A Class of Stochastic Nonlinear Integral Equations on $L^p$ Spaces and its Application to Optimal Control

N. U. Ahmed

Department of Electrical Engineering, University of Ottawa, Ottawa 2, Canada

In this paper certain useful properties of a class of stochastic nonlinear integral equations on $L^p(p \geq 1)$ spaces are considered. Sufficient conditions for boundedness, continuity and compactness in probability on $L^p$ spaces of these random operators are presented. By use of these properties and certain fundamental results of (Hans 1, 2 1957) stochastic versions of Schauder and Banach fixed point theorems are proved. An application of these results to a class of stochastic optimal control problems is briefly considered. Extension of the methods to Orlicz spaces are also briefly indicated.

1. INTRODUCTION

In certain problems of optimal control (Ahmed, 1968) of stochastic systems and in stochastic boundary value problems (Bharucha-Reid, 1960a, 1959, 1960b, 1964) in Physics and engineering one is faced with the study of the properties of stochastic integral equations on a concrete Banach space. After the abstract theory of stochastic operators on abstract Banach spaces were extensively developed by Hans (Hans, 1957a, 1957b) and others, Bharucha-Reid studied in a series of interesting papers stochastic linear Fredholm integral equations on Orlicz spaces (Bharucha-Reid, 1960a) and other concrete Banach spaces like $L^2$ space. Existence of resolvents of linear Fredholm integral equations on abstract Banach spaces, and their properties of measurability were studied in (Bharucha-Reid, 1959). Stochastic nonlinear integral equations of Uryson type were also studied by Bharucha-Reid (Bharucha-Reid, 1964). Stochastic differential equations and many associated control problems were studied by Ito, Wonham, Kozin, Kailath and others. For this an excellent bibliography is given in (Kailath et al., 1968).

In this paper we propose to study random solutions in $L^p (p \geq 1)$ spaces of stochastic integral equations of the following class:

$$u = v + \lambda A_s u = B_s u$$

(1.1)
where

\[(A_\sigma u)(t) = \sum_{n=1}^{\infty} \int_{I^n} \cdots \int_{I^n} L_n(\sigma | t; \tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(\sigma, \tau_i) \, d\tau_1 \cdots d\tau_n\]

\[t \in I = [t_0, T]\]

and \(\sigma\) is an element of a probability (measure) space \((\Sigma, S, \mu)\). Here \(\Sigma\) is an abstract set of parameter values and \(S\) is the Borel field of sets in \(\Sigma\) and \(\mu\) is an appropriate probability measure on \(S\) with \(\mu(\Sigma) = 1\). We shall assume throughout that the probability measure \(\mu\) is complete, that is, a subset of a set of \(\mu\) measure zero is also an element of the \(\sigma\) algebra \(S\).

The integral equation 1.1 arises in the study (Ahmed, 1968) of optimal control of a class of stochastic nonlinear integral operators of Volterra type defined by \(y = F_\sigma u\) where

\[(F_\sigma u)(t) = \sum_{n=1}^{\infty} \int_{t_0}^{t} \cdots \int_{t_0}^{t} K_n(\sigma | t; \tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(\tau_i) \, d\tau_1 \cdots d\tau_n\]  

\[t \in I \text{ and } \sigma \in \Sigma.\]  

This is discussed by the author elsewhere and in Section 3 of this paper. In Section 2 we present certain proofs of some basic properties, for example, boundedness, continuity and compactness in probability of the random operator \(\{A_\sigma\}\) in \(L^p (p \geq 1)\) spaces. The question of existence and uniqueness of a solution of eq. 1.1 is considered after "random solution" is defined.

Application to a class of optimal control problems is briefly considered in Section 3.

It will be assumed throughout the paper that for each \(n \in J^+\) (the set of positive integers) the components \(\{A_n(\sigma), n \geq 1\}\) of the operator \(A_\sigma\) are measurable transforms in the sense of Hans: (1957a) \((X, \mathcal{F})\) is a measurable space, \(X\) is a separable Banach space and \(\mathcal{F}\) is the \(\sigma\) algebra of Borel subsets of \(X\); \((\Sigma, S, \mu)\) is a probability measure space; \(A_n : \Sigma \times X \rightarrow X\) is called a random (or measurable) transform if the inclusion

\[
\{\{\sigma : A_n(\sigma)u \in D\} : u \in T, D \in \mathcal{F}\} \subset S
\]

holds where \(T\) is an appropriate subset of \(X\).

