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Ground state bands of the E(5) and X(5) critical symmetries obtained from Davidson potentials through a variational procedure

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Abstract

Davidson potentials of the form $\beta^2 + \beta_0^4/\beta^2$, when used in the original Bohr Hamiltonian for γ-independent potentials bridge the U(5) and O(6) symmetries. Using a variational procedure, we determine for each value of angular momentum L the value of β_0 at which the derivative of the energy ratio $R_L = E(L)/E(2)$ with respect to β_0 has a sharp maximum, the collection of R_L values at these points forming a band which practically coincides with the ground state band of the E(5) model, corresponding to the critical point in the shape phase transition from U(5) to O(6). The same potentials, when used in the Bohr Hamiltonian after separating variables as in the X(5) model, bridge the U(5) and SU(3) symmetries, the same variational procedure leading to a band which practically coincides with the ground state band of the X(5) model, corresponding to the critical point of the U(5) to SU(3) shape phase transition. A new derivation of the Holmberg–Lipas formula for nuclear energy spectra is obtained as a by-product.

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Keywords: Davidson potential; Shape phase transition; Variational method; E(5) model; X(5) model

1. Introduction

The recently introduced E(5) [1] and X(5) [2] models are supposed to describe shape phase transitions in atomic nuclei, the former being related to the transition from U(5) (vibrational) to O(6) (γ-unstable) nuclei, and the latter corresponding to the transition from U(5) to SU(3) (rotational) nuclei. In both cases the original Bohr collective Hamiltonian [3] is used, with an infinite well potential in the collective β-variable. Separation of variables is achieved in the E(5) case by assuming that the potential is independent of the collective γ-variable, while in the X(5) case the potential is assumed to be of the form $u(\beta) + u(\gamma)$. We are going to refer to these two cases as "the E(5) framework" and "the X(5) framework", respectively. The selection of an infinite well potential in the β-variable in both cases is justified by the fact that the potential is expected to be flat around the point at which a shape phase transi-

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tion occurs. Experimental evidence for the occurrence of the E(5) and X(5) symmetries in some appropriate nuclei is growing ([4,5] and [6,7], respectively).

In the present Letter we examine if the choice of the infinite well potential is the optimum one for the description of shape phase transitions. For this purpose, we need one-parameter potentials which can span the U(5)–O(6) region in the E(5) framework, as well as the U(5)–SU(3) region in the X(5) framework. It turns out that the exactly soluble [8,9] Davidson potentials [10]

$$u(\beta) = \beta^2 + \frac{\beta_0^4}{\beta^2}, \tag{1}$$

where β_0 is the position of the minimum of the potential, do possess this property. Taking into account the fact that various physical quantities should change most rapidly at the point of the shape phase transition [11], we locate for each value of the angular momentum L the value of β_0 for which the rate of change of the ratio $R_L = E(L)/E(2)$, a widely used measure of nuclear collectivity, is maximized. It turns out that the collection of R_L ratios formed in this way in the case of a potential independent of the γ-variable correspond to the E(5) model, while in the case of the $u(\beta) + u(\gamma)$ potential lead to the X(5) model, thus proving that the choice of the infinite well potential made in Refs. [1,2] is the optimum one. The variational procedure used here is analogous to the one used in the framework of the variable moment of inertia (VMI) model [12], where the energy is minimized with respect to the (angular momentum dependent) moment of inertia for each value of the angular momentum L separately.

In Section 2 the E(5) case is considered, while the X(5) case is examined in Section 3, in which a new derivation of the Holmberg–Lipas formula [13] for nuclear energy spectra is obtained as a by-product. Finally, Section 4 contains a discussion of the present results and plans for further work.

2. Davidson potentials in the E(5) framework

The original Bohr Hamiltonian [3] is

$$H = -\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4\beta^2}\sum_{k=1,2,3}\frac{Q_k^2}{\sin^2(\gamma - \frac{2}{3}\pi k)}\right] + V(\beta,\gamma), \tag{2}$$

where β and γ are the usual collective coordinates describing the shape of the nuclear surface, Q_k ($k = 1, 2, 3$) are the components of angular momentum, and B is the mass parameter.

