Asymptotic Proportionality (Weak Ergodicity) and Conditional Asymptotic Equality of Solutions to Time-Heterogeneous Sublinear Difference and Differential Equations

HORST R. THIEME*

SFB 123, Universität Heidelberg, Im Neuenheimer Feld 294.
D-6900 Heidelberg, West Germany

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The concept of asymptotic proportionality and conditional asymptotic equality which is presented here aims at making global asymptotic stability statements for time-heterogeneous difference and differential equations. For such non-autonomous problems (apart from special cases) no prominent special solutions (equilibria, periodic solutions) exist which are natural candidates for the asymptotic behaviour of arbitrary solutions. One way out of this dilemma consists in looking for conditions under which any two solutions to the problem (with different initial conditions) behave in a similar or even the same way as time tends to infinity. We study a general sublinear difference equation in an ordered Banach space and, for illustration, time-heterogeneous versions of several well-known differential equations modelling the spread of gonorrhea in a heterogeneous population, the spread of a vector-borne infectious disease, and the dynamics of a logistically growing spatially diffusing population.

1. INTRODUCTION

The solutions of the celebrated logistic differential equation (Verhulst, 1838)

\[ \dot{x} = x(\alpha - \beta x), \quad t > 0, \]

with \( \alpha \in \mathbb{R}, \beta > 0 \), exhibit the following well-known asymptotic behaviour:

- If \( \alpha \leq 0, \ x(0) \geq 0 \), then \( x(t) \to 0 \) for \( t \to \infty \).
- If \( \alpha > 0, \ x(0) > 0 \), then \( x(t) \to \alpha/\beta \) for \( t \to \infty \).

Asymptotic statements of this kind depend on the existence of prominent special solutions. In this case it is the equilibrium solution \( \alpha/\beta \) to (1.1) the

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existence of which is due to the time-independence of the parameters \( \alpha \) and \( \beta \).

What can be said about the asymptotic behaviour of solutions to (1.1) if the parameters \( \alpha \) and \( \beta \) depend on time \( t \) in a continuous, but otherwise irregular fashion? Transforming (1.1) into an integral equation we find

\[
x(t) = x(0) \exp \left( \int_0^t \alpha(\tau) \, d\tau - \int_0^t \beta(\tau) x(\tau) \, d\tau \right).
\]

Let \( y \) be another solution to (1.1) with \( 0 < y(0) < x(0) \). An easy comparison argument shows that \( 0 < y(t) \leq x(t) \) for \( t > 0 \). From (1.2) we find that

\[
1 \geq \frac{y(t)}{x(t)} = \frac{y(0)}{x(0)} \exp \left( \int_0^t \beta(s) [x(s) - y(s)] \, ds \right).
\]

i.e., \( y(t)/x(t) \) is monotone non-decreasing and bounded; in particular

\[
y(t)/x(t) \to \text{const} > 0 \quad \text{for} \quad t \to \infty.
\]

Note that the statement (1.4) requires no assumption for \( \alpha(t) \), \( \beta(t) \) except of continuity and \( \alpha(t) \in \mathbb{R} \), \( \beta(t) \geq 0 \). It even provides interesting information in the case of constant coefficients, namely if \( \alpha \leq 0 \). Further we can immediately derive the following alternative:

\begin{itemize}
  \item Either
    \[
    \int_0^\infty \beta(s) x(s) \, ds < \infty
    \]
    for all solutions \( x \) to (1.1) with \( x(0) \geq 0 \)
    
  \item or
    \[
    \int_0^\infty \beta(s) x(s) \, ds = \infty
    \]
    for all solutions \( x \) to (1.1) with \( x(0) > 0 \), and,
    
  \item for any pair \( x, y \) of such solutions,
    \[
    x(t)/y(t) \to 1 \quad \text{for} \quad t \to \infty.
    \]
\end{itemize}

This paper aims at generalizing this result to systems of differential equations, e.g., to

\[
x_j(t) = (1 - x_j) \sum_{k=1}^m \beta_{jk}(t) x_k - \alpha_j(t) x_j,
\]
ASYMPTOTIC PROPORTIONALITY

and further to diffusion equations

\[
(\partial_t - A_\sigma) x(t, \sigma) = \alpha(t, \sigma) x(t, \sigma) \left\{ \beta(t, \sigma) - x(t, \sigma) \right\} \quad \text{for } t > 0, \sigma \in \Omega,
\]

\[
x(t, \sigma) = 0 \quad \text{for } t > 0, \sigma \in \partial \Omega.
\]

(1.7)

The system (1.5) has been used to model the dynamics of gonorrhea (see [1, 12, 15, 18], e.g., for this subject and further references) and (1.6) to model the dynamics of vector-borne diseases (see [3, 6, 18], e.g.). (1.7) describes the dynamics of a diffusing population with the individuals locally obeying a logistic reproduction/mortality law. We will show that all non-negative non-trivial solutions of either (1.5), (1.6) or (1.7) are asymptotically proportional, i.e., that any pair \( x, y \) of non-negative non-trivial solutions of (1.7), e.g., satisfies

\[
x(t, \sigma)/y(t, \sigma) \to \text{const} > 0
\]

for \( t \to \infty \) uniformly in \( \sigma \in \Omega \) without the constant depending on \( \sigma \).

Further we show conditional asymptotic equality of solutions, i.e., the validity of the following alternative: either all solutions tend to zero as time tends to infinity or all non-negative non-trivial solutions are asymptotically equal, i.e., for any pair \( x, y \) of non-negative solutions to (1.7), e.g., we have

\[
x(t, \sigma)/y(t, \sigma) \to 1 \quad \text{for } t \to \infty,
\]

uniformly in \( \sigma \in \Omega \).

For (1.5) the relation (1.9) generalizes results obtained in [1].

Our method consists in reducing (1.5), (1.6), (1.7), respectively, to difference equations

\[
u_{n+1} = A_n u_n, \quad n \in \mathbb{N},
\]

with vectors \( u_n \) in the positive cone \( X_+ \) of an ordered Banach space \( X \) and sublinear order preserving operators \( A_n \) on \( X_+ \). Asymptotic proportionality of solutions to (1.10) is then proved by elementary estimates which employ some notation from ordered Banach space theory.

Since we do not require that the operators \( A_n \) be strictly sublinear, linear problems are also included. In the linear case our results are not new, however, and they were obtained as early as 1926 for linear Volterra integral equations by Norton [16]. Asymptotic proportionality
(traditionally, but emptily, called weak ergodicity) later attracted some interest from mathematical demography. See [19], e.g., for some references. We should mention that conditional asymptotic equality implies global asymptotic stability of equilibria for a special case of Eq. (1.5), e.g., namely

\[ x_j'(t) = x_j(t) \left\{ (1 - x_j) \sum_{k=1}^{n} \beta_{jk} x_k - x_j \right\} \]  

(1.11)

with time-independent parameters \( \beta_{jk} \). See Section 4.3, Corollary 4.11, for more details.

For the reader's convenience we study only the specific examples (1.5), (1.6), (1.7). The theory for (1.10), which is developed in Section 3, allows one to handle (1.5), (1.6), (1.7) with general right-hand sides which satisfy certain (quasi)monotonicity and (quasi)sublinearity conditions. It is also possible to handle combinations of (1.5), (1.6), (1.7). We do not do so here, because each of the equations contains specific technical difficulties which, for clarity of presentation, we do not want to accumulate.

One possible practical application of asymptotic proportionality/conditional asymptotic equality is the following: If you make computer simulations for the long-run behaviour of the gonorrhea model (1.5), e.g., our results tell you that the outcome will not depend significantly on the initial conditions you choose. Unfortunately our theory does not indicate the time after which the system will have forgotten the initial conditions (either completely or up to a positive constant). In the linear case corresponding estimates can be obtained by using Hilbert's projective metric. See [2, 4, 5, 8, 9, 17, 19]. This tool may also be helpful in the general sublinear case.

2. A COUNTEREXAMPLE

The following example illustrates that asymptotic proportionality for solutions to (1.5), (1.6), or (1.7) is not a matter of course:

\[ \dot{x}_j = (1 - x_j) e^{-t} \sum_{k=1}^{n} x_k, \]  

(2.1)

\( j = 1, \ldots, m \). Setting

\[ z = \sum_{k=1}^{n} x_k \]  

(2.2)

we obtain

\[ \dot{z} = e^{-t} (m \quad z) z. \]  

(2.3)
Hence \( z = \sum x_k \) is bounded. Integrating (2.1) yields
\[
x_j(t) = 1 - (1 - x_j(0)) \exp \left( - \int_0^t e^{-s} z(s) \, ds \right).
\]
Thus for any pair of solutions \( x, y \) to (2.1) we have
\[
x_j(t)/y_j(t) \to \frac{1 - (1 - x_j(0)) \varepsilon}{1 - (1 - y_j(0)) \delta}
\]
with \( 0 < \delta, \varepsilon < 1 \). Though we have convergence of the quotients, the limits depend on \( j \).