2. RANDOM OPERATOR EQUATIONS IN \(L^p\) \((p \geq 1)\) SPACES

The following basic assumption will be made about the kernels of the operator \(A_\sigma \cdot H_1\): For each \(n \in J^+\) (the set of positive integers) and for almost all \(\sigma \in \Sigma, L_n(\sigma | t; \tau_1, \cdots, \tau_n) \in L^p(I^n)\) with respect to the
variables \( (\tau_1, \tau_2, \tau_n) \in I^n \) and the function \( \mathcal{L}_n(\sigma | t) \) defined by

\[
\mathcal{L}_n(\sigma | t) = \left( \int_{I^n} \left| L_n(\sigma | t; \tau_1 \cdots \tau_n) \right|^q \, d\tau_1 \cdots d\tau_n \right)^{1/q}
\]

(2.1)

belongs to \( L^p(p \geq 1, p^{-1} + q^{-1} = 1) \) for almost all \( \sigma \in \Sigma \) and that the function \( \| \mathcal{L}_n \|_p \) is a \( \mu \) measurable function of \( \sigma \).

**Lemma 1.** For almost all \( \sigma \in \Sigma \) the operator \( A_n(\sigma): L^p \to L^p \); where \( A_n(\sigma) \) is defined by

\[
(A_n(\sigma)u)(t) = \int_{I^n} \cdots \int_{I^n} L_n(\sigma | t, \tau_1 \cdots \tau_n) \prod_{i=1}^n u(\tau_i) \, d\tau_i, \quad t \in I.
\]

(2.2)

That is, \( A_n: \Sigma \times L^p \to L^p \) \( \mu \)-almost everywhere.

**Proof.** The proof follows from Hölder's inequality and the basic assumption about the kernels \( \{L_n\} \).

**Lemma 2.** Let for each \( n \in J \) \( \| \mathcal{L}_n \|_p \) be a measurable function of \( \sigma \in \Sigma \) and let \( f_n(\sigma) = \sqrt[p]{\| \mathcal{L}_n \|_p} \). Then \( \lim \) \( f_n(\sigma) \) is also a measurable function.

**Proof.** By hypothesis, for an arbitrary \( a \in \mathbb{R}^+ \) (the set of nonnegative real numbers) \( \{ \sigma : f_k(\sigma) < a \} \) is a measurable set. Therefore

\[
\{ \sigma : \lim f_n(\sigma) < a \} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{ \sigma : f_k(\sigma) < a \}
\]

is also a measurable set being the countable intersection of countable union of measurable sets. Hence \( \lim f_n(\sigma) \) is a measurable function.

**Lemma 3.** Let for each \( r \in \mathbb{R}^+ \), \( C_r = \{ \sigma : \lim f_k(\sigma) \leq 1/r \} \). Let \( r_0 \) be the largest positive real number in the extended real number system for which \( \mu(\Sigma \setminus C_{r_0}) = 0 \). Then the stochastic operator \( A_\sigma \) is bounded and uniformly continuous with probability one on \( Q_r = \{ u \in L^p : \| u \|_p \leq r \} \) for every \( r < r_0 \).

**Proof.** It follows from Hölder and Minkowski inequalities that

\[
\| A_\sigma u \|_p \leq \sum_{n=1}^{\infty} \| \mathcal{L}_n(\sigma | t) \|_p \| u \|_p^n
\]

(2.3)

\[
\leq \sum_{n=1}^{\infty} (f_n(\sigma)r)^n \text{ for } u \in Q_r (r < r_0)
\]

and for all \( \sigma \in \Sigma \). Let \( 0 < \epsilon < (1/r) - (1/r_0) \) be chosen then since for
all $\sigma \in C_{r_0}$ \( \lim_{n \to \infty} f_n(\sigma) \leq 1/r_0 \) there exists an integer $n_0(\epsilon) < \infty$ such that $f_n(\sigma) < 1/r_0 + \epsilon$ for all $n \geq n_0$. Let $\beta = r(1/r_0 + \epsilon)$ then for all $\sigma \in C_{r_0}$ and $u \in S_r(r < r_0)$

$$\| A_{\sigma} u \|_p \leq \sum_{n=1}^{n_0-1} (f_n(\sigma)r)^n + \frac{\beta}{1 - \beta}. \quad (2.4)$$

