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On the Concept of Virtual States

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The technique of the decomposed Feynman propagator is used to establish the equivalence between the Feynman and field theoretic formalisms. It is shown that for an nth order process, each of the 2^{n-1} decomposed Feynman diagrams is equivalent to a certain group in the $n!$ field theoretic diagrams. This is demonstrated for the fourth order Compton scattering of an electron by identifying the energy denominators in the two formalisms.

INTRODUCTION

It is well known that the concepts of virtual states in nonrelativistic wave mechanics, Feynman formulation, and field theory are in principle the same though the detailed structure of the "state" is different in the three descriptions.[1]

1. In nonrelativistic wave mechanics, we exclude negative energy states—in other words we do not envisage the creation and annihilation of pairs. In the temporal evolution of the system, the number of fermions and anti-fermions are assumed to be separately conserved. In calculating the matrix elements we perform spatial integration first and the temporal evolution of the states in momentum representation is studied. It is therefore possible to speak of the state of a system at a particular time t.

2. In the Feynman formalism we include particles in the negative energy states, but the sequence of events in a perturbation expansion is not temporally ordered. We usually take the four-dimensional transform and, in this case, we can speak of the initial and final states being connected by a Feynman sequence of intermediate states.

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[1] In this paper we use the usual Feynman notation and $\mathbf{P} = p_\mu \gamma_\mu = E\gamma_\mu - \vec{p} \cdot \vec{\gamma}$ we have also set $c = \hbar = 1$.

It is to be noted that the fourth component of the Feynman four-vector of any term within brackets such as $[\mathbf{p}_1 + \overset{+}{\mathbf{q}} - \mathbf{q}_1]$ is given by

$$[(\vec{p}_1 + \vec{q} - \vec{q}_1)^2 + m^2]^{1/2}.$$

3. In field theory, we envisage the creation and annihilation of particles and antiparticles. If, in the calculation of matrix elements the spatial integration is performed first, the situation is the same as in (1) except that we can have a multiplicity of particles. If however, we wish to integrate space and time together, we have first to rearrange the operators suitably and this leads to the Feynman matrix element. It was considered that the essential merit of the four-dimensional integration was the inherent covariance of the theory at every stage, while separate integration with respect to time leads to energy denominators. This seems to have been accepted for so long that no effort was made to find if it was possible to decompose the Feynman matrix element into relativistically invariant components. *We now find that this is indeed possible in such a manner that some of the revealing features of the temporal ordering are preserved while at the same time the elegance of relativistic invariance is not lost.* In fact if this is done the concept of virtual states becomes identical in all three formalisms.

The nth order matrix element for the scattering of an electron from-momentum p_1' to momentum p_2' is given by

$$M = \int d^4x_n \cdots \int d^4x_1 \, \bar{\psi}_{p_2'}(x_n) \, K(x_n, x_{n-1}) \cdots \psi_{p_1'}(x_1) \tag{1}$$

In the Feynman formalism, the matrix element in momentum representation obtained by performing the four-dimensional integration is given by

$$M = \bar{u}_{(p_2')} \, \mathbf{e}_n \frac{1}{\mathbf{p}_n - m} \mathbf{e}_{n-1} \cdots \mathbf{e}_2 \frac{1}{\mathbf{p}_1 - m} \mathbf{e}_1 u(p_1') \tag{2}$$

where the $\mathbf{p}_i$ refer to intermediate virtual states with energy

$$p_4 \neq E_{p_i} = \sqrt{\vec{p}_i^{\,2} + m^2}.$$

For a given order in the sequence of perturbations $\mathbf{e}_n \cdots \mathbf{e}_1$ (i.e., for a single Feynman diagram), the above can be decomposed into 2^{n-1} terms which are individually covariant as follows: The space and time integrations of (1) are separately performed, the former leading to conservation of three momentum at every vertex, and the time integration which is subsequently performed is split into two parts corresponding to the ranges $t = -\infty$ to 0 with energy $-E$ and $t = 0$ to $+\infty$ with energy $+E$ respectively. It was shown in an earlier paper that this leads to the decomposition of the Feynman propagator

$$\frac{1}{\mathbf{p} + \mathbf{q} - m} = \frac{1}{2(E_{p+q})} \left\{ \frac{\overset{+}{\mathbf{P}} + m}{E_p + E_q - E_{p+q}} - \frac{\overset{-}{\mathbf{P}} + m}{E_p + E_q + E_{p+q}} \right\} \tag{3}$$

where $E_{p+q}^2 = (\vec{p} + \vec{q})^2 + m^2$ and $\overset{+}{\mathbf{P}}$ is the Feynman four vector with energy

