Range-kernel orthogonality of the elementary operator $X \rightarrow \sum_{i=1}^{n} A_i X B_i - X$

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Abstract

Let $H$ be a separable infinite dimensional complex Hilbert space and let $B(H)$ denote the algebra of operators on $H$ into itself. Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be $n$-tuples in $B(H)$. Define the elementary operators $\Delta_{AB}$ and $\Delta^*_{AB} : B(H) \rightarrow B(H)$ by $\Delta_{AB}(X) = \sum_{i=1}^{n} A_i X B_i - X$ and $\Delta^*_{AB}(X) = \sum_{i=1}^{n} A_i^* X B_i^* - X$. This note considers the range-kernel orthogonality of the restrictions of $\Delta_{AB}$ and $\Delta^*_{AB}$ to Schatten $p$-classes $\mathcal{C}_p$. It is proved that:

(a) if $1 < p < \infty$, $S \in \mathcal{C}_p$ and $\sum_{i=1}^{n} A_i^* A_i$, $\sum_{i=1}^{n} A_i A_i^*$, $\sum_{i=1}^{n} B_i^* B_i$ and $\sum_{i=1}^{n} B_i B_i^*$ are all $\leq 1$, then $\min\{\|\Delta_{AB}(X) + S\|_p, \|\Delta^*_{AB}(X) + S\|_p\} \geq \|S\|_p$ for all $X \in \mathcal{C}_p$ if and only if $\Delta_{AB}(S) = 0 = \Delta^*_{AB}(S)$;

(b) if $p = 2$ and $S \in \mathcal{C}_2$, then $\|\Delta_{AB}(X) + S\|_2 = \|\Delta_{AB}(X)\|_2 + \|S\|_2$ and $\|\Delta^*_{AB}(X) + S\|_2 = \|\Delta^*_{AB}(X)\|_2 + \|S\|_2$ if and only if $\Delta_{AB}(S) = 0 = \Delta^*_{AB}(S)$; and

(c) if $A$ and $B$ are the $n$-tuples of (a) such that $\Delta_{BB}(S) = 0 = \Delta^*_{BB}(S)$ for some injective $S \in \mathcal{C}_1$, then the inequality of (a) holds (with $p = 1$ and) for all $X \in \mathcal{C}_1$ if and only if $\Delta_{AB}(S) = 0 = \Delta^*_{AB}(S)$.

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1. Introduction

Let $H$ be a separable infinite dimensional complex Hilbert space and let $B(H)$ denote the algebra of operators (= bounded linear transformations) on $H$ into itself. Given $A, B \in B(H)$, the generalised derivation $\delta_{A,B} : B(H) \to B(H)$ (the elementary operator $\delta_{A,B} : B(H) \to B(H)$) is defined by $\delta_{A,B}(X) = AX - BX$ (respectively, $\delta_{A,B}(X) = AXB - X$). Let $d_{A,B}$ denote either $\delta_{A,B}$ or $\delta_{A,B}$. Recall that if $M$ and $N$ are subspaces of a Banach space $V$ with norm $\| \cdot \|$, $M$ is said to be orthogonal to $N$ if $\| m + n \| \geq \| n \|$ for all $m \in M$ and $n \in N$. The range-kernel orthogonality of the operator $d_{A,B}$ has been considered by a number of authors in the recent past (see [4,5,8,9] and some of the references cited there), with the first such result proved by Anderson in [2]. Anderson [2] proved that if $A \in B(H)$ is a normal operator and $S \in B(H)$ is in the commutant of $A$, then $\| \delta_{AA}(X) + S \| \geq \| S \|$ for all $X \in B(H)$. This result has a $\triangle$ analogue: indeed it is known that if $A$ and $B^*$ satisfy a normality-like hypothesis and $d_{AB}(S) = 0$ for some $S \in B(H)$, then $\|d_{AB}(X) + S\| \geq \| S \|$ for all $X \in B(H)$ (see [4,8] for further details).