Assuming that the potential depends only on the variable β, i.e., $V(\beta,\gamma) = U(\beta)$, one can proceed to separation of variables in the standard way [3,14], using the wavefunction $\Psi(\beta,\gamma,\theta_i) = f(\beta)\Phi(\gamma,\theta_i)$, where θ_i ($i = 1, 2, 3$) are the Euler angles describing the orientation of the deformed nucleus in space.

In the equation involving the angles, the eigenvalues of the second order Casimir operator of SO(5) occur, having the form $\Lambda = \tau(\tau + 3)$, where $\tau = 0, 1, 2, \ldots$ is the quantum number characterizing the irreducible representations (irreps) of SO(5), called the "seniority" [15]. This equation has been solved by Bes [16].

The "radial" equation can be simplified by introducing [1] reduced energies $\epsilon = \frac{2B}{\hbar^2}E$ and reduced potentials $u = \frac{2B}{\hbar^2}U$, leading to

$$\left[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{\tau(\tau+3)}{\beta^2} + u(\beta)\right]f(\beta) = \epsilon f(\beta). \tag{3}$$

When plugging the Davidson potentials of Eq. (1) in the above equation, the β_0^4/β^2 term is combined with the $\tau(\tau + 3)/\beta^2$ term appearing there and the equation is solved exactly [8,9], the eigenfunctions being Laguerre polynomials of the form

$$F_n^\tau(\beta) = \left[\frac{2n!}{\Gamma(n + p + \frac{5}{2})}\right]^{1/2}\beta^p L_n^{p+\frac{3}{2}}(\beta^2)e^{-\beta^2/2}, \tag{4}$$

where $\Gamma(n)$ stands for the Γ-function, while p is determined by [8]

$$p(p+3) = \tau(\tau+3) + \beta_0^4, \tag{5}$$

leading to

$$p = -\frac{3}{2} + \left[\left(\tau + \frac{3}{2}\right)^2 + \beta_0^4\right]^{1/2}. \tag{6}$$

The energy eigenvalues are then [8,9] (in $\hbar\omega = 1$ units)

$$E_{n,\tau} = 2n + p + \frac{5}{2} = 2n + 1 + \left[\left(\tau + \frac{3}{2}\right)^2 + \beta_0^4\right]^{1/2}. \tag{7}$$

For $\beta_0 = 0$ the original solution of Bohr [3], which corresponds to a 5-dimensional (5D) harmonic oscillator characterized by the symmetry U(5) ⊃ SO(5) ⊃ SO(3) ⊃ SO(2) [17], is obtained. The values of angular momentum L contained in each irrep of SO(5) (i.e., for each value of τ) are given by the algorithm [18] $\tau = 3\nu_\Delta + \lambda$, where $\nu_\Delta = 0, 1, \ldots$ is the missing quantum number in the reduction SO(5) ⊃ SO(3), and $L = \lambda, \lambda + 1, \ldots, 2\lambda - 2, 2\lambda$ (with $2\lambda - 1$ missing).

The levels of the ground state band are characterized by $L = 2\tau$ and $n = 0$. Then the energy levels of the ground state band are given by

$$E_{0,L} = 1 + \frac{1}{2}\left[(L+3)^2 + 4\beta_0^4\right]^{1/2}, \tag{8}$$

while the excitation energies of the levels of the ground state band relative to the ground state are

$$E_{0,L,\mathrm{exc}} = E_{0,L} - E_{0,0} = \frac{1}{2}\left(\left[(L+3)^2 + 4\beta_0^4\right]^{1/2} - \left[9 + 4\beta_0^4\right]^{1/2}\right). \tag{9}$$

