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Artificial Intelligence 168 (2005) 162–210

Artificial Intelligence

www.elsevier.com/locate/artint

SCC-recursiveness: a general schema for argumentation semantics

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Received 12 January 2005; received in revised form 18 May 2005; accepted 31 May 2005

Available online 14 July 2005

Abstract

In argumentation theory, Dung's abstract framework provides a unifying view of several alternative semantics based on the notion of extension. In this context, we propose a general recursive schema for argumentation semantics, based on decomposition along the strongly connected components of the argumentation framework. We introduce the fundamental notion of SCC-recursiveness and we show that all Dung's admissibility-based semantics are SCC-recursive, and therefore a special case of our schema. On these grounds, we argue that the concept of SCC-recursiveness plays a fundamental role in the study and definition of argumentation semantics. In particular, the space of SCC-recursive semantics provides an ideal basis for the investigation of new proposals: starting from the analysis of several examples where Dung's preferred semantics gives rise to questionable results, we introduce four novel SCC-recursive semantics, able to overcome the limitations of preferred semantics, while differing in other respects.

Keywords: Argumentation semantics; Extensions; Defeat cycles

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0004-3702/$ – see front matter
doi:10.1016/j.artint.2005.05.006

1. Introduction

Argumentation theory is a framework for practical and uncertain reasoning, where arguments supporting conclusions are progressively constructed and compared in order to identify the set of conclusions that should be considered justified according to the current state of available knowledge. Since the construction of arguments proceeds by exploiting incomplete and uncertain information, conflicts between them may arise and their management is regarded as an essential aspect of the theory. The ability to deal in this way with uncertain and conflicting information plays an important role in a variety of application contexts, such as legal reasoning [19,21,26], intelligent agents [17], planning [16], inconsistency handling [1], negotiation and dialogue [3]. As a consequence, argumentation theory is receiving an increasing interest both from the theoretical and application viewpoints, and a variety of approaches have been proposed in the literature, e.g., [2,6,13,14, 25,27].

An extensive survey of this research field is provided in [22], where the literature proposals are examined along five dimensions, i.e., the underlying logical language, the definition of what an argument is, the criteria for identifying conflict between arguments, the definition of the relevant relation of defeat between them, and, finally, the definition of the justification status of arguments. As to the last point, each proposal is based on an *argumentation semantics* which defines in a declarative way the criteria to determine, given a set of interacting arguments, which ones of them should emerge as justified from the conflict. To this purpose, almost all of the argumentation semantics rely on the notion of *extension*, roughly consisting in a set of non-conflicting arguments: an argument is considered as justified if it belongs to all extensions prescribed by the semantics. As pointed out in [22], two alternative approaches can be followed in this respect: in the *unique-status approach* a single extension is always identified, while in the *multiple-status approach* several extensions may exist for a given set of arguments. Moreover, specific proposals also differ in the form the underlying semantics is introduced. For instance, in [6,9,27] a fixed point definition is exploited, while in [13] the semantics is defined inductively by means of the notion of level.

A unifying framework, able to encompass a large variety of proposals, has been proposed by Dung in [9]. Abstraction is achieved by leaving unspecified the origin and the structure of arguments, and by modeling the interaction between them simply as a binary relation indicating that an argument attacks another one. This way, Dung's approach is generic with respect to the first four dimensions of the classification proposed in [22], and, as pointed out in [1], it allows one to focus exclusively on semantics issues, without getting entangled in the details of what arguments are. Thanks to its generality, Dung's proposal has been recognized as a unifying framework encompassing most of the existing approaches to argumentation and has also inspired subsequent proposals of argumentation systems, e.g., [21,27]. Moreover, Dung's theory is relevant in several fields where conflict management plays a central role, ranging from logic programming to nonmonotonic reasoning and game theory.

As far as semantics is concerned, Dung's framework captures several alternative semantics which are introduced in [9] by means of fixed point definitions, and are all based on the notion of *admissible set*. Among them, *preferred semantics* is regarded as the most satisfactory approach, able to overcome the limitations of the previously proposed *grounded* and *stable* semantics. However, we show in this paper that preferred semantics is not exempted from producing questionable results in some cases concerning cyclic attack relationships (see [4,5] for a preliminary discussion on this problem). In the search of alternative proposals, able to retain the advantages of preferred semantics and, at the same time, to support alternative treatments of such problematic cases, we carry out a broad conceptual analysis aimed at identifying a set of basic principles that can be regarded as universally acceptable in argumentation semantics and, in particular, underly all the approaches encompassed by Dung's framework.

This analysis leads to the introduction of a novel general recursive schema for the definition of extensions, based on the graph-theoretic notion of strongly connected components of an argumentation framework. Semantics adhering to this schema feature the property of *SCC-recursiveness*, which entails that a specific semantics can be characterized in terms of a *base function*, which plays the role of a parameter in the recursive schema. SCC-recursiveness can be assumed as a basic unifying concept in argumentation theory for two reasons: on the one hand, all semantics captured by Dung's framework satisfy the SCC-recursiveness property and, on the other hand, the SCC-recursive schema supports in a rather straightforward way the definition of new semantics, since basic desirable properties of extensions—and therefore of the argumentation semantics itself—can be guaranteed by simple requirements on the base function. On these grounds, we introduce, to exemplify the potential of the approach, four novel SCC-recursive semantics overcoming the above mentioned limitations of the preferred semantics, while differing in other aspects.

The paper is organized as follows. In Section 2, we recall the basic notions of Dung's theory and carry out a survey about argumentation semantics in this context, focusing on the intuitive concepts underlying formal definitions and properties of extensions. Problematic cases that point out some questionable behaviors of preferred semantics are then presented in Section 3. The general SCC-recursive schema is introduced in Section 4, while Section 5 illustrates its role as a unifying concept in argumentation semantics, showing that traditional semantics adhere to the schema. Section 6 analyzes some general properties of SCC-recursive semantics, while in Section 7 four novel SCC-recursive semantics are introduced and compared. Finally, Section 8 concludes the paper.

2. Dung's theory

2.1. Argumentation framework

The general theory proposed by Dung [9] is based on the primitive notion of *argumentation framework*:

Definition 1. An argumentation framework is a pair $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, where $\mathcal{A}$ is a set, and $\rightarrow \subseteq (\mathcal{A} \times \mathcal{A})$ is a binary relation on $\mathcal{A}$, called attack relation.

The idea is that arguments are simply conceived as the elements of the set $\mathcal{A}$, whose origin and structure are not specified, and the interaction between them is modeled by the binary relation of attack. An argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ can be represented as a directed graph, called *defeat graph*, where nodes are the arguments and edges correspond to the elements of the attack relation.

In the following, the nodes that attack a given argument α are called *defeaters* of α and form a set which is denoted as $\mathrm{parents}_{\mathrm{AF}}(\alpha)$:[1]

Definition 2. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a node $\alpha \in \mathcal{A}$, we define $\mathrm{parents}_{\mathrm{AF}}(\alpha) = \{\beta \in \mathcal{A} \mid \beta \rightarrow \alpha\}$. If $\mathrm{parents}_{\mathrm{AF}}(\alpha) = \emptyset$, then α is called an *initial* node.

Since we will frequently consider properties of sets of arguments, it is useful to extend the notations defined for the nodes:

Definition 3. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, a node $\alpha \in \mathcal{A}$ and two sets $S, P \subseteq \mathcal{A}$, we define:

$$S \rightarrow \alpha \equiv \exists \beta \in S : \beta \rightarrow \alpha$$

$$\alpha \rightarrow S \equiv \exists \beta \in S : \alpha \rightarrow \beta$$

$$S \rightarrow P \equiv \exists \alpha \in S, \beta \in P : \alpha \rightarrow \beta$$

$$\mathrm{outparents}_{\mathrm{AF}}(S) = \{\alpha \in \mathcal{A} \mid \alpha \notin S \wedge \alpha \rightarrow S\}$$

In Dung's framework, an argumentation semantics is defined by specifying the criteria for deriving, for a generic argumentation framework, the set of all possible extensions, each one representing a set of arguments considered to be acceptable together. Given a generic argumentation semantics $\mathcal{S}$, the set of extensions prescribed by $\mathcal{S}$ for a given argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ is denoted as $\mathcal{E}_{\mathcal{S}}(\mathrm{AF})$.

The set of extensions is then used to compute the justification status of the arguments, according to the following definition:

Definition 4. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, the arguments of $\mathcal{A}$ can be partitioned, with reference to a given argumentation semantics $\mathcal{S}$, into three sets:

- the set of *undefeated* arguments $\mathrm{U}_{\mathcal{S}}(\mathrm{AF}) = \{\alpha \in \mathcal{A} \mid \forall E \in \mathcal{E}_{\mathcal{S}}(\mathrm{AF})\ \alpha \in E\}$;
- the set of *defeated* arguments $\mathrm{D}_{\mathcal{S}}(\mathrm{AF}) = \{\alpha \in \mathcal{A} \mid \forall E \in \mathcal{E}_{\mathcal{S}}(\mathrm{AF})\ \alpha \notin E\}$;
- the set of *provisionally defeated* arguments $\mathrm{P}_{\mathcal{S}}(\mathrm{AF}) = \{\alpha \in \mathcal{A} \mid \exists E_1, E_2 \in \mathcal{E}_{\mathcal{S}}(\mathrm{AF}): \alpha \in E_1 \wedge \alpha \notin E_2\}$.

[1] We use the graph-theoretical term *parents* instead of *attackers* since, in the following, we will need to resort to other related graph-theoretical notions, in particular that of ancestors.

2.2. Argumentation semantics: a focused survey

In this subsection, we carry out a conceptual analysis of the basic intuitive principles underlying several extension-based argumentation semantics and identify their formal counterpart by referring to definitions and properties in Dung's theory [9].

Starting from the intuition that an extension is a set of arguments considered to be acceptable together, one may envisage as a first requirement the fact that no conflict is allowed between arguments belonging to the same extension, since it should represent an internally consistent choice (among possibly many) over the whole set of available arguments. This amounts to require that an extension is conflict-free.

Definition 5. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, a set $E \subseteq \mathcal{A}$ is *conflict-free* if and only if $\nexists \alpha, \beta \in E$ such that $\alpha \rightarrow \beta$.

Clearly not all conflict-free sets are reasonable candidates for the notion of extension. In fact, simply identifying the extensions with the conflict-free sets leads to consider the empty set, which is of course conflict-free, as an extension and, therefore, according to Definition 4, no argument would ever be justified in any argumentation framework. In order to prevent this degenerate behavior, a completeness requirement is needed, to ensure that the largest consistent choices of arguments are taken into account for the determination of the justification status. In the case of conflict-free sets, this amounts to require that an extension is also maximal with respect to set inclusion.

Definition 6. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, a set $E \subseteq \mathcal{A}$ is *maximal conflict-free* if and only if it is maximal (with respect to set inclusion) among the conflict-free sets of AF. The set made up of all the maximal conflict-free sets of AF will be denoted as $\mathcal{MCF}_{\mathrm{AF}}$.

It is however easy to see that identifying extensions with maximal conflict-free sets gives rise to undesired behaviors even in very simple cases. For instance, in the case of an argumentation framework consisting of a defeat chain (see Fig. 1) it is widely accepted that the initial node, which has no defeaters (α in our example), and all other nodes in odd positions, whose defeaters are defeated by undefeated nodes (only γ in our example) should be regarded as undefeated, while nodes in even positions (β and δ) should be regarded as defeated. This is an instance of the *reinstatement* principle [22]. However, the maximal conflict-free sets in this example are $\{\alpha, \gamma\}$, $\{\alpha, \delta\}$, and $\{\beta, \delta\}$: no argument would be included in all extensions and therefore the status of provisionally defeated would be assigned to all arguments.

A first intuition to solve this problem is based on the idea that an extension should not only be internally consistent but also able to reject the arguments that are outside the

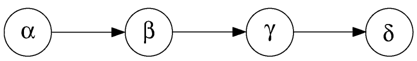


Fig. 1. A chain of four nodes.

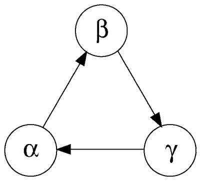


Fig. 2. A three-length cycle.

extension, namely if an argument is not in an extension then it should be attacked by the extension itself. This reasoning leads to the notion of stable extension [9,23].

Definition 7. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, a set $E \subseteq \mathcal{A}$ is a *stable extension* of AF if and only if

$$E \text{ is conflict-free } \wedge \forall \alpha \in \mathcal{A}\colon \alpha \notin E,\ E \rightarrow \alpha$$

The set of all the stable extensions of AF will be denoted as $\mathcal{SE}(\mathrm{AF})$.