For a compact operator $X$, let $s_1(X) \geq s_2(X) \geq \cdots \geq 0$ denote the singular values of $X$ (i.e., the eigenvalues of $|X| = |X^*X|^{1/2}$) arranged in their decreasing order. The operator $X$ is said to belong to the Schatten $p$-class $\mathcal{C}_p$ if

$$\| X \|_p = \left\{ \left( \sum_{j=1}^{\infty} s_j(X)^p \right)^{1/p} = \text{tr}(|X|^p) \right\}^{1/p} < \infty, \quad 1 \leq p < \infty,$$

where ‘tr’ denotes the trace functional. The range-kernel orthogonality of $d_{A,B}|\mathcal{C}_p$, the restriction of $d_{A,B}$ to $\mathcal{C}_p$, and more generally for the class of unitarily invariant norms, has been considered in a number of papers (see [5,8,9] for further references), and it is known that if $S \in \mathcal{C}_p$ for some $1 < p < \infty$, then min$\{\| d_{AB}(X) + S \|_p \}$, $\| d_{AB}(S) \|_p \geq \| S \|_p$ for all $X \in \mathcal{C}_p$ if and only if $d_{AB}(S) = 0 = d_{AB}^*(S)$ [5, Theorem (iii)].

Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be $n$-tuples of operators and define the elementary operators $\triangle_{AB}$ and $\triangle_{AB}^* : B(H) \to B(H)$ (of length $n + 1$) by $\triangle_{AB}(X) = \sum_{i=1}^{n} A_iXB_i - X$ and $\triangle_{AB}^*(X) = \sum_{i=1}^{n} A_i^*XB_i^* - X$. The range-kernel orthogonality of $\triangle_{AB}|\mathcal{C}_p$, $1 \leq p < \infty$, has recently been considered by Turnšek in [10], where it is shown that if $\sum_{i=1}^{n} A_i^*A_i$, $\sum_{i=1}^{n} A_iA_i^*$, $\sum_{i=1}^{n} B_iB_i^*$ and $\sum_{i=1}^{n} B_iB_i^*$ are all $\leq 1$, and if $\| d_{AB}(S) = 0 \| \| \triangle_{AB}^*(X) + S \|_p \geq \| S \|_p$ for all $X \in \mathcal{C}_p$. Here, the stronger result that $\| \triangle_{AB}(X) + S \|_2^2 = \| \triangle_{AB}(X) \|_2^2 + \| S \|_2^2 = \| \triangle_{AB}^*(X) + S \|_2^2$ holds in the case in which $(p = 2)$ and the $n$-tuples $A, B$ consist of mutually commuting normal operators.

The purpose of this note is to extend these results to prove the following necessary and sufficient condition.
Theorem 1.

(i) Let 1 < p < ∞ and let S ∈ ℂ_p. Suppose that the n-tuples A and B are such that Σ^n_{i=1} A^* A_i, Σ^n_{i=1} A_i A^* i, Σ^n_{i=1} B^*_i B_i, and Σ^n_{i=1} B_i B^*_i are all ≤ 1. Then

\[ \min \{ \| \Delta_{AB}(X) + S \|_p, \| \Delta^*_AB(X) + S \|_p \} \geq \| S \|_p \]

for all X ∈ ℂ_p if and only if Δ_{AB}(S) = 0 = Δ^*_AB(S).

(ii) Let p = 2 and let S ∈ ℂ_2. Then

\[ \| \Delta_{AB}(X) + S \|_2^2 = \| \Delta_{AB}(X) \|_2^2 + \| S \|_2^2, \]

\[ \| \Delta^*_AB(X) + S \|_2^2 = \| \Delta^*_AB(X) \|_2^2 + \| S \|_2^2 \]

if and only if Δ_{AB}(S) = 0 = Δ^*_AB(S).

(iii) Let p = 2 and let S ∈ ℂ_2. If A and B are n-tuples of mutually commuting normal operators, then \( \| \Delta_{AB}(X) + S \|_2^2 = \| \Delta^*_AB(X) \|_2^2 + \| S \|_2^2 = \| \Delta^*_AB(X) \|_2^2 + \| S \|_2^2 \) for all X ∈ ℂ_2 if and only if Δ_{AB}(S) = 0.

(iv) If A and B are n-tuples of part (i) and S ∈ ℂ_1 is injective, then inequality (1) holds (with p = 1 and) for all X ∈ ℂ_1 if and only if Δ_{AB}(U) = 0 = Δ^*_AB(U).

If also Δ_{BB}(S) = 0 = Δ^*_BB(S), then inequality (1) holds (with p = 1 and) for all X ∈ ℂ_1 if and only if Δ_{AB}(S) = 0 = Δ^*_AB(S).

(v) If A and B are n-tuples of part (i) such that Δ_{AB}(S) = 0 = Δ^*_AB(S) for some S ∈ ℂ_1, then inequality (1) holds (with p = 1 and) for all X ∈ ℂ_1.