$+ E_{p+q}$ and $\bar{\mathbf{P}}$ has the fourth component $- E_{p+q}$. It was however not realized at that time that each of these terms is relativistically invariant. In fact, the first term corresponding to positive energy is nothing but the transform of

$$\frac{1}{(2\pi)^4} \int_{c_+} \frac{u\bar{u}\, e^{-ip.x}}{p^2 - m^2}\, d^4p \qquad \text{for} \qquad x_4 > 0 \tag{4}$$

and can be obtained from the contour omitting the pole at $p_4 = - E$; in a similar way, the second term can be represented by the transform of the invariant function

$$\frac{1}{(2\pi)^4} \int_{c_-} \frac{e^{-ip.x}\, u\bar{u}}{p^2 - m^2}\, d^4p \qquad \text{for} \qquad x_4 < 0 \tag{5}$$

so that both the terms are relativistically invariant.

If we use this propagator it is more convenient to think of the energy of the "virtual" particle to be $+ E_{p+q}$ with momentum $\vec{p} + \vec{q}$. It is virtual in the sense that its energy does not correspond to $E_p + E_q$, the energy of the system before its creation. In a similar way, $- E_{p+q}$ corresponds to a negative energy "virtual" particle. These two parts are taken together in the Feynman formalism where we attribute an energy p_4 to the virtual particle.

The main advantage of this decomposed propagator is that it lends itself to a method of comparison with field theory due to the presence of the energy denominators so that the equivalence between Feynman and field theoretic formalism can be established even in the old fashioned manner, that is, after space integration.

In a field theoretic picture for a given order in the sequence of perturbations, the nth order term has $n!$ diagrams, each of which will give different energy denominators. This is because the position of every vertex relative to all other $n - 1$ vertices is important since the time integration is performed in a temporarily ordered way. Thus every new complexion gives a different energy denominator and a sum over intermediate states implies a sum over all such diagrams.

If, on the other hand, we employ the method of the decomposed Feynman propagator, the position of every vertex on the time axis with respect to the previous (in the Feynman sense) one is relevant since the integration over interval $t_n - t_{n-1}$ is divided into two ranges, positive and negative, or whether the nth vertex lies "above" or "below" the $n - 1$th vertex. Thus, since for an nth order process there are $n - 1$ propagators each of which can be split into two parts, it follows that we can have 2^{n-1} diagrams. It now remains to be shown that the $n!$ diagrams of field theory are equivalent to the 2^{n-1} such diagrams, $n! \geqslant 2^{n-1}$ for all $n \geqslant 2$).

Calculations

We here demonstrate explicitly the equivalence for a fourth order process. We have for definiteness considered the Compton scattering of an electron.

We shall consider a particular sequence in which the initial photon of four momentum $q(\omega, \vec{q})$ is absorbed by an electron at rest and three photons of four momenta $q_1(\omega_1, \vec{q}_1)$, $q_2(\omega_2, \vec{q}_2)$ and $q_3(\omega_3, \vec{q}_3)$ are emitted in this order along the Feynman path, the final electron having momentum $p_2(E_2, \vec{p}_2)$. The calculation of the entire matrix element would of course involve all permutations of the above sequence. However for the present purpose it is sufficient if we consider a particular sequence only.

A. Field Theoretic Formalism

The matrix element in field theory for this fourth order process is

$$M_4 = \sum \frac{H_{\mathrm{fIII}}\, H_{\mathrm{III\,II}}\, H_{\mathrm{II\,I}}\, H_{\mathrm{Ii}}}{(E_0 - E_{\mathrm{III}})\,(E_0 - E_{\mathrm{II}})\,(E_0 - E_{\mathrm{I}})} \tag{6}$$