Note that the above definition implies that a stable extension is a maximal conflict-free set. In the example of Fig. 1 there is only one stable extension, namely $\{\alpha, \gamma\}$, and the desired result is thus achieved. However there are argumentation frameworks where no stable extension exists, as in the case of odd-length cycles. In fact, considering the simple argumentation framework of Fig. 2, it is easy to see that none of the conflict-free sets $\emptyset$, $\{\alpha\}$, $\{\beta\}$, $\{\gamma\}$ attacks all the arguments outside it. Therefore, a semantics based on stable extensions fails to assign a justification status to arguments in these cases. A practical example of this kind of problem has been pointed out by Pollock in [14] and concerns the case of three witnesses (Smith, Jones, and Robertson) which question each other reliability in the following way: Jones says that Smith is unreliable, Smith says that Robertson is unreliable, and Robertson says that Jones is unreliable. In a defeat graph representation, this corresponds to the three-length cycle shown in Fig. 3(a). Moreover suppose that Smith says that it is raining (node "rain" in Fig. 3(a)). Resorting to stable extensions, it is impossible to assign a justification status to the nodes of this graph. However, if we have four rather than three witnesses (in general, an even-length rather than an odd-length cycle) stable extensions exists. In fact, considering Fig. 3(b) there are two stable extensions, namely $\{S, R, \mathit{rain}\}$ and $\{P, J\}$, yielding all of the arguments provisionally defeated, as intuitively should be. Pollock points out that this is a serious drawback since "surely, it should make

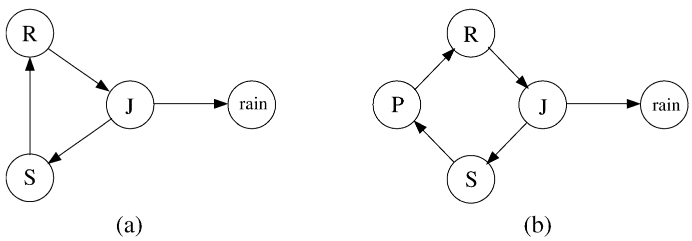


Fig. 3. The witnesses example.

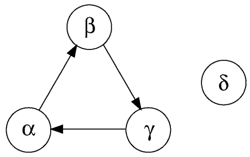


Fig. 4. A problematic defeat graph for stable semantics.

no difference that the defeat cycle is of odd-length rather than even-length. We should get the same result in either case" [14].

One might be tempted to apply a local correction to stable semantics by prescribing that when no stable extension exists, then the empty set should be considered as the unique stable extension. Such a solution would however be unsatisfactory for several reasons. First of all, it still gives different results for odd-length and even-length cycles (defeated vs. provisionally defeated for all nodes). Moreover, it gives incorrect results in graphs where nodes which should be undefeated are present along with odd-length cycles. For instance, in the graph shown in Fig. 4 no stable extension exists, entailing that all nodes are not justified, however node δ, which has no defeaters and is not involved in the cycle, should be undefeated. A more appropriate way to solve this problem consists in taking a different perspective: rather than imposing the "aggressive" condition of attacking all external arguments, it is enough to require that, more pacifically, the extension is just able to defend itself from external attacks. This intuition has been formalized in [9] by introducing the notions of acceptable argument and admissible set.

Definition 8. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, an argument $\alpha \in \mathcal{A}$ is *acceptable* with respect to a set $E \subseteq \mathcal{A}$ if and only if

$$\forall \beta \in \mathcal{A}\colon\ \beta \rightarrow \alpha,\ E \rightarrow \beta$$

The set of the arguments acceptable with respect to a set E will be denoted using the characteristic function $\mathrm{F}_{\mathrm{AF}}(E)$:

Definition 9. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, the function

$$\mathrm{F}_{\mathrm{AF}} : 2^{\mathcal{A}} \rightarrow 2^{\mathcal{A}}$$

which, given a set $E \subseteq \mathcal{A}$, returns the set of the acceptable arguments with respect to E is called the *characteristic function* of AF.

Definition 10. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, a set $E \subseteq \mathcal{A}$ is *admissible* if and only if

$$E \text{ is conflict-free and } \forall \beta \in \mathcal{A}\colon\ \beta \rightarrow E,\ E \rightarrow \beta$$

namely it is conflict-free and each argument in E is acceptable with respect to E. The set made up of all the admissible sets of AF will be denoted as $\mathcal{AS}(\mathrm{AF})$.

Building on these definitions, the notion of complete extension can be introduced, by imposing that an admissible set also satisfies a completeness requirement. Complete exten-

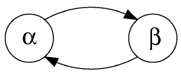


Fig. 5. The 'Nixon diamond' example.

sions play a key role in Dung's theory, since all semantics encompassed by his framework select their extensions among the complete ones. Intuitively, a complete extension is an admissible set such that no argument outside the set is acceptable with respect to the set itself.

Definition 11. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, a set $E \subseteq \mathcal{A}$ is a *complete extension* if and only if E is admissible and every argument of $\mathcal{A}$ which is acceptable with respect to E belongs to E, i.e.,

$$E \in \mathcal{AS}(\mathrm{AF}) \wedge \forall \alpha \in \mathrm{F}_{\mathrm{AF}}(E), \quad \alpha \in E$$

The set of complete extensions of AF will be denoted as $\mathcal{CE}(\mathrm{AF})$.

One might guess that the notion of complete extension entails maximality, since no acceptable argument is left outside the extension. However, this is not the case. In fact, the above property only states that nodes already defended by the extension are included, but it does not impose that nodes (or sets of nodes) that defend themselves are added to the extension. For instance, in the case of Fig. 5, the empty set is a complete extension, while either of the nodes defends itself and therefore also $\{\alpha\}$ and $\{\beta\}$ are complete extensions.

As a consequence, a further notion of extension, called preferred extension, has been introduced in [9] by directly requiring maximality of admissible sets.

Definition 12. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, a set $E \subseteq \mathcal{A}$ is a *preferred extension* of AF if and only if it is a maximal (with respect to set inclusion) admissible set, i.e., a maximal element of $\mathcal{AS}(\mathrm{AF})$. The set of preferred extensions of AF will be denoted as $\mathcal{PE}(\mathrm{AF})$.

Preferred semantics is able to cope with the problematic examples involving odd-length cycles presented above: in fact, in the graph of Fig. 2 there is a preferred extension (the empty set), while for the case of Fig. 4 the only preferred extension is $\{\delta\}$ as desired. It can be noted however that the treatment of odd and even-length cycles is still unequal (as pointed out for instance by Pollock in [18]).

Note that all the proposals surveyed above belong to the area of *multiple-status* approaches, where multiple extensions may exist for a given argumentation framework. An alternative research line has focused on *unique-status* approaches, that prescribe the existence of exactly one extension for each argumentation framework. Grounded semantics [9, 13] is probably the most representative proposal among unique-status approaches and has played an important role in the development of argumentation theory.

First of all, an alternative definition of justification status is required [13], since Definition 4 is not appropriate in case of a unique-status approach:[2]

Definition 13. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, the arguments of $\mathcal{A}$ can be partitioned, with reference to a given unique-status argumentation semantics $\mathcal{S}$, into three sets on the basis of the unique extension $E \in \mathcal{E}_{\mathcal{S}}(\mathrm{AF})$:

- the set of *undefeated* arguments $\mathrm{U}_{\mathcal{S}}(\mathrm{AF}) = \{\alpha \in \mathcal{A} \mid \alpha \in E\}$;
- the set of *defeated* arguments $\mathrm{D}_{\mathcal{S}}(\mathrm{AF}) = \{\alpha \in \mathcal{A} \mid \alpha \notin E \wedge E \rightarrow \alpha\}$;
- the set of *provisionally defeated* arguments $\mathrm{P}_{\mathcal{S}}(\mathrm{AF}) = \{\alpha \in \mathcal{A} \mid \alpha \notin E \wedge E \not\rightarrow \alpha\}$.

Formally, the (unique) extension of the grounded semantics, called the grounded extension, can be defined as the least fixed point of the characteristic function.

Definition 14. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, the *grounded extension* of AF, denoted as $\mathrm{GE}(\mathrm{AF})$, is the least fixed point (with respect to set inclusion) of F_{AF}.

In more intuitive terms, the computation of the grounded extension can be understood as the process of labeling the nodes of the defeat graph starting from the initial ones. As a first step, initial nodes are labeled as undefeated and the nodes attacked by them are labeled as defeated. Then the already labeled nodes are suppressed and the step is repeated on the resulting subgraph, and so on. If in an iteration no initial node is found, all the unlabeled nodes are labeled as provisionally defeated, and the process terminates. Note in particular that, since there are no initial nodes in the graphs of Fig. 3(a) and Fig. 3(b), all nodes are labeled as provisionally defeated in the first step. In Fig. 4 node δ is labeled undefeated in the first step, then all other nodes are labeled provisionally defeated, as desired.

According to the above analysis, grounded semantics fits well all the basic intuitions about the assignment of justification status and represents a sort of reference as far as undefeated and defeated arguments are concerned. In a sense, these assignments are unquestionable and should be agreed with by any alternative proposal. In Dung's framework, this reference role has a formal counterpart in the following property: the grounded extension is contained in any complete extension, and therefore in the extensions of any semantics (in particular in any preferred extension).

Proposition 15. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$:

- *the grounded extension* $\mathrm{GE}(\mathrm{AF})$ *is the least* (*with respect to set inclusion*) *complete extension of* AF;
- *the preferred extensions in* $\mathcal{PE}(\mathrm{AF})$ *are the maximal* (*with respect to set inclusion*) *complete extensions of* AF.

[2] The existence of two alternative definitions of justification status is actually unsatisfactory. In [8], we have proposed a unified definition overcoming this limitation and introducing a more articulated classification. The considerations and the results presented in this paper are however independent of the choice of justification status definition.

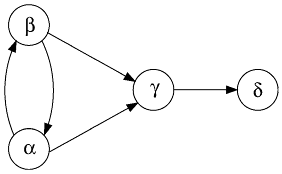


Fig. 6. Argumentation framework with a floating argument.

For these reasons, as well as for its computational advantages, variants of grounded semantics have been considered in several works [10,13,21].

Any unique-status approach is affected however by a limitation concerning the treatment of the so-called *floating defeat*, as pointed out in [12,24]. Consider the defeat graph presented in Fig. 6: as it has no initial nodes, the grounded semantics directly labels all nodes as provisionally defeated. On the other hand, preferred semantics (as well as stable semantics) prescribes two extensions $\{\alpha, \delta\}$ and $\{\beta, \delta\}$, thus yielding δ undefeated, α and β provisionally defeated, and γ defeated. This seems to be the intuitively correct result since both α and β, while preventing each other to be justified, defeat γ, thus enabling in any case the acceptance of δ.

3. Preferred semantics: problematic behaviors

As discussed in the previous section, Dung's work has played an influential role on argumentation research in recent years, due to its generality, to the proposed unifying view able to capture the most significant existing approaches, and to the importance of the specific results presented. In particular, preferred semantics is able to overcome the limitations of stable semantics as far as the existence of extensions in presence of defeat cycles is concerned, and the limitations of grounded semantics in the treatment of floating defeat.

However, while preferred semantics can be considered a significant advancement with respect to previous proposals, one can notice that the treatment of cycles does not appear completely satisfactory, since it is not as elegant as the one of grounded semantics. As a matter of fact, if the nodes of a defeat graph are arranged in a cycle of attack relationships, then they are not justified: this seems to be the intuitively right result, since all arguments in a cycle should be treated equally for obvious symmetry reasons and considering them all justified would yield a contradiction. However, this result is obtained in rather different ways in the two semantics. In the context of the grounded semantics, all arguments forming a cycle are directly labeled as provisionally defeated, since the grounded extension turns out to be the empty set. On the other hand, the preferred semantics features a sort of asymmetry, since it treats odd-length cycles differently from the even-length ones. Considering the argumentation framework of Fig. 5, consisting of a two-length cycle, two preferred extensions exist, namely $\{\alpha\}$ and $\{\beta\}$, therefore both arguments are provisionally defeated according to preferred semantics. With reference to the argumentation framework of Fig. 2, consisting of a three-length cycle, Definition 12 identifies the empty set as the unique preferred extension, therefore all the arguments are defeated. More generally, with odd-length cycles there is a unique empty extension, and then all arguments are defeated, while with

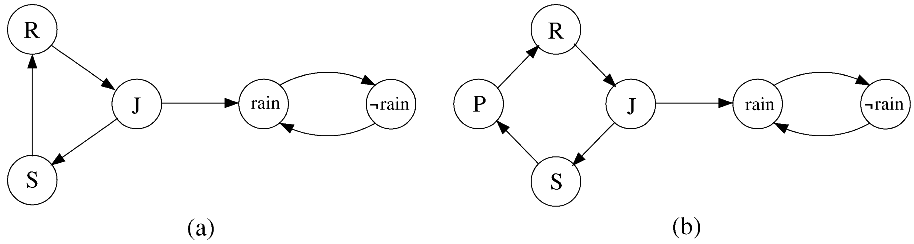


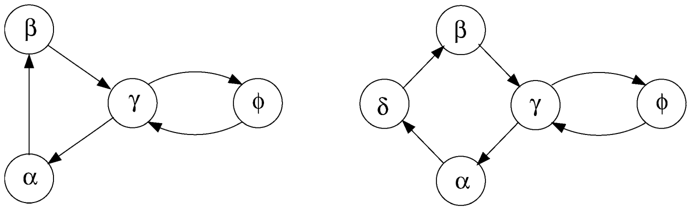
Fig. 7. A simple variant of the witnesses example.

even-length cycles non-empty extensions exist but their intersection is empty, and then all arguments are provisionally defeated. This peculiar way of assigning a justification status to odd-length cycles has recently been indicated as "puzzling" by Pollock [18].