By hypothesis $H_1$ there exists a set $\Sigma_0 \subset S$ such that $\mu(\Sigma \setminus \Sigma_0) = 0$ and $f_n(\sigma) < \infty$ for all $\sigma \in \Sigma_0$ for all $n \leq n_0 - 1$. Thus for all $\sigma \in (\Sigma_0 \cap C_{r_0})$ and $u \in S_r(r < r_0)$, $\| A_{\sigma} u \|_p < \infty$. Clearly $\mu(\Sigma_0 \setminus C_{r_0}) = 0$ and hence for all $u \in S_r, \mu(\sigma : \| A_{\sigma} u \|_p < \infty) = 1$. This proves the boundedness of the operator $A_{\sigma}$ with probability one and that

$$\{ A_{\sigma} u : \sigma \in \Sigma_0 \cap C_{r_0}, u \in S_r \} \subset L^p.$$

For continuity let $\{ u_n \} \subset S_r$ and let $u_n$ converge strongly to an element $v_0 \in S_r$. It can be easily shown that for $u, v \in S_r(r < r_0)$ and for all $\sigma \in \Sigma$ we have $\| A_{\sigma} u - A_{\sigma} v \|_p \leq \alpha_r(\sigma) \| u - v \|_p$ where $\alpha_r(\sigma)$ given by $\alpha_r(\sigma) = \sum_{n=1}^{\infty} n \| L_n(\sigma | t) \|_p r^{n-1}$ is a measurable function finite almost everywhere on $\Sigma$ for $r < r_0$. Let for each $n \in J^+ D_n = \{ \sigma : \| A_{\sigma} u_n - A_{\sigma} v_0 \|_p > 1/n \}$ and $B_n = \{ \sigma : \alpha_r(\sigma) \| u_n - v_0 \|_p > 1/n \}$. Then $D_n \subset B_n$ for all $n \in J^+$ and $B_n$ is a contracting sequence of measurable sets. Since $u_n \to v_0 \subset S_r \subset L^p$ and $\alpha_r(\sigma)$ is finite almost everywhere $\lim_{n \to \infty} \mu(D_n) = 0$ and hence $\lim_{n \to \infty} \mu(B_n) = 0$. Thus $D_n$ converges to the null set as $n$ approaches infinity. Since $v_0 \in S_r \subset L^p$ is arbitrary, the stochastic operator $A_{\sigma}$ is (uniformly) continuous with probability one on $S_r$ for every $r < r_0$. This completes the proof of the lemma.

**Lemma 4.** The boundedness of the operator $A_{\sigma}$ on $S_r(r < r_0)$ with probability one along with the hypothesis $H_1$ implies its compactness on $S_r(r < r_0)$ in probability.

**Proof.** Let us consider the $L^p$ norm of the quantity $(A_{\sigma} u)(t + h) - (A_{\sigma} u)(t)$ for $u \in S_r$. For this purpose let us define $S_m(\sigma, h)$ and $R_m(\sigma, h)$ by

$$S_m(\sigma, h) = \sum_{n=1}^{m} \| L_n(\sigma | t + h) - L_n(\sigma | t) \|_p r^n \quad (2.5)$$

and $R_m(\sigma, h) = \sum_{n=m+1}^{\infty} \| L_n(\sigma | t + h) \|_p r^n$
where

\[ \| L_n(\sigma | t + h) - L_n(\sigma | t) \|_p \]

\[ = \left( \int_I \left( \int_I \cdots \int_I | L_n(\sigma | t + h; \tau_1, \cdots \tau_n) - L_n(\sigma | t; \tau_1 \cdots \tau_n) |^q \, d\tau_1 \cdots d\tau_n \right)^{1/q} \, dt \right)^{1/p} \]  

(2.6)

and

\[ \| \hat{L}_n(\sigma | t + h) \|_p \]

\[ = \left( \int_I \left( \int_I \cdots \int_I | L_n(\sigma | t + h; \tau_1, \cdots \tau_n) |^q \, d\tau_1 \cdots d\tau_n \right)^{1/q} \, dt \right)^{1/p}. \]  

(2.7)