3. ASYMPTOTIC PROPORTIONALITY FOR SUBLINEAR HETEROGENEOUS
DIFFERENCE EQUATIONS IN ORDERED BANACH SPACES

In this section we consider difference equations
\[
u_{n+1} = A_n u_n, \quad n \in \mathbb{N}
\]
in an ordered Banach space \( X \). Here \( u_n, n \in \mathbb{N} \), are vectors in \( X_+ \) and \( A_n \) are sublinear operators which preserve the order. More precisely we assume that the Banach space \( X \) has a closed convex cone \( X_+ \) such that
\[
x \leq y \iff y - x \in X_+
\]
generates a (sometimes called "partial") order on \( X \) which is compatible with the linear and topological structure of \( X \). Well-known examples are \( \mathbb{R}^n, C(\Omega), L^p(\Omega), 1 \leq p \leq \infty \), with the coordinate-wise or point-wise order. See [13], e.g., for more details.

A subset \( Y \) of \( X_+ \setminus \{0\} \) is called a sublinear manifold iff
\[
\gamma x \in Y \iff 0 < \gamma < 1, x \in Y.
\]
An operator \( A \) mapping a sublinear manifold \( Y \) of \( X_+ \) into itself is called order preserving (monotone) on \( Y \) iff
\[
A x \geq A y \quad \text{for} \quad x \geq y, x, y \in Y.
\]
\( A \) is called sublinear on \( Y \) iff
\[
A(\gamma x) \geq \gamma A x \quad \text{for} \quad 0 < \gamma < 1, x \in Y.
\]
In view of relation (1.8) we define asymptotic proportionality in this context as follows:
DEFINITION 3.1. Two sequences \((u_n), (v_n)\) in \(X_+\) are called asymptotically proportional (with factor \(\gamma > 0\)) iff there are convergent sequences \((\gamma_j), (\tilde{\gamma}_j)\) of non-negative real numbers such that

\[
\gamma_j u_j \leq v_j \leq \tilde{\gamma}_j u_j \quad \text{for} \quad j \in \mathbb{N}
\]

and

\[
0 < \lim_{j \to \infty} \gamma_j = \lim_{j \to \infty} \tilde{\gamma}_j =: \gamma.
\]

\((u_j), (v_j)\) are called asymptotically equal iff they are asymptotically proportional with factor \(\gamma = 1\).

Remark. One can assume that the proportionality factor \(\gamma\) satisfies \(\gamma \leq 1\). Otherwise change the roles of \((u_n), (v_n)\).

Linear autonomous examples already illustrate that sublinearity and monotonicity of \(A_n\) are not sufficient to make any pair of solutions to (3.1) asymptotically proportional. Let, e.g.,

\[
A_n = A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then \(u_n \neq v_n = u_{n+1} = u_{n-1}\).

Before we formulate conditions for \(A_n\), we must introduce additional notation.

For \(u, v \in X_+, v \neq 0\), we set

\[
[u/v]^* := \inf\{\alpha \geq 0, u \leq \alpha v\}
\]

\[
[u/v]_* := \sup\{\alpha \geq 0, u \geq \alpha v\}
\]

\[
\psi(u; v) = \frac{[u/v]^*}{[u/v]_*} \quad \text{if} \quad u \neq 0, \quad \psi(0; v) = 1.
\]

We do not exclude the fact that \([u/v]^*\) or \(\psi(u; v)\) may be infinite. In (3.9) we use the convention that \(0/0 = 1, \infty/\infty = \infty\) for \(0 \leq c < \infty\). If \(u, v\) are vectors in \((0, \infty)^m\), then \([u/v]^* = \sup\{u_j/v_j; j = 1, \ldots, m\}\) and \([u/v]_*\) is the corresponding infimum. Note that \(d(u, v) = \lim \psi(u; v)\) provides Hilbert's projective metric. See [2, 4, 5, 8, 9, 17, 19].

We list some properties which are used later without further mention:

(i) \([u_1 + u_2/v]_* \geq [u_1/v]_* + [u_2/v]_*\).

(ii) \([u/w]_* \geq [u/v]_* [v/w]_*\).
(iii) \([xu/v]_\# = x[u/v]_\# \) for \(x \geq 0\).

(iv) \([u/u]_\# = 1\).

These relations also hold for \([u/v]_\#^*\) with reversed inequality in (i), (ii).

(v) \(\psi(xu; v) = \psi(u; v) \) for \(x > 0\).

(vi) \(\psi(u; w) \leq \psi(u; v) \psi(v; w)\).

**Definition 3.2.** Let \(w_n, n \in \mathbb{N}, \) be elements in \(X_+, \) \(w_n \neq 0. \) Let \(Y\) be a sublinear manifold in \(X_+ \setminus \{0\}\).

(a) A family of operators \(A_n, n \in \mathbb{N}, \) on \(Y\) is called **uniformly positive** on \(Y\) iff there exists some constant \(c > 0\) such that

\[\psi(A_n u; w_{n+1}) \leq c < \infty\]

for all \(n \in \mathbb{N}, u \in Y.\)

(b) A family of order preserving operators \(A_n, n \in \mathbb{N}, \) on \(Y\) is called **uniformly monotone** iff for any constant \(c > 0\) there is a constant \(\bar{c} > 0\) such that

\[\psi(A_n u - A_n v; w_{n+1}) \leq \bar{c}\]

for all \(n \in \mathbb{N}, u \geq v, u, v \in Y\) with \(\psi(u; w_n) + \psi(v; w_n) \leq c.\)

(c) A family of sublinear operators \(A_n, n \in \mathbb{N}, \) on \(Y\) is called **uniformly sublinear** on \(Y\) iff for any constant \(c > 0\) there is a constant \(\bar{c} > 0\) such that

\[\psi(A_n(yu) - yA_n u; w_{n+1}) \leq \bar{c}\]

for all \(n \in \mathbb{N}, 0 < y < 1, u \in Y\) with \(\psi(u; w_n) \leq c.\)

**Theorem 3.3.** Let the operators \(A_n, n \in \mathbb{N}, \) form a uniformly positive, uniformly monotone and uniformly sublinear family on the sublinear manifold \(Y\) in \(X_+ \setminus \{0\}\). Then any two solutions to (3.1) with initial values in \(Y\) are asymptotically proportional.

**Proof.** Let \((u_n), (v_n)\) be two solutions to (3.1) with elements in \(Y.\) We have to show that \([u_n/v_n]_\#^*\) and \([u_n/v_n]_\#^*\) converge and have the same limit.

**Step 1.** We define

\[
\gamma_n := \min(1, [u_n/v_n]_\#^*) \\
\bar{\gamma}_n := \min(1, [v_n/u_n]_\#^*). \tag{3.10}
\]

This implies that

\[
u_n \geq \gamma_n v_n, \quad v_n \geq \bar{\gamma}_n u_n. \tag{3.11}\]
Since the operators \( A_n \) are order preserving and sublinear, we easily derive that
\[
y_n \leq y_{n+1}, \quad \tilde{y}_n \leq \tilde{y}_{n+1} \quad \text{for } n \in \mathbb{N} \tag{3.12}
\]
such that
\[
y_n \nearrow \gamma \leq 1, \quad \tilde{y}_n \nearrow \tilde{\gamma} \leq 1, \quad \text{for } n \to \infty. \tag{3.13}
\]
Obviously the proof is finished if
\[
\gamma = \tilde{\gamma} = 1.
\]

**Step 2.** Without restriction we assume that
\[
\gamma < 1 \tag{3.14}
\]
and hence
\[
\gamma_n = \left\lfloor u_n/v_n \right\rfloor \ast < 1 \tag{3.15}
\]
and claim that
\[
\left\lfloor u_n/v_n \right\rfloor \ast \to \gamma \quad \text{for } n \to \infty.
\]
Note that
\[
u_{n+1} = \gamma_n v_{n+1} + y_n + z_n \tag{3.16}
\]
with
\[
y_n = u_{n+1} - A_n(\gamma_n v_n) = A_n u_n - A_n(\gamma_n v_n)
\]
\[
z_n = A_n(\gamma_n v_n) - \gamma_n A_n v_n = A_n(\gamma_n v_n) - \gamma_n v_{n+1}.
\tag{3.17}
\]
This implies
\[
\gamma_{n+1} = \left\lfloor u_{n+1}/v_{n+1} \right\rfloor \ast \\
\geq \gamma_n + \left\lfloor y_n/v_{n+1} \right\rfloor \ast + \left\lfloor z_n/v_{n+1} \right\rfloor \ast.
\]
As \((\gamma_n)\) converges we find that
\[
\left\lfloor y_n/v_{n+1} \right\rfloor \ast \to 0 \quad \text{for } n \to \infty.
\]
\[
\left\lfloor z_n/v_{n+1} \right\rfloor \ast \to 0 \quad \text{for } n \to \infty. \tag{3.18}
\]
We claim that (3.18) implies
\[
\begin{align*}
[y_n/v_{n+1}]^* & \to 0 \quad \text{for } n \to \infty, \\
[z_n/v_{n+1}]^* & \to 0 \quad \text{for } n \to \infty.
\end{align*}
\tag{3.19}
\]

The proof is completed by (3.19) because (3.16) implies that
\[
\gamma_{n+1} \leq [u_{n+1}/v_{n+1}]^* \leq \gamma_n + [y_n/v_{n+1}]^* + [z_n/v_{n+1}]^*
\]
and so \([u_n/v_n]^* \to \gamma\) for \(n \to \infty\) follows.