So far, this difference might be considered as a mere question of symmetry and elegance. However, as we show considering a simple variant of the witnesses example of Fig. 3, it can be recognized that the different treatment of odd-length cycles is a real problem since it gives rise to counter-intuitive results. Let us suppose that an additional source of information, e.g., a weather report, suggests that it is not raining, contradicting the claim made by Smith; in the case of three witnesses, the resulting argumentation framework is shown in Fig. 7(a). In this case, it turns out that $\mathcal{PE}(\mathrm{AF}) = \{\{\neg rain\}\}$, therefore $\neg rain$ is undefeated while all the other arguments are not justified. On the other hand, in the case of four witnesses we obtain the argumentation framework shown in Fig. 7(b): it admits several preferred extensions whose intersection is empty, and therefore all arguments are provisionally defeated. Notice that similar situations arise, in general, by replacing the three-length and four-length cycle with any odd-length and even-length cycle, respectively. Therefore, the justification status of the argument supported by the weather report turns out to depend (in an alternating way) on the number of conflicting witnesses. Note that this difference arises because an odd-length cycle has no extensions besides the empty one: as a consequence, in the argumentation framework of Fig. 7(a) there is no extension where node *rain* is in and $\neg rain$ is out and, therefore, $\neg rain$ emerges as the only justified argument. Instead, such an extension exists with an even-length cycle and, therefore, both *rain* and $\neg rain$ turn out to be provisionally defeated.

This seems to be rather questionable: as remarked in [14] the length of the leftmost cycle should not affect the justification status. More generally, it is counter-intuitive that different results in conceptually similar situations depend on the length of the cycle: symmetry reasons suggest that all cycles should be treated equally and should yield the same results.

Notice that, in the above example, the odd-length cycle is in a sense stronger than the even-length one, since in the case of Fig. 7(a) the status of *rain* is the same as if it would be attacked by an initial node. The opposite happens however in the variant of this example shown in Fig. 8. Considering the argumentation framework of Fig. 8(a), it turns out that there is only one preferred extension, namely $\{\alpha, \phi\}$, therefore both α and ϕ are justified according to the preferred semantics. In fact, the absence of non-empty extensions for the three-length cycle prevents the existence of extensions where ϕ is out and γ is in. Since ϕ attacks γ, also α survives: as a consequence, α and ϕ emerge (questionably) undefeated.



(a) (b)

Fig. 8. Problematic argumentation frameworks for preferred semantics.

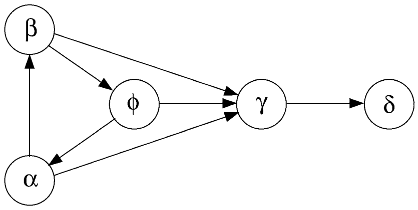


Fig. 9. A case of floating defeat and floating acceptance.

On the other hand, by replacing the three-length cycle with a four-length cycle, we obtain the argumentation framework of Fig. 8(b), whose arguments are all provisionally defeated (and a similar result is obtained with any other even-length cycle). In this case, therefore, the odd-length cycle is in a sense weaker than the even-length one, since it is not able to prevent ϕ from being justified.

In summary, we notice that odd-length cycles are problematic for preferred semantics from two points of view: first of all, they give radically different results with respect to even-length cycles in situations where such a difference does not seem justified, moreover they change their capability of defeating other arguments depending on the topology of the defeat graph.

These problems have been first pointed out in our previous works [4,7]. Other specific difficulties of preferred semantics related to the treatment of odd-length cycles have been noticed in argumentation literature. In [22], a problem in the treatment of the argumentation framework shown in Fig. 9 is pointed out. This is a case of floating defeat against argument γ by the nodes in the three-length cycle, namely it can be regarded as a variant of the argumentation framework shown in Fig. 6, where the even-length attack loop has been replaced by an odd-length loop. Again, regarding as irrelevant the distinction between even-length and odd-length cycles would yield to consider γ defeated and δ justified: although δ is attacked by γ, it is reinstated by arguments α, β and ϕ since any of them attacks γ. However, it turns out that preferred semantics admits as preferred extension only the empty set and, therefore, no argument is justified. The authors underline that "this seems one of the main unsolved problems in argumentation-based semantics" [22].

Problems also arise when considering the extreme case of odd-length cycles, namely self-defeating arguments, which have received a special attention in the literature [14,22]. The argumentation framework AF shown in Fig. 10 is yet another problematic case for preferred semantics, as observed by Dung himself [9]. In fact, the only preferred extension

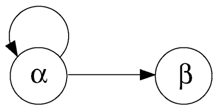


Fig. 10. The case of a self-defeating argument.

here is the empty set, though one can argue that since α attacks itself, β should be justified. Actually, both solutions are reasonable in some sense, since different treatments of self-defeating arguments may be appropriate in different contexts: a discussion and some examples about this point are provided in [15].

Before drawing any conclusion from the previous discussion, it is fair to recognize that different points of view about the intuitive interpretation of the defeat graphs presented above are possible as well. In [22], it is suggested that, while these examples are problematic if one adheres to an intuitive symmetry requirement, an alternative view can be conceived where "odd defeat loops are of an essentially different kind than even defeat loops". For instance, one might state that odd-length cycles are like paradoxes, i.e., situations where nothing can be believed, while even-length cycles are like dilemmas, i.e., situations where a choice needs to be made. While, in our opinion, this remark is not applicable to the above examples concerning witnesses, we agree that the matter is far from admitting a univocal solution. In this perspective, one should not look for just *one* right semantics, since the correct behavior is a matter of interpretation. Similarly, in [11] examples are pointed out where the behavior of grounded semantics appears more appropriate than the one of preferred semantics in the treatment of so-called floating conclusions. From a more general stance, in [20] it is remarked that using intuition about specific examples to derive general considerations about defeasible reasoning may be risky and inappropriate. This is due, in particular, to the specific nature of defeasible reasoning where it is a fundamental standpoint that inferences are never conclusive and are always subject to be invalidated in the light of additional information. For these reasons, alternative solutions to some puzzling examples can be obtained by making explicit some information left implicit in their original formulation. Moreover, it can be argued that in these problematic cases the topology of the defeat graph does not determine a univocal solution per se, since the same abstract structure may give rise to different intuitively plausible justification status assignments when instantiated on distinct common sense reasoning examples: this is a downside of the generality of abstract frameworks. As a consequence, Prakken suggests that "it is better to use intuitions not as critical tests but as generators for further investigation" [20].

The work described in the present paper adheres to this suggestion. The problematic cases discussed above point out that different solutions can be considered reasonable in different contexts (or even in the same context by different people) and, in particular, preferred semantics may be regarded as problematic in some cases. As a conclusion of this survey, we aim at investigating an extension of Dung's theory able to capture a larger variety of argumentation semantics, so that none of the reasonable intuitions arising in different contexts is excluded. The framework we are looking for should therefore be able to include the traditional grounded, stable, and preferred semantics, as well as alternative treatments of the problematic examples discussed above. To achieve such a level of generality, we need to single out a few basic principles which can be regarded as a sort of insuppressible conceptual core for any argumentation semantics. The relevant analysis is carried out in the following section.

4. A general recursive schema for argumentation semantics

In our search for a minimal set of fundamental principles of argumentation semantics, we depart from the common practice of characterizing extensions by means of a set of global properties they should respect: rather, we adopt a sort of constructive approach, where the structure of the defeat graph drives the incremental definition of extensions. In this perspective, an argumentation semantics can be viewed as the definition of a mechanism for constructing all possible extensions of an argumentation framework, and this, in turn, can be understood as an incremental process that step-wise chooses which nodes of an argumentation framework should be included into an extension.

We draw inspiration from the way the justification status can be computed according to the grounded semantics: as described in Section 2.2, computation proceeds from the frontier of the defeat graph towards the inside. Considering, as an example, the chain shown in Fig. 1, the initial node α is assigned the status of undefeated, causing β, which is attacked by α, to be assigned the status of defeated; this in turn causes γ to be assigned the status of undefeated, and so on. This node labeling procedure suggests that edges in the defeat graph represent not only the attack relation, but also a dependency relation in the assignment of justification status: roughly, the status of a node depends on those of its defeaters. However, this intuition has to be refined in order to cope with the case of cyclic attack relations, where mutual dependence would prevent this reasoning to be applicable. To this purpose, let us consider the case of floating defeat shown in Fig. 6. In a sense, the subgraph $\{\alpha, \beta\}$ plays the role of initial node with respect to γ; in fact, the status assignment within this subgraph determines the justification status of γ. For instance, in the case of preferred semantics the construction of the extensions might proceed by selecting either α or β within the subgraph, and then propagating the effect of this choice on the subsequent nodes. For each of these choices γ is attacked and then left out from the extension; as a consequence, the attack from γ to δ is ineffective and δ is included in both the extensions.

This example brings to light a fundamental aspect: the dependency relation introduced at the level of single nodes may also hold at the level of subgraphs, that play, in a sense, the role of single virtual nodes in the propagation of defeat. However, in order to derive a concrete result from this intuition, it is necessary to identify a decomposition of the defeat graph which appropriately reflects the dependency relation mentioned above. It turns out that such decomposition is provided by the graph-theoretical notion of *strongly connected components*.

Definition 16. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, the binary relation of path-equivalence between nodes, denoted as $PE_{\mathrm{AF}} \subseteq (\mathcal{A} \times \mathcal{A})$, is defined as follows:

- $\forall \alpha \in \mathcal{A}, (\alpha, \alpha) \in PE_{\mathrm{AF}}$,
- given two distinct nodes $\alpha, \beta \in \mathcal{A}$, $(\alpha, \beta) \in PE_{\mathrm{AF}}$ if and only if there is a path from α to β and a path from β to α.

The *strongly connected components* of AF are the equivalence classes of nodes under the relation of path-equivalence. The set of the strongly connected components of AF is denoted as $\mathrm{SCCS}_{\mathrm{AF}}$. Given a node $\alpha \in \mathcal{A}$, the strongly connected component α belongs to is denoted as $\mathrm{SCC}_{\mathrm{AF}}(\alpha)$.

A particular case, to be considered in the following, is represented by the empty argumentation framework: when $\mathrm{AF} = \langle \emptyset, \emptyset \rangle$ we assume $\mathrm{SCCS}_{\mathrm{AF}} = \{\emptyset\}$.

To exemplify Definition 16, in the graph of Fig. 1 there are four strongly connected components each consisting of a single node (i.e., $\mathrm{SCCS}_{\mathrm{AF}} = \{\{\alpha\}, \{\beta\}, \{\gamma\}, \{\delta\}\}$), since there is not any couple of mutually reachable nodes. On the other hand, the graphs of Fig. 2, Fig. 5, Fig. 8 consist of exactly one strongly connected component coinciding with the whole set of nodes, since they are all mutually reachable. The graphs in Fig. 3, Fig. 4 and Fig. 10 include two strongly connected components: one consisting of a cycle (a degenerate one in the case of Fig. 10), the other one of a single node. Three strongly connected components are present in the graph of Fig. 6, namely $\{\alpha, \beta\}$, $\{\gamma\}$, and $\{\delta\}$, and a similar decomposition applies to the graph of Fig. 9. Finally two distinct strongly connected components can easily be identified in the graphs of Fig. 7: for instance in the graph of Fig. 7(a), the nodes R, S, and J form a first strongly connected component S_1, since they are mutually reachable being arranged in a cycle, while the nodes *rain* and $\neg$*rain* form a second distinct strongly connected component S_2, since they are mutually reachable while there is not any path leading from them to any node of S_1.