It follows from the above inequalities that \( \| A_\sigma u(t + h) - A_\sigma u(t) \|_p \leq S_m(\sigma, h) + R_m(\sigma, h) + R_m(\sigma, 0) \) for all \( u \in S_r(r < r_0) \). Since the operator \( A_\sigma \) is bounded on \( S_r \) with probability one (Lemma 3) for every \( \epsilon > 0 \) and \( \beta > 0 \) there exists a set \( C_0 \) such that \( \mu(C_0^c) < \beta/2 \) and \( R_m(\sigma, h) < \epsilon/4 \) for all \( \sigma \in C_0 \) whenever \( n \geq m_0(\epsilon, \beta) \) independent of \( h \). \( C_0^c \) is the complement of the set \( C_0 \). As usual \( L_n(\sigma | t + h; \tau_1 \cdots \tau_n) \) can be taken equal to zero whenever \( t + h \in I \). Further, since the Kernels \( \{L_\tau\} \) satisfy \( H_1 \), for the given \( \epsilon \) and \( \beta > 0 \) there exists (Hartman and Mikusinski 1961) a \( \delta(\epsilon, \beta, n) \) for each \( n \in \{1, 2, \cdots, m_0\} \) such that for \( |h| < \delta \)

\[ \| L_n(\sigma | t + h) - L_n(\sigma | t) \|_p \leq \epsilon/2m_0 r^n \]

for all \( \sigma \in C_1 \in S \) where \( \mu(C_1^c) < \beta/2 \).

Let \( \delta_0(\epsilon, \beta) = \min_{n \in \{1, 2, \cdots, m_0\}} \delta(\epsilon, \beta, n) \) then for every \( |h| < \delta_0(\epsilon, \beta) \), \( S_{m_0}(\sigma, h) < \epsilon/2 \) for all \( \sigma \in C_1 \). Let \( C = \{\sigma: \| A_\sigma u(t + h) - A_\sigma u(t) \|_p \leq \epsilon, |h| < \delta_0\} \). Clearly, \( C_0 \cap C_1 \subseteq C \) and \( \mu(C^c) \leq \mu(C_0^c) + \mu(C_1^c) \). Therefore \( \mu(C^c) < \beta \) for all \( |h| < \delta_0(\epsilon, \beta) \), that is, \( \mu(\sigma: \| A_\sigma u(t + h) - A_\sigma u(t) \|_p > \epsilon) < \beta \) for all \( h \) such that \( |h| < \delta_0(\epsilon, \beta) \). Thus for every \( \epsilon > 0 \lim_{|h| \to 0} \mu(\sigma: \| A_\sigma u(t + h) - A_\sigma u(t) \|_p > \epsilon) = 0 \) for all \( u \in S_r \) and hence by the previous lemma and a wellknown theorem of M. Riesz (Kantorovich, 1964) it follows that the set
\{ A_\sigma u : u \in S_r (r < r_0) \subset L^p \} \text{ is a compact subset of } L^p \text{ in probability. This completes the proof of the lemma.}

**Definition 1.** An operator \( A \) mapping a Banach space \( B_1 \) into a Banach space \( B_2 \) is said to be completely continuous on \( D \subset B_1 \) if it is continuous on \( D \) and maps every bounded subset \( D_0 \subset D \) into a compact set in \( B_2 \).

**Proposition 1.** Let the sequence of kernels \( \{ L_n \} \) corresponding to the stochastic operator \( A_\sigma, \sigma \in (\Sigma, S, \mu) \) satisfy \( H_1 \) and let for each \( n \| \hat{L}_n \| = \mu \) measurable function of \( \sigma \) with \( \lim \sqrt[n]{\hat{L}_n} \| p \leq 1/r_0 \) with probability one. Then the operator \( A_\sigma, \sigma \in (\Sigma, S, \mu) \) is completely continuous on \( S_r (r < r_0) \) in probability.

Proof. The proof follows from the above definition (suitably modified for the stochastic situation) and the Lemmas 3 and 4.

Before considering the question of existence of a solution of the stochastic integral eq. 1.1 it is important to define what is meant by a solution of a random equation.