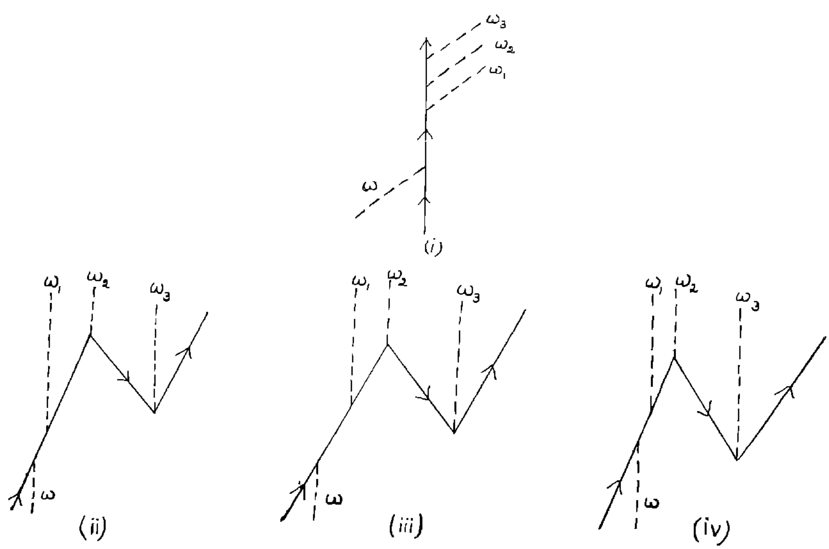
where f and i refer to final and initial states respectively and the summation is over all intermediate states III, II, and I.

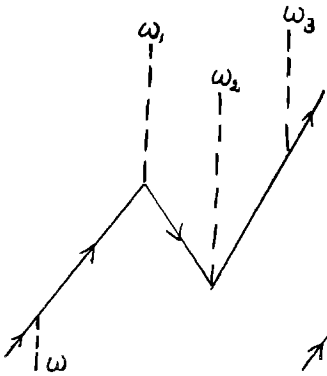
$$H_{\mathrm{fIII}} = (\Phi_{\mathrm{f}}, H_{\mathrm{int}}\, \Phi_{\mathrm{III}}) \text{ etc.} \tag{7}$$

and

$$H_{\mathrm{int}} = G \int \bar{\psi}\gamma_\mu \psi\phi \, d^3x \tag{8}$$

where ψ and ϕ are the electron and photon field operators respectively.





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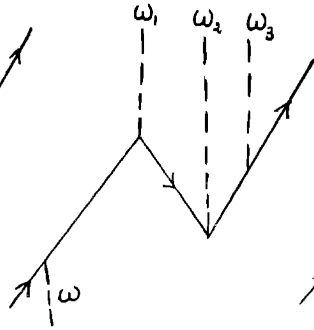

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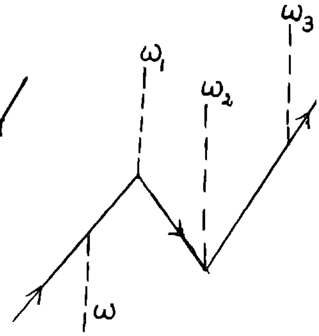

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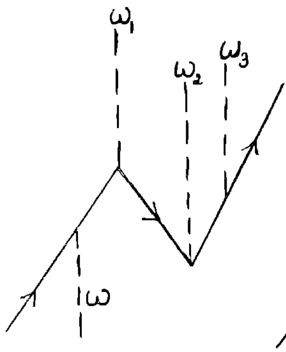

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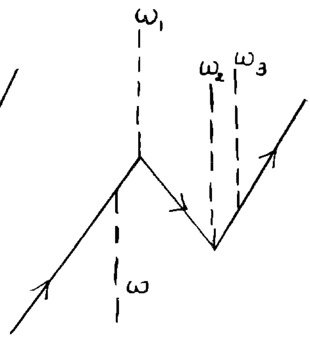

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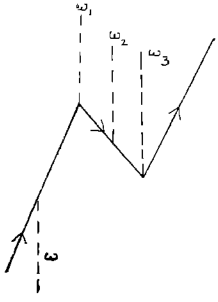

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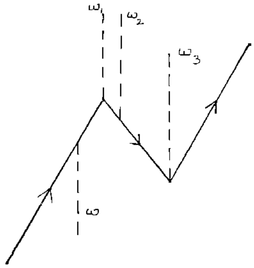

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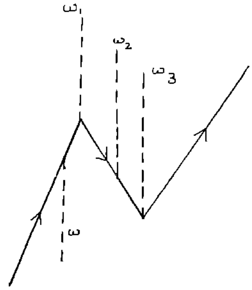

(xii)