**Step 3.** The system (3.18) implies (3.19)! First we conclude from the uniform positivity of \((A_n)\) that
\[
\psi(u_n; w_n) \leq c,
\]
\[
\psi(v_n; w_n) \leq c
\tag{3.20}
\]
and also
\[
\psi(\alpha v_n; w_n) = \psi(v_n; w_n) \leq c
\tag{3.21}
\]
for all \(\alpha \geq 0, n \in \mathbb{N}\), with some constant \(c > 0\). Now let
\[
[y_n/v_{n+1}]^* \to 0 \quad \text{for } n \to \infty.
\]
As
\[
[y_n/v_{n+1}]^* \geq [y_n/w_{n+1}]^*[w_{n+1}/v_{n+1}]^*
\]
\[
\frac{[y_n/w_{n+1}]^*}{[v_{n+1}/w_{n+1}]^*},
\]
we conclude from (3.17), (3.20), (3.21) and the uniform monotonicity of \((A_n)\) that
\[
[y_n/v_{n+1}]^* \geq c^{-1} \tilde{c}^{-1} \frac{[y_n/w_{n+1}]^*}{[v_{n+1}/w_{n+1}]^*}
\]
\[
= c^{-1} \tilde{c}^{-1} [y_n/w_{n+1}]^* [w_{n+1}/v_{n+1}]^*
\]
\[
\geq c^{-1} \tilde{c}^{-1} [y_n/v_{n+1}]^*.
\]
So
\[
[y_n/v_{n+1}]^* \to 0 \quad \text{for } n \to \infty.
\]
The second part of (3.19) is proved analogously by using the uniform sublinearity of \((A_n)\).
Remark. The proof of Theorem 3.3 can be refined such that the constant in Definition 3.2(c) may depend on $\gamma$. But this refinement is not needed for the applications presented in this paper.

As the theorem has a technical look we give an easy application:

$$u_{n+1}(x) = \int_0^1 b_n(x, y) f(u_n(y)) \, dy$$

(3.22)

for $0 \leq x \leq 1$, $n \in \mathbb{N}$.

Here $f$ is a monotone non-decreasing function on $[0, \infty)$ with $f(u)/u$ monotone non-increasing, $f(0) = 0$ and $f(u) > 0$ for $u > 0$. The integral kernels $b_n$ are assumed to be continuous and non-negative on $[0, 1]^2$. The basic assumption is the following,

$$w_{n+1}(x) g(y) \leq b_n(x, y) \leq cw_{n+1}(x) g(y),$$

(3.23)

for all $n \in \mathbb{N}$, $x, y \in [0, 1]$ with the functions $w_n$, $g$ on $[0, 1]$ having the following properties:

$g$ is a strictly positive continuous function on $[0, 1]$.

The $w_n$ are non-negative continuous functions on $[0, 1]$ which are not identically zero. We stress the fact that the constant $c$ in (3.23) must be independent of $n$.

Let $A_n$ be defined via the right-hand side of (3.22). Then for $u \geq v$, $u, y \in X_+ \setminus \{0\} =: Y$, we have

$$(A_n u - A_n v)(x) \geq w_{n+1}(x) \int_0^1 \left[ f(u(y)) - f(v(y)) \right] g(y) \, dy$$

and

$$(A_n u - A_n v)(x) \leq cw_{n+1}(x) \int_0^1 \left[ f(u(y)) - f(v(y)) \right] g(y) \, dy.$$  

Hence

$$\Psi(A_n u - A_n v, w_{n+1}) \leq c.$$  

As $A_n 0 = 0$ by $f(0) = 0$, this consideration proves uniform positivity and uniform monotonicity of $(A_n)$. Uniform sublinearity is proved in a similar way.

**Corollary 3.4.** Let $f$ and $b_n$, $n \in \mathbb{N}$, in (3.22) satisfy the above-mentioned assumptions. Then any pair of solutions $u_n$, $v_n$ to (3.22) with non-negative initial values $u_0$, $v_0 \neq 0$ is asymptotically proportional, i.e.,

$$u_n(x)/v_n(x) \to \text{const} > 0$$

for $n \to \infty$ uniformly in $0 \leq x \leq 1$, without the constant depending on $x$. 
There is an easy, but useful, extension of Theorem 3.3.

**Theorem 3.5.** Let $A_n, n \in \mathbb{N}$, form a family of order preserving sublinear operators on the sublinear manifold $Y$ in $X^* \setminus \{0\}$. Let there exist some $m \in \mathbb{N}$ such that, for any $k = 1, \ldots, m$, the operators

$$B^k_n = A_{(n+1)m+k-1} \cdots A_{nm+k}$$

form uniformly positive, uniformly monotone, and uniformly sublinear families $(B^k_n)_{n \in \mathbb{N}}$. Then any two solutions to (3.1) with initial values in $Y$ are asymptotically proportional.

**Proof.** Let $(u_n), (v_n)$ be two solutions to (3.1) with elements in $Y$. Let

$$\tilde{u}_n = u_{nm+k}, \quad \tilde{v}_n = v_{nm+k}, \quad n \in \mathbb{N}.$$ 

Then

$$\tilde{u}_{n+1} = B^k_n \tilde{u}_n, \quad \tilde{v}_{n+1} = B^k_n \tilde{v}_n.$$ 

So $(u_{nm+k})_n$ and $(v_{nm+k})_n$ are asymptotically proportional with factors $\gamma_k, k \in \mathbb{N}$. Obviously

$$\gamma_m = \gamma_0.$$ 

Let us assume without restriction that $\gamma_0 \leq 1$. See the remark following Definition 3.2. From the sublinearity of the operators $A_n$ we find inductively that

$$\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_k \leq 1.$$ 

Finally we have

$$\gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_m = \gamma_0.$$ 

This implies the assertion.

4. **The Multigroup Gonorrhea Model**

In this section we study the asymptotic proportionality of solutions to the multigroup gonorrhea model

$$\dot{x}_j = (1 - x_j) \sum_{k=1}^{m} \beta_{jk}(t) x_k - x_j(t) x_j$$ \hspace{1cm} (4.1)

for $j = 1, \ldots, m$. 
We refer to \([1, 11, 12, 15, 18]\) for background information concerning this epidemic model and for further references. See \([1]\) for first results in the direction of this paper.

4.1. General Properties of the Flow

We define \(U(t, s), t \geq s\), to be the operators on \(X_+ = [0, \infty)^m\) which map a vector \(x \in X_+\) to the solution \(x(t)\) to (4.1) for \(t > s\) with initial values \(x(s) = x\). For the moment we only assume that \(\alpha_j\) and \(\beta_{jk}\) are non-negative continuous functions of \(t \geq 0\). It is useful to introduce the following notation: If \(x, y \in X = \mathbb{R}^m\), then

\[ x > y \quad \text{iff} \quad x \geq y, \ x \neq y \]

and

\[ x \gg y \quad \text{iff} \quad x - y \in (0, \infty)^m \quad \text{iff} \quad x_j > y_j \quad \text{for} \quad j = 1, \ldots, m. \]

**Lemma 4.1.** The operators \(U(t, s), t \geq s\), are order preserving and sublinear on \(X_+\).

**Proof.** Obviously \(U(t, s)\) maps \(X_+\) into itself. The monotonicity of \(U(t, s)\) follows from the quasimonotonicity of the right-hand side of (4.1) and a standard comparison argument. A similar comparison argument provides the sublinearity of \(U(t, s)\). Let us first assume that \(f_{ijk}(t) > 0\) for all \(j, k = 1, \ldots, m, t > 0\). Further let

\[ x(s) \gg y(s) > 0 \]  

(4.2)

with some \(0 < \gamma < 1\) and let \(x(t), y(t)\) be solutions to (4.1) for \(t > s\) and initial values \(x(s), y(s)\). Obviously \(x(t), y(t) > 0\) for \(t \geq s\). Let \(t\) be the first instant at which \(x_j(t) = \gamma y_j(t)\) for some \(t \geq s, j \in \{1, \ldots, m\}\). Obviously \(t > s\) and \(x_j(t) \leq \gamma y_j(t)\). But, on the other hand,

\[ x_j(t) \geq (1 - \gamma y_j(t)) \sum \beta_{jk} \gamma y_k(t) - \alpha_j(t) \gamma y_j(t) \]

\[ > \gamma y_j(t). \]

It follows from this contradiction that (4.2) implies \(x(t) \gg y(t)\) for all \(t \geq s\). As the solutions to (4.1) depend continuously on the initial values and the parameters the sublinearity of \(U(t, s)\) follows.