For $u(\beta)$ being a 5D infinite well

$$u(\beta) = \begin{cases} 0, & \text{if } \beta \leqslant \beta_W, \\ \infty, & \text{for } \beta > \beta_W \end{cases} \tag{10}$$

one obtains the E(5) model of Iachello [1] in which the eigenfunctions are Bessel functions $J_{\tau+3/2}(z)$ (with $z = \beta k$, $k = \sqrt{\epsilon}$), while the spectrum is determined by the zeros of the Bessel functions

$$E_{\xi,\tau} = \frac{\hbar^2}{2B} k_{\xi,\tau}^2, \quad k_{\xi,\tau} = \frac{x_{\xi,\tau}}{\beta_W}, \tag{11}$$

where $x_{\xi,\tau}$ is the ξth zero of the Bessel function $J_{\tau+3/2}(z)$. The spectra of the E(5) and Davidson cases become directly comparable by establishing the formal correspondence $n = \xi - 1$.

It is instructive to consider the ratios

$$R_L = \frac{E_{0,L} - E_{0,0}}{E_{0,2} - E_{0,0}}, \tag{12}$$

where the notation $E_{n,L}$ is used.

Table 1
R_L ratios (defined in Eq. (12)) for the ground state band of the Davidson potentials in the E(5) framework (Eq. (8)) for different values of the parameter β_0, compared to the O(6) exact results

L	R_L $\beta_0 = 5$	R_L $\beta_0 = 10$	R_L O(6)
4	2.494	2.500	2.500
6	4.475	4.498	4.500
8	6.935	6.996	7.000
10	9.861	9.991	10.000
12	13.242	13.483	13.500
14	17.064	17.471	17.500
16	21.312	21.954	22.000
18	25.969	26.930	27.000
20	31.020	32.398	32.500

For $\beta_0 = 0$ it is clear that the original vibrational model of Bohr [3] (with $R_4 = 2$) is obtained, while for large β_0 the O(6) limit of the Interacting Boson Model (IBM) [18] (with $R_4 = 2.5$) is approached [8]. The latter fact can be seen in Table 1, where the R_L ratios for two different values of the parameter β_0 are shown, together with the O(6) predictions (which correspond to $E(L) = AL(L+6)$, with A constant [19]). It is clear that the O(6) limit is approached as β_0 is increased, the agreement being already quite good at $\beta_0 = 5$.

It is useful to consider the ratios R_L, defined above, as a function of β_0. As seen in Fig. 1, where the ratios R_4, R_{12} and R_{20} are shown, these ratios increase with β_0, the increase becoming very steep at some value $\beta_{0,\max}$ of β_0, where the first derivative $dR_L/(d\beta_0)$ reaches a maximum value, while the second derivative $d^2R_L/(d\beta_0^2)$ vanishes. Numerical results for $\beta_{0,\max}$ are shown in Table 2, together with the values of R_L occurring at these points, which are compared to the R_L ratios occurring in the ground state band of the E(5) model [1]. Very close agreement of the values determined by the procedure described above with the E(5) values is observed in Table 2, as well as in Fig. 2, where these ratios are also shown, together with the corresponding ratios of the U(5) and O(6) limits.

The work performed in this section is reminiscent of a variational procedure. Wishing to determine the critical point in the shape phase transition from U(5) to O(6), one chooses a potential (the Davidson potential) with a free parameter (β_0), which helps in covering the whole range of interest. Indeed, for $\beta_0 = 0$ the U(5) picture is obtained, while large values of β_0 lead to the

