^{\pi}\left| 2^{1/2}\sin\tfrac{1}{2}\theta\, f(\cos\theta) - \sum_{k=0}^{n} 2^{1/2} C_k \sin\left(k+\tfrac{1}{2}\right)\theta\right|\mathrm{d}\theta \\ &= 2\int_{0}^{\pi/2}\left| g(\phi) - \sum_{k=0}^{n} 2^{1/2} C_k \sin(2k+1)\phi\right|\mathrm{d}\phi, \end{aligned} \tag{9}$$

where $g(\phi) = 2^{1/2}\sin\phi\, f(\cos 2\phi)$. Now, if we form the natural extension of $g(\phi)$ to $[-\pi, \pi]$ of ϕ, by defining it to be even about $\phi = \frac{1}{2}\pi$ and odd about $\phi = 0$, then $g(\phi)$ has a Fourier series expansion in ϕ with terms only in $\sin(2k+1)\phi$.

Hence, the right-hand side of (9) is the L_1 norm of the error in the Fourier partial sum of an L_2 function, and this tends to zero with n (since such a series is L_2 convergent and hence L_1 convergent). Thus, $\| h - h_n \|_\infty \to 0$, as $n \to \infty$, and the approximation method is *uniformly convergent*. We have therefore proved the following theorem.

Theorem 3.1. *The indefinite integral from* -1 *to* x *of* $(1+x)^{-1/2}$ *times the partial sum of the expansion of* $f(x)$ *in Chebyshev series of the fourth kind converges uniformly to the indefinite integral of* $(1+x)^{-1/2}f(x)$, *provided the latter function is* L_2 *integrable.*

We note that the coefficients C_k in (6), which are Fourier series coefficients of $g(\phi)$, may be determined by a fast Fourier transform technique. Alternatively, we can expect to obtain comparably accurate results by using, in place of f_n, the polynomial which interpolates f in the zeros of $W_{n+1}(x)$. This can be rapidly determined by a discrete Fourier transform technique.

In the special case in which $f(x)$ is a monic polynomial of degree $n+1$, $h-h_n$ is a constant multiple of

$$(1+x)^{1/2}V_{n+1}(x).$$

From Property 1.1 this is a minimax approximation to zero, and hence the integration method is optimal in this case.

4. Integral transforms

4.1. Hilbert-type kernels

It is well known that the Chebyshev polynomials of first and second kinds are integral transforms of each other with respect to weighted Hilbert kernels, as follows:

$$\int_{-1}^{1}(1-x^2)^{-1/2}\frac{T_n(x)}{x-y}\,\mathrm{d}x=\pi U_{n-1}(y), \tag{10}$$

$$\int_{-1}^{1}(1-x^2)^{1/2}\frac{U_{n-1}(x)}{x-y}\,\mathrm{d}x=-\pi T_n(y). \tag{11}$$

Here the integral $\int$ is to be interpreted as a Cauchy principal value integral. These two formulae correspond, under the transformation $x=\cos\theta$, $y=\cos\phi$ to the trigonometric formulae

$$\int_0^{\pi}\frac{\cos n\theta}{\cos\theta-\cos\phi}\,\mathrm{d}\theta=\pi\frac{\sin n\phi}{\sin\phi},\qquad \int_0^{\pi}\frac{\sin n\theta\sin\theta}{\cos\theta-\cos\phi}\,\mathrm{d}\theta=-\pi\cos n\phi,$$

which may readily be proved by induction.

It is further known (see, e.g., [1]) that the third- and fourth-kind polynomials are similarly related:

$$\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/2}\frac{V_n(x)}{x-y}\,\mathrm{d}x=\pi W_n(y), \tag{12}$$

$$\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{1/2}\frac{W_n(x)}{x-y}\,\mathrm{d}x=-\pi V_n(y). \tag{13}$$

Note that (10)–(13) all adopt a Hilbert kernel with a weight function, and that the latter weight is precisely that with respect to which the left-hand side Chebyshev polynomial system is orthogonal. The formulae (12) and (13) are easily obtained from (10) and (11) by using (3). For example, setting $x = 2u^2 - 1$ and $y = 2v^2 - 1$, where $u = \cos \frac{1}{2}\theta$ and $v = \cos \frac{1}{2}\phi$,

$$\begin{aligned}
\int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{1/2} \frac{V_n(x)}{x-y} \mathrm{d}x &= \int_{0}^{1} \frac{u}{(1-u^2)^{1/2}} \frac{T_{2n+1}(u)}{u^2 - v^2} 2\,\mathrm{d}u \\
&= \tfrac{1}{2} \int_{-1}^{1} (1-u^2)^{-1/2} T_{2n+1}(u) \left(\frac{1}{u+v} + \frac{1}{u-v} \right) \mathrm{d}u \\
&= \int_{-1}^{1} (1-u^2)^{-1/2} \frac{T_{2n+1}(u)}{u-v} \mathrm{d}u = \pi U_{2n}(v), \quad \text{by (10)}, \\
&= \pi W_n(y).
\end{aligned}$$

The four formulae (10)–(13) suggest obvious orthogonal expansion techniques for obtaining Hilbert-type transforms for "arbitrary" functions. Indeed, provided all relevant expansions are convergent, we may link f and g by Chebyshev series expansions, as follows.