2. The proof

In addition to the notation already introduced, the following further notation will be used. We shall denote the set of real numbers by ℜ, the set of complex numbers by ℂ and the real part of a complex number λ by Re λ. The index conjugate to the index p will be denoted by p' (i.e., 1/p + 1/p' = 1). The closure of the range (the orthogonal complement of the kernel) of an operator X will be denote by ran X (respectively, ker X). The operator X is said to be a quasi-affinity if both X and X* have dense range.

Before going on to prove the theorem we state a couple of complementary results. To this end recall that if V is a Banach space with norm \( \| \cdot \| \), then \( \| \cdot \| \) is said to be Gateaux-differentiable at a non-zero \( x \in V \) if

\[ \lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t} = \text{Re} \ D_x(y), \]

exists for all \( y \in V \). Here \( D_x \) is the unique support functional in the dual space \( V^* \) of V such that \( \| D_x \| = 1 \) and \( \| D_x(x) \| = \| x \| \). The Gateaux-differentiability of \( \| \cdot \| \) at \( x \) implies that \( x \) is a smooth point of the sphere with radius \( \| x \| \). Since \( ℂ_p, 1 <
If \( p < \infty \), is a uniformly convex space, each \( S \in \mathcal{C}_p \) is a smooth point. Let \( S \in \mathcal{C}_p \) and \( 1 < p < \infty \), have the polar decomposition \( S = U|S| \). Then \( |S|^{p-1}U^* \in \mathcal{C}_p' \) and \( D_S(Y) = \text{tr}(|S|^{p-1}U^*Y/\|S\|_p^{p-1}) \) for any \( Y \in \mathcal{C}_p \) [1, Theorem 2.3]. Recall now that if \( u \) and \( v \) are elements of \( V \) and \( u \) is a smooth point of \( V \), then \( \|u + tv\| \geq \|u\| \) for all \( t \in C \) (i.e., \( u \) is orthogonal to \( v \)) if and only if \( D_u(v) = 0 \) (see [7] or [9, Lemma 1]).

**Lemma 2.** Let \( C \) denote the \( n \)-tuple of operators \( (C_1, C_2, \ldots, C_n) \) and let \( \triangle_C : B(H) \to B(H) \) be the elementary operator \( \triangle_C(X) = \sum_{i=1}^n C_i XC_i - X \). Suppose that \( S = U|S| \in \mathcal{C}_p ; 1 < p < \infty \). Then

\[
\|\triangle_C(X) + S\|_p \geq \|S\|_p
\]

for all \( X \in B(H) \) such that \( \triangle_C(X) \in \mathcal{C}_p \) \((1 < p < \infty)\) if and only if \( \text{tr}(|S|^{p-1}U^*\triangle_C(X)) = 0 \).

The proof of the lemma is an immediate consequence of the above. (See also [9, Theorem 1] and [5, Lemma 2].)

Suppose now that the \( n \)-tuple \( C \) of Lemma 2 satisfies the property that \( \sum_{i=1}^n C_i C_i^* \leq 1 \) and \( \sum_{i=1}^n C_i^* C_i \leq 1 \). Let \( T \) be a compact operator such that \( \triangle_C(T) = 0 = \triangle_C^*(T) \). Let

\[
E_1 = [C_1 \ C_2 \ \cdots \ C_n], \quad E_2 = [C_1 \ C_2 \ \cdots \ C_n]^t, \\
F_1 = [C_1^* \ C_2^* \ \cdots \ C_n^*], \quad F_2 = [C_1^* \ C_2^* \ \cdots \ C_n^*]^t,
\]

and let \( l_n \) denote the identity of \( M_n(C) \). Then

\[
|T|^2 \leq E_2^*(|T|^2 \otimes l_n)E_2, \quad |T|^2 \leq F_2^*(|T|^2 \otimes l_n)F_2
\]

and it follows from a generalisation of [3, Theorem 8], see [10, Lemma 2.3], that the eigenspaces corresponding to distinct non-zero eigenvalues of the compact positive operator \(|T|^2\) reduce each \( C_i \). In particular, we have:

**Lemma 3.** Suppose that \( \sum_{i=1}^n C_i C_i^* \leq 1 \) and \( \sum_{i=1}^n C_i^* C_i \leq 1 \). If \( \triangle_C(T) = 0 = \triangle_C^*(T) \) for some compact operator \( T \), then \( |T| \) commutes with \( C_i \) for all \( 1 \leq i \leq n \).