4.2. Asymptotic Proportionality

Asymptotic proportionality (equality) of two solutions to (4.1) can be defined in analogy to Definition 3.1 concerning sequences solving (3.1). The
following relation follows from Lemma 4.1 in the same way as Theorem 3.5
follows from Theorem 3.3.

**Lemma 4.2.** Let \( x, y \) be two solutions to (4.1) with initial values in \( X_+ \)
and \((t_n) \to \infty \) for \( n \to \infty \). Then \( x, y \) are asymptotically proportional iff the
sequences \((x(t_n))\) and \((y(t_n))\) are asymptotically proportional.

In order to avoid technicalities we formulate stronger assumptions than
actually needed for proving asymptotic proportionality of solutions. These
assumptions are to hold throughout this section.

**Assumptions 4.3.** The non-negative continuous functions \( \beta_{jk} \) and \( \alpha_j \) on
\([0, \infty)\) satisfy the following conditions:

(a) \( \beta_{jk} \) and \( \alpha_j \) are bounded on \([0, \infty)\).

(b) There exists some \( \epsilon > 0 \) such that \( \alpha_j(t) \geq \epsilon > 0 \) for all \( t \geq 0 \).

(c) There exists an irreducible matrix \( B = (\beta_{jk}) \) such that
\[ \beta_{jk}(t) \geq \beta_{jk} > 0 \] for all \( t \geq 0 \).

We recall that a non-negative \( m \times m \) matrix \( B \) is irreducible iff the matrix
\[ \sum_{k=1}^{m} B^k \]
has positive entries only. This implies that the population is
epidemiologically connected, i.e., that the disease will spread to all sub-
groups of the population in whatever subgroup it has started. We note that
\[ \sum_{k=1}^{m} \beta_{jk} > 0 \] for \( j = 1, \ldots, m \) if \( \beta \) is irreducible.

In order to apply the results of Section 3 we choose
\[ A_n = U(n + 1, n), \quad n \in \mathbb{N}. \quad (4.3) \]

We next look for a sublinear manifold \( Y \) in \( X_+ \setminus \{0\} \) which attracts all
solutions to (4.1) starting from initial values in \( X_+ \setminus \{0\} \) and on which the
family \((A_n)\) is uniformly monotone and uniformly sublinear. It will turn out
that
\[ Y = (0, 1 - \epsilon)^m \quad (4.4) \]
is a good choice for some appropriate \( 0 < \epsilon < 1 \).

**Lemma 4.4.** If \( \epsilon \in (0, 1) \) is properly chosen, \( Y \) is invariant for solutions to
(4.1) and attracts all solutions to (4.1) starting from initial values in
This means: If \( x(s) \in Y \) and \( x(t) \) solves (4.1) for \( t > s \), then \( x(t) \in Y \) for \( t > s \). Further, if \( x(s) \in X_+ \setminus \{0\} \), then \( x(t) \in Y \) for sufficiently large \( t > s \).

**Proof.** Obviously the set \([0, 1]^m\) is invariant for solutions to (4.1) and attracts all solutions to (4.1) starting from initial values in \( X_+ \). Without restriction we can confine our consideration to solutions with values in \([0, 1]^m\). By Assumption 4.3(a, b) we find some \( c > 0 \) such that

\[
\sum_{k=1}^{m} \beta_{jk}(t) \leq c \alpha_j(t) \quad \text{for} \quad t \geq 0.
\]

Hence

\[
x_j \leq \alpha_j(t) \{ (1-x_j) c - x_j \} < 0,
\]

if \( x_j > c/(c+1) \). If \( 1 > 1 - \varepsilon > c/(c+1) \), \([0, 1-\varepsilon]^m\) is invariant and attracting for solutions starting in \( X_+ \). Recall Assumption 4.3(b). Without restriction we can now confine our attention to solutions with values in \([0, 1-\varepsilon]^m\). So

\[
\dot{x} \geq \varepsilon B x - \dot{c} x
\]

by Assumption 4.3(a, c) with a suitably chosen constant \( \dot{c} > 0 \). Hence

\[
x(t) \geq \exp(-\dot{c}(t-s)) \exp((t-s) \varepsilon B) x(s).
\]

As \( B \) is irreducible, \( \exp((t-s) \varepsilon B) \) is a strictly positive matrix for \( t > s \). Hence \( x(t) \geq 0 \) for \( t > s \). Clearly \( x(t) \geq 0 \) for \( t \geq s \), if \( x(s) \geq 0 \).

Lemmas 4.2 and 4.4 tell us that, in order to prove asymptotic proportionality of solutions \( x, y \) to (4.1) starting in \( X_+ \setminus \{0\} \), it is sufficient to prove asymptotic proportionality of solutions

\[
x(n+1) = A_n x(n) \\
y(n+1) = A_n y(n)
\]

with \( x(0), y(0) \in Y \).

In order to apply Theorem 3.3, i.e., to prove that the operators \( A_n = U(n+1, n) \) form a uniformly positive, uniformly monotone and uniformly sublinear family of operators on \( Y \), we choose

\[
w_n = w = (1, ..., 1) \in \mathbb{R}^m.
\]

**Lemma 4.5.** The operators \( A_n, n \in \mathbb{N} \), are uniformly positive on \( Y \).
Proof. It follows from the boundedness of $\beta_{jk}$ that
\[ x(n + 1) \leq c \sum_{k=1}^{m} x_k(n) w \]
for any solution $x$ to (4.1) for $t > n$ with initial value $x(n)$. The constant $c > 0$ does not depend on $n$. By Lemma 4.4 and Assumption 4.3(c)
\[ x'(t) \geq \varepsilon B x - cx. \]
Hence
\[ x(n + 1) \geq e^{-c} e^B x(n). \]
As $B$ is irreducible, $e^{sB}$ is a strictly positive matrix, thus
\[ x(n + 1) \geq \delta \sum_{n=1}^{m} x_k(n) w \]
with some $\delta > 0$ not depending on $n$. We obtain
\[ \psi(x(n + 1); w) \leq c/\delta. \]

**Lemma 4.6.** The operators $A_n$, $n \in \mathbb{N}$, are uniformly monotone on $Y$.

Proof. Consider two solutions $x$ and $y$ to (4.1) for $t > n$ with initial values $x(n) \geq y(n)$. Set
\[ z = x - y \geq 0. \tag{4.6} \]
Then
\[ \hat{z}_j = (1 - x_j) \sum_{k=1}^{m} \beta_{jk}(t) z_k - \left\{ \sum_{k=1}^{m} \beta_{jk}(t) y_k + \alpha_j(t) \right\} z_j. \]
By the boundedness of $\beta_{jk}$
\[ z(n + 1) \leq c \sum_{k=1}^{m} z_k(n) w. \tag{4.7} \]
As $y_k \leq 1$, by Assumption 4.3 and Lemma 4.4,
\[ z(n + 1) \geq \exp(-c) e^{-sB} z(n). \]
As $B$ is irreducible,

$$z(n + 1) \geq \delta \sum_{k=1}^{m} z_k(n) w. \quad (4.8)$$

Thus, by (4.6), (4.7), and (4.8),

$$\psi(x(n + 1) - y(n + 1); w) \leq c/\delta$$

with $c/\delta$ not depending on $n$.

Before we study $(A_n)$ for uniform sublinearity we show the following

**Lemma 4.7.** Let $x$ be a solution to (4.1) for $t > n$ with initial value $x(n)$. Then

$$\sup \{ x_j(t); j = 1, \ldots, m, n \leq t \leq n + 1 \}$$

$$\inf \{ x_j(t); j = 1, \ldots, m, n \leq t \leq n + 1 \}$$

$$\leq \text{const} \psi(x(n); w)$$

for $n \in \mathbb{N}$ without the constant depending on $n$.

**Proof.** By Assumption 4.3, for $n \leq t \leq n + 1$,

$$x(t) \leq c \sum_k x_k(n) w \leq c [x(n)/w] * mw.$$

On the other hand

$$x(t) \geq e^{-ct} x(n) \geq [x(n)/w] \ast e^{-c} w$$

for $n \leq t \leq n + 1$. This implies the assertion.