We extend to strongly connected components the notion of parents, denoting the set of the other strongly connected components that attack a strongly connected component S as $\mathrm{sccparents}_{\mathrm{AF}}(S)$, and we introduce the definition of *proper ancestors*, denoted as $\mathrm{sccanc}_{\mathrm{AF}}(S)$:

Definition 17. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a strongly connected component $S \in \mathrm{SCCS}_{\mathrm{AF}}$, we define

$$\mathrm{sccparents}_{\mathrm{AF}}(S) = \{P \in \mathrm{SCCS}_{\mathrm{AF}} \mid P \neq S \text{ and } P \rightarrow S\}$$

and

$$\mathrm{sccanc}_{\mathrm{AF}}(S) = \mathrm{sccparents}_{\mathrm{AF}}(S) \cup \bigcup_{P \in \mathrm{sccparents}_{\mathrm{AF}}(S)} \mathrm{sccanc}_{\mathrm{AF}}(P)$$

A strongly connected component S such that $\mathrm{sccparents}_{\mathrm{AF}}(S) = \emptyset$ is called *initial*.

It is well-known that the graph obtained by considering strongly connected components as single nodes is acyclic: this confirms that considering a dependency relation at the level of $\mathrm{SCCS}_{\mathrm{AF}}$ is a sound starting point. Recalling the basic example of Fig. 1, where the strongly connected components consist of single nodes, clearly the dependency among justification states of nodes has a direct counterpart in terms of strongly connected components. Turning to the example of Fig. 6, a similar consideration applies: the subgraph $\{\alpha, \beta\}$ is actually the only initial strongly connected component of the graph. The choices concerning extension construction carried out in this first strongly connected component clearly do not depend on those concerning the other ones and directly affect the choice about the subsequent strongly connected component (actually consisting of $\{\gamma\}$), which in turn affects the last strongly connected component, namely $\{\delta\}$. Similar considerations can be easily applied to other examples.

Generalizing this intuition, we assume a first basic principle, called *directionality principle*: with reference to a given extension E, for any strongly connected component S of AF, the choice of the subset of S to be included in E (i.e., $(E \cap S)$) only depends on the choices made in the strongly connected components of $\mathrm{sccanc}_{\mathrm{AF}}(S)$, i.e., those that are antecedent to S in the acyclic graph made up of strongly connected components.

In particular, the choices in the antecedent strongly connected components determine a partition of the nodes of S into three subsets:[3]

Definition 18. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, a set $E \subseteq \mathcal{A}$ and a strongly connected component $S \in \mathrm{SCCS}_{\mathrm{AF}}$, we define:

- $D_{\mathrm{AF}}(S, E) = \{\alpha \in S \mid (E \cap \mathrm{outparents}_{\mathrm{AF}}(S)) \rightarrow \alpha\}$;
- $P_{\mathrm{AF}}(S, E) = \{\alpha \in S \mid (E \cap \mathrm{outparents}_{\mathrm{AF}}(S)) \not\rightarrow \alpha \wedge \exists \beta \in (\mathrm{outparents}_{\mathrm{AF}}(S) \cap \mathrm{parents}_{\mathrm{AF}}(\alpha))\colon E \not\rightarrow \beta\}$;
- $U_{\mathrm{AF}}(S, E) = S \setminus (D_{\mathrm{AF}}(S, E) \cup P_{\mathrm{AF}}(S, E)) = \{\alpha \in S \mid (E \cap \mathrm{outparents}_{\mathrm{AF}}(S)) \not\rightarrow \alpha \wedge \forall \beta \in (\mathrm{outparents}_{\mathrm{AF}}(S) \cap \mathrm{parents}_{\mathrm{AF}}(\alpha)) E \rightarrow \beta\}$.

In words, the set $D_{\mathrm{AF}}(S, E)$ consists of the nodes of S attacked by E from outside S, the set $U_{\mathrm{AF}}(S, E)$ consists of the nodes of S that are not attacked by E from outside S and are defended by E (i.e., their defeaters from outside S are all attacked by E), and $P_{\mathrm{AF}}(S, E)$ consists of the nodes of S that are not attacked by E from outside S and are not defended by E (i.e., at least one of their defeaters from outside S is not attacked by E). It is easy to verify that $D_{\mathrm{AF}}(S, E)$, $P_{\mathrm{AF}}(S, E)$ and $U_{\mathrm{AF}}(S, E)$ are determined only by the elements of E that belong to the strongly connected components in $\mathrm{sccanc}_{\mathrm{AF}}(S)$.

To exemplify the above definitions, consider again the defeat graph presented in Fig. 7(a), which consists of two strongly connected components $S_1 = \{R, S, J\}$ and $S_2 = \{\mathit{rain}, \neg\mathit{rain}\}$, where clearly S_1 precedes S_2. Accordingly, let us now show how different choices of $E \cap S_1$ determine different partitions of S_2.

Let us first consider the case that $(E \cap S_1) = \{J\}$: then the node *rain* receives an external attack coming from a node included in E. Therefore it satisfies the condition for membership in $D_{\mathrm{AF}}(S_2, E)$, while the node $\neg$*rain* does not receive external attacks from E, and thus satisfies the condition for membership in $U_{\mathrm{AF}}(S_2, E)$. Therefore it turns out that $D_{\mathrm{AF}}(S_2, E) = \{\mathit{rain}\}$ and $U_{\mathrm{AF}}(S_2, E) = \{\neg\mathit{rain}\}$. Note that the condition about $\neg$*rain* does not depend on the choice of $E \cap S_1$, therefore it will always be the case that $\neg\mathit{rain} \in U_{\mathrm{AF}}(S_2, E)$.

Let us turn to the case $(E \cap S_1) = \{S\}$: now *rain* does not receive external attacks from E, but is attacked from outside S_2 by the node J, which is not included in E nor

[3] The symbols D, P and U are meant to recall the terms “Defeated”, “Provisionally defeated” and “Undefeated” respectively. It has to be noted, however, that here they refer to a relationship between arguments and a particular extension E, rather than to the overall justification status of arguments.

is attacked by E. Thus *rain* satisfies the condition for membership in $P_{\mathrm{AF}}(S_2, E)$ yielding $P_{\mathrm{AF}}(S_2, E) = \{rain\}$, and, of course, again $U_{\mathrm{AF}}(S_2, E) = \{\neg rain\}$.

Finally, if $(E \cap S_1) = \{R\}$, the node *rain* is defended by E since its only external attacker J is in turn attacked by $R \in E$, so *rain* satisfies the condition for membership in $U_{\mathrm{AF}}(S_2, E)$, yielding $U_{\mathrm{AF}}(S_2, E) = \{rain, \neg rain\}$.

Now, we need to investigate further principles which drive the selection of $E \cap S$ on the basis of the above three sets. First of all, as already discussed in Section 2.2, a *conflict-free principle* is universally accepted: an extension cannot include conflicting arguments. This entails that arguments in $D_{\mathrm{AF}}(S, E)$, being attacked by nodes in E, cannot be chosen in the construction of the extension E (i.e., do not belong to $E \cap S$). Selection of arguments to be included in E is therefore restricted to $(S \setminus D_{\mathrm{AF}}(S, E)) = (U_{\mathrm{AF}}(S, E) \cup P_{\mathrm{AF}}(S, E))$, which, for ease of notation, will be denoted in the following as $UP_{\mathrm{AF}}(S, E)$.

As far as this selection is concerned, we recall that it is generally agreed that undefeated nodes are able to make ineffective the attacks of nodes they attack, i.e., of defeated nodes. This *reinstatement principle* prescribes that nodes defeated by an extension E play no role in the selection of nodes to be included in E. Taking into account the decomposition into strongly connected components, the application of this principle can be examined considering separately, for a given strongly connected component S, the nodes defeated by E inside and outside S. Inside S, the principle entails that the nodes in $D_{\mathrm{AF}}(S, E)$ and the relevant attack relations can be suppressed. This implies that the selection within S of the nodes to be included in an extension E has to be carried out on a sort of reduced argumentation framework, consisting in $UP_{\mathrm{AF}}(S, E)$, completely neglecting the nodes in $D_{\mathrm{AF}}(S, E)$.

To formalize this concept, we provide the definition of *restriction* of an argumentation framework to a given subset of its nodes:

Definition 19. Let $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ be an argumentation framework, and let $S \subseteq \mathcal{A}$ be a set of arguments. The *restriction* of AF to S is the argumentation framework $\mathrm{AF}\downarrow_S = \langle S, \rightarrow \cap (S \times S)\rangle$.

Combining this definition with the reinstatement principle, we obtain that the selection of nodes within a strongly connected component S has to be carried out on the restricted argumentation framework $\mathrm{AF}\downarrow_{UP_{\mathrm{AF}}(S,E)}$ without taking into account the attacks coming from $D_{\mathrm{AF}}(S, E)$.

Let us turn to the suppression of nodes defeated by E outside S. For ease of description, let us call in the following *outer attacker* of a strongly connected component S, with respect to an extension E, any node α such that $\alpha \notin S$, $\alpha \notin E$, $\alpha \rightarrow UP_{\mathrm{AF}}(S, E)$. Note that any outer attacker α of S necessarily belongs to a strongly connected component parent of S, i.e., $\mathrm{SCC}_{\mathrm{AF}}(\alpha) \in \mathrm{sccparents}_{\mathrm{AF}}(S)$. We will also refer to the outer attackers of a node $\beta \in S$ to denote the subset of the outer attackers of S attacking β. In general some of the outer attackers of S with respect to E are in turn attacked by E, while others are not.

As to the first ones, according to the reinstatement principle the nodes of $UP_{\mathrm{AF}}(S, E)$ should be treated as if the outer attackers that are attacked by E did not exist. To clarify this point, consider the particular case where all outer attackers of S are in turn attacked by E (formally, $UP_{\mathrm{AF}}(S, E) = U_{\mathrm{AF}}(S, E)$): according to the above considerations, the selection of nodes to be included in the extension is carried out in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$ neglecting the suppressed outer attacks and therefore following the same principles which are applied to an unrestricted argumentation framework, i.e., selection is carried out in the same way as if the argumentation framework was not the result of a restriction. For instance, using again the example of Fig. 7(a), when $UP_{\mathrm{AF}}(S_2, E) = U_{\mathrm{AF}}(S_2, E) = \{rain, \neg rain\}$ the reinstatement principle entails that the selection of $E \cap S_2$ is analogous to the case of Fig. 5, namely to the case of a defeat graph featuring the same topology and not resulting from restriction.

On the other hand, outer attackers that are not attacked by E may play a role in the construction of the extensions. Accordingly, the nodes of $UP_{\mathrm{AF}}(S, E)$ can be partitioned into *defended nodes*, i.e., nodes which have no outer attackers or whose outer attackers are all attacked by E, and *undefended nodes*, such that at least one of their outer attackers is not attacked by E.

Summing up, on the basis of the principles we have identified, the selection of $(E \cap S)$ turns out to depend only on:

- the restricted argumentation framework $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$;
- the distinction between defended and undefended nodes within $UP_{\mathrm{AF}}(S, E)$.

A direct way to formalize these ideas consists in stating that the nodes to be included in the extension are selected by means of a generic *selection function*, which will be denoted as $\mathcal{GF}$. The function $\mathcal{GF}$ takes in input two parameters:

- a generic, possibly restricted, argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ to which selection has to be applied;
- the set $C \subseteq \mathcal{A}$ of defended nodes[4]

and gives as output a set of subsets of $\mathcal{A}$, which represents all possible choices for $E \cap \mathcal{A}$. Accordingly, we will use the notation $\mathcal{GF}(\mathrm{AF}, C)$ for the function. For the sake of generality, besides specifying its input and output, we do not make any a priori assumption about $\mathcal{GF}$ and, in particular, about whether and how the parameter C is taken into account.

Now, the next step is to define $\mathcal{GF}(\mathrm{AF}, C)$ for a generic argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a set $C \subseteq \mathcal{A}$, representing the defended nodes of AF: two cases have to be considered in this respect.

If AF consists of exactly one strongly connected component, it does not admit a decomposition where to apply the directionality principle, therefore it has to be assumed that $\mathcal{GF}(\mathrm{AF}, C)$ coincides in this case with a *base function*, denoted as $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, C)$, that must be assigned in order to characterize a particular argumentation semantics $\mathcal{S}$. The definition of this base function is, at least in principle, unconstrained.

On the other hand, if AF can be decomposed into several strongly connected components, then, according to the directionality and reinstatement principles, $\mathcal{GF}(\mathrm{AF}, C)$

[4] This is just one of the ways of formalizing the distinction between defended and undefended nodes: this formalization turned out to enable a more elegant and compact technical treatment with respect to other alternatives.

is obtained by applying recursively $\mathcal{GF}$ to each strongly connected component of AF, deprived of the nodes in $D_{\mathrm{AF}}(S,E)$. Formally, this means that for any $S \in \mathrm{SCCS_{AF}}$, $(E \cap S) \in \mathcal{GF}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, C')$, where C' represents the set of defended nodes of the restricted argumentation framework $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$. The set C' can be determined taking into account both the attacks coming from outside AF (which can be actually present only if AF results in turn from restriction) and those coming from other strongly connected components of AF, namely from $\mathrm{sccparents_{AF}}(S)$. Since the set C consists of the nodes defended from the former kind of attacks at the level of AF, while $U_{\mathrm{AF}}(S,E)$ consists of those defended from the latter, it turns out that $C' = U_{\mathrm{AF}}(S,E) \cap C$.