We now evaluate M_4 for each of the $n!/2 = 12$ diagrams. The other half is just an exact counterpart with all positive energy intermediate states replaced by negative energies and vice versa. Thus, corresponding to diagram (i), we have

$$M_{4(\mathrm{i})} = \frac{\sum H_{\mathrm{f\,III}}\, H_{\mathrm{III\,II}}\, H_{\mathrm{II\,I}}\, H_{\mathrm{I\,i}}}{(m + \omega - E_{p_1+q})\,(m + \omega - \omega_1 - E_{p_1+q-q_1})\,(E_2 + \omega_3 - E_{p_2+q_3})}$$

$$= \frac{\left(\begin{array}{l}\langle \Phi_{\mathrm{f}} \mid H_{\mathrm{int}}(x_4) \mid \Phi_{\mathrm{III}}\rangle \langle \Phi_{\mathrm{III}} \mid H_{\mathrm{int}}(x_3) \mid \Phi_{\mathrm{II}}\rangle \\ \qquad\qquad \times \langle \Phi_{\mathrm{II}} \mid H_{\mathrm{int}}(x_2) \mid \Phi_{\mathrm{I}}\rangle \langle \Phi_{\mathrm{I}} \mid H_{\mathrm{int}}(x_1) \mid b_{p_1}^{\dagger}\rangle_0\end{array}\right)}{(m + \omega - E_{p_1+q})\,(m + \omega - \omega_1 - E_{p_1+q-q_1})\,(E_2 + \omega_3 - E_{p_2+q_3})} \tag{9}$$

where $b_{p_1}^{\dagger}$ is the creation operator of the electron with momentum $p_1(0, m)$. The photon field operators are all omitted in what follows since they always commute and are hence not relevant for our arguments. We shall also omit numerical factors for convenience. Conservation of energy implies

$$m + \omega = E_2 + \omega_1 + \omega_2 + \omega_3. \tag{10}$$

and the energy denominators in (9) correspond to diagram (i). Expanding H_{int}'s, inserting the photon operators, and integrating over the space variables $x_1, x_2, \cdots, x_4$ leads ultimately to an over-all δ-function which implies momentum conservation. Thus the electron operators in the numerator of (9) reduce to

$$\sum_{\substack{\text{all mom}\\ \text{indices}}} \langle \Phi_{\mathrm{f}} \mid (b^{\dagger} b)\,(b^{\dagger} b)\,(b^{\dagger} b)\,(b^{\dagger} b) \mid b_{p_1}^{\dagger}\rangle_0\, \bar{u}_{p_2}[u\bar{u}\; u\bar{u}\; u\bar{u}]\, u_{p_1} \tag{11}$$

where the b's are the annihilation operators for electrons.

Making use of

$$b_{p'} b_p^{\dagger}\rangle_0 = \delta(p - p') \tag{12}$$

and the over-all δ-function resulting from the space integration, (q) becomes

$$M_{4(\mathrm{i})} = \frac{\bar{u}_{p_2}[u\bar{u}_{p_2+q_3}]\,[u\bar{u}_{p_1+q-q_1}]\,[u\bar{u}_{p_1+q}]\, u_{p_1}}{(m + \omega - E_{p_1+q})\,(m + \omega - \omega_1 - E_{p_1+q-q_1})\,(E_2 + \omega_3 - E_{p_2+q_3})} \tag{13}$$

(ii) Similarly for the diagram (ii) we have

$$M_{4(\mathrm{ii})} = \frac{\left(\begin{array}{l}\langle \Phi_{\mathrm{f}} \mid H_{\mathrm{int}}(x_4) \mid \Phi_{\mathrm{III}}\rangle \langle \Phi_{\mathrm{III}} \mid H_{\mathrm{int}}(x_3) \mid \Phi_{\mathrm{II}}\rangle \\ \qquad\qquad \times \langle \Phi_{\mathrm{II}} \mid H_{\mathrm{int}}(x_2) \mid \Phi_{\mathrm{I}}\rangle \langle \Phi \mid H_{\mathrm{int}}(x_1) \mid b_{p_1}^{\dagger}\rangle_0\end{array}\right)}{(m + \omega - E_{p_1+q})\,(m + \omega - \omega_1 - E_{p_1+q-q_1})\,(\omega_2 - E_{p_1+q-q_1} - E_{p_2+q_3})} \tag{14}$$

and the numerator

$$= \sum \langle \Phi_{\mathrm{f}} \mid (db)\,(b^{\dagger}d^{\dagger})\,(b^{\dagger}b)\,(b^{\dagger}b) \mid b_{p_1}^{\dagger}\rangle_0\, \bar{u}_{p_2}(v\bar{v})\,(u\bar{u})\,(u\bar{u})\,u_{p_1}$$

where d and $d^{\dagger}$ are the annihilation and creation operators of the positron and v the corresponding spinor; this is now rearranged as