**Lemma 4.8.** The operators $A_n, n \in \mathbb{N}$, are uniformly sublinear on $Y$.

**Proof.** Let $x$ be a solution to (4.1) for $t > n$ with initial value $x(n) \in Y$. Let $y$ be the solution to (4.1) for $t > n$ with $y(n) = \gamma x(n), 0 < \gamma < 1$. Set

$$z = y - \gamma x \geq 0. \quad (4.9)$$

Recall Lemma 4.1. Now

$$z(n) = 0 \quad (4.10)$$

and

$$z_j = (1 - y_j) \sum \beta_{jk} z_k + \gamma(1 - \gamma) x_j \sum \beta_{jk} x_k - \gamma z_j \sum \beta_{jk} x_k - \alpha_j z_j. \quad (4.11)$$
Let
\[ x^* = \inf \{ x_j(t); j = 1, \ldots, m, n \leq t \leq n + 1 \} \]  
(4.12)
and let \( x^* \) be the corresponding supremum. Then
\[ x^* \leq c \sum_k z_k w + c\gamma(1 - \gamma)(x^*)^2 w. \]
Hence, as \( z(n) = 0 \) by (4.10),
\[ z(n + 1) \leq c\gamma(1 - \gamma)(x^*)^2 w. \]
(4.13)
On the other hand
\[ \dot{z} \geq \delta\gamma(1 - \gamma)(x^*)_2 w - cz. \]
This implies
\[ z(n + 1) \geq \delta\gamma(1 - \gamma)(x^*)^2 w. \]
(4.14)
By Lemma 4.7 and (4.12), \( x^*/x^* \leq \text{const } \psi(x(n); w). \) Together with (4.13) and (4.14) this implies
\[ \psi(A_n(yx(n)) - \gamma A_n x(n); w) \]
\[ = \psi(z(n + 1); w) \leq c/\delta \psi(x(n); w)^2. \]
From Theorem 3.3 we now derive the following result:

**Theorem 4.9.** Let Assumptions 4.3 be satisfied. Then any two solutions \( x, y \) to (4.1) with initial values in \([0, \infty)^m \setminus \{0\}\) are asymptotically proportional, i.e.,
\[ x_j(t)/y_j(t) \to \text{const } > 0 \]
for \( t \to \infty, j = 1, \ldots, m, \) without the constant depending on \( j. \)

4.3. Conditional Asymptotic Equality

Let us consider two solutions \( x, y \) to (4.1) which are asymptotically proportional with factor \( \gamma > 0. \) This implies that
\[ x(t) - \gamma y(t) \to 0 \quad \text{for } t \to \infty. \]
(4.15)
We suppose that \( \gamma \neq 1. \) Integrating (4.1) from \( t \) to \( t + 1 \) and using (4.15) then yields
\[ \int_t^{t+1} y_j(s) \sum_{k=1}^m \beta_{jk}(s) y_k(s) ds \to 0 \quad \text{for } t \to \infty. \]
(4.16)
Suppose that

\[ \limsup_{t \to \infty} \sum_{k=1}^{m} y_k(t) > 0. \]  

(4.17)

Then there exist a sequence \( t_n \to \infty \) and some \( \delta > 0 \) such that

\[ \sum_{k=1}^{m} y_k(t_n) \geq \delta > 0. \]  

(4.18)

If \( 1 \leq t \leq 2 \), by Assumption 4.3 and Lemma 4.4

\[ y(t_n + t) \geq \delta \exp(c \mathbf{B}) y(t_n) \]

\[ \quad \geq \delta \sum_{k=1}^{m} y_k(t_n) w \]

with \( w = (1, \ldots, 1) \in \mathbb{R}^m \). So (4.16) and Assumption 4.3(c) imply that

\[ \left( \sum_{k=1}^{m} y_k(t_n) \right)^2 \sum_{j,k} \beta_{jk} \to 0 \quad \text{for} \quad n \to \infty. \]

As \( \mathbf{B} \) is irreducible, \( \sum_{j,k} \beta_{jk} > 0 \) and we obtain a contradiction to (4.18).

Thus \( \gamma = 1 \) holds or (4.17) is not valid.

So we have obtained the following result:

**Theorem 4.10.** Let Assumptions 4.3 be satisfied. If there exists at least one solution to (4.1) which does not converge to zero for \( t \to \infty \), then all solutions to (4.1) which start in \([0, \infty)^m \setminus \{0\}\) are asymptotically equal, i.e., for any pair of solutions \( x, y \) to (4.1) with \( x(0), y(0) \in [0, \infty)^m \setminus \{0\} \) we have

\[ x_j(t)/y_j(t) \to 1 \quad \text{for} \quad t \to \infty, j = 1, \ldots, m. \]

A complete discussion is possible for the following special case of (4.1).

\[ x_j = \alpha_j(t) \left\{ (1 - x_j) \sum_{k=1}^{m} \beta_{jk} x_k - x_j \right\}, \quad j = 1, \ldots, m, \]  

(4.19)

with the entries \( \beta_{jk} \) independent of \( t \).

**Corollary 4.11.** Let \( \mathbf{B} = (\beta_{jk}) \) be an irreducible non-negative \((m \times m)\) matrix. Let the functions \( \alpha_j(t) \) be positive, bounded, and bounded away from zero. Then the following hold:

(a) If the spectral radius of \( \mathbf{B} \) is \( \leq 1 \), then \( x(t) \to 0 \) for \( t \to \infty \).
(b) If the spectral radius of $B$ is $> 1$, then there exists a unique equilibrium solution

$$y_j = (1 - y_j) \sum_{k=1}^{m} \theta_{jk} y_k, \quad j = 1, ..., m$$

in $(0, 1)^m$ and any solution $x(t)$ to (4.1) which starts in $[0, \infty)^m \setminus \{0\}$ converges towards $y$ for $t \to \infty$.

Proof. (a) Assume that $\bar{x} = \lim_{t \to \infty} \sup x(t) \neq 0$. It follows that $\bar{x}_j \geq \varepsilon > 0$, $j = 1, ..., m$. As the right-hand side of (4.19) is quasimonotone, we have $0 \leq (1 - \varepsilon) B \bar{x} - \bar{x}$ in contradiction to spectral radius of $B \leq 1$.

(b) The existence of $y$ follows from a result in [15]. Also see [11]. Now (b) is an immediate consequence of Theorem 4.10.

5. AN SIS-HOST-VECTOR-DISEASE

Here we study the asymptotic proportional stability and conditional equality of solutions $x$ to the delay-differential equation

$$x'(t) = (1 - x(t)) \beta(t) x(t - 1) - \alpha(t) x(t); \quad t > 0,$$

$$x(t) = x_0(t), \quad -1 \leq t \leq 0. \quad (5.1)$$

Here $x_0$ is a given continuous non-negative function on $[-1, 0]$.

This equation describes the spread of an infectious disease in a population which splits the individuals into susceptibles and infectives only and is transmitted by vectors. See [3, 6] or [18] for a more thorough explanation of the model and for further references.

The existence, uniqueness and continuous dependence on data of solutions to (5.1) follow from standard arguments. See [10] or [18].

ASSUMPTIONS

5.1. Let $\alpha(t), \beta(t)$ be continuous non-negative functions on $[0, \infty)$ with the following additional properties:

(a) $0 < \beta(t) \leq \text{const} \alpha(t)$ for $t \geq 0$.

(b) $\int_0^\infty \alpha(t) \, dt = \infty$.

(c) The elements $\int_0^n \alpha(t) \, dt$ and $\int_0^n \beta(t) \, dt$, $n \in \mathbb{N}$, form bounded sequences.

(d) The function $\int_0^t \beta(s) \, ds$ of $t \geq 0$ is bounded away from zero.

5.1 Preliminaries

As usual we associate a flow on the cone $X_+$ of non-negative functions in $X = C[-1, 0]$ with the solutions to (5.1). We recall the following useful
notation: If \( x \) is a continuous function which is at least defined on some interval \([t-1, t]\) then the definition
\[
x_r(s) = x(t + s), \quad -1 \leq s \leq 0
\] (5.2)

provides an element \( x_r \in X \). Conversely any element \( y \in X \) generates an element \( x \in C[t-1, t] \) with \( x_r = y \). We define a sublinear manifold \( Y \) in \( X_+ \setminus \{0\} \) by
\[
Y = \{ y \in X; 0 < y(s) < 1 - \varepsilon, -1 \leq s \leq 0 \} \tag{5.3}
\]
for some \( \varepsilon \in (0, 1) \) which is chosen according to the following lemma.

**Lemma 5.2.** There is some \( 0 < \varepsilon < 1 \) such that \( Y \) is invariant and attractive for the flow generated by (5.1), i.e.:

- If \( x \) solves (5.1) for \( t > s \) and \( x_s \in Y \), then \( x_r \in Y \) for \( t \geq s \).
- If \( x_s \in X_+ \setminus \{0\} \), then \( x_r \in Y \) for \( t > s \) being sufficiently large.

**Proof.** The following four statements easily follow from Assumptions 5.1:

(i) If \( x \) solves (5.1) for \( t > s \geq 0 \) and \( x(s) > 0 \), then \( x(t) > 0 \) for \( t \geq s \).
(ii) If \( x \) solves (5.1) for \( t \geq s \) and \( x(s) \leq 1 \), then \( x(t) \leq 1 \) for \( t \geq s \).
(iii) If \( x \) solves (5.1) for \( t > s \), then \( x(t) \leq 1 \) for some \( t > s \).
(iv) If \( x \) solves (5.1) for \( t > s \) and \( x_s \in X_+ \setminus \{0\} \), then \( x(t) > 0 \) for some \( t > s \).

These four statements together allow us, for proving the lemma, to restrict our attention to solutions \( x \) to (5.1) for \( t > s \) with \( 0 < x(r) \leq 1 \) for \( s - 1 \leq r \leq s \). By Assumption 5.1(a) we have
\[
\dot{x} \leq \alpha(t) \{ c(1 - x) - x \}, \quad t > s
\]
for some suitable constant \( c > 1 \). If \( x(t) > (\delta + c)/(1 + c) \),
\[
\dot{x}(t) < -\delta \alpha(t).
\]
So any \( \varepsilon \) with \( 1 > 1 - \varepsilon > c(1 + c) \) will work.