**Definition 2.** For each fixed \( v \in L^p \) (eq. 1.1) the mapping \( u \) of \( \Sigma \times I \) to the scalars is a random solution of the random equation 1.1 if (i) \( \{ u(\sigma, \cdot) : \sigma \in (\Sigma, S, \mu) \} \subset L^p \) (ii) for each arbitrary but fixed \( \sigma \in (\Sigma \backslash \Sigma_0) \) where \( \mu(\Sigma_0) = 0 \) the sample function \( u(\sigma, \cdot) \in L^p \) is a solution of eq. 1.1 and (iii) for almost all but fixed \( t \in I u (\cdot, t) \) is \( \mu \) measurable in \( \sigma \).

Thus a random solution of the stochastic integral eq. 1.1 is a measurable generalized random variable (Hans, 1957a; Barucha-Reid, 1960a) with values in \( L^p \).

**Proposition 2.** Let the hypotheses of Proposition 1 hold and let

(i) \( \mu\{ \sigma : \| v \|_p \leq b < r_0 \} = 1 \)

(ii) \( \mu\{ \sigma : \| \sup_{u \in S_r} A_\sigma u \|_p \leq a \} = 1 \)

Then the eq. 1.1 has a random fixed point in \( S_r (r < r_0) \) provided \( \lambda < r - b/a \).

Proof. Since by Proposition 1 we have \( \mu\{ \sigma : A_\sigma \text{ completely continuous on } S_r \} = 1 \) and by hypothesis of the present proposition it follows that \( \mu\{ \sigma : \sup_{u \in S_r} B_\sigma u \|_p < r \} = 1 \) Schauder's fixed point theorem applies. This completes the proof of the proposition.

For practical application a constructive existence theorem is desirable.
Under certain conditions a random Banach fixed point theorem can be proved.

**Proposition 3.** For a fixed \( \lambda \) in the field of scalars if \( \mu(\sigma \in \Sigma: | \lambda | \alpha_r(\sigma) < 1) = 1 \) (where \( \alpha_r(\sigma) \) is a measurable function defined in Lemma 3) then there exists in probability a unique random solution of the random integral eq. 1.1 provided also \( \mu(\sigma \in \Sigma: ||v||_p < r(1 - |\lambda | \alpha_r(\sigma))) = 1 \).

**Proof.** Let \( u_0 \in S_r \subset L^p \) be chosen arbitrarily and let us define a sequence \( \{u_n\} \) of generalized random variables by

\[
(2.8) \quad u_{n+1} = u + \lambda \alpha_r u_n, \quad n = 0, 1, 2 \cdots .
\]

For the convergence of the sequence \( \{u_n\} \) to a random solution of the stochastic integral eq. 1.1 it is firstly necessary to show that \( \{u_n\} \) is a Cauchy sequence in \( L^p \) in probability. In fact it will be shown that \( \{u_n\} \) is a Cauchy sequence in \( L^p \) with probability one. For any fixed positive integer \( m \geq 1 \) and for any positive integer \( n \) let us define the sets \( F_n \), \( G_n \) and \( H_n \in S \) by

\[
F_n = \left\{ \sigma: \|u_{n+m} - u_n\|_p > \frac{1}{n} \right\},
\]

\[
G_n = \left\{ \sigma: (|\lambda | \alpha_r(\sigma))^n \|u_1 - u_0\|_p > \frac{1}{n} (1 - |\lambda | \alpha_r(\sigma)) \right\},
\]

and

\[
H_n = \left\{ \sigma: n(|\lambda | \alpha_r(\sigma))^n > \frac{(1 - |\lambda | \alpha_r(\sigma))}{2r} \right\}
\]

respectively.