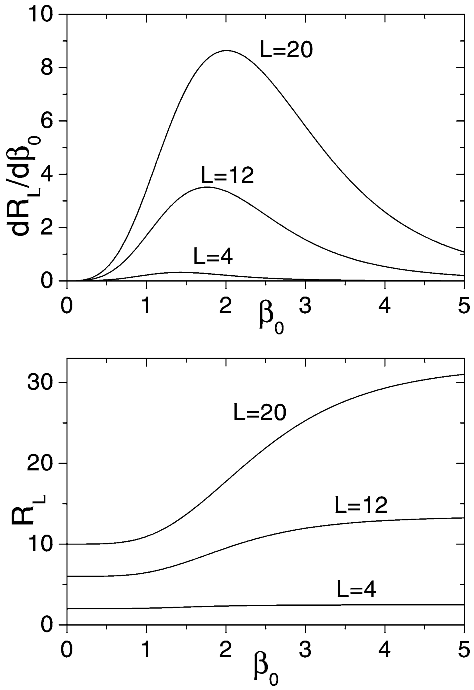

Fig. 1. The R_L ratios (defined in Eq. (12)) for $L = 4, 12, 20$ and their derivatives $dR_L/d\beta_0$ vs. the parameter β_0, calculated using Davidson potentials (Eq. (1)) in the E(5) framework. The R_L curves also demonstrate the evolution from the U(5) symmetry (on the left) to the O(6) limit (on the right). See Section 2 for further details.

Table 2
Parameter values $\beta_{0,\max}$ where the first derivative of the energy ratios R_L (defined in Eq. (12)) in the E(5) framework has a maximum, while the second derivative vanishes, together with the R_L ratios obtained at these values (labeled by "var") and the corresponding ratios of the E(5) model, for several values of the angular momentum L

L	$\beta_{0,\max}$	R_L var	R_L E(5)
4	1.421	2.185	2.199
6	1.522	3.549	3.590
8	1.609	5.086	5.169
10	1.687	6.793	6.934
12	1.759	8.667	8.881
14	1.825	10.705	11.009
16	1.888	12.906	13.316
18	1.947	15.269	15.799
20	2.004	17.793	18.459

O(6) limit. One then needs a physical quantity which can serve as a "measure" of collectivity. For this purpose one considers the ratios R_L, encouraged by the

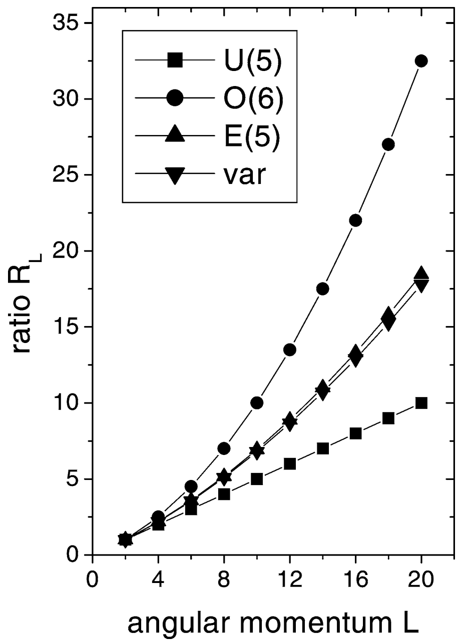

Fig. 2. Values of the ratio R_L (defined in Eq. (12)) obtained through the variational procedure (labeled by "var") using Davidson potentials in the E(5) framework, compared to the values provided by the U(5), O(6), and E(5) models. See Section 2 for further details.

fact that these ratios are well-known indicators of collectivity in nuclear structure [20]. Since at the critical point (if any) one expects the collectivity to change very rapidly, one looks, for each R_L ratio separately, for the value of the parameter at which the change of R_L is maximum. Indeed, the first derivative of the ratio R_L with respect to the parameter β_0 exhibits a sharp maximum, which is then a good candidate for being the critical point for this particular value of the angular momentum L. The R_L values at the critical points corresponding to each value of L form a collection, which should correspond to the behaviour of the ground state band of a nucleus at the critical point. The infinite well potential used in E(5) succeeds in reproducing all these "critical" R_L ratios in the ground state band for all values of the angular momentum L, *without using any free parameter*. It is therefore proved that the infinite well potential is indeed the optimum choice for describing the ground state bands of nuclei at the critical point of the U(5) to O(6) shape phase transition.