The flow of solutions to (5.1) is completely described by the operators \( A_n, n \in \mathbb{N}, \) on \( X_+ \): Take \( y_0 \in X_+ \), solve (5.1) for \( t > n \) with \( x_n = y_0 \) and set
\[
A_n y_0 = x_{n+1}. \tag{5.4}
\]

One easily checks the following relation by induction.
LEMMA 5.3. Let $x$ be a solution to (5.1) for $t > 0$ with $x_0 \in X_+$. Then

$$x_{n+1} = A_n x_n.$$ 

Comparison arguments similar to those in Lemma 4.1 provide the following properties of the operators $A_n$, $n \in \mathbb{N}$.

LEMMA 5.4. The operators $A_n$, $n \in \mathbb{N}$, are order preserving and sublinear on $X_+$.

5.2. Asymptotic Proportionality

Obviously two solutions $x, y$ to (5.1) are asymptotically proportional, i.e.,

$$x(t)/y(t) \to \text{const} > 0 \quad \text{for} \quad t \to \infty,$$

iff the sequences $(x_n)$, $(y_n)$ in $X_+$ are asymptotically proportional. Recall Definition 3.1. By Lemma 5.2 it is sufficient to consider sequences with elements in the sublinear manifold $Y$ in (5.3) with $\varepsilon \in (0, 1)$ chosen accordingly. It follows from Lemmas 5.2 and 5.4 that the order preserving sublinear operators $A_n$, $n \in \mathbb{N}$, map $Y$ into itself. We want to apply Theorem 3.5 and claim that the operators $(A_{n+1}, A_n)_{n \in \mathbb{N}}$ form a uniformly monotone and uniformly sublinear family of operators on $Y$. To this end we choose

$$w_n(s) = w(s) = 1 \quad \text{for} \quad -1 \leq s \leq 0. \quad (5.6)$$

LEMMA 5.5. The operators $A_{n+1} A_n$, $n \in \mathbb{N}$, are uniformly positive on $Y$.

Proof. Consider a solution $x$ to (5.1) for $t > n$ with $x_n \in Y$.

As

$$\dot{x}(t) \leq \beta(t) x(t-1), \quad n \leq t \leq n + 2, \quad (5.7)$$

we have, for $n \leq t \leq n + 2$, that

$$x(t) \leq x(n) + \int_n^t \beta(s) x(s-1) \, ds.$$ 

Hence, for $n + 1 \leq t \leq n + 2$,

$$x(t) \leq x(n) + \int_n^{n+1} \beta(s) x(s-1) \, ds + \int_{n+1}^t \beta(s) x(n) \, ds$$

$$+ \int_{n+1}^t \beta(s) \left( \int_n^{r-1} \beta(r) x(r-1) \, dr \right) ds$$

$$\leq \left\{ 1 + \int_{n+1}^{n+2} \beta(s) \, ds \right\} \cdot \left\{ x(n) + \int_n^{n+1} \beta(r) x(r-1) \, dr \right\}.$$
On the other hand, by Lemma 5.2,

\[ i(t) \geq e^{\beta(t)} x(t) - \alpha(t) x(t). \]  

So, for \( n + 1 \leq t \leq n + 2 \),

\[ x(t) \geq \exp \left( - \int_{n}^{n+2} \alpha(\tau) d\tau \right) \cdot \left\{ x(n) + \varepsilon \int_{n}^{n+1} \beta(r) x(r-1) dr \right\}. \]

Hence

\[ \psi(x_{n+2}; w) \leq \frac{1}{\varepsilon} \left\{ 1 + \int_{n+1}^{n+2} \beta(s) ds \right\} \cdot \exp \left( \int_{n}^{n+2} \alpha(\tau) d\tau \right). \]

**Lemma 5.6.** The operators \( A_{n+1} A_{n} \), \( n \in \mathbb{N} \), are uniformly monotone.

**Proof.** Consider two solutions \( x \) and \( y \) to (5.1) for \( t > n \) with prescribed data \( x_{n}, y_{n} \in Y, x_{n} \geq y_{n} \). Set

\[ z = x - y \geq 0. \]  

Recall that the operators \( A_{n} \) are monotone. We have

\[ \dot{z}(t) = (1 - x(t)) \beta(t) z(t - 1) - \alpha(t) z(t). \]

Thus

\[ \dot{z} \leq \beta(t) z(t - 1) \]

and

\[ \dot{z} \geq e^{\beta(t)} z(t - 1) - \left\{ \alpha(t) + \beta(t) \right\} z(t), \]

for \( n \leq t \leq n + 2 \). So we essentially have the same situation as in Lemma 5.5. See (5.7) and (5.8). Hence

\[ \psi(x_{n+2} - y_{n+2}; w) \leq \frac{1}{\alpha} \left\{ 1 + \int_{n+1}^{n+2} \beta(s) ds \right\} \cdot \exp \left( \int_{n}^{n+2} \left[ \alpha(s) + \beta(s) \right] ds \right). \]

Before we can turn to the uniform sublinearity of \( A_{n+1} A_{n} \), we need the following auxiliary result.
LEMMA 5.7. Let \( x(t) \) be a solution to (5.1) for \( t > n \) with \( x \) given on \([n - 1, n]\) such that \( x_n \in Y \). Then

\[
\frac{\sup x([n-1, n+2])}{\inf x([n-1, n+2])} \leq \text{const } \psi(x_n; w)
\]

without the constant depending on \( n \).

Proof. We see from the proof of Lemma 5.5 that

\[
\sup x([n-1, n+2]) \leq \left\{1 + \int_{n+1}^{n+2} \beta(s) \, ds\right\} \cdot \left\{1 + \int_n^{n+1} \beta(s) \, ds\right\} [x_n/w]_*.
\]

Further

\[
\inf x([n-1, n+2]) \geq \exp \left(-\int_{n-1}^{n+2} \alpha(\tau) \, d\tau\right) [x_n/w]_*.
\]

LEMMA 5.8. The operators \( A_{n+1} A_n \), \( n \in \mathbb{N} \), are uniformly sublinear.

Proof. Let \( x \) be a solution to (5.1) for \( t > n \) with prescribed values on \([n - 1, n]\) such that \( x_n \in Y \). Let \( y \) be the solution to (5.1) for \( t > n \) with

\[
y(t) = \gamma x(t); \quad n - 1 \leq t \leq n,
\]

with \( 0 < \gamma < 1 \). Let

\[
z = y - \gamma x \geq 0. \tag{5.10}
\]

Recall that the operators \( A_n \) are sublinear. Then

\[
\dot{z}(t) = (1 - y(t)) \beta(t) z(t - 1)
- \gamma(t) \beta(t) \gamma x(t - 1) - \alpha(t) \gamma(t)
+ \gamma(1 - \gamma) x(t) \beta(t) x(t - 1)
\]

for \( t > n \), and

\[
z(t) = 0 \quad \text{for} \quad n - 1 \leq t \leq n. \tag{5.11}
\]

Hence

\[
\dot{z}(t) \leq \beta(t) z(t - 1) + \gamma(1 - \gamma) x(t) \beta(t) x(t - 1) \tag{5.12}
\]

and

\[
\dot{z}(t) \geq \gamma(1 - \gamma) x(t) \beta(t) x(t - 1)
- \{\beta(t) + \alpha(t)\} z(t) \tag{5.13}
\]
for $t > n$. By (5.11) this implies
\[
z(t) \leq \int_{n+1}^{t} \beta(s) z(s-1) \, ds \\
+ \int_{n}^{t} \gamma(1-\gamma) x(s) \beta(s) x(s-1) \, ds
\]
for $t \geq n$; hence, again by (5.11), for $t \geq n + 1$,
\[
z(t) \leq \int_{n}^{t} \gamma(1-\gamma) x(s) \beta(s) x(s-1) \, ds \\
+ \int_{n+1}^{t} \beta(s) \left( \int_{n}^{s-1} \gamma(1-\gamma) x(r) \beta(r) \cdot x(r-1) \, dr \right) \, ds.
\]

For $n + 1 \leq t \leq n + 2$ we now have
\[
z(t) \leq \left[ 1 + \int_{n+1}^{n+2} \beta(s) \, ds \right] \cdot \int_{n}^{n+2} \gamma(1-\gamma) x(s) \beta(s) \cdot x(s-1) \, ds. \tag{5.14}
\]

On the other hand, by (5.13),
\[
z(t) \geq \int_{n}^{n+1} \gamma(1-\gamma) x(s) \beta(s) x(s-1) \, ds \\
\cdot \exp \left( -\int_{n}^{n+2} \{ x(s) + \beta(s) \} \, ds \right), \tag{5.15}
\]
for $n + 1 \leq t \leq n + 2$.

