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Choice number and energy of graphs

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Abstract

The energy of a graph G, denoted by $E(G)$, is defined as the sum of the absolute values of all eigenvalues of the adjacency matrix of G. It is proved that $E(G) \geqslant 2(n - \chi(\overline{G})) \geqslant 2(\mathrm{ch}(G) - 1)$ for every graph G of order n, and that $E(G) \geqslant 2\mathrm{ch}(G)$ for all graphs G except for those in a few specified families, where $\overline{G}$, $\chi(G)$, and $\mathrm{ch}(G)$ are the complement, the chromatic number, and the choice number of G, respectively.

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1. Introduction

All the graphs that we consider in this paper are finite, simple and undirected. Let G be a graph. Throughout this paper the *order* of G is the number of vertices of G. If $\{v_1, \ldots, v_n\}$ is the set of vertices of G, then the *adjacency matrix* of G, $A = [a_{ij}]$, is an $n \times n$ matrix where $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. Thus A is a symmetric matrix with zeros on the diagonal, and all the eigenvalues of A are real and are denoted by $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$. By the eigenvalues of G we mean those of its adjacency matrix. The *energy* $E(G)$ of a graph G is defined as the sum of the absolute values of all eigenvalues of G, which is twice the sum of the positive eigenvalues since the sum of all the eigenvalues is zero. For a survey on the energy of graphs, see [7].

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For a graph G, the *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Suppose that to each vertex v of a graph G we assigned a set L_v of k distinct elements. If for any such assignment of sets L_v it is possible, for each $v \in V(G)$, to choose $\ell_v \in L_v$ so that $\ell_u \neq \ell_v$ if u and v are adjacent, then G is said to be *k-choosable*. The *choice number* $\mathrm{ch}(G)$ of G is the smallest k such that G is k-choosable.

We denote by $A_{n,t}$, $1 \leqslant t \leqslant n-1$, the graph obtained by joining a new vertex to t vertices of the complete graph K_n. If we add two pendant vertices to a vertex of K_n, the resulting graph has order $n+2$ and we denote it by B_n.

In [1], it is proved that apart from a few families of graphs, $E(G) \geqslant 2\max(\chi(G), n-\chi(\overline{G}))$ (see the following theorem). Our goal in this paper is to extend this result to the choice number of graphs.

Theorem A. *Let G be a graph. Then* $E(G) < 2\chi(G)$ *if and only if G is a union of some isolated vertices and one of the following graphs*:

(i) *the complete graph* K_n;
(ii) *the graph* B_n;
(iii) *the graph* $A_{n,t}$ *for* $n \leqslant 7$, *except when* $(n,t) = (7,4)$, *and also for* $n \geqslant 8$ *and* $t \in \{1, 2, n-1\}$;
(iv) *a triangle with two pendant vertices adjacent to different vertices*.

The following is our main result.

Theorem 1. *Let G be a graph. Then* $E(G) < 2\mathrm{ch}(G)$ *if and only if G is a union of some isolated vertices and one of the following graphs*:
(i)–(iv) *as in Theorem A*;
(v) *the complete bipartite graph* $K_{2,4}$.

2. Proofs

In this section, we present a proof for Theorem 1. To do so we need some preliminaries.

A well-known theorem of Nordhaus and Gaddum [8] states that for every graph G of order n, $\chi(G) + \chi(\overline{G}) \leqslant n+1$. This inequality can be extended to the choice number. The graphs attaining equality are characterized in [3]. It is proved that there are exactly three types of such graphs defined as follows:

- A graph G is of *type* F_1 if its vertex set can be partitioned into three sets S_1, T, S_2 (possibly, $S_2 = \emptyset$) such that $S_1 \cup S_2$ is an independent set of G, every vertex of S_1 is adjacent to every vertex of T, every vertex of S_2 has at least one non-neighbor in T, and $|S_1|$ is sufficiently large that the choice number of the induced subgraph on $T \cup S_1$ is equal to $|T|+1$. This implies that $\mathrm{ch}(G) = |T|+1$ also. Theorem 1 of [6] states that if T does not induce a complete graph, then $|S_1| \geqslant |T|^2$; we will use this result later.
- A graph is of *type* $\overline{F}_1$ if it is the complement of a graph of type F_1.
- A graph is of *type* F_2 if its vertex set can be partitioned into a clique K, an independent set S, and a five-cycle C such that every vertex of C is adjacent to every vertex of K and to no vertex of S.

Theorem B. (a) [4] $\mathrm{ch}(G) + \mathrm{ch}(\overline{G}) \leqslant n + 1$ *for every graph G of order n.*
(b) [3] *Equality holds in (a) if and only if G is of type* F_1, $\overline{F}_1$ *or* F_2.

Lemma 1. *For every graph G of order n,*

$$E(G) \geqslant 2(n - \chi(\overline{G})) \geqslant 2(n - \mathrm{ch}(\overline{G})) \geqslant 2(\mathrm{ch}(G) - 1).$$

Proof. As remarked in [1], the first inequality follows from Theorem 2.30 of [5], which states that $n - \chi(\overline{G}) \leqslant \lambda_1(G) + \cdots + \lambda_{\chi(\overline{G})}(G)$. The second inequality holds because $\mathrm{ch}(G) \geqslant \chi(G)$ for every graph G, and the third inequality holds by Theorem B(a). □

Lemma 2. *For every graph G*, $\mathrm{ch}(G) \leqslant \lambda_1(G) + 1$.