The above considerations suggest to introduce a new notion for argumentation semantics, called *SCC-recursiveness*:

Definition 20. A given argumentation semantics $\mathcal{S}$ is *SCC-recursive* if and only if for any argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, $\mathcal{E}_{\mathcal{S}}(\mathrm{AF}) = \mathcal{GF}(\mathrm{AF}, \mathcal{A})$, where for any $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and for any set $C \subseteq \mathcal{A}$, the function $\mathcal{GF}(\mathrm{AF}, C) \subseteq 2^{\mathcal{A}}$ is defined as follows: for any $E \subseteq \mathcal{A}$, $E \in \mathcal{GF}(\mathrm{AF}, C)$ if and only if

- in case $|\mathrm{SCCS_{AF}}| = 1$, $E \in \mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, C)$,
- otherwise, $\forall S \in \mathrm{SCCS_{AF}}$ $(E \cap S) \in \mathcal{GF}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S,E) \cap C)$,

where $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, C)$ is a function, called *base function*, that, given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ such that $|\mathrm{SCCS_{AF}}| = 1$ and a set $C \subseteq \mathcal{A}$, gives a subset of $2^{\mathcal{A}}$.

In particular, the set of all the extensions of the original unrestricted argumentation framework AF coincides with $\mathcal{GF}(\mathrm{AF}, \mathcal{A})$, since obviously there are no attacks from outside and, therefore, the set C coincides with the set $\mathcal{A}$ of all arguments.

Note that the definition of $\mathcal{GF}(\mathrm{AF}, C)$ is recursive with respect to the decomposition of AF into strongly connected components. Since $\mathcal{GF}(\mathrm{AF}, C)$ is applied to progressively more restricted argumentation frameworks, the definition is well founded: in particular the base of the recursion is given by the function $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, C)$, which returns the extensions of a generic argumentation framework consisting of a unique strongly connected component. Therefore, in order to define a SCC-recursive semantics, it is sufficient to specify its behavior only on single-SCC argumentation frameworks.

The definition given above has also a straightforward constructive interpretation: it suggests an effective (recursive) procedure for computing all the extensions of an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ according to a given SCC-recursive semantics, once a specific base function is assigned. A particular role in this context is played by the initial strongly connected components. In fact, for any initial strongly connected component I, since by definition there are no outer attacks, $UP_{\mathrm{AF}}(I,E) = U_{\mathrm{AF}}(I,E) = I$ for any E and the set of defended nodes coincides with I. This gives rise to the invocation $\mathcal{GF}(\mathrm{AF}{\downarrow}_I, I)$, for any initial strongly connected component I. Since $\mathrm{AF}{\downarrow}_I$ obviously consists of a unique strongly connected component, according to Definition 20 the base function $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}{\downarrow}_I, I)$ is invoked, which returns the extensions of $\mathrm{AF}{\downarrow}_I$ according to the semantics $\mathcal{S}$. Therefore, the base function can be first computed on the initial strongly connected components, where it directly returns the extensions prescribed by the seman-

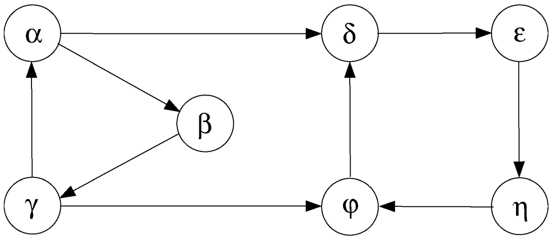


Fig. 11. An argumentation framework with two strongly connected components.

tics, then the results of this computation are used to identify, within the subsequent strongly connected components, the restricted argumentation frameworks on which the procedure is recursively invoked.

To support a better understanding of the concepts introduced above, we describe in detail their application to the argumentation framework presented in Fig. 11, which consists of two strongly connected components, namely $S_1 = \{\alpha, \beta, \gamma\}$ and $S_2 = \{\delta, \varepsilon, \eta, \varphi\}$. In the example we refer to a generic unspecified semantics $\mathcal{S}$. The set of extensions of the whole argumentation framework is given by $\mathcal{GF}(\mathrm{AF}, \mathcal{A})$ and, since AF consists of more than one strongly connected component, the function $\mathcal{GF}$ is invoked recursively on them (with $C = \mathcal{A}$), following their order. Formally, we have that $E \in \mathcal{E}_{\mathcal{S}}(\mathrm{AF})$ iff

- $(E \cap S_1) \in \mathcal{GF}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S_1,E)}, U_{\mathrm{AF}}(S_1, E))$,
- $(E \cap S_2) \in \mathcal{GF}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S_2,E)}, U_{\mathrm{AF}}(S_2, E))$.

As explained above, for the initial strongly connected component S_1 it holds that $UP_{\mathrm{AF}}(S_1, E) = U_{\mathrm{AF}}(S_1, E) = S_1$ for any E: this gives rise to the invocation $\mathcal{GF}(\mathrm{AF}{\downarrow}_{S_1}, S_1)$. Since $\mathrm{AF}{\downarrow}_{S_1}$ consists of a unique strongly connected component, the base function $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}{\downarrow}_{S_1}, S_1)$ is invoked, which returns the extensions of $\mathrm{AF}{\downarrow}_{S_1}$ according to the semantics $\mathcal{S}$. For the sake of the example, let us assume that $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}{\downarrow}_{S_1}, S_1) = \{\{\alpha\}, \{\beta\}, \{\gamma\}\}$, i.e., that, according to $\mathcal{S}$, the extensions of $\mathrm{AF}{\downarrow}_{S_1}$ are the singletons included in S_1. At the level of the whole argumentation framework, they represent the alternative choices for $E \cap S_1$. Now, each of these alternative choices for $E \cap S_1$ has an impact on the subsequent strongly connected component S_2 and determines a distinct $UP_{\mathrm{AF}}(S_2, E)$, since, for any E, $UP_{\mathrm{AF}}(S_2, E) = UP_{\mathrm{AF}}(S_2, E \cap S_1)$. Thus, $\mathcal{GF}$ has to be applied to $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S_2,\{\alpha\})}$, $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S_2,\{\beta\})}$, and $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S_2,\{\gamma\})}$, to determine which subsets of S_2 can be joined with $\{\alpha\}$, $\{\beta\}$, and $\{\gamma\}$ respectively to form extensions at the level of the whole graph. If other strongly connected components were present, the same reasoning would apply to them.

To continue our example, let us now consider the case where

$$E \cap S_1 = \{\alpha\}$$

As to the partition induced on S_2 by this choice, since only node δ is attacked by $\alpha \in E$, $D_{\mathrm{AF}}(S_2, \{\alpha\}) = \{\delta\}$ and, therefore, $UP_{\mathrm{AF}}(S_2, \{\alpha\}) = \{\varepsilon, \eta, \varphi\}$, which determines the first parameter for the recursive invocation of $\mathcal{GF}$. Within $UP_{\mathrm{AF}}(S_2, \{\alpha\})$, the only node which receives an outer attack is φ. Since its attacker γ is not included in $E \cap S_1$ nor is attacked by $E \cap S_1$, we have $P_{\mathrm{AF}}(S_2, \{\alpha\}) = \{\varphi\}$, which entails $U_{\mathrm{AF}}(S_2, \{\alpha\}) = \{\varepsilon, \eta\}$. Therefore, the recursive invocation on the second strongly connected component of the graph has the form $(E \cap S_2) \in \mathcal{GF}(\mathrm{AF}', \{\varepsilon, \eta\})$, where $\mathrm{AF}' = \mathrm{AF}{\downarrow_{\{\varepsilon,\eta,\varphi\}}}$.

Now, let us identify the sets $E' \in \mathcal{GF}(\mathrm{AF}', \{\varepsilon, \eta\})$ which represent the possible values for $(E \cap S_2)$: the function $\mathcal{GF}$ is applied to AF' following the same reasoning lines as above. First, note that AF' is actually a chain of three nodes, therefore it consists of three simple strongly connected components $S'_1 = \{\varepsilon\}$, $S'_2 = \{\eta\}$, $S'_3 = \{\varphi\}$. S'_1 is the only initial strongly connected component of AF', therefore $UP_{\mathrm{AF}'}(S'_1, E') = U_{\mathrm{AF}'}(S'_1, E') = S'_1 = \{\varepsilon\}$ for any extension E' of AF'. Following Definition 20, the possible values for $(E' \cap S'_1)$ are given by $\mathcal{GF}(\mathrm{AF}'{\downarrow_{\{\varepsilon\}}}, \{\varepsilon\} \cap \{\varepsilon, \eta\}) = \mathcal{GF}(\langle\{\varepsilon\}, \emptyset\rangle, \{\varepsilon\})$. Since $\langle\{\varepsilon\}, \emptyset\rangle$ consists of a single strongly connected component, the base function $\mathcal{BF}_{\mathcal{S}}(\langle\{\varepsilon\}, \emptyset\rangle, \{\varepsilon\})$ is in turn invoked: let us (reasonably) assume that it returns $\{\{\varepsilon\}\}$ as result. Then, as it was intuitively evident, there is only one possibility for $E' \cap S'_1$, whose effects on subsequent strongly connected components of AF' have to be determined. It is immediate to note that $D_{\mathrm{AF}'}(S'_2, E' \cap S'_1) = S'_2 = \{\eta\}$, therefore $UP_{\mathrm{AF}'}(S'_2, E' \cap S'_1) = \emptyset$, and also the set of defended nodes is empty. Formally, the possible values for $E' \cap S'_2$ are given by $\mathcal{GF}(\mathrm{AF}'{\downarrow_{\emptyset}}, \emptyset \cap \{\varepsilon, \eta\}) = \mathcal{GF}(\langle\emptyset, \emptyset\rangle, \emptyset)$. The base function is then invoked on the empty argumentation framework, giving reasonably $\{\emptyset\}$ as result. Now we are ready to consider the situation of S'_3: within AF' it receives an attack only from S_2 and it is clear from above that $E' \cap S_2 = \emptyset$. Therefore $U_{\mathrm{AF}'}(S'_3, E') = U_{\mathrm{AF}'}(S'_3, \emptyset) = S'_3 = \{\varphi\}$. Then $D_{\mathrm{AF}'}(S'_3, E') = P_{\mathrm{AF}'}(S'_3, E') = \emptyset$ and $UP_{\mathrm{AF}'}(S'_3, E') = U_{\mathrm{AF}'}(S'_3, E')$. Applying Definition 20, the possible values for $E' \cap S'_3$ are given by $\mathcal{GF}(\mathrm{AF}'{\downarrow_{S'_3}}, S'_3 \cap \{\varepsilon, \eta\}) = \mathcal{GF}(\langle\{\varphi\}, \emptyset\rangle, \emptyset)$. Note that, in this case, the second parameter of the function $\mathcal{GF}$, namely the set of defended nodes, differs from $U_{\mathrm{AF}'}(S'_3, E')$, due to the attack coming from γ and considered at a previous level of recursion. Again, $\mathrm{AF}'{\downarrow_{S'_3}}$ consists of a single strongly connected component, therefore the possible values of $E' \cap S'_3$ are given by $\mathcal{BF}_{\mathcal{S}}(\langle\{\varphi\}, \emptyset\rangle, \emptyset)$. In this case, the result of the function is no more obvious since different semantics might ascribe a different role to the set of defended nodes, as it will be discussed later. Supposing that the result is $\{\{\varphi\}\}$, we have a unique $E' \in \mathcal{GF}(\mathrm{AF}', \{\varepsilon, \eta\})$, and therefore a unique value for $(E \cap S_2)$, i.e., $\{\varepsilon, \varphi\}$. Summing up, we obtain $\{\alpha, \varepsilon, \phi\}$ as an extension of the original AF.

For the sake of brevity we do not carry out such a detailed analysis for the cases $E \cap S_1 = \{\beta\}$, and $E \cap S_1 = \{\gamma\}$, rather a sketch is provided for the interested reader.