It is clear that \( F_n \subset G_n \). Further \( G_n \setminus M \subset H_n \) where, by hypothesis on \( v \in L^p \), \( M = \{\sigma: \|v\|_p \geq r(1 - |\lambda | \alpha_r(\sigma))\} \) has zero measure. From these it follows that \( F_n \setminus (M \cup N) \subset G_n \setminus (M \cup N) \subset H_n \setminus N \) where \( N = \{\sigma: |\lambda | \alpha_r(\sigma) \geq 1\} \) has measure zero. \( \{H_n\} \) is a sequence of measurable sets and \( \lim_{n \to \infty} \mu(H_n) = 0 \). Since the measure \( \mu \) is assumed to be complete we have \( \lim_{n \to \infty} \mu(F_n \setminus (M \cup N)) = 0 \) and \( \lim_{n \to \infty} (F_n \setminus (M \cup N) = F \setminus (M \cup N) \) is a measurable set with \( \mu(F \setminus (M \cup N)) = 0 \). This implies that \( \mu(F) = 0 \) and that for every \( m \geq 1 \mu(\sigma: \lim_{n \to \infty} \|u_{n+m} - u_n\|_p = 0) = 1 \). Therefore, \( \{u_n\} \) is a Cauchy sequence in \( L^p \) with probability one and consequently also in probability. Thus there exists a mapping \( u^0 \) of the set \( \Sigma \) into the space \( L^p (p \geq 1) \) such that \( \mu(\sigma: \lim_{n \to \infty} \|u_n - u^0\| = 0) = 1 \). By Theorem 4. (Hans, 1957a) and our assumption that \( \mu \) is complete it follows that \( u^0 \) itself is a generalized random variable.
It will be shown that $u^0$ is a unique solution (in probability) of the stochastic integral eq. 1.1. For every $\epsilon > 0$ let $D_\epsilon$ be defined by

$$D_\epsilon = \{ \sigma : \| u^0 - B_\sigma u^0 \| > \epsilon \}.$$ 

Then $D_\epsilon = \{ \sigma : \| u^0 - B_\sigma u_n + B_\sigma u_n - B_\sigma u^0 \|_p > \epsilon \}$ for any positive integer $n$ and

$$D_\epsilon' \supset \{ \sigma : \| u^0 - B_\sigma u_n \|_p + \lambda \| \alpha_r(\sigma) \| u_n - u^0 \|_p \leq \epsilon \}.$$ 

This implies $D_\epsilon' \supset \{ \sigma : \| u^0 - u_{n+1} \|_p + \lambda \| \alpha_r(\sigma) \| u_n - u^0 \|_p \leq \epsilon \}$ and further $D_\epsilon' \supset \bigcup_{s \in \Delta} \{ \sigma : \| u^0 - u_{n+1} \|_p \leq (\epsilon/2) - s \} \cap \{ \sigma : \lambda \| \alpha_r(\sigma) \| \| u_n - u^0 \|_p \leq (\epsilon/2) + s \}$ where $D_\epsilon'$ is the complement of the set $D_\epsilon$ and $\Delta$ is the set of rationals in the interval $[-(\epsilon/2), + (\epsilon/2)]$. Thus the inclusion relation

$$D_\epsilon \subset \bigcap_{s \in \Delta} \{ \sigma : \| u^0 - u_{n+1} \|_p > (\epsilon/2) - s \} \cup \{ \sigma : \lambda \| \alpha_r(\sigma) \| u^0 - u_n \|_p > (\epsilon/2) + s \}$$

holds and this in turn implies that

$$D_\epsilon \subset \{ \sigma : \| u^0 - u_{n+1} \|_p > (\epsilon/2) \} \cup \{ \sigma : \lambda \| \alpha_r(\sigma) \| u^0 - u_n \|_p > (\epsilon/2) \}.$$ 

Therefore for every $\epsilon > 0$ and for any positive integer $n$

$$\mu(D_\epsilon) \leq \mu(\{ \sigma : \| u^0 - u_{n+1} \|_p > (\epsilon/2) \}) + \mu(\{ \sigma : \lambda \| \alpha_r(\sigma) \| u^0 - u_n \|_p > (\epsilon/2) \}).$$ 

Since $\lambda \| \alpha_r(\sigma) \|$ is bounded with probability one and $\lim_n \mu(\{ \sigma : \| u^0 - u_n \|_p > \epsilon \}) = 0$ for any $\epsilon > 0$. Thus $u^0$ is a generalized random solution of the stochastic integral equation $u = B_\sigma u$. That the solution $u^0$ is unique (in probability) is proved by contradiction. Let $v^0$ be another generalized random solution of 1.1; then for any $\epsilon > 0$, $\mu(\{ \sigma : \| u_n - v^0 \|_p > \epsilon \}) \to 0$. Let $\epsilon > 0$ be arbitrary and let $B_\epsilon \subset \Sigma$ be defined by

$$B_\epsilon = \{ \sigma : \| u^0 - v^0 \|_p > \epsilon \}.$$ 

Then it can be shown by a similar procedure as before that for any positive integer $n$

$$B_\epsilon \subset \{ \sigma : \| u_n - u^0 \|_p > \epsilon/2 \} \cup \{ \sigma : \| u_n - v^0 \|_p > \epsilon/2 \}.$$ 