By Lemma 5.7 and (5.14), (5.15) we find
\[
\psi(z_{n+2}; w) \leq \text{const} \left[ 1 + \int_{n+1}^{n+2} \beta(s) \, ds \right] \\
\cdot \exp \left( \int_{n}^{n+2} \{ x(s) + \beta(s) \} \, ds \right) \\
\cdot \left\{ 1 + \int_{n+1}^{n+2} \beta(s) \, ds \right\} (\psi(x_{n}; w))^2.
\]

Hence
\[
\psi(A_{n+1}, A_{n} x_{n} - \gamma A_{n+1} A_{n} x_{n}; w) \leq \text{const}(\psi(x_{n}; w))^2
\]
by Assumption 5.1.
ASYMPTOTIC PROPORTIONALITY

From Theorem 3.5 we now derive the following result.

**Theorem 5.9.** Let Assumptions 5.1 be satisfied. Then any two solutions \( x, y \) to (5.1) with initial data \( x_0, y_0 \in X_+ \setminus \{0\}, \ X = C[-1,0], \) are asymptotically proportional, i.e.,

\[
x(t)/y(t) \to \text{const} > 0
\]

for \( t \to \infty. \)

5.3. Conditional Asymptotic Equality

Let \( x, y \) be two solutions to (5.1) which are asymptotically proportional, but not asymptotically equal. In a manner similar to that in Section 4.3 we conclude that

\[
\int_t^{t+1} y(s) \beta(s) y(s-1) \, ds \to 0
\]

for \( t \to \infty \). As

\[
y(s) \geq y(r) \exp \left( - \int_r^s \alpha(\tau) \, d\tau \right)
\]

for \( s \geq r \), we conclude that

\[
\int_t^{t+1} \beta(s) ds (y(t+1)) \exp \left( -2 \int_t^{t+1} \alpha(s) \, ds \right)
\to 0 \quad \text{for } t \to \infty.
\]

From Assumptions 5.1(c, d) we derive the following result.

**Theorem 5.10.** Let Assumptions 5.1 be satisfied. If there exists at least one solution to (5.1) which does not converge to zero for \( t \to \infty \), then

\[
x(t)/y(t) \to 1, \quad t \to \infty
\]

for any pair of solutions \( x, y \) to (5.1) with initial data \( x_0, y_0 \in X_+ \setminus \{0\} \).

6. Time-Heterogeneous Logistic Growth with Diffusion

In this section we consider a diffusing population on a bounded habitat \( \Omega \) which locally satisfies logistic growth:

\[
\partial_t u(t, x) - \Delta_x u(t, x) = \alpha(t, x) u(t, x) \{ \beta(t, x) - u(t, x) \}
\]

for \( t > 0, \ x \in \Omega, \)

\[
\partial_t u(t, \cdot) = 0 \quad \text{for } t > 0, \ x \in \partial \Omega.
\]

(6.1)

\( u(t, \cdot) \) denotes the spatial density of individuals at time \( t. \)
In stating our assumptions we do not aim at maximum generality but at giving a flavour of how the theory in Section 3 can be applied.

**Assumptions 6.1.** (a) \( \alpha \) and \( \beta \) are continuously differentiable in \( t \) and \( x \) with bounded derivatives. \( \alpha \) and \( \beta \) are bounded, positive and even bounded away from zero on \([0, \infty) \times \Omega\).

(b) \( \Omega \) is a bounded open domain in \( \mathbb{R}^m \) with sufficiently smooth boundary \( \partial \Omega \) (let us say \( C^2 \)).

For function \( u, v \in C(\Omega) \) we introduce the following notation:

\[
\begin{align*}
  u > v & \quad \text{iff} \quad u \geq v, u \neq v \\
  u \geq v & \quad \text{iff} \quad u(x) > v(x) \quad \text{for all} \quad x \in \Omega.
\end{align*}
\]

Comparison and maximum principles easily provide the following properties of solutions to (6.1):

**Lemma 6.2.** If \( u \) is a solution to (6.1) for \( t > s \) and \( u(s, \cdot) \) is continuous on \( \Omega \) and \( u(s, \cdot) > 0 \) for \( t > s \). Further there exists a constant \( c_1 > 0 \) with the following properties: For any \( u(s, \cdot) > 0, u(t, x) < c_1 \) for \( x \in \Omega \) and \( t > s \) sufficiently large. Moreover \( u(t, x) < c_1 \) for \( x \in \Omega, t > s \), if \( u(s, x) < c_1 \) for \( x \in \Omega \).

**Proof.** The first statement is standard. In order to prove the second, we construct a supersolution \( v \) in the form

\[
v(t, x) = \tilde{c}_1 + c_2 e^{-\epsilon(t-s)}.
\]

Choosing \( \tilde{c}_1 > \beta(t, x) + 1 \) for \( t \geq 0, x \in \Omega \) and \( c_2 + \tilde{c}_1 > u(s, x) \) for \( x \in \Omega \) and \( \epsilon \) small, we find

\[
(\partial_t - \Delta_x) v(t, x) > \alpha(t, x) v(t, x) \left\{ \beta(t, x) - v(t, x) \right\} \quad \text{for} \quad t > s, x \in \Omega,
\]

\[
v(t, x) > 0 = u(t, x) \quad \text{for} \quad t > 0, x \in \partial \Omega,
\]

\[
v(s, x) > u(s, x) \quad \text{for} \quad x \in \Omega.
\]

A standard comparison argument implies

\[
u(t, x) < v(t, x) \quad \text{for} \quad t > s, x \in \Omega.
\]

Let \( G \) be the Green's function associated with the problem

\[
(\partial_t - \Delta_x) u(t, x) = 0 \quad \text{for} \quad t > 0, x \in \Omega, \quad u(t, x) = 0 \quad \text{for} \quad t > 0, x \in \partial \Omega \] and \( w \) the solution of

\[
-\Delta_x w(x) = 1, \quad x \in \Omega \]

\[
w(x) = 0, \quad x \in \partial \Omega.
\]

See [7, 3.7, 3.8; 14, IV.16].
Essentially the same proof as in [13, 7.2.2, Lemma 7.2] provides the following result. See [7, 2.5, Theorem 14].

**Lemma 6.3.** Let $\Omega$ be an open non-empty subset of $\Omega$. Then, for every $t > 0$, there exist $c > \epsilon > 0$ such that

$$\epsilon w(x) \leq \int_{\Omega} G(t, x, y) \, dy = \int_{\Omega} G(t, y, x) \, dy \leq cw(x)$$

for $x \in \Omega$.

**Lemma 6.4.** (a) For any $t > 0$ there exist $\epsilon, c > 0$ such that

$$\epsilon w(x) w(y) \leq G(t, x, y) \leq cw(x) w(y)$$

for all $x \in \Omega$.

(b) $\int_{0}^{t} \int_{\Omega} G(s, x, y) \, ds \, dy \leq w(x)$ for $x \in \Omega$, $t > 0$.

(c) For any open non-empty subset $\Omega'$ of $\Omega$ there exists some $\delta > 0$ such that

$$\int_{\Omega'}^{1} \int_{\Omega} G(s, x, y) \, ds \, dy \geq \delta w(x)$$

for $x \in \Omega$.

**Proof:** (a) As $A$ generates a semigroup, we have

$$G(t, x, y) = \int_{\Omega} \int_{\Omega} G(t/3, x, z) \cdot G(t/3, z, y) \, dz \, dz'.$$

Let $\Omega \subset \Omega$. Then $G(t/3, u, v) \geq \delta > 0$ for $u, v \in \Omega$ and

$$G(t, x, y) \geq \delta \int_{\Omega} G(t/3, x, z) \, dz \int_{\Omega} G(t/3, \bar{z}, y) \, d\bar{z} \geq \epsilon w(x) w(y)$$

by Lemma 6.3. As $G(t/3, z, \bar{z}) \leq c$ for $z, \bar{z} \in \Omega$, the other estimate follows analogously.

(b) $-A \int_{0}^{t} \int_{\Omega} G(s, x, y) \, ds \, dy = 1 - \int_{\Omega} G(t, x, y) \, dy < 1$ for $x \in \Omega$ and $\int_{0}^{1} \int_{\Omega} G(s, x, y) \, ds \, dy = 0$ for $x \in \partial \Omega$ together with (6.2) and a maximum principle provide the statement.

(c) $\int_{1/2}^{1/2} \int_{\Omega} G(s, x, y) \, ds \, dy = \int_{0}^{1/2} \int_{\Omega} (\int_{\Omega} G(s, x, z) \, dz) \, ds \, dy \geq \int_{\Omega} G(\frac{1}{2}, x, z) \, dz \inf_{x \in \Omega} \int_{1/2}^{1/2} \int_{\Omega} G(s, z, y) \, ds \, dy$ with some $\Omega \subset \Omega$. Lemma 6.3 now implies the assertion.
Let $U(t, s)$ be the solution operator on the cone $X_+, X = C(\Omega)$ which maps an element $y \in X_+$ to $x(t)$ with $x$ the solution to (6.1) for $t > s$, $x(s) = y$. Standard comparison principles imply the following:

**Lemma 6.5.** The operators $U(t, s)$, $t \geq s \geq 0$, are order preserving and sublinear on $X_+$.

### 6.2. Asymptotic Proportionality

As in Section 4.2 we can conclude from Lemma 6.5 that the asymptotic proportionality of two solutions $u, v$ to (6.1) is equivalent to the asymptotic proportionality of the sequences $(u(n, \cdot))$, $(v(n, \cdot))$. Obviously

$$u_{n+1} = A_n u_n$$

with

$$u_n = u(n, \cdot), \quad A_n = U(n+1, n).$$

By Lemma 6.2 we can restrict our consideration to sequences $(u_n)$ in the sublinear manifold $Y$ of $X_+$,

$$Y = \{ u \in X_+, 0 < u(x) < c_1 \text{ for } x \in \Omega \}$$

with $c_1$ chosen accordingly.