In other words, starting from the Davidson potentials and using a variational procedure, according to which the rate of change of the R_L ratios as a function of the parameter β_0 is maximized for each value of the angular momentum L separately, one forms the collection of critical values of R_L which corresponds to the ground state band of the E(5) model, which is supposed to describe nuclei at the critical point.

Variational procedures in which each value of the angular momentum L is treated separately are not unheard of in nuclear physics. An example is given by the variable moment of inertia (VMI) model [12], in which the energy of the nucleus is minimized with respect to the (angular momentum dependent) moment of inertia for each value of the angular momentum separately. From the cubic equation obtained from this condition, the moment of inertia is uniquely determined (as a function of angular momentum) in each case. The collection of energy levels occurring by using in the energy formula the appropriate value of the moment of inertia for each value of the angular momentum L forms the ground state band of the nucleus.

Some comparison of the variational procedure used here with the standard Ritz variational method used in quantum mechanics ([21], for example) is in place. In the (simplest version of the) Ritz variational method a trial wave function containing a parameter is chosen and subsequently the energy is minimized with respect to this parameter, thus determining the parameter value and, after the relevant substitution, the energy value. In the present case a trial potential containing a parameter is chosen and subsequently the rate of change of the physical quantity (here the rate of change of the energy ratios) is maximized with respect to this parameter, thus determining the parameter value and, after the relevant calculation, the value of the physical quantity (here the energy ratios). The main similarity between the two methods is the use of a parameter-dependent trial wave function/trial potential, respectively. The main difference between the two methods is that in the former the relevant physical quantity (the energy) is minimized with respect to the parameter, while in the latter the rate of change of the physical quantity (the energy ratios) is maximized with respect to the parameter.

3. Davidson potentials in the X(5) framework

Starting again from the original Bohr Hamiltonian of Eq. (2), one seeks solutions of the relevant Schrödinger equation having the form $\Psi(\beta,\gamma,\theta_i) = \phi_K^L(\beta,\gamma)\mathcal{D}^L_{M,K}(\theta_i)$, where θ_i ($i = 1, 2, 3$) are the Euler angles, $\mathcal{D}(\theta_i)$ denote Wigner functions of them, L are the eigenvalues of angular momentum, while M and K are the eigenvalues of the projections of angular momentum on the laboratory-fixed z-axis and the body-fixed z'-axis, respectively.

As pointed out in Ref. [2], in the case in which the potential has a minimum around $\gamma = 0$ one can write the last term of Eq. (2) in the form

$$\sum_{k=1,2,3} \frac{Q_k^2}{\sin^2(\gamma - \frac{2\pi}{3}k)} \approx \frac{4}{3}\big(Q_1^2 + Q_2^2 + Q_3^2\big) + Q_3^2\bigg(\frac{1}{\sin^2\gamma} - \frac{4}{3}\bigg). \tag{13}$$

Using this result in the Schrödinger equation corresponding to the Hamiltonian of Eq. (2), introducing reduced energies $\epsilon = 2BE/\hbar^2$ and reduced potentials $u = 2BV/\hbar^2$, and assuming that the reduced potential can be separated into two terms, one depending on β and the other depending on γ, i.e., $u(\beta,\gamma) = u(\beta) + u(\gamma)$, the Schrödinger equation can be separated into two equations [2], the “radial” one being

$$\bigg[-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{4\beta^2}\frac{4}{3}L(L+1) + u(\beta)\bigg]\xi_L(\beta) = \epsilon_\beta \xi_L(\beta). \tag{14}$$

When plugging the Davidson potentials of Eq. (1) in this equation, the β_0^4/β^2 term of the potential is combined with the $L(L+1)/3\beta^2$ term appearing there and the equation is solved exactly, the eigenfunctions being Laguerre polynomials of the form

$$F_n^L(\beta) = \bigg[\frac{2n!}{\Gamma(n+a+\frac{5}{2})}\bigg]^{1/2} \beta^a L_n^{a+\frac{3}{2}}\big(\beta^2\big)e^{-\beta^2/2}, \tag{15}$$

where a is given by

$$a = -\frac{3}{2} + \bigg[\frac{1}{3}L(L+1) + \frac{9}{4} + \beta_0^4\bigg]^{1/2}. \tag{16}$$