Thus $\mu(B_\epsilon) = 0$ for every $\epsilon > 0$ which proves the uniqueness. This completes the proof of the proposition.
Remark. If \( \mu(A) = 1 \) for every \( A \in S \) containing an element \( \{\sigma_0\} \) in its interior where \( \{\sigma_0\} \) is a special element of the probability space \( (\Sigma, S, \mu) \) then the stochastic operator \( A_\sigma \) reduces to a deterministic one. In that situation the results presented here will reduce to those given in (Ahmed, 1968).

3. APPLICATION TO OPTIMAL CONTROL

Let the input and output of a physical system be related through a stochastic nonlinear operator of Volterra type as defined by eq. 1.2. For an example we consider the following simple optimal control problem: Let a fixed element \( z \in L^2(I) \) be the desired output and let \( \alpha > 0 \) and \( \beta > 0 \) be the indices of cost of control and cost of error respectively and let the measure of performance of the system be described by the following cost functional \( G(u) \) defined by

\[
G(u) = \alpha(u, u) + \beta(F_\sigma u - z, F_\sigma u - z)
\]  

(3.1)

where \( F_\sigma: L^2 \to L^2 \) for each \( \sigma \in \Sigma \).

It is desired to find a generalized random variable \( u(\sigma, t) \in D^0(F_\sigma) \) (the interior of the domain \( D(F_\sigma) \)) that minimizes the cost functional \( G(u) \). Necessary and sufficient conditions for the existence of a locally optimal control which is a generalized random variable in \( L^2 \) can be proved in a similar way as in the deterministic case (Ahmed, 1968). The following results are stated without presenting the details.

**Proposition 4.** A necessary condition that a generalized random variable \( u_0(\sigma, t) \in D^0(F_\sigma) \subset L^2(I) \) be an extremal control is that for almost all \( \sigma \in \sum F_\sigma \) has a linear Gateaux derivative at each point in \( D^0(F_\sigma) \) and \( u_0 \) satisfy the functional equation

\[
u_0(\sigma, \tau) = \lambda \int_\tau^T F'_{u_0}(\sigma | t, \tau)(z(t) - (F_\sigma u_0)(t)) dt
\]

(3.2)

for almost all \( \sigma \in \Sigma \) and \( \tau \in I \) where \( F_{u'} \) is the Gateaux gradient of \( F_\sigma \) at \( u \) and \( \lambda = \beta/\alpha \).

**Proposition 5.** A necessary and sufficient condition that a locally extremal control \( u_0(\sigma, t) \in D^0(F_\sigma) \) be a locally optimal control is that \( F_\sigma \) has the second Gateaux derivative at \( u_0 \) and that \( (G''_{u_0}(\sigma, \cdot)h, h) \geq 0 \) with probability one for all \( h \in L^2 \).

These results are the stochastic counterparts of the deterministic case (Ahmed, 1968).
Using Fubini’s theorem wherever an interchange of the order of integration is involved and simply rearranging the terms in eq. 3.2 one obtains the following stochastic integral equation

\[ u(\sigma, t) = \lambda v(\sigma, t) + \lambda \sum_{n=1}^{\infty} \int_{I} \int L_n(\sigma \mid t; \tau_1 \cdots \tau_n) \prod_{i=1}^{n} u(\sigma, \tau_i) \, d\tau_i \]  

(3.3)

where

\[ v(\sigma, t) = \int_{t}^{T} K_1(\sigma \mid \tau, t)z(\tau) \, d\tau, \quad t \in I \]  

(3.4)

\[ L_n(\sigma \mid t; \tau_1 \cdots \tau_n) = ((n + 1)R_n(\sigma \mid t; t_1 \cdots t_n) \]  

\[ - R_n'(\sigma \mid t; \tau_1 \cdots \tau_n)) \]  

(3.5)

\[ R_n(\sigma \mid t; \tau_1 \cdots \tau_n) = \int_{\max(\xi, \tau_1, \cdots, \tau_n)}^{T} K_{n+1}(\sigma \mid \xi; t, \tau_1 \cdots \tau_n)z(\xi) \, d\xi \]  