**Lemma 6.6.** The operators $A_n$, $n \in \mathbb{N}$, are uniformly positive on $Y$.

**Proof.** Let $u$ be a solution to (6.1) for $t > n$, $u_n \in Y$. Then $u(t, \cdot) \in Y$ for $t > n$ by Lemma 6.2 and

$$-cu \leq (\partial_t - A_x) u \leq cu$$

for some constant $c > 0$ not depending on $n$. Hence

$$e^{ct} u(n + t, x) \geq \int_{\Omega} G(t, x, y) u(n, y) \, dy$$

$$\geq e^{-ct} u(n + t, x)$$

for $t > n$, $x \in \Omega$. By Lemma 6.4 we find $\delta, \bar{c} > 0$ such that

$$\delta u(n + 1, x) \leq w(x) \int_{\Omega} u(n, y) w(y) \, dy$$

$$\leq \bar{c} u(n + 1, x),$$

(6.7)
thus

\[ \psi(u_{n+1}, w) \leq \frac{c}{\delta}. \]

**Lemma 6.7.** The operators \( A_n, n \in \mathbb{N}, \) are uniformly monotone on \( Y. \)

**Proof.** Let \( u, v \) be solutions to (6.1) for \( t > n, \) \( u(n, \cdot), v(n, \cdot) \in Y, \) \( u(n, \cdot) \geq v(n, \cdot). \) By Lemmas 6.2 and 6.5, \( u(t, \cdot), v(t, \cdot) \in Y \) and \( u(t, \cdot) \geq v(t, \cdot) \) for \( t \geq n. \) Set

\[ z = u - v \geq 0. \]

Then

\[ (\delta_t - A_x) z = \alpha(t, x) (\beta(t, x) - u) - \alpha(t, x) v. \]

This is essentially the same situation as in Lemma 6.6, (6.6), and the statement follows by the same arguments.

**Lemma 6.8.** The operators \( A_n, n \in \mathbb{N}, \) are uniformly sublinear on \( Y. \)

**Proof.** Let \( u \) be a solution to (6.1) for \( t > n \) with \( u(n, \cdot) \in Y. \) Then \( u(t, \cdot) \in Y \) for \( t \geq n. \) Let \( v \) be the solution to (6.1) with

\[ v(n, \cdot) = \gamma u(n, \cdot), \]

\( 0 < \gamma < 1. \) Set

\[ z = v - \gamma u \geq 0. \]

(6.8)

See Lemma 6.5. So

\[ z(n, \cdot) = 0 \]

and

\[ (\delta_t - A_x) z = \alpha(t, x) \gamma (1 - \gamma) u^2 + \alpha(t, x) \gamma (\beta(t, x) - v - \gamma u), \]

Hence

\[ -cz + \delta \gamma (1 - \gamma) u^2 \leq (\delta_t - A_x) z \leq cz + c \gamma (1 - \gamma) u^2 \]

for appropriate constants \( 0 < \delta < c. \) This implies

\[ ez(n + 1, x) \leq \int_{n}^{n+1} \int_{\Omega} \gamma (1 - \gamma) u^2(s, y) G(n + 1 - s, x, y) \, ds \, dy \]

\[ \leq \tilde{c} z(n + 1, x). \]
Set
\[ u_\bullet = \inf\{ [u(s, \cdot)/w]_\bullet; n \leq s \leq n + 1 \} \quad (6.9) \]
and let \( u^* \) be the corresponding supremum. Then
\[
\varepsilon z(n + 1, x)(u^*)^{-2} \leq \gamma (1 - \gamma) \int_0^1 \int_\Omega (w(y))^2 \cdot G(s, x, y) \, ds \, dy
\leq \varepsilon z(n + 1, x)(u_\bullet)^{-2}.
\]
It follows from Lemma 6.4(b,c) that
\[
\psi((n + 1, \cdot); w) \leq \text{const}(u^*/u_\bullet)^2.
\]
The formulas (6.7), (6.8), (6.9) now imply
\[
\psi(A_n \gamma u_n - \gamma A_n u_n; w) \leq \text{const}
\]
with the constant not depending on \( n, u_n = u(n, \cdot) \). This implies the assertion.

From Theorem 3.3 we now obtain the following result.

**Theorem 6.9.** Let Assumptions 6.1 be satisfied. Then any two solutions \( u, v \) to (6.1) for \( t > 0 \) with initial data in \( X_+ \setminus \{0\}, X_+ = C(\overline{\Omega}) \), are asymptotically proportional, i.e.,
\[
u(t, x) / v(t, x) \to \text{const} > 0
\]
for \( t \to \infty \), uniformly in \( x \in \Omega \), without the constant depending on \( x \in \Omega \).

6.3. Conditional Asymptotic Equality

Using the Green's function we can integrate (6.1) and obtain
\[
u(t + 2, x) = \int_\Omega G(t, x, y) u(t, y) \, dy
\]
\[
\quad + \int_t^{t + 2} \int_\Omega \alpha(s, y) u(s, y) \beta(s, y)
\]
\[
- u(s, y) \} G(t + 2 - s, x, y) \, ds \, dy
\]
\[
(6.10)
\]
for \( t \geq 0, x \in \Omega \).

Let \( u, v \) be two solutions to (6.10) which are asymptotically proportional, but not asymptotically equal. Then, by Assumption 6.1(a)
\[
\int_t^{t + 2} \int_\Omega (v(s, y))^2 G(t + 2 - s, x, y) \, ds \, dy \to 0
\]
for \( t \to \infty \).
Let us suppose that $v(t, x)$ does not converge to zero for $t \to \infty$, uniformly in $x \in \Omega$. Then
\begin{equation}
    v(t_n, x_n) \geq \varepsilon > 0 \quad \text{for } n \in \mathbb{N}
    \tag{6.12}
\end{equation}
with suitably chosen $\varepsilon > 0$, $x_n \in \Omega$, $t_n \to \infty$ ($n \to \infty$). Hence, as we may assume that
\[-cv \leq (\partial_x - \Delta_x)v \leq cv\]
by Lemma 6.2 and (6.1), we have
\begin{equation}
    \varepsilon \leq v(t_n, x_n) \leq \text{const} \int_{\Omega} G(1, x_n, y) \, v(t_n - 1, y) \, dy. \tag{6.13}
\end{equation}
We can assume that, after a subsequence is chosen, $x_n \to x_0 \in \Omega$. By (6.13), $x_0 \in \Omega$. The properties of the Green’s function also allow us to choose some $\varepsilon', \delta > 0$ such that
\[\varepsilon' \leq \int_{\Omega} G(1, x, y) \, v(t_n - 1, y) \, dy\]
for $|x - x_0| < \delta$, $n \in \mathbb{N}$. Now
\[v(t_n + 1 + s, x) = \hat{\varepsilon} \int_{\Omega} \int_{\Omega} G(1 + s, x, z) G(1, z, y) \, v(t_n - 1, y) \, dz \, dy \geq \bar{\varepsilon} > 0\]
for $x$ in a neighbourhood $\Omega$ of $x_0$, $0 \leq s \leq 1$, $n \in \mathbb{N}$. Thus
\[\int_{t_n + 1}^{t_n + 2} \int_{\Omega} \left(v(r, x) \right)^2 G(t_n + 3 - r, x_0, x) \, dr \, dx \geq \int_1^2 (\bar{\varepsilon})^2 \left(\int_{\Omega} G(r, x_0, x) \, dx\right) \, dr \geq \text{const} > 0\]
for all $n \in \mathbb{N}$, in contradiction to (6.11).

So we have proven the following result:

**Theorem 6.10.** Let Assumptions 6.1 be satisfied. If there exists at least one solution to (6.1) which does not converge to zero for $t \to \infty$ uniformly on $\Omega$, then
\[u(t, x) \in \Omega, \text{ for any two solutions } u, v \text{ to (6.1) with initial data in } X_+ \setminus \{0\}, X = C(\overline{\Omega}).\]
Note added in proof. After having submitted this paper the author learned of the work by Fujimoto and Krause [20]. (See also the literature cited there which supplements the literature given in this article.) They prove asymptotic proportionality for time-heterogeneous difference equations with sublinear monotone operators which are ray-preserving. They use Hilbert's projective metric. The operators arising in our applications do not seem to be ray-preserving.

REFERENCES