(i) If

$$f(x) \sim \sum_{k=1}^{\infty} a_k T_k(x) \quad \text{and} \quad g(y) \sim \pi \sum_{k=1}^{\infty} a_k U_{k-1}(y),$$

then

$$\int_{-1}^{1} (1-x^2)^{-1/2} \frac{f(x)}{x-y} \mathrm{d}x = g(y). \tag{14}$$

(ii) If

$$f(x) \sim \sum_{k=1}^{\infty} b_k U_{k-1}(x) \quad \text{and} \quad g(y) \sim \pi \sum_{k=1}^{\infty} b_k T_k(y),$$

then

$$\int_{-1}^{1} (1-x^2)^{1/2} \frac{f(x)}{x-y} \mathrm{d}x = -g(y). \tag{15}$$

(iii) If

$$f(x) \sim \sum_{k=0}^{\infty} c_k V_k(x) \quad \text{and} \quad g(y) \sim \pi \sum_{k=0}^{\infty} c_k W_k(y),$$

then

$$\int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{1/2} \frac{f(x)}{x-y} \mathrm{d}x = g(y). \tag{16}$$

(iv) If

$$f(x) \sim \sum_{k=0}^{\infty} d_k W_k(x) \quad \text{and} \quad g(y) \sim \pi \sum_{k=0}^{\infty} d_k V_k(y),$$

then

$$\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{1/2}\frac{f(x)}{x-y}\,\mathrm{d}x=-g(y). \tag{17}$$

These provide us with procedures for determining, in principle, either $g(y)$ from $f(x)$ or $f(x)$ from $g(y)$.

For practical implementation the given function $f(x)$ may be replaced by the partial sum of degree n of the relevant expansion, and $g(y)$ may then be defined similarly. Alternatively, the polynomial of degree n interpolating at the zeros of the relevant Chebyshev polynomial of degree $n+1$ may be adopted and expressed in the form of a sum of Chebyshev polynomials. Analogous procedures may be adopted if $g(y)$ is the given function.

4.2. Logarithmic kernels

If the formulae (10)–(13) are integrated with respect to y, then new results, also linking Chebyshev polynomials, are obtained.

Theorem 4.1. *The integral equation*

$$\int_{-1}^{1}(1-x^2)^{-1/2}\phi(x)K(x,y)\,\mathrm{d}x=\lambda\phi(y) \tag{18}$$

has the following eigensolutions ϕ and corresponding eigenvalues λ for the following kernels K:

(i) $\phi(x)=T_n(x), \quad \lambda=\dfrac{-\pi}{n},$

$K=K_1(x,y)=\log|x-y|,$

(ii) $\phi(x)=(1-x^2)^{1/2}U_{n-1}(x), \quad \lambda=\dfrac{\pi}{n},$

$K=K_2(x,y)=\log|x-y|-\log\left|1-xy-(1-x^2)^{1/2}(1-y^2)^{1/2}\right|,$

(iii) $\phi(x)=(1+x)^{1/2}V_n(x), \quad \lambda=\dfrac{\pi}{n+\frac{1}{2}},$

$K=K_3(x,y)=\log|x-y|-\log\left|2+x+y-2(1+x)^{1/2}(1+y)^{1/2}\right|,$

(iv) $\phi(x)=(1-x)^{1/2}W_n(x), \quad \lambda=\dfrac{\pi}{n+\frac{1}{2}},$

$K=K_4(x,y)=\log|x-y|-\log\left|2-x-y-2(1-x)^{1/2}(1-y)^{1/2}\right|.$

Proof. (i) Integrating (10) with respect to y from -1 to y gives (18) for $\phi=T_n(x)$, once we have observed that the values at -1 match each other exactly. This result is well known, see [3, p.337], for example. (The order of integration may be reversed, if the integrals are regarded in a Lebesgue sense.)

(ii) Multiplying (11) by $(1-y^2)^{-1/2}$, and integrating from -1 to y, using (4), we deduce (18) for $\phi(x)=(1-x^2)^{1/2}U_{n-1}(x)$ with $\lambda=\pi/n$ and

$$K(x, y)=(1-x^2)^{1/2}\int_{-1}^{y}(1-y^2)^{-1/2}(x-y)^{-1}\,\mathrm{d}y.$$

Writing $x=\cos 2\phi$, $y=\cos 2\psi$, we may deduce after some algebra that

$$K(x, y)=\log\left|\frac{\sin(\phi+\Psi)}{\sin(\phi-\Psi)}\right|=\log\left|\frac{1-xy+(1-x^2)^{1/2}(1-y^2)^{1/2}}{x-y}\right|$$

$$=\log\left|\frac{x-y}{1-xy-(1-x^2)^{1/2}(1-y^2)^{1/2}}\right|=K_2(x, y).$$

(iii) Multiplying (12) by $(1+y)^{-1/2}$, and integrating, using (4), we deduce (18) for $\phi(x)=(1+x)^{1/2}V_n(x)$ with $\lambda=\pi/(n+\frac{1}{2})$ and