(3.6)

and

\[ R_n'(\sigma \mid t; \tau_1 \cdots \tau_n) = \sum_{i=1}^{n} (n + 1 - s) \]  

\[ \cdot \int_{\max(\xi, \tau_1, \cdots, \tau_n)}^{T} K_{s}(\sigma \mid \xi, \tau_1 \cdots \tau_s)K_{n+1-s}(\sigma \mid \xi; t, \tau_{s+1} \cdots \tau_n) \, d\xi \]  

(3.7)

with \((t, \tau_1, \cdots, \tau_n) \in I^{n+1}\) and \(n \geq 1\). The expression for \(R_n'\) uses the convention that for any \(m \geq 1\) \(K_m(\sigma \mid t_1; \cdots, t_{m+r}) = K_m(\sigma \mid t_1, t_2, \cdots, t_{m+r})\) for all \(r \geq 1\), that is the number of arguments of \(K_m\) (excluding \(\sigma\)) must equal \(m + 1\).

Remark. It is important to mention that in case the integral operator \(F_\sigma\) in 1.2 has its upper limits of integrations fixed (as in the boundary value problems) or if the Kernels satisfy the property \(K_n(\sigma \mid t, \tau_1 \cdots \tau_n) \equiv 0\) for \(t < \tau_i\) for every \(i = 1, 2, \cdots, n\) then the lower limits of integration in 3.4, 3.6 and 3.7 will reduce to \(t_0\).

The integral eq. 3.3 is exactly similar to the integral eq. 1.1 except for the factor \(\lambda\) with \(v\).

From the above example it is clear that ideally it would be desirable to determine the random fixed point. This is due to the fact that when an experiment is performed on a physical system it is a sample from the space of random operators that is realized and not its expected value. For certain applications it may be often satisfactory to be able to deter-
mine the expected value of the random fixed point. Thus if \( u^0(\sigma, t) \) is a random fixed point of the stochastic operator 1.1 and if the expected value \( Eu^0 \), that is the Bochner integral (Hans, 1957a, 1957b)

\[
\hat{u}(\cdot) = \int_{\Sigma} u^0(\sigma, \cdot) \, d\mu(\sigma)
\]

exists as an element of \( L^p \) then \( \hat{u}(\cdot) \) is the expected value of the "random fixed point". Further if the random operator \( A_\tau \) is a uniformly reducing random transform (Hans, 1957a) (that is \( |\lambda| \alpha_r(\sigma) < 1 \) for all \( \sigma \in \Sigma \)) and the Bochner integral

\[
T(x) = \int_{\Sigma} B_{\sigma} x \, d\mu(\sigma)
\]

exists for every \( x \in S_\tau \subset L^p \) then the mapping \( T \) of the space \( L^p \) into itself is reducing and therefore has a unique fixed point. This fixed point of the mapping \( T \) is called "fixed point of the expected value" of the uniformly reducing random transform \( B_\tau \). Except for very simple random transforms the two fixed points do not coincide. From computational point of view it seems that 3.9 is easier to compute than 3.8 since the latter requires solving the integral eq. 1.1 (for each fixed \( \sigma \in \Sigma \)) explicitly. But the fixed point of \( T \), in general, is neither the required solution nor its expected value.

4. EXTENSION TO ORLICZ SPACES

It appears that the methods presented in Section 2 can be generalized to Orlicz spaces provided the \( N \)-function \( M \) (Krasnoselskii, 1961) defining the Orlicz space \( L_M \) satisfies the \( \Delta_2 \) condition. This is a much stronger condition than requiring \( \Delta_2 \) conditions. This situation arises because of the presence of terms of the form \( \prod_{i=1}^n u(\sigma, \tau_i) \) in the description of our operator \( A \). This extension work will not be attempted here.

CONCLUSION

In this paper we have considered the stochastic fixed point of a large class of stochastic nonlinear integral operators on \( L^p (p \geq 1) \) spaces. Stochastic versions of both the Schauder and Banach fixed point theorems were presented in Propositions 2 and 3 on the basis of Lemmas 1–4 and Proposition 1.

Application to a specific class of stochastic optimal control problems was considered, though clearly the methods will apply to any problem
described by an integral equation of the form 1.1. Extension to multi-variable systems is immediate.

Extension of the methods presented here to Orlicz spaces were only briefly indicated.

Received September 23, 1968; Revised April 15, 1969

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