**Proof of Theorem 1.** (i) Define \( C_i \) \((1 \leq i \leq n)\), \( \hat{X} \) and \( \hat{S} \in B(\hat{H}) \), \( \hat{H} = H \oplus H \), by

\[
C_i = \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \quad \hat{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{S} = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}.
\]

Then (both) \( \hat{X} \) and \( \hat{S} \in \mathcal{C}_p(\hat{H}) \); \( 1 < p < \infty \). Let \( \triangle_C \) and \( \triangle_C^* : B(\hat{H}) \to B(\hat{H}) \) be the elementary operators defined by \( \triangle_C(T) = \sum_{i=1}^n C_i T C_i - T \) and \( \triangle_C^*(T) = \sum_{i=1}^n C_i^* T C_i^* - T \). Then inequality (1) holds if and only if

\[
\min\{\|\triangle_C(\hat{X}) + \hat{S}\|_p, \|\triangle_C^*(\hat{X}) + \hat{S}\|_p\} \geq \|\hat{S}\|_p.
\]
Let $S$ have the polar decomposition $S = U|S|$ and define the operator $\tilde{S}$ by

$$
\tilde{S} = \begin{bmatrix}
0 & 0 \\
|S|^{p-1}U^* & 0
\end{bmatrix}.
$$

Then $\tilde{S} \in \mathcal{C}_p'(\hat{H})$ and it follows from Lemma 2 that (3) holds if and only if

$$
\text{tr}(\tilde{S} \triangle C(\hat{X})) = 0 = \text{tr}(\tilde{S} \triangle^* C(\hat{X})).
$$

Choosing $X \in \mathcal{C}_p$ to be the rank-1 operator $x \otimes y$ it follows from (4) that if (1) holds, then

$$
\text{tr}(\tilde{S} \triangle C(\hat{X})) = \text{tr} \left( \sum_{i=1}^n \tilde{S} C_i \hat{X} C_i - \tilde{S} \hat{X} \right)
$$

$$
= \text{tr} \left( \sum_{i=1}^n C_i \tilde{S} C_i - \tilde{S} \right) \hat{X}
$$

$$
= \text{tr} \left( \sum_{i=1}^n B_i |S|^{p-1}U^* A_i - |S|^{p-1}U^* \right)(x \otimes y)
$$

$$
= \left( \sum_{i=1}^n B_i |S|^{p-1}U^* A_i x, y \right) - (|S|^{p-1}U^* x, y) = 0
$$

and

$$
\text{tr}(\tilde{S} \triangle^* C(\hat{X})) = \left( \sum_{i=1}^n B_i^* |S|^{p-1}U^* A_i^* x, y \right) - (|S|^{p-1}U^* x, y) = 0
$$

for all $x, y \in H$. Hence

$$
\sum_{i=1}^n B_i |S|^{p-1}U^* A_i - |S|^{p-1}U^* = 0 = \sum_{i=1}^n B_i^* |S|^{p-1}U^* A_i^* - |S|^{p-1}U^*,
$$

or,

$$
\triangle_{AB}(U|S|^{p-1}) = 0 = \triangle^*_{AB}(U|S|^{p-1}).
$$

Let $T \in B(\hat{H})$ be defined by

$$
T = \begin{bmatrix}
0 & U|S|^{p-1} \\
0 & 0
\end{bmatrix}.
$$
Then \( T \) is compact and \( \triangle_C(T) = 0 = \triangle_C^*(T) \). Applying Lemma 3 it now follows that \(|T|\) commutes with \( C_i \) for all \( 1 \leq i \leq n \). This implies that \(|S|^{p-1}\), and so also \(|S|\), commutes with \( B_i \) for all \( 1 \leq i \leq n \); hence, by (5),

\[
\triangle_{AB}(S) = 0 = \triangle_{AB}^*(S).
\] (6)