$$K(x, y)=\int_{-1}^{y}\left(\frac{1+x}{1+y}\right)^{1/2}(x-y)^{-1}\,\mathrm{d}y\quad(\text{set } x=2u^2-1,\ y=2v^2-1)$$

$$=\int_0^v\frac{2u}{u^2-v^2}\mathrm{d}v=\log\left|\frac{u+v}{u-v}\right|=\log\left|\frac{u^2+v^2+2uv}{u^2-v^2}\right|$$

$$=\log\left|\frac{2+x+y+2(1+x)^{1/2}(1+y)^{1/2}}{x-y}\right|$$

$$=\log\left|\frac{x-y}{2+x+y-2(1+x)^{1/2}(1+y)^{1/2}}\right|=K_3(x, y).$$

(iv) follows similarly. □

Note that $K_2(x, y)$ has not only a (log) singularity on the line $x=y$, but also two additional point (log) singularities at $x=y=\pm 1$.

Note also that $K_3(x, y)$ has not only a (log) singularity on the line $x=y$, but also an additional point (log) singularity at $x=y=-1$, while K_4 is similar but has its additional point singularity at $x=y=+1$.

The results (i)–(iv) of Theorem 4.1 suggest an obvious orthogonal expansion technique for obtaining log-type transforms of "arbitrary" functions. Again, provided that all of the relevant expansions converge, we may link f and g by formal Chebyshev series expansions, as follows.

Corollary 4.2. (i) *If*

$$f(x)\sim\sum_{k=1}^{\infty}a_kT_k(x)\quad\textit{and}\quad g(y)\sim-\pi\sum_{k=1}^{\infty}\frac{a_kT_k(y)}{k},$$

then

$$\int_{-1}^{1}(1-x^2)^{-1/2}f(x)K_1(x, y)\,\mathrm{d}x=g(y).$$

(ii) *If*

$$f(x) \sim \sum_{k=1}^{\infty} b_k U_{k-1}(x) \quad \textit{and} \quad g(y) \sim \pi(1-y^2)^{1/2} \sum_{k=1}^{\infty} \frac{b_k U_{k-1}(y)}{k},$$

then

$$\int_{-1}^{1} f(x) K_2(x, y)\, \mathrm{d}x = g(y).$$

(iii) *If*

$$f(x) \sim \sum_{k=0}^{\infty} c_k V_k(x) \quad \textit{and} \quad g(y) \sim \pi(1+y)^{1/2} \sum_{k=0}^{\infty} \frac{c_k V_k(y)}{k+\frac{1}{2}},$$

then

$$\int_{-1}^{1} (1-x)^{-1/2} f(x) K_3(x, y)\, \mathrm{d}x = g(y).$$

(iv) *If*

$$f(x) \sim \sum_{k=0}^{\infty} d_k W_k(x) \quad \textit{and} \quad g(y) \sim \pi(1-y)^{1/2} \sum_{k=0}^{\infty} \frac{d_k W_k(y)}{k+\frac{1}{2}},$$

then

$$\int_{-1}^{1} (1+x)^{-1/2} f(x) K_4(x, y)\, \mathrm{d}x = g(y).$$

As in Section 4.1, for a practical implementation, each of the series (in Corollary 4.2) may be replaced by an appropriate series partial sum, or alternatively f (or g) may be replaced by that Chebyshev sum (i.e., polynomial) of degree n which interpolates f (or g) at the zeros of the corresponding Chebyshev polynomial. Precisely the latter approach has been adopted in [7].

4.3. Hypersingular equations

We may also obtain a set of results by differentiating (10)–(13), after premultiplying by $(1-y^2)^{1/2}$, 1, $(1-y)^{1/2}$, $(1+y)^{1/2}$, respectively. However, these do not give simple kernels, except in the second-kind case, where we obtain

$$\oint_{-1}^{1} (1-x^2)^{1/2} \frac{U_{n-1}(x)}{(x-y)^2} \mathrm{d}x = -\pi n U_{n-1}(y). \tag{19}$$

Moreover, if

$$f(x) \sim \sum_{k=1}^{\infty} a_k U_{k-1}(x), \qquad g(y) \sim -\pi \sum_{k=1}^{\infty} k a_k U_{k-1}(y),$$

then

$$\oint_{-1}^{1} (1-x^2)^{1/2} \frac{f(x)}{(x-y)^2} \mathrm{d}x = g(y). \tag{20}$$

The integrals in (19) and (20) are to be interpreted as Hadamard finite-part integrals.

For a practical implementation we may again replace each series by a partial series sum or by a Chebyshev sum which interpolates in Chebyshev zeros. Indeed, in [8], such an approach is successfully adopted.

Appendix

After completion of the paper, we noticed that result (ii) of Theorem 4.1 appears in a modified form as [4, equation (4.7)] where also (19) is quoted. (However, results (iii) and (iv) of Theorem 4.1 remain original.)

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