In the case $E \cap S_1 = \{\beta\}$, there are no attacks from E to S_2 and β defends φ, therefore it turns out that $D_{\mathrm{AF}}(S_2, E) = \emptyset$, $P_{\mathrm{AF}}(S_2, E) = \{\delta\}$, $U_{\mathrm{AF}}(S_2, E) = \{\varepsilon, \eta, \varphi\}$, which coincides with the set of defended nodes within S_2. Then $UP_{\mathrm{AF}}(S_2, E) = S_2$ giving rise to the following invocation: $\mathcal{GF}(\mathrm{AF}{\downarrow_{S_2}}, \{\varepsilon, \eta, \varphi\})$. Now, $\mathrm{AF}{\downarrow_{S_2}}$ consists of a unique strongly connected component, therefore the possible completions of extensions at the level of the whole graph are given by $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}{\downarrow_{S_2}}, \{\varepsilon, \eta, \varphi\})$, whose outcome depends on the specific semantics $\mathcal{S}$ considered and is not specified here: joining the elements of $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}{\downarrow_{S_2}}, \{\varepsilon, \eta, \varphi\})$ with $\{\beta\}$, we obtain a set of extensions $\mathcal{B} \subset \mathcal{E}_{\mathcal{S}}(\mathrm{AF})$, such that $\forall B \in \mathcal{B}$, $B \cap S_1 = \{\beta\}$.

Finally, in the case $E \cap S_1 = \{\gamma\}$, E attacks the node φ and defends the node δ within S_2. Therefore $D_{\mathrm{AF}}(S_2, E) = \{\varphi\}$, $P_{\mathrm{AF}}(S_2, E) = \emptyset$, and $U_{\mathrm{AF}}(S_2, E) = UP_{\mathrm{AF}}(S_2, E) = \{\delta, \varepsilon, \eta\}$, which coincides also with the set of defended nodes. Letting

$\mathrm{AF}' = \mathrm{AF}{\downarrow_{\{\delta,\varepsilon,\eta\}}}$, $\mathcal{GF}(\mathrm{AF}', \{\delta, \varepsilon, \eta\})$ is invoked. AF' is a chain of three nodes, therefore its strongly connected components are the singletons $S'_1 = \{\delta\}$, $S'_2 = \{\varepsilon\}$, and $S'_3 = \{\eta\}$. S'_1 is the initial one and we have $\mathcal{GF}(\mathrm{AF}'{\downarrow_{S'_1}}, S'_1 \cap \{\delta, \varepsilon, \eta\}) = \mathcal{GF}(\langle\{\delta\}, \emptyset\rangle, \{\delta\}) = \mathcal{BF}_{\mathcal{S}}(\langle\{\delta\}, \emptyset\rangle, \{\delta\})$, which reasonably gives $\{\{\delta\}\}$ as only possible choice for $E' \cap S'_1$. Taking this into account, $UP_{\mathrm{AF}'}(S'_2, E') = \emptyset$ and then we have $\mathcal{GF}(\mathrm{AF}'{\downarrow_{\emptyset}}, \emptyset) = \mathcal{GF}(\langle\emptyset, \emptyset\rangle, \emptyset) = \mathcal{BF}_{\mathcal{S}}(\langle\emptyset, \emptyset\rangle, \emptyset)$, which reasonably gives $\{\emptyset\}$ as only possible choice for $E' \cap S'_2$. This entails $UP_{\mathrm{AF}'}(S'_3, E') = S'_3$ and we have $\mathcal{GF}(\mathrm{AF}'{\downarrow_{S'_3}}, S'_3 \cap \{\delta, \varepsilon, \eta\}) = \mathcal{GF}(\langle\{\eta\}, \emptyset\rangle, \{\eta\}) = \mathcal{BF}_{\mathcal{S}}(\langle\{\eta\}, \emptyset\rangle, \{\eta\})$, which reasonably gives $\{\{\eta\}\}$ as only possible choice for $E' \cap S'_3$. Summing up, we obtain $\{\gamma, \delta, \eta\}$ as an extension of the original AF, and finally $\mathcal{E}_S(\mathrm{AF}) = \{\{\alpha, \varepsilon, \phi\}, \{\gamma, \delta, \eta\}\} \cup \mathcal{B}$.

Partly due to their recursive nature, Definition 20 and its detailed application may appear complex.[5] We argue, however, that the underlying basic ideas are relatively simple and can be summarized as follows:

(1) the argumentation framework is partitioned into its strongly connected components; they form a partial order which encodes the dependencies existing among them according to the directionality principle;
(2) the possible choices for extensions within each initial strongly connected component are determined using a semantic-specific base function which returns the extensions of argumentation frameworks consisting of a single strongly connected component;
(3) for each possible choice determined at step 2, according to the reinstatement principle, the nodes directly attacked within subsequent strongly connected components are suppressed and the distinction between defended and undefended nodes is (possibly) taken into account;
(4) the steps 1–3 above are applied recursively on the restricted argumentation frameworks obtained at step 3.

One may now wonder whether the property of SCC-recursiveness characterizes a suitable family of semantics. On the one hand, such family should be general enough to include all traditional approaches to argumentation semantics in Dung's framework, on the other hand it should be constrained enough to support the definition of novel proposals based on reasonable definitions of the base function.

As far as the first requirement is concerned, it is reasonable to expect a positive answer since SCC-recursiveness has been derived using a very essential set of largely accepted principles in argumentation semantics. This intuition is formally backed up in the following section, where it is shown that all the semantics encompassed by Dung's theory are SCC-recursive. As to the second requirement, in Section 6 we show that SCC-recursive semantics satisfies two desirable properties under very general constraints on the base function, while in Section 7 we exploit SCC-recursiveness to introduce four novel semantics able to cope in different ways with the problematic cases illustrated in Section 3.

[5] Other examples of application of the recursive schema will be given in Section 7.

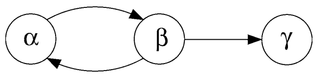


Fig. 12. A ‘Nixon diamond’ attacking a node.

5. SCC-recursive characterization of traditional semantics

5.1. Generalizing Dung’s theory

In order to develop an SCC-recursive characterization of traditional semantics, it is necessary, first of all, to redefine Dung’s theory in generalized terms, by restating its fundamental concepts with reference to a specific subset $C \subseteq \mathcal{A}$, from which acceptable arguments (that compose the extensions) are selected, since it represents the subset of defended nodes within $\mathcal{A}$, as explained in the previous section. Original Dung’s definitions can be recovered letting $C = \mathcal{A}$. Proofs are omitted throughout this subsection as they are straightforward extensions of those in [9].

Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a set $C \subseteq \mathcal{A}$, we define admissible sets in C as follows:

Definition 21. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a set $C \subseteq \mathcal{A}$, a set $E \subseteq \mathcal{A}$ is an *admissible set in C* if and only if $E \subseteq C \wedge E \in \mathcal{AS}(\mathrm{AF})$. The set of admissible sets in C is denoted as $\mathcal{AS}(\mathrm{AF}, C)$.

Note that, in general, $\mathcal{AS}(\mathrm{AF}, C) \neq \mathcal{AS}(\mathrm{AF}{\downarrow_C})$. For instance, in the argumentation framework of Fig. 12, with $C = \{\gamma\}$ we have that $\mathcal{AS}(\mathrm{AF}, C) = \{\emptyset\}$, since γ is not an admissible set because of the attack coming from β, while $\mathcal{AS}(\mathrm{AF}{\downarrow_C}) = \{\{\gamma\}\}$.

We introduce now the notion of stable, complete and preferred extensions in the context of the generalized framework.

Definition 22. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a set $C \subseteq \mathcal{A}$, a set $E \subseteq \mathcal{A}$ is a *stable extension in C* if and only if $E \subseteq C$ and $E \in \mathcal{SE}(\mathrm{AF})$. The set of stable extensions in C is denoted as $\mathcal{SE}(\mathrm{AF}, C)$.

Definition 23. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a set $C \subseteq \mathcal{A}$, a set $E \subseteq \mathcal{A}$ is a *complete extension in C* if and only if $E \in \mathcal{AS}(\mathrm{AF}, C)$, and every argument $\alpha \in C$ which is acceptable with respect to E belongs to E, i.e., $\forall \alpha \in C$: $\alpha \in F_{\mathrm{AF}}(E)$, $\alpha \in E$. The set of complete extensions in C is denoted as $\mathcal{CE}(\mathrm{AF}, C)$.

Definition 24. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a set of arguments $C \subseteq \mathcal{A}$, a *preferred extension in C* is a maximal element (with respect to set inclusion) of $\mathcal{AS}(\mathrm{AF}, C)$. The set of preferred extensions in C is denoted as $\mathcal{PE}(\mathrm{AF}, C)$.

In other terms, $E \in \mathcal{PE}(\mathrm{AF}, C)$ if and only if E is a maximal set such that $E \subseteq C$ and E is admissible in AF.

Given these definitions, an important question concerns the existence of a preferred extension for any argumentation framework AF and for any set $C \subseteq \mathcal{A}$. The following theorem provides a positive answer to this question, extending Dung's results:

Theorem 25. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set* $C \subseteq \mathcal{A}$:

- *The elements of* $\mathcal{AS}(\mathrm{AF}, C)$, *i.e., the admissible subsets of* C, *form a complete partial order.*
- *For all* $F \in \mathcal{AS}(\mathrm{AF}, C)$, *there is* $E \in \mathcal{PE}(\mathrm{AF}, C)$ *such that* $F \subseteq E$.

Corollary 26. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set* $C \subseteq \mathcal{A}$, $\mathcal{PE}(\mathrm{AF}, C)$ *is non empty, i.e., there is always a preferred extension* $E \in \mathcal{PE}(\mathrm{AF}, C)$.

Also in the generalized framework, the grounded semantics can be defined in terms of the least fixed point of the characteristic function.

Definition 27. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a set of arguments $C \subseteq \mathcal{A}$, the function

$$\mathrm{F}_{\mathrm{AF},C} : 2^C \rightarrow 2^C$$
$$\mathrm{F}_{\mathrm{AF},C}(Q) = \{\alpha \mid \alpha \in C, \alpha \text{ acceptable with respect to } Q\}$$

is called the *characteristic function of* AF *in* C.

It is easy to see that $\mathrm{F}_{\mathrm{AF},C}$ is monotonic (with respect to set inclusion).

Definition 28. Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a set $C \subseteq \mathcal{A}$, the *grounded extension of* AF *in* C, denoted as $\mathrm{GE}(\mathrm{AF}, C)$, is the least (with respect to set inclusion) fixed point of $\mathrm{F}_{\mathrm{AF},C}$.

Notice that by definition $\mathrm{GE}(\mathrm{AF}, C) \subseteq C$. Also in this case we provide a positive result concerning the existence of the grounded extension:

Lemma 29. *For any argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and for all sets* $C \subseteq \mathcal{A}$, $\mathrm{GE}(\mathrm{AF}, C)$ *exists and is unique.*

Finally, the following relations between grounded, preferred and complete extensions can be drawn:

Proposition 30. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set of arguments* $C \subseteq \mathcal{A}$, $\mathrm{GE}(\mathrm{AF}, C)$ *is the least* (*with respect to set inclusion*) *complete extension in* C (*i.e., the least element in* $\mathcal{CE}(\mathrm{AF}, C)$).

Proposition 31. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set of arguments* $C \subseteq \mathcal{A}$, *the preferred extensions* $\mathcal{PE}(\mathrm{AF}, C)$ *are the maximal* (*with respect to set inclusion*) *complete extensions in* C (*i.e., the maximal elements in* $\mathcal{CE}(\mathrm{AF}, C)$).

Since Dung's original definitions are recovered by letting $C = \mathcal{A}$, a SCC-recursive formulation of the extended definitions also covers the original ones: this is achieved in the following subsections, where we show that all the traditional semantics covered by Dung's theory allow a definition of extensions recursively characterized along strongly connected components.

5.2. Stable semantics

The following proposition shows that stable extensions are in correspondence with a decomposition along strongly connected components: the intersection of a stable extension with any strongly connected component S is itself a stable extension of the restriction of the argumentation framework AF to $UP_{\mathrm{AF}}(S, E)$.

Proposition 32. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set of arguments* $E \subseteq \mathcal{A}$, $E \in \mathcal{SE}(\mathrm{AF})$ *if and only if* $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$

$$(E \cap S) \in \mathcal{SE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)})$$

Proof. First, let us prove that if E is a stable extension of AF then it satisfies local stability relevant to a generic strongly connected component $S \in \mathrm{SCCS}_{\mathrm{AF}}$, corresponding to the following conditions:

(1) $(E \cap S) \subseteq UP_{\mathrm{AF}}(S, E)$;
(2) $(E \cap S)$ is conflict-free;
(3) $\forall \alpha \in UP_{\mathrm{AF}}(S, E)$: $\alpha \notin (E \cap S)$, $(E \cap S) \rightarrow \alpha$.