Conversely, assume that (6) is satisfied. Then an argument similar to the one above shows that \(|S|\) commutes with \( B_i \) for all \( 1 \leq i \leq n \), and hence that (5) is satisfied. Recall that if \( Y \in \mathcal{C}_p \) and \( Z \in \mathcal{C}_p' \), then \( YZ \in \mathcal{C}_1 \) and \( \text{tr}(YZ) = \text{tr}(ZY) \). Since \( X \in \mathcal{C}_p \) and \(|S|^{p-1}U^* \in \mathcal{C}_p' \), it follows that

\[
\text{tr}(\tilde{\theta} \triangle_C(\hat{X})) = \text{tr} \left( |S|^{p-1}U^* \sum_{i=1}^{n} A_i X B_i - |S|^{p-1}U^* X \right) = 
\text{tr} \left( \sum_{i=1}^{n} |S|^{p-1}U^* A_i X B_i \right) - \text{tr}(|S|^{p-1}U^* X) = 
\text{tr} \left( X \sum_{i=1}^{n} B_i |S|^{p-1}U^* A_i \right) - \text{tr}(X|S|^{p-1}U^*) = 
\text{tr} \left( X \left\{ \sum_{i=1}^{n} B_i |S|^{p-1}U^* A_i - |S|^{p-1}U^* \right\} \right) = 0
\]

and

\[
\text{tr}(\tilde{\theta} \triangle_C^*(\hat{X})) = \text{tr} \left( X \left\{ \sum_{i=1}^{n} B_i^* |S|^{p-1}U^* A_i^* - |S|^{p-1}U^* \right\} \right) = 0.
\]

This completes the proof. \( \square \)

**The case** \( p = 2 \). In the case in which \( p = 2 \), \( \mathcal{C}_2(\hat{H}) \) has a Hilbert space structure with inner product \( \langle Y, Z \rangle = \text{tr}(Z^*Y) \). Since \(|S|^{p-1}U^* = S^* \) in this case, it follows from (4) that (1) holds (with \( p = 2 \)) if and only if

\[
\langle \triangle_{AB}(S), X \rangle = 0 = \langle \triangle_{AB}^*(S), X \rangle
\]

for all \( X \in \mathcal{C}_2 \). Consequently, (1) holds in this case for all \( n \)-tuples \( A \) and \( B \) if and only if \( \triangle_{AB}(S) = 0 = \triangle_{AB}^*(S) \).

To prove part (ii) of Theorem 1, we notice that

\[
\|\triangle_{AB}(X) + S\|_2^2 = \|\triangle_{AB}(X)\|_2^2 + \|S\|_2^2 + 2 \text{ Re} \langle \triangle_{AB}(X), S \rangle = \|\triangle_{AB}(X)\|_2^2 + \|S\|_2^2 + 2 \text{ Re} \langle X, \triangle_{AB}^*(S) \rangle
\]

and

\[
\|\triangle_{AB}^*(X) + S\|_2^2 = \|\triangle_{AB}^*(X)\|_2^2 + \|S\|_2^2 + 2 \text{ Re} \langle X, \triangle_{AB}(S) \rangle.
\]

Hence if \( \triangle_{AB}(S) = 0 = \triangle_{AB}^*(S) \), then (2) holds.
Consider now the case in which the \( n \)-tuples \( \mathbf{A} \) and \( \mathbf{B} \) consist of mutually commuting normal operators. Then

\[
\|\triangle_{\mathbf{AB}}(X)\|_2^2 = \|\triangle_{\mathbf{AB}}^*(X)\|_2^2
\]

for all \( X \in \mathcal{C}_2 \) (see [11, Theorem 1]). The proof of Theorem 1(iii) thus follows from part (ii).

The case \( p = 1 \). We recall here that the norm of \( \mathcal{C}_1 \) is Gateaux-differentiable at an \( S \in \mathcal{C}_1 \) if and only if either \( S \) or \( S^* \) is injective, and then \( DS(Y) = \text{tr}(U^*Y) \) or \( \text{tr}(UY) \), depending upon whether \( S \) or \( S^* \) is injective, for all \( Y \in \mathcal{C}_1 \) [1, Theorem 2.2]. Thus if \( S \in \mathcal{C}_1 \) is injective, then a necessary and sufficient condition for (1) to hold (with \( p = 1 \) and all \( X \in \mathcal{C}_1 \)) is \( \text{tr}(U^*\triangle_{\mathbf{AB}}(X)) = 0 = \text{tr}(U^*\triangle_{\mathbf{AB}}^*(X)) \).