$$-\sum \langle \Phi_{\mathrm{f}} \mid b^{\dagger}(dd^{\dagger})\,(bb^{\dagger})\,(bb^{\dagger})\,b \mid b_{p_1}^{\dagger}\rangle_0\, \bar{v}_{p_2}(v\bar{v})\,(u\bar{u})\,(u\bar{u})\,u_{p_1}$$

so that we can now apply (12) and we have

$$M_{4(\mathrm{ii})} = \frac{-\,\bar{u}_{p_2}[v\bar{v}_{p_2+q_3}]\,[u\bar{u}_{p_1+q-q_1}]\,[u\bar{u}_{p_1+q}]\,u_{p_1}}{\left\{\begin{array}{l}(m+\omega-E_{p_1+q})\,(m+\omega-\omega_1-E_{p_1+q-q_1})\\ \qquad\qquad\times\,(\omega_2-E_{p_1+q-q_1}-E_{p_2+q_3})\end{array}\right\}} \tag{15}$$

(iii) In the case of diagram (iii), which differs from (ii) in that the vertex at which ω_3 is emitted is "below" the vertex at which ω_1 is emitted, the energy denominators are obviously different. We have

$$M_{4(\mathrm{iii})} = \frac{\sum \langle \Phi_{\mathrm{f}} \mid (db)\,(b^{\dagger}b)\,(b^{\dagger}d^{\dagger})\,(b^{\dagger}b) \mid b_{p_1}^{\dagger}\rangle_0\, \bar{u}_{p_2}(v\bar{v})\,(u\bar{u})\,(u\bar{u})\,u_{p_1}}{\left\{\begin{array}{l}(m+\omega-E_{p_1+q})\,(\omega_1+\omega_2-E_{p_1+q}-E_{p_2+q_3})\\ \qquad\qquad\times\,(\omega_2-E_{p_1+q-q_1}-E_{p_2+q_3})\end{array}\right\}} \tag{16}$$

The numerator when rearranged is

$$-\,\langle \Phi_{\mathrm{f}} \mid b_{p_2}^{\dagger}(dd^{\dagger})\,(bb^{\dagger})\,(bb^{\dagger}) \mid b_{p_1}^{\dagger}\rangle_0$$

and again

$$M_{4(\mathrm{iii})} = \frac{-\,\bar{u}_{p_2}[v\bar{v}_{p_2+q_3}]\,[u\bar{u}_{p_1+q-q_1}]\,[u\bar{u}_{p_1+q}]\,u_{p_1}}{\left\{\begin{array}{l}(m+\omega-E_{p_1+q})\,(\omega_1+\omega_2-E_{p_1+q}-E_{p_2+q_3})\\ \qquad\qquad\times\,(\omega_2-E_{p_1+q-q_1}-E_{p_2+q_3})\end{array}\right\}} \tag{17}$$

We note that the numerators in $M_{4(\mathrm{ii})}$ and $M_{4(\mathrm{iii})}$ are identical though the denominators are different.

(iv) In a similar way the vertex at which ω_3 is emitted can be still further "lowered" which gives rise to the diagram (iv), and it can be easily verified that the numerator is the same as in the previous two cases.

We thus have

$$M_{4(\mathrm{iv})} = \frac{-\bar{u}_{p_2}[v\bar{v}_{p_2+q_3}]\,[u\bar{u}_{p_1+q-q_1}]\,[u\bar{u}_{p_1+q}]\,u_{p_1}}{\left\{\begin{array}{r}(-E_2-\omega_3-E_{p_2+q_3})\,(\omega_1+\omega_2-E_{p_1+q}-E_{p_2+q_3}) \\ \times\,(\omega_2-E_{p_1+q-q_1}-E_{p_2+q_3})\end{array}\right\}} \tag{18}$$

The diagrams (ii), (iii), and (iv) form a group which will later be shown to be equivalent to one Feynman diagram.