Notice that, by definition, the second and third conditions should be verified with reference to the argumentation framework $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$, however it is easy to see that, according to the first condition and the definition of $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$, it is sufficient to verify them in AF. As for the first condition, we have that $\forall \alpha \in (E \cap S)$ $\alpha \notin D_{\mathrm{AF}}(S, E)$, otherwise by the definition of $D_{\mathrm{AF}}(S, E)$ we would have that $E \rightarrow \alpha$, therefore E would not be conflict-free contradicting the hypothesis that E is a stable extension of AF. As far as the second condition is concerned, it directly follows from the fact that E is conflict-free, entailed by the hypothesis that E is a stable extension of AF. As for the third condition, let us consider a generic argument $\alpha \in UP_{\mathrm{AF}}(S, E)$ such that $\alpha \notin (E \cap S)$. Since $\alpha \in S$ and $\alpha \notin (E \cap S)$, $\alpha \notin E$, therefore by the hypothesis that E is a stable extension of AF we have that $E \rightarrow \alpha$, i.e., $\exists \beta \in E : \beta \rightarrow \alpha$. Since $\alpha \in UP_{\mathrm{AF}}(S, E)$, taking into account the definitions of $UP_{\mathrm{AF}}(S, E)$ it turns out that $\nexists \gamma \in (E \cap \mathrm{outparents}_{\mathrm{AF}}(S))$ such that $\gamma \rightarrow \alpha$. As a consequence, it must be the case that $\exists \beta \in E, \beta \in S : \beta \rightarrow \alpha$, thus $(E \cap S) \rightarrow \alpha$, and the first part of the proof is complete.

Turning to the other direction of the proof, we have to show that, with reference to the argumentation framework AF:

(1) E is conflict-free;
(2) $\forall \alpha \in \mathcal{A}$: $\alpha \notin E$, $E \rightarrow \alpha$.

Let us prove the first condition by contradiction, assuming that $\exists \alpha, \beta \in E$: $\alpha \to \beta$, and let us indicate $\mathrm{SCC}_{\mathrm{AF}}(\beta)$ as S. Let us notice that α cannot belong to S, since in this case $(E \cap S)$ would not be conflict-free in AF as well as in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$, thus contradicting the hypothesis that $(E \cap S) \in \mathcal{SE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)})$. As a consequence, $\alpha \in E$ and $\alpha \notin S$, therefore $\beta \in D_{\mathrm{AF}}(S, E)$ by the definition of $D_{\mathrm{AF}}(S, E)$. However, this contradicts the fact that $\beta \in (E \cap S)$, which by the hypothesis is contained in $UP_{\mathrm{AF}}(S, E)$.

Let us finally turn to the second condition, considering a generic $\alpha \in \mathcal{A}$: $\alpha \notin E$, and let us indicate $\mathrm{SCC}_{\mathrm{AF}}(\alpha)$ as S. We can distinguish two cases for α. If $\alpha \in D_{\mathrm{AF}}(S, E)$, then $E \to \alpha$ directly follows from the definition of $D_{\mathrm{AF}}(S, E)$. In the other case, $\alpha \in UP_{\mathrm{AF}}(S, E)$. Since $\alpha \notin E$, we have that $\alpha \notin (E \cap S)$, therefore the hypothesis $(E \cap S) \in \mathcal{SE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)})$ entails that $(E \cap S) \to \alpha$ in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$. Obviously, $(E \cap S) \to \alpha$ holds also in AF, and the proof is complete. $\square$

Exploiting the following lemma, the above result can be extended to the generalized definition of stable extensions, yielding the desired recursive characterization in Proposition 34.

Lemma 33. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \to \rangle$ *and a stable extension* E *of* AF, $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$, $P_{\mathrm{AF}}(S, E) = \emptyset$.

Proof. Let us assume by contradiction that $\exists S \in \mathrm{SCCS}_{\mathrm{AF}}$: $P_{\mathrm{AF}}(S, E) \neq \emptyset$, i.e., $\exists \alpha \in P_{\mathrm{AF}}(S, E)$. According to the definition of $P_{\mathrm{AF}}(S, E)$, we have that $\exists \beta \notin E, \beta \to \alpha$: $E \not\to \beta$, contradicting the hypothesis that E is a stable extension of AF. $\square$

Proposition 34. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \to \rangle$ *and a set of arguments* $E \subseteq \mathcal{A}$, $\forall C \subseteq \mathcal{A}$, $E \in \mathcal{SE}(\mathrm{AF}, C)$ *if and only if* $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$

$$(E \cap S) \in \mathcal{SE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big)$$

Proof. Let us start from the first direction of the proof, assuming that $E \in \mathcal{SE}(\mathrm{AF}, C)$. According to the definition of $\mathcal{SE}(\mathrm{AF}, C)$, we have that $E \in \mathcal{SE}(\mathrm{AF})$, thus Proposition 32 entails that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$

$$(E \cap S) \in \mathcal{SE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)})$$

In order to prove the thesis, we have only to show that $(E \cap S) \subseteq (U_{\mathrm{AF}}(S, E) \cap C)$. First, the hypothesis that E is stable and therefore conflict-free entails that $(E \cap D_{\mathrm{AF}}(S, E)) = \emptyset$. Moreover, according to Lemma 33 $P_{\mathrm{AF}}(S, E) = \emptyset$. As a consequence, it must be the case that $(E \cap S) \subseteq U_{\mathrm{AF}}(S, E)$. Furthermore, according to the hypothesis $E \subseteq C$, it holds that $(E \cap S) \subseteq (U_{\mathrm{AF}}(S, E) \cap C)$.

Let us turn to the other direction of the proof, assuming that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$

$$(E \cap S) \in \mathcal{SE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big)$$

Taking into account the definition of $\mathcal{SE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C)$, Proposition 32 entails that $E \in \mathcal{SE}(\mathrm{AF})$. Moreover, since $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$ $(E \cap S) \subseteq C$ we have that $E \subseteq C$, thus $E \in \mathcal{SE}(\mathrm{AF}, C)$, and the proof is complete. $\square$

It can be noted that, on the basis of Lemma 33, the same result would have been achieved by adopting an alternative definition of $\mathcal{SE}(\mathrm{AF}, C)$, namely $\mathcal{SE}(\mathrm{AF}, C) \equiv \mathcal{SE}(\mathrm{AF})$. In fact, in the case of stable extensions C has no concrete role, since all arguments outside an extension are attacked by the extension itself, and therefore all arguments within a strongly connected component S are defended against attacks from outside S.

5.3. Admissible sets

Since admissible sets play a key role in Dung's theory, their characterization according to SCC-recursiveness is needed as a basis for the analysis of other semantics. This is achieved by Proposition 38, which requires three preliminary lemmas.

Lemma 35 (from [9, pp. 327]). *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, an admissible set of arguments* $E \in \mathcal{AS}(\mathrm{AF})$*, and an argument* $\alpha \in \mathrm{F}_{\mathrm{AF}}(E)$ *acceptable with respect to* E*,* $E \cup \{\alpha\}$ *is admissible, i.e.,* $(E \cup \{\alpha\}) \in \mathcal{AS}(\mathrm{AF})$.

Lemma 36. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, an admissible set of arguments* $E \in \mathcal{AS}(\mathrm{AF})$*, and an argument* $\alpha \in \mathrm{F}_{\mathrm{AF}}(E)$ *acceptable with respect to* E*, denoting* $\mathrm{SCC}_{\mathrm{AF}}(\alpha)$ *as* S*, it holds that*:

- $\alpha \in U_{\mathrm{AF}}(S, E)$; *and*
- α *is acceptable with respect to* $(E \cap S)$ *in the argumentation framework* $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$*, i.e.,* $\alpha \in \mathrm{F}_{\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}}(E \cap S)$.

Proof. First of all, on the basis of Lemma 35 $(E \cup \{\alpha\}) \in \mathcal{AS}(\mathrm{AF})$, and in particular $(E \cup \{\alpha\})$ is conflict-free: as a consequence $\alpha \notin D_{\mathrm{AF}}(S, E)$, otherwise by the definition of $D_{\mathrm{AF}}(S, E)$ it would be the case that $E \rightarrow \alpha$. Moreover, $\alpha \notin P_{\mathrm{AF}}(S, E)$, otherwise by the definition of $P_{\mathrm{AF}}(S, E)$ we would have that $\exists \beta \in E$: $\beta \rightarrow \alpha$ and $E \not\rightarrow \beta$, thus contradicting the fact that $\alpha \in \mathrm{F}_{\mathrm{AF}}(E)$, namely the acceptability of α with respect to E. As a consequence, the only possibility for α is that $\alpha \in U_{\mathrm{AF}}(S, E)$.

Turning to the second part of the proof, let us first note that, on the basis of the hypothesis that $E \in \mathcal{AS}(\mathrm{AF})$, all the elements of E are acceptable with respect to E, i.e., $\forall \gamma \in E, \gamma \in \mathrm{F}_{\mathrm{AF}}(E)$. Thus, the above result entails that $(E \cap S) \subseteq U_{\mathrm{AF}}(S, E)$, therefore $(E \cap S)$ is actually a set of arguments in the argumentation framework $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$. Let us consider now a generic argument β that attacks α in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$: we have to prove that $(E \cap S) \rightarrow \beta$ holds in this argumentation framework. Of course, $\beta \rightarrow \alpha$ also in AF, therefore the acceptability of α with respect to E, i.e., $\alpha \in \mathrm{F}_{\mathrm{AF}}(E)$, entails that there is $\gamma \in E$ such that $\gamma \in \mathrm{parents}_{\mathrm{AF}}(\beta)$. Now, since $\beta \in UP_{\mathrm{AF}}(S, E)$, by definition of $UP_{\mathrm{AF}}(S, E)$ all of its defeaters outside S do not belong to E, thus $\gamma \in S$ and therefore $\gamma \in (E \cap S)$. As a consequence, $(E \cap S) \rightarrow \beta$ holds in AF, and obviously it also holds in the restricted argumentation framework $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$. $\square$

Lemma 37. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, let* $E \subseteq \mathcal{A}$ *be a set of arguments such that,* $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$

$$(E \cap S) \in \mathcal{AS}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E)\big)$$

Given a strongly connected component $\widehat{S} \in \mathrm{SCCS}_{\mathrm{AF}}$ *and an argument* $\alpha \in U_{\mathrm{AF}}(\widehat{S}, E)$ *which is acceptable with respect to* $(E \cap \widehat{S})$ *in the argumentation framework* $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}$*, i.e.,* $\alpha \in \mathrm{F}_{\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}}(E \cap \widehat{S})$*,* α *is acceptable with respect to* E *in* AF*, i.e.,* $\alpha \in \mathrm{F}_{\mathrm{AF}}(E)$*.*

Proof. With reference to the argumentation framework AF, we have to prove that $\forall \beta \in \mathcal{A}$: $\beta \to \alpha$, then also $E \to \beta$. We distinguish two cases for β.

First, let us suppose that $\mathrm{SCC}_{\mathrm{AF}}(\beta) = \mathrm{SCC}_{\mathrm{AF}}(\alpha) = \widehat{S}$. If $\beta \in D_{\mathrm{AF}}(\widehat{S}, E)$, then $E \to \beta$ holds in AF by definition of $D_{\mathrm{AF}}(\widehat{S}, E)$. If, on the other hand, $\beta \in UP_{\mathrm{AF}}(\widehat{S}, E)$, then $\beta \to \alpha$ holds in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}$, therefore according to the hypothesis of acceptability concerning α it must be the case that $(E \cap \widehat{S}) \to \beta$ holds in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}$. Obviously, such relation holds also in AF, and entails that $E \to \beta$.

Let us consider the other case, i.e., $\mathrm{SCC}_{\mathrm{AF}}(\beta) \neq \mathrm{SCC}_{\mathrm{AF}}(\alpha) = \widehat{S}$. In this case, $\beta \in (\mathrm{outparents}_{\mathrm{AF}}(\widehat{S}) \cap \mathrm{parents}_{\mathrm{AF}}(\alpha))$, while by the hypothesis $\alpha \in U_{\mathrm{AF}}(\widehat{S}, E)$: on the basis of the definition of $U_{\mathrm{AF}}(\widehat{S}, E)$, it must be the case that $E \to \beta$ in AF, and the proof is complete. □

Proposition 38. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \to \rangle$ *and a set of arguments* $E \subseteq \mathcal{A}$*,* $\forall C \subseteq \mathcal{A}$*,* $E \in \mathcal{AS}(\mathrm{AF}, C)$ *if and only if* $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$

$$(E \cap S) \in \mathcal{AS}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big)$$

Proof. First, let us prove that if E is admissible then it satisfies the conditions relevant to a generic strongly connected component $S \in \mathrm{SCCS}_{\mathrm{AF}}$. According to the definition of $\mathcal{AS}(\mathrm{AF}, C)$, $E \subseteq C$ and $\forall \alpha \in E$, α is acceptable with respect to E, i.e., $\alpha \in \mathrm{F}_{\mathrm{AF}}(E)$. As a consequence, on the basis of Lemma 36 we have that $\forall \alpha \in (E \cap S), \alpha \in U_{\mathrm{AF}}(S, E)$, therefore $(E \cap S) \subseteq (U_{\mathrm{AF}}(S, E) \cap C)$. Moreover, by the same lemma α is acceptable with respect to $(E \cap S)$ in the argumentation framework $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$. This, as well as the fact that E is admissible and therefore conflict-free, entails that $(E \cap S)$ is admissible in the argumentation framework $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$, and therefore that $(E \cap S) \in \mathcal{AS}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C)$.