The energy eigenvalues are then (in $\hbar\omega = 1$ units)

$$E_{n,L} = 2n + a + \frac{5}{2} = 2n + 1 + \bigg[\frac{1}{3}L(L+1) + \frac{9}{4} + \beta_0^4\bigg]^{1/2}. \tag{17}$$

The levels of the ground state band are characterized by $n = 0$. Then the excitation energies relative to the ground state are given by

$$E_{0,L,\mathrm{exc}} = \left[\frac{1}{3}L(L+1) + \frac{9}{4} + \beta_0^4\right]^{1/2} - \left[\frac{9}{4} + \beta_0^4\right]^{1/2}, \tag{18}$$

which can easily be put into the form

$$E'_{0,L,\mathrm{exc}} = \frac{E_{0,L,\mathrm{exc}}}{[\frac{9}{4} + \beta_0^4]^{1/2}} = \left[1 + \frac{L(L+1)}{3(\frac{9}{4} + \beta_0^4)}\right]^{1/2} - 1, \tag{19}$$

which is the same as the Holmberg–Lipas formula [13]

$$E_H(L) = a_H\left(\sqrt{1 + b_H L(L+1)} - 1\right), \tag{20}$$

with $a_H = 1$

$$b_H = \frac{1}{3(\frac{9}{4} + \beta_0^4)}. \tag{21}$$

It is clear that the Holmberg–Lipas formula gives rotational spectra for small values of b_H, at which one can keep only the first L-dependent term in the Taylor expansion of the square root appearing in Eq. (20), leading to energies proportional to $L(L+1)$. From Eq. (21) it is then clear that rotational spectra are expected for large values of β_0. This can be seen in Table 3, where the R_L ratios occurring for two different values of β_0 are shown, together with the predictions of the SU(3) limit of IBM, which correspond to the pure rotator with $E(L) = AL(L+1)$, where A constant [18]. The agreement to the SU(3) results is quite good already at $\beta_0 = 5$. On the other hand, the case $\beta_0 = 0$ corresponds to an exactly soluble model with $R_4 = 2.646$, which has been called the X(5)–β^2 model [22].

Table 3
R_L ratios (defined in Eq. (12)) for the ground state band of the Davidson potentials in the X(5) framework (Eq. (17)) for different values of the parameter β_0, compared to the SU(3) exact results

L	R_L $\beta_0 = 5$	R_L $\beta_0 = 10$	R_L SU(3)
4	3.327	3.333	3.333
6	6.967	6.998	7.000
8	11.897	11.993	12.000
10	18.087	18.317	18.333
12	25.503	25.968	26.000
14	34.102	34.941	35.000
16	43.839	45.233	45.333
18	54.665	56.841	57.000
20	66.530	69.760	70.000

It is worth remarking at this point that the Holmberg–Lipas formula can be derived [19] by assuming that the moment of inertia I in the energy expression of the rigid rotator ($E(L) = L(L+1)/2I$) is a function of the excitation energy, i.e., $I = \alpha + \beta E(L)$, where α and β are constants, the latter being proportional to b_H and acquiring positive values. It is therefore clear that the Holmberg–Lipas formula, as well as the spectrum of the Davidson potentials derived in this section, have built-in the concept of the variable moment of inertia (VMI) model [12], according to which the moment of inertia is an increasing function of the angular momentum.