Proof. Wilf ([9], see also [2, p. 90]) proved that every graph G has a vertex with degree at most $\lambda_1(G)$, and so does every induced subgraph of G. He deduced from this that $\chi(G) \leqslant \lambda_1(G) + 1$, and the same argument also proves that $\mathrm{ch}(G) \leqslant \lambda_1(G) + 1$. □

Lemma 3. *Suppose G has* $2K_2$ *as an induced subgraph. Then* $E(G) \geqslant 2\mathrm{ch}(G)$.

Proof. By the Interlacing Theorem (Theorem 0.10 of [2]), $\lambda_2(G) \geqslant \lambda_2(2K_2) = 1$, and so $E(G) \geqslant 2(\lambda_1(G) + \lambda_2(G)) \geqslant 2(\lambda_1(G) + 1) \geqslant 2\mathrm{ch}(G)$ by Lemma 2. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let G be a graph such that $E(G) < 2\mathrm{ch}(G)$. We may assume that G has at least one edge, since otherwise G is the union of some isolated vertices and K_1, which is permitted by (i) of Theorem 1. Since removing isolated vertices does not change the value of $E(G)$ or $\mathrm{ch}(G)$, we may assume that G has no isolated vertices. If $\mathrm{ch}(G) + \mathrm{ch}(\overline{G}) \leqslant n$, then $E(G) \geqslant 2\mathrm{ch}(G)$ by Lemma 1; this contradiction shows that $\mathrm{ch}(G) + \mathrm{ch}(\overline{G}) = n + 1$, which means that G has one of the types F_1, $\overline{F}_1$ and F_2 by Theorem B(b). We consider these three cases separately.

Case 1. G has type F_1. Then G has $G[T] \vee \overline{K}_k$ as an induced subgraph, where $G[T]$ is the subgraph induced by G on T, $k = |S_1|$, and $\vee$ denotes 'join'. Let $|T| = t$, so that $\mathrm{ch}(G) = t + 1$. If $G[T]$ is a complete graph, then $\chi(G) = t + 1 = \mathrm{ch}(G)$, so that $E(G) < 2\chi(G)$ and G is one of the graphs listed in Theorem A. So we may assume that $G[T]$ is not a complete graph. In this case, as remarked after the definition of type F_1, $k = |S_1| \geqslant |T|^2 \geqslant t^2$. Thus

$$\lambda_1(G[T] \vee \overline{K}_k) \geqslant \lambda_1(K_{t,t^2}) = t\sqrt{t} \geqslant t + 1,$$

provided $t \geqslant 3$; since $\mathrm{ch}(G) = t + 1$, we have $E(G) \geqslant 2\mathrm{ch}(G)$. So we may assume that $t \leqslant 2$, when $G[T] = \overline{K}_2$ and $k \geqslant t^2 = 4$. For $k \geqslant 5$, we have $\lambda_1(K_{2,k}) \geqslant \sqrt{10} > 3 = \mathrm{ch}(K_{2,k})$, thus $E(G) \geqslant 2\mathrm{ch}(G)$. So we may assume that $k = 4$. If $G \neq K_{2,4}$, then either $|S_1| \geqslant 5$ or $|S_2| > 0$; thus G has either $K_{2,5}$ or H as an induced subgraph, where H is formed from $K_{2,4}$ by adding an extra vertex joined to one of the vertices of degree 4. We have $E(K_{2,5}) = 2\sqrt{10} > 6$. The graph H has a P_4 as an induced subgraph so $\lambda_2(H) \geqslant \lambda_2(P_4) > 0.6$. On the other hand $\lambda_1(H) \geqslant \lambda_1(K_{2,4}) = 2\sqrt{2}$. Therefore, $E(H) > 2(2\sqrt{2} + 0.6) > 6$. Hence $E(G) > 6 = 2\mathrm{ch}(G)$ if $G \neq K_{2,4}$. Therefore, $G = K_{2,4}$.

Case 2. G has type $\overline{F}_1$. So $\overline{G}$ is of type F_1 with the associated partition $\{S_1, T, S_2\}$. Let $t = |T|$ and $k = |S_1|$. If $\overline{G}[T]$ is not a complete graph, then $k \geqslant t^2 > 1$ as in Case 1; hence G has $2K_2$ as an induced subgraph, which gives a contradiction by Lemma 3. So $\overline{G}[T]$ is a complete graph.

Let J be the set of those vertices of T that are adjacent to all vertices of S_2 in G. Let v be a vertex of S_1. Then G is a graph of type F_1 with the associated partition $\{S_1', T', S_2'\}$, in which

$$S_1' = \{v\}, \quad T' = S_2 \cup (S_1 \setminus \{v\}), S_2' = T, \quad \text{if } k \geqslant 2;$$
$$S_1' = J \cup \{v\}, \quad T' = S_2, \quad S_2' = T \setminus J, \quad \text{if } k = 1.$$

Therefore, the result follows by Case 1.

Case 3. G has type F_2. Thus G has a 5-cycle as an induced subgraph. So $\lambda_2(G) + \lambda_3(G) \geqslant \lambda_2(C_5) + \lambda_3(C_5) > 1$. Hence, by Lemma 2, we obtain

$$E(G) \geqslant 2(\lambda_1 + \lambda_2 + \lambda_3) > 2(1 + \lambda_1) \geqslant 2\text{ch}(G). \qquad \square$$

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