(v) A group of diagrams can now be obtained by having the second Feynman intermediate state of negative energy while the first and third have positive energies. The procedure for the reduction of the numerator is quite similar to the previous ones and in future we shall merely write down the matrix element with the appropriate energy denominators. We have for diagram (v).

$$M_{4(\mathrm{v})} = \frac{-\bar{u}_{p_2}[u\bar{u}_{p_2+q_3}]\,[v\bar{v}_{p_1+q-q_1}]\,[u\bar{u}_{p_1+q}]\,u_{p_1}}{\left\{\begin{array}{r}(m+\omega-E_{p_1+q})\,(m+\omega-\omega_2-E_{p_1+q}-E_{p_2+q_3}-E_{p_1+q-q_1}) \\ \times\,(E_2+\omega_3-E_{p_2+q_3})\end{array}\right\}} \tag{19}$$

(vi) This diagram can be obtained from (v) by lowering the last vertex below the second (on the time axis) and we have

$$M_{4(\mathrm{vi})} = \frac{-\bar{u}_{p_2}[u\bar{u}_{p_2+q_3}]\,[v\bar{v}_{p_1+q-q_1}]\,[u\bar{u}_{p_1+q}]\,u_{p_1}}{\left\{\begin{array}{r}(m+\omega-E_{p_1+q})\,(m+\omega-E_{p_1+q-q_1}-\omega_2-E_{p_2+q_3}-E_{p_1+q}) \\ \times\,(\omega_1-E_{p_1+q}-E_{p_1+q-q_1})\end{array}\right\}} \tag{20}$$

(vii) In this the second vertex in (vi) becomes the first and vice versa so that

$$M_{4(\mathrm{vii})} = \frac{-\bar{u}_{p_2}[u\bar{u}_{p_2+q_3}]\,[v\bar{v}_{p_1+q-q_1}]\,[u\bar{u}_{p_1+q}]\,u_{p_1}}{\left\{\begin{array}{r}(m+\omega-E_{p_1+q}-E_{p_1+q-q_1}-\omega_2-E_{p_2+q_3}) \\ \times\,(-E_{p_1+q-q_1}-\omega_2-E_{p_2+q_3})\,(E_2+\omega_3-E_{p_2+q_3})\end{array}\right\}} \tag{21}$$

(viii) Here again the last vertex of (vii) becomes the third and

$$M_{4(\text{viii})} = \frac{-\bar{u}_{p_2}[u\bar{u}_{p_2+q_3}]\,[v\bar{v}_{p_1+q-q_1}]\,[u\bar{u}_{p_1+q}]\,u_{p_1}}{\left\{\begin{array}{l}(-E_{p_1+q-q_1}-\omega_2-E_{p_2+q_3})\\ \quad\times(m+\omega-E_{p_1+q-q_1}-E_{p_1+q}-\omega_2-E_{p_2+q_3})\\ \quad\times(\omega_1-E_{p_1+q}-E_{p_1+q-q_1})\end{array}\right\}} \tag{22}$$

(ix) The last diagram of the class (v ⋯ viii) is obtained by making the second vertex in viii the third and vice versa so that

$$M_{4(\text{ix})} = \frac{-\bar{u}_{p_2}(u\bar{u}_{p_2+p_3})\,(v\bar{v}_{p_1+q-q_1})\,(u\bar{u}_{p_1+q})\,u_{p_1}}{\left\{\begin{array}{l}(-E_{p_1+q-q_1}-\omega_2-E_{p_2+q_3})\,(-E_{p_1+q-q_1}-\omega_2-\omega_3-E_2)\\ \quad\times(\omega_1-E_{p_1+q}-E_{p_1+q-q_1})\end{array}\right\}} \tag{23}$$

This exhausts all such possible diagrams and the five diagrams can be shown to be equivalent to a single Feynman diagram. Another class of diagrams will be with the first Feynman intermediate state of positive energy and the other two of negative energy. It is easily seen that we have three such diagram (x ⋯ xii) which can be shown to be equivalent to one Feynman diagram. We have

$$M_{4(\text{x})} = \frac{\bar{u}_{p_2}(v\bar{v}_{p_2+q_3})\,(v\bar{v}_{p_1+q-q_1})\,(u\bar{u}_{p_1+q})\,u_{p_1}}{\left\{\begin{array}{l}(m+\omega-E_{p_1+q})\,(\omega_1+\omega_2-E_{p_2+q_3}-E_{p_1+q})\\ \quad\times(\omega_1-E_{p_1+q-q_1}-E_{p_1+q})\end{array}\right\}} \tag{24}$$

$$M_{4(\text{xi})} = \frac{\bar{u}_{p_2}(v\bar{v}_{p_2+q_3})\,(v\bar{v}_{p_1+q-q_1})\,(u\bar{u}_{p_1+q})\,u_{p_1}}{\left\{\begin{array}{l}(-E_2-\omega_3-E_{p_2+q_3})\,(\omega_1+\omega_2-E_{p_1+q}-E_{p_2+q_3})\\ \quad\times(\omega_1-E_{p_1+q-q_1}-E_{p_1+q})\end{array}\right\}} \tag{25}$$