For $u(\beta)$ being a 5D infinite well potential (see Eq. (10)) one obtains the X(5) model of Iachello [2], in which the eigenfunctions are Bessel functions $J_\nu(k_{s,L}\beta)$ with

$$\nu = \left(\frac{L(L+1)}{3} + \frac{9}{4}\right)^{1/2}, \tag{22}$$

while the spectrum is determined by the zeros of the Bessel functions, the relevant eigenvalues being

$$\epsilon_{\beta;s,L} = (k_{s,L})^2, \quad k_{s,L} = \frac{x_{s,L}}{\beta_W}, \tag{23}$$

where $x_{s,L}$ is the sth zero of the Bessel function $J_\nu(k_{s,L}\beta)$. The spectra of the X(5) and Davidson cases become directly comparable by establishing the formal correspondence $n = s - 1$.

It is useful to consider the ratios R_L, defined in the previous section, as a function of β_0. As seen in Fig. 3, these ratios again increase with β_0, the increase becoming very steep at some value $\beta_{0,\max}$ of β_0, where the first derivative $dR_L/(d\beta_0)$ reaches a maximum value, while the second derivative $d^2R_L/(d\beta_0^2)$ vanishes. Numerical results for $\beta_{0,\max}$ are shown in Table 4, together with the values of R_L occurring at these points, which are compared to the R_L ratios occurring in the ground state band of the X(5) model [2]. Very close agreement of the values determined by the procedure described above with the X(5) values is observed.

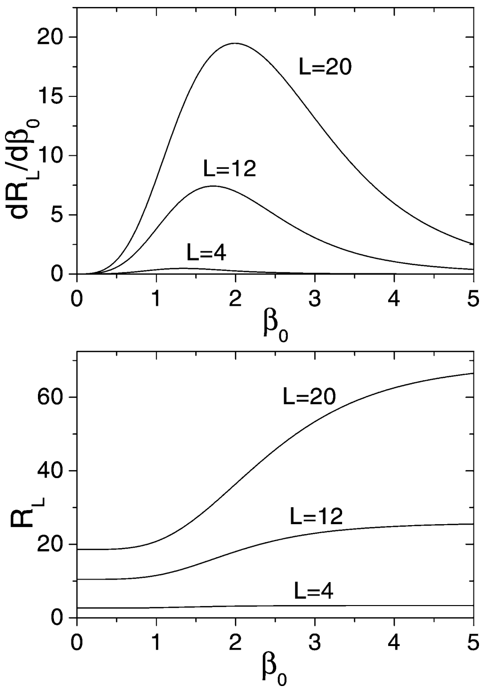

Fig. 3. The R_L ratios (defined in Eq. (12)) for $L = 4$, 12, 20 and their derivatives $dR_L/d\beta_0$ vs. the parameter β_0, calculated using Davidson potentials (Eq. (1)) in the X(5) framework. The R_L curves also demonstrate the evolution from the X(5)–β^2 symmetry (on the left) to the SU(3) limit (on the right). See Section 3 for further details.

Table 4
Parameter values $\beta_{0,\max}$ where the first derivative of the energy ratios R_L (defined in Eq. (12)) in the X(5) framework has a maximum, while the second derivative vanishes, together with the R_L ratios obtained at these values (labeled by "var") and the corresponding ratios of the X(5) model, for several values of the angular momentum L

L	$\beta_{0,\max}$	R_L var	R_L X(5)
4	1.334	2.901	2.904
6	1.445	5.419	5.430
8	1.543	8.454	8.483
10	1.631	11.964	12.027
12	1.711	15.926	16.041
14	1.785	20.330	20.514
16	1.855	25.170	25.437
18	1.922	30.442	30.804
20	1.985	36.146	36.611