Choosing \( X \in \mathcal{C}_1 \) to be the rank-1 operator \( x \otimes y \) (as in the proof of part (i)) it now follows that a necessary and sufficient condition for (1) to hold (with \( p = 1 \) and all \( X \in \mathcal{C}_1 \)) is that

\[
\triangle_{\mathbf{AB}}(U) = 0 = \triangle_{\mathbf{AB}}^*(U).
\]

Now if \( \triangle_{\mathbf{BB}}(S) = 0 = \triangle_{\mathbf{BB}}^*(S) \), then (by Lemma 3) \( |S| \) commutes with \( B_i \) for all \( 1 \leq i \leq n \); hence (1) holds (with \( p = 1 \) and) for all \( X \in \mathcal{C}_1 \) if and only if \( \triangle_{\mathbf{AB}}(S) = 0 = \triangle_{\mathbf{AB}}^*(S) \). This proves (iv).

To prove (v), assume that \( \triangle_{\mathbf{AB}}(S) = 0 = \triangle_{\mathbf{AB}}^*(S) \) for some \( S \in \mathcal{C}_1 \). Then an argument similar to the one used before (apply the argument preceding the statement of Lemma 3 to \( \hat{S}^2 \leq E_2^2(|\hat{S}|^2 \otimes 1_n)E_2 \) and \( |\hat{S}^*|^2 \leq E_1(|\hat{S}^*|^2 \otimes 1_n)E_1^* \)) shows that \( |S| \) commutes with \( B_i \) and \( |S^*| \) commutes with \( A_i \) for all \( 1 \leq i \leq n \). In particular, each \( A_i \) (respectively, \( B_i \)) has a direct sum decomposition \( A_i = A_{i1} \oplus A_{i2} \) (respectively, \( B_i = B_{i1} \oplus B_{i2} \)) with respect to the decomposition \( H = \overline{\text{ran} \, S} \oplus \overline{\text{ran} \, S}^\perp \) (respectively, \( H = \ker \perp S \oplus \ker S \)). Let \( S_1 : \ker \perp S \rightarrow \overline{\text{ran} \, S} \) be the quasi-affinity defined by setting \( S_1x = Sx \) for each \( x \in \ker \perp S \). Then \( S_1 \in \mathcal{C}_1(\ker \perp S \rightarrow \overline{\text{ran} \, S}) \) and

\[
\triangle_{\mathbf{AB}}(S) = (\sum_{i=1}^n A_{i1}S_1B_{i1} - S_1) \oplus 0 = 0 = (\sum_{i=1}^n A_{i1}^*S_1^*B_{i1}^* - S_1) \oplus 0 = \triangle_{\mathbf{AB}}^*(S), \text{i.e., } \sum_{i=1}^n A_{i1}S_1B_{i1} - S_1 = 0 = \sum_{i=1}^n A_{i1}^*S_1^*B_{i1}^* - S_1.
\]

The operator \( S_1 \) being a quasi-affinity, \( U_1 \) in the polar decomposition \( S_1 = U_1|S_1| \) is a unitary. Hence

\[
\min \left\{ \left\| \sum_{i=1}^n A_{i1}X_{11}B_{i1} - X_{11} + S_1 \right\|_1, \left\| \sum_{i=1}^n A_{i1}^*X_{11}B_{i1}^* - X_{11} + S_1 \right\|_1 \right\} 
\]

for all \( X_{11} \in \mathcal{C}_1(\ker \perp S \rightarrow \overline{\text{ran} \, S}) \). Let \( X = [X_{jk}]_{j,k=1}^n \) be an operator in \( \mathcal{C}_1 \). Then

\[
\|\triangle_{\mathbf{AB}}(X) + S\|_1 = \left\| \begin{bmatrix} \sum_{i=1}^n A_{i1}X_{11}B_{i1} - X_{11} + S_1 \\ & \ast \end{bmatrix} \right\|_1
\]
\[ \geq \left\| \sum_{i=1}^{n} A_{i1} X_{11} B_{i1} - X_{11} + S_{1} \right\|_{1} \geq \| S_{1} \|_{1} = \| S \|_{1} \]

and

\[ \| \triangle_{AB}^{*}(X) + S \|_{1} = \left\| \begin{bmatrix} \sum_{i=1}^{n} A_{i1}^{*} X_{11} B_{i1}^{*} - X_{11} + S_{1} & * \\ * & * \end{bmatrix} \right\|_{1} \geq \left\| \sum_{i=1}^{n} A_{i1}^{*} X_{11} B_{i1}^{*} - X_{11} + S_{1} \right\|_{1} \geq \| S_{1} \|_{1} = \| S \|_{1}. \]

(Here we have used the fact that the norm of a matrix is greater than or equal to the norm of an entry along the main diagonal of the matrix [6].) This completes the proof. □

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