$$M_{4(\text{xii})} = \frac{\bar{u}_{p_2}(v\bar{v}_{p_2+q_3})\,(v\bar{v}_{p_1+q-q_1})\,(u\bar{u}_{p_1+q})\,u_{p_1}}{\left\{\begin{array}{l}(-E_2-\omega_3-E_{p_2+q_3})\,(-E_2-\omega_3-\omega_2-E_{p_1+q-q_1})\\ \quad\times(\omega_1-E_{p_1+q-q_1}-E_{p_1+q})\end{array}\right\}} \tag{26}$$

B. Feynman Formalism

The Feynman matrix elements for a given type of diagram can be immediately written down with the use of the decomposed propagator (3) and we have for the four equivalent diagrams in the order discussed in Section A the following:

$$M_{4,1}^F = \frac{\bar{u}_{p_2}\{[\mathbf{p}_2 \overset{+}{+} \mathbf{q}_3] + m\}\,\{[\mathbf{p}_1 + \overset{+}{\mathbf{q}} - \mathbf{q}_1] + m\}\,\{[\mathbf{p}_1 \overset{+}{+} \mathbf{q}] + m\}\,u_{p_1}}{(m + \omega - E_{p_1+q})\,(m + \omega - \omega_1 - E_{p_1+q-q_1})\,(E_2 + \omega_3 - E_{p_2+q_3})} \tag{27}$$

$$M_{4,2}^F = \frac{\bar{u}_{p_2}\{[\mathbf{p}_2 \overset{-}{+} \mathbf{q}_3] + m\}\,\{[\mathbf{p}_1 + \overset{+}{\mathbf{q}} - \mathbf{q}_1] + m\}\,\{[\mathbf{p}_1 \overset{+}{+} \mathbf{q}] + m\}\,u_{p_1}}{(E_2 + \omega_3 + E_{p_2+q_3})\,(m + \omega - \omega_1 - E_{p_1+q-q_1})\,(m + \omega - E_{p_1+q})} \tag{28}$$

$$M_{4,3}^F = \frac{\bar{u}_{p_2}\{[\mathbf{p}_2 \overset{+}{+} \mathbf{q}_3] + m\}\,\{[\mathbf{p}_1 + \overset{-}{\mathbf{q}} - \mathbf{q}_1] + m\}\,\{[\mathbf{p}_1 \overset{+}{+} \mathbf{q}] + m\}\,u_{p_1}}{(E_2 + \omega_3 - E_{p_2+q_3})\,(m + \omega - \omega_1 + E_{p_1+q-q_1})\,(m + \omega - E_{p_1+q})} \tag{29}$$

$$M_{4,4}^F = \frac{\bar{u}_{p_2}\{[\mathbf{p}_2 \overset{-}{+} \mathbf{q}_3] + m\}\,\{[\mathbf{p}_1 + \overset{-}{\mathbf{q}} - \mathbf{q}_1] + m\}\,\{[\mathbf{p}_1 \overset{+}{+} \mathbf{q}_1] + m\}\,u_{p_1}}{(E_2 + \omega_3 + E_{p_2+q_3})\,(m + \omega - \omega_1 + E_{p_1+q-q_1})\,(m + \omega - E_{p_1+q})} \tag{30}$$

C. Equivalence

It now remains to be shown that the sum of the individual expressions in Section A for a given type of diagram reduces to the corresponding expression in Section B.

1. $M_{4(\mathrm{i})} \equiv M_{4,1}^F$ as is seen from (27) and (13).

Since

$$\sum_{\text{spins}} u\bar{u}_{p_2+q_3} = \{[\mathbf{p}_2 \overset{+}{+} \mathbf{q}_3] + m\} \text{ etc.}$$

2. To show

$$M_{4(\mathrm{ii})} + M_{4(\mathrm{iii})} + M_{4(\mathrm{iv})} = M_{4,2}^F \tag{31}$$