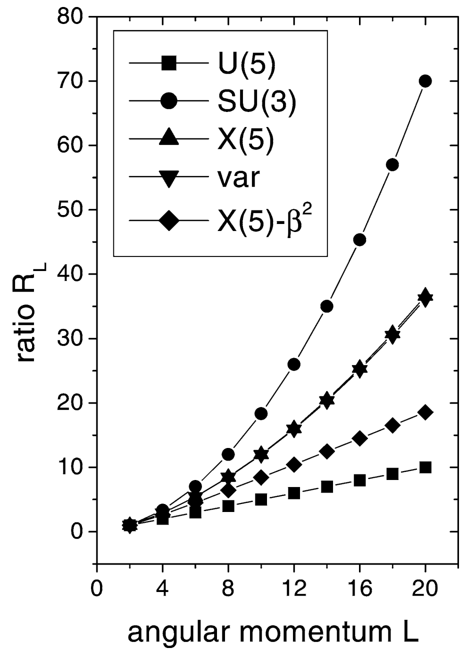

Fig. 4. Values of the ratio R_L (defined in Eq. (12)) obtained through the variational procedure (labeled by "var") using Davidson potentials in the X(5) framework, compared to the values provided by the U(5), SU(3), X(5), and X(5)–β^2 models. See Section 3 for further details.

The work performed here is reminiscent of a variational procedure, as in the previous section. Wishing to determine the critical point in the shape phase transition from U(5) to SU(3), one chooses a potential (the Davidson potential) with a free parameter (β_0), which serves in spanning the range of interest. For large values of β_0 the SU(3) limit is obtained, while for $\beta_0 = 0$ the X(5)–β^2 picture is obtained [22], which is not the U(5) limit, but it is located between U(5) and X(5), on the way from U(5) to SU(3). Thus the region of interest around X(5) is covered from X(5)–β^2 to SU(3). Then the values of β_0 at which the first derivative $dR_L/d\beta_0$ exhibits a sharp maximum are determined for each value of the angular momentum L separately, the collection of R_L ratios at these values of β_0 forming a band, which turns out to be in very good agreement with the ground state band of X(5), the model supposed to be appropriate for describing nuclei at the critical point in the transition from U(5) to SU(3), thus indicating that the choice of the infinite well potential used in the X(5) model is the optimum one. The results are depicted in Fig. 4, where in addition to the bands provided by the variational procedure and the X(5) model, the bands corresponding to the U(5), X(5)–β^2, and SU(3) cases are shown.

4. Discussion

The main results and conclusions obtained in the present Letter are listed here:

(1) A variational procedure for determining the values of physical quantities at the point of shape phase transitions in nuclei has been suggested. Using one-parameter potentials spanning the region between the two limiting symmetries of interest, the parameter values at which the rate of change of the physical quantity becomes maximum are determined for each value of the angular momentum separately and the corresponding values of the physical quantity at these parameter values are calculated. The values of the physical quantity collected in this way represent its behaviour at the critical point.

(2) The method has been applied in the shape phase transition from U(5) to O(6), using one-parameter Davidson potentials [10] and considering the energy ratios $R_L = E(L)/E(2)$ within the ground state band as the relevant physical quantity, leading to a band which practically coincides with the ground state band of the E(5) model [1]. It has also been applied in the same way in the shape phase transition from U(5) to SU(3), leading to a band which practically coincides with the ground state band of the X(5) model [2].

(3) It should be emphasized that the application of the method was possible because the Davidson potentials correctly reproduce the U(5) and O(6) symmetries in the former case (for small and large parameter values, respectively), as well as the relevant X(5)–β^2 [22] and SU(3) symmetries in the latter case (for small and large parameter values, respectively).

(4) As a by-product, a derivation of the Holmberg–Lipas formula [13] has been achieved using Davidson potentials in the X(5) framework.

It is clearly of interest to apply the variational procedure introduced here to physical quantities other than the energy ratios in the ground state band. Energy ratios involving levels of excited bands, ratios of B(E2) transition rates (both intraband and interband), and ratios of quadrupole moments are obvious choices. Work in these directions is in progress, using the Davidson potentials, since they possess the appropriate limiting behaviour for small and large parameter values. However, any other potential/Hamiltonian bridging the relevant pairs of symmetries (U(5)–O(6) and U(5)–SU(3)) should be equally appropriate.

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