Now from (15) and (17)

$$M_{4(\mathrm{iii})} + M_{4(\mathrm{iv})}$$

$$= \left[\frac{N}{(\omega_1 + \omega_2 - E_{p_1+q} - E_{p_2+q_3})(\omega_2 - E_{p_1+q-q_1} - E_{p_2+q_3})}\right]$$

$$\times \left\{\frac{1}{m + \omega - E_{p_1+q}} - \frac{1}{E_2 + \omega_3 + E_{p_2+q_3}}\right\}$$

$$= -\left[\frac{N}{(\omega_2 - E_{p_1+q-q_1} - E_{p_2+q_3})(m + \omega - E_{p_1+q})(E_2 + \omega_3 + E_{p_2+q_3})}\right]$$

and

$$M_{4(\mathrm{ii})} + M_{4(\mathrm{iii})} + M_{4(\mathrm{iv})}$$

$$= \left[\frac{N}{(m + \omega - E_{p_1+q})(\omega_2 - E_{p_1+q-q_1} - E_{p_2+q_3})}\right]$$

$$\times \left\{\frac{1}{m + \omega - \omega_1 - E_{p_1+q-q_1}} - \frac{1}{E_2 + \omega_3 + E_{p_2+q_3}}\right\}$$

$$= -\frac{N}{(m + \omega - E_{p_1+q})(m + \omega - \omega_1 - E_{p_1+q-q_1})(E_2 + \omega_3 + E_{p_2+q_3})}$$

$$\equiv M^F_{4,2}$$

since

$$-\sum_{\mathrm{spins}} v\bar{v}_{p_2+q_3} = -\sum_{\mathrm{spins}} u_{-E}\bar{u}_{-E(p_2+q_3)} = \{[\mathbf{p}_2 \mp \mathbf{q}_3] + m\} \text{ etc.}$$

3. We now demonstrate

$$M_{4(\mathrm{v})} + M_{4(\mathrm{vi})} + M_{4(\mathrm{vii})} + M_{4(\mathrm{viii})} + M_{4(\mathrm{ix})} \equiv M^F_{4,3} \tag{32}$$

The numerators can similarly be shown to be the same as that of (29)

$$\alpha = M_{4(\mathrm{v})} + M_{4(\mathrm{vi})}$$

$$= \frac{N}{(m + \omega - E_{p_1+q})(E_2 + \omega_3 - E_{p_2+q_3})(\omega_1 - E_{p_1+q} - E_{p_1+q-q_1})}$$

$$\beta = M_{4(\mathrm{vii})} + M_{4(\mathrm{viii})}$$

$$= \frac{N}{(-\omega_2 - E_{p_1+q-q_1} - E_{p_2+q_3})(\omega_3 + E_2 - E_{p_2+q_3})(\omega_1 - E_{p_1+q} - E_{p_1+q-q_1})}$$

$$\gamma = M_{4(\text{vii})} + M_{4(\text{viii})} + M_{4(\text{ix})}$$

$$= -\frac{N}{\left\{\begin{array}{r}(\omega_1 - E_{p_1+q} - E_{p_1+q-q_1})(\omega_3 + E_2 - E_{p_2+q_3}) \\ \times (E_{p_1+q-q_1} + \omega_2 + E_2 + \omega_3)\end{array}\right\}}$$

Therefore

$$\alpha + \gamma = -\frac{N}{(E_2 + \omega_3 - E_{p_2+q_3})(m + \omega - E_{p_1+q})(\omega_2 + E_2 + \omega_3 + E_{p_1+q-q_1})}$$

$$= -\frac{N}{(E_2 + \omega_3 - E_{p_2+q_3})(m + \omega - E_{p_1+q})(m + \omega - \omega_1 + E_{p_1+q-q_1})}$$

$$\equiv M^F_{4,3}.$$

4. Similarly

$$M_{(\text{x})} + M_{(\text{xi})} + M_{(\text{xii})} \equiv M^F_{4,4} \tag{33}$$

$$M_{(\text{x})} + M_{(\text{xi})}$$

$$= \frac{N}{(\omega_1 - E_{p_1+q-q_1} - E_{p_1+q})(-E_2 - \omega_3 - E_{p_2+q_3})(m + \omega - E_{p_1+q})}$$

and

$$M_{(\text{x})} + M_{(\text{xi})} + M_{(\text{xii})}$$

$$= \frac{N}{(E_2 + \omega_3 + E_{p_2+q_3})(m + \omega - E_{p_1+q})(E_2 + \omega_3 + \omega_2 + E_{q_1+q-q_1})}$$

$$= \frac{N}{(E_2 + \omega_3 + E_{p_2+q_3})(m + \omega - E_{p_1+q})(m + \omega - \omega_1 + E_{p_1+q-q_1})} = M^F_{4,4}.$$

We have thus demonstrated that with the use of the decomposed propagator the equivalence between Feynman and the field theory can be established in a simple and straightforward manner.

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