As far as the other direction of the proof is concerned, we first notice that, by the hypothesis, $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$ $(E \cap S) \subseteq (U_{\mathrm{AF}}(S, E) \cap C) \subseteq (S \cap C)$, therefore $E \subseteq C$: in order to prove the claim, we have only to show that E is admissible in AF.

Let us first show that E is conflict-free by reasoning by contradiction, i.e., let us suppose that $\exists \alpha, \beta \in E$: $\beta \to \alpha$. Let us denote $\mathrm{SCC}_{\mathrm{AF}}(\alpha)$ as S. Clearly, it cannot be the case that $\mathrm{SCC}_{\mathrm{AF}}(\alpha) = \mathrm{SCC}_{\mathrm{AF}}(\beta)$, since in this case $(E \cap S)$ would not be conflict-free, thus contradicting the hypothesis concerning its admissibility in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$. As a consequence, $\beta \in (E \cap \mathrm{outparents}_{\mathrm{AF}}(S))$, therefore $\alpha \in D_{\mathrm{AF}}(S, E)$ by the definition of $D_{\mathrm{AF}}(S, E)$. However, this contradicts the fact that $\alpha \in (E \cap S)$, which according to the hypothesis is contained in $U_{\mathrm{AF}}(S, E)$.

In order to complete the proof, we have to prove that a generic $\alpha \in E$ is acceptable with respect to E, i.e., $\alpha \in \mathrm{F}_{\mathrm{AF}}(E)$. If we denote $\mathrm{SCC}_{\mathrm{AF}}(\alpha)$ as $\widehat{S}$, we have that $\alpha \in (E \cap \widehat{S})$, and by the hypothesis $(E \cap \widehat{S}) \in \mathcal{AS}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\hat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$. Therefore, $\alpha \in U_{\mathrm{AF}}(\widehat{S}, E)$, and α is acceptable with respect to $(E \cap \widehat{S})$ in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}$. Since the hypothesis entails that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$ $(E \cap S) \in \mathcal{AS}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E))$, Lemma 37 can be applied to α, entailing that α is acceptable with respect to E in AF, i.e., $\alpha \in \mathrm{F}_{\mathrm{AF}}(E)$. □

5.4. Complete semantics

Exploiting the results in previous subsection, the following proposition shows that also complete extensions are in correspondence with a recursive decomposition along strongly connected components.

Proposition 39. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set of arguments* $E \subseteq \mathcal{A}$, $\forall C \subseteq \mathcal{A}$, $E \in \mathcal{CE}(\mathrm{AF}, C)$ *if and only if* $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$

$$(E \cap S) \in \mathcal{CE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big)$$

Proof. As for the first direction of the proof, if $E \in \mathcal{CE}(\mathrm{AF}, C)$ then in particular $E \in \mathcal{AS}(\mathrm{AF}, C)$, therefore Proposition 38 entails that

$$\forall S \in \mathrm{SCCS}_{\mathrm{AF}} \quad (E \cap S) \in \mathcal{AS}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big) \tag{1}$$

As a consequence, we have only to show that $\forall \alpha \in (U_{\mathrm{AF}}(S, E) \cap C)$ such that α is acceptable with respect to $(E \cap S)$ in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$, $\alpha \in (E \cap S)$. First, we notice that Lemma 37 can be applied to α, since (1) entails that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$ $(E \cap S) \in \mathcal{AS}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E))$. On the basis of this lemma, α is acceptable with respect to E in AF, i.e., $\alpha \in \mathrm{F}_{\mathrm{AF}}(E)$. Moreover, $\alpha \in (U_{\mathrm{AF}}(S, E) \cap C)$, therefore in particular $\alpha \in C$. As a consequence, from the hypothesis that $E \in \mathcal{CE}(\mathrm{AF}, C)$ it follows that $\alpha \in E$ and therefore $\alpha \in (E \cap S)$.

As for the other direction of the proof, according to Definition 23 we have that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$ the following conditions hold:

$$(E \cap S) \in \mathcal{AS}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big) \tag{2}$$

$$\forall \alpha \in \big(U_{\mathrm{AF}}(S, E) \cap C\big)\colon\ \alpha \in \mathrm{F}_{\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}}(E \cap S), \quad \alpha \in (E \cap S) \tag{3}$$

Thus, on the basis of (2) Proposition 38 entails that $E \in \mathcal{AS}(\mathrm{AF}, C)$, therefore we have only to prove that $\forall \alpha \in C$ such that α is acceptable with respect to E, $\alpha \in E$, i.e., $\forall \alpha \in C$: $\alpha \in \mathrm{F}_{\mathrm{AF}}(E)$, $\alpha \in E$. Denoting $\mathrm{SCC}_{\mathrm{AF}}(\alpha)$ as S, on the basis of Lemma 36 we have that $\alpha \in U_{\mathrm{AF}}(S, E)$, so that $\alpha \in (U_{\mathrm{AF}}(S, E) \cap C)$, and α is acceptable with respect to $(E \cap S)$ in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$, i.e., $\alpha \in \mathrm{F}_{\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}}(E \cap S)$. Then, taking into account (3) we have that $\alpha \in (E \cap S)$, therefore $\alpha \in E$. □

5.5. Preferred semantics

Also preferred extensions fit the decomposition schema along strongly connected components, as shown by Proposition 41 based on the following lemma.

Lemma 40. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, *an admissible set* $E \in \mathcal{AS}(\mathrm{AF})$, *and a strongly connected component* $S \in \mathrm{SCCS}_{\mathrm{AF}}$, *let* $\widehat{E}$ *be a set of arguments such that*:

- $(E \cap S) \subseteq \widehat{E} \subseteq U_{\mathrm{AF}}(S, E)$;
- $\widehat{E}$ *is admissible in the argumentation framework* $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$, *i.e.*, $\widehat{E} \in \mathcal{AS}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)})$.

It holds that $(E \cup \widehat{E})$ *is admissible in* AF, *i.e.*, $(E \cup \widehat{E}) \in \mathcal{AS}(\mathrm{AF})$.

Proof. First, we prove that $(E \cup \widehat{E})$ is conflict-free. Of course, $\widehat{E}$ is conflict-free in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$ by the hypothesis, and, as a consequence, $\widehat{E}$ is conflict-free also in AF. Since also E is conflict-free in AF by the hypothesis of admissibility, we have to prove that $\widehat{E} \not\to E$ and $E \not\to \widehat{E}$. Since E is admissible in AF, $\widehat{E} \to E$ entails that $E \to \widehat{E}$, therefore we have only to prove that $E \not\to \widehat{E}$. Since $\widehat{E} \subseteq U_{\mathrm{AF}}(S, E)$, $(E \cap \mathrm{outparents}_{\mathrm{AF}}(S)) \not\to \widehat{E}$, therefore $E \to \widehat{E}$ only if $(E \cap S) \to \widehat{E}$. However, this situation is not possible since $(E \cap S) \subseteq \widehat{E}$ and $\widehat{E}$ is conflict-free.

Now, with reference to the argumentation framework AF we have to prove that $\forall \beta \in \mathcal{A}$ such that $\beta \to (E \cup \widehat{E})$, it is the case that $(E \cup \widehat{E}) \to \beta$. In case $\beta \to E$, the conclusion follows from admissibility of E. On the other hand, if $\beta \to \widehat{E}$, we have that $\beta \in (\mathrm{outparents}_{\mathrm{AF}}(S) \cup S)$ since $\widehat{E} \subseteq S$; we distinguish three cases for β:

(1) if $\beta \in \mathrm{outparents}_{\mathrm{AF}}(S)$, then, taking into account that $\beta \to \widehat{E}$ and $\widehat{E} \subseteq U_{\mathrm{AF}}(S, E)$, it must be the case according to the definition of $U_{\mathrm{AF}}(S, E)$ that $E \to \beta$;
(2) if $\beta \in D_{\mathrm{AF}}(S, E)$, then according to the definition of $D_{\mathrm{AF}}(S, E)$ it must be the case that $E \to \beta$;
(3) if $\beta \in UP_{\mathrm{AF}}(S, E)$, then $\beta \to \widehat{E}$ holds also in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$. As a consequence, the hypothesis that $\widehat{E} \in \mathcal{AS}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)})$ entails that $\widehat{E} \to \beta$ holds in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$, and, of course, that $\widehat{E} \to \beta$ also holds in AF.

In any case, $(E \cup \widehat{E}) \to \beta$, and the proof is completed. □

Proposition 41. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \to \rangle$ *and a set of arguments* $E \subseteq \mathcal{A}$, $\forall C \subseteq \mathcal{A}$, $E \in \mathcal{PE}(\mathrm{AF}, C)$ *if and only if* $\forall S \in \mathrm{SCCS_{AF}}$

$$(E \cap S) \in \mathcal{PE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big)$$

Proof. As far as the first direction of the proof is concerned, let us assume that $E \in \mathcal{PE}(\mathrm{AF}, C)$. By definition, $E \in \mathcal{AS}(\mathrm{AF}, C)$, therefore, on the basis of Proposition 38, we have that $\forall S \in \mathrm{SCCS_{AF}}$

$$(E \cap S) \in \mathcal{AS}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big)$$

Let us now reason by contradiction, assuming that $\exists \widehat{S} \in \mathrm{SCCS_{AF}}$ such that $(E \cap \widehat{S})$ is not maximal among the sets included in $\mathcal{AS}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$. According to Theorem 25, there must be a set $\widehat{E}$ such that

- $(E \cap \widehat{S}) \subsetneq \widehat{E} \subseteq (U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$, and
- $\widehat{E} \in \mathcal{AS}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$.

Taking into account that, according to the definition of $\mathcal{AS}(\mathrm{AF}, C)$, $E \in \mathcal{AS}(\mathrm{AF})$, Lemma 40 entails that the set $E' \triangleq E \cup \widehat{E}$ is admissible in AF. Moreover, since both E and $\widehat{E}$ are contained in C we have that $E' \subseteq C$, therefore $E' \in \mathcal{AS}(\mathrm{AF}, C)$. However, it is easy to see that E is strictly contained in E', contradicting the maximality of E among the sets of $\mathcal{AS}(\mathrm{AF}, C)$.

Let us turn now to the other direction of the proof, assuming that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$, $(E \cap S) \in \mathcal{PE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C)$. On the basis of Proposition 38, $E \in \mathcal{AS}(\mathrm{AF}, C)$: in order to prove that E is also a preferred extension, we reason again by contradiction, supposing that $\exists E' \subseteq C, E \subsetneq E'$: $E' \in \mathcal{PE}(\mathrm{AF}, C)$ (notice that the existence of E' is imposed by Theorem 25). Since $E \subsetneq E'$, there must be at least a strongly connected component $S \in \mathrm{SCCS}_{\mathrm{AF}}$ such that $(E \cap S) \subsetneq (E' \cap S)$: taking into account the acyclicity of the strongly connected components, there exists in particular $\widehat{S} \in \mathrm{SCCS}_{\mathrm{AF}}$ such that

$$\forall S \in \mathrm{SCCS}_{\mathrm{AF}}\colon\ S \in \mathrm{sccanc}_{\mathrm{AF}}(\widehat{S}), \quad (E' \cap S) = (E \cap S) \tag{4}$$

$$(E \cap \widehat{S}) \subsetneq (E' \cap \widehat{S}) \tag{5}$$

Note that condition (4) is trivially verified if $\widehat{S}$ is initial.

Since $E' \in \mathcal{AS}(\mathrm{AF}, C)$, on the basis of Proposition 38 we have that $(E' \cap \widehat{S}) \in \mathcal{AS}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E')}, U_{\mathrm{AF}}(\widehat{S}, E') \cap C)$. Taking into account (4), it is easy to see that $U_{\mathrm{AF}}(\widehat{S}, E') = U_{\mathrm{AF}}(\widehat{S}, E)$ and $P_{\mathrm{AF}}(\widehat{S}, E') = P_{\mathrm{AF}}(\widehat{S}, E)$, therefore $(E' \cap \widehat{S}) \in \mathcal{AS}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$. However, on the basis of (5) we have that $(E \cap \widehat{S}) \subsetneq (E' \cap \widehat{S})$, and this contradicts the hypothesis that $(E \cap \widehat{S}) \in \mathcal{PE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$. □

5.6. Grounded semantics

Finally, in this subsection, we prove that the decomposition schema also holds for grounded semantics, as shown by the following Proposition 42.

Proposition 42. *Given an argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *and a set of arguments* $E \subseteq \mathcal{A}$, $\forall C \subseteq \mathcal{A}$, $E = \mathrm{GE}(\mathrm{AF}, C)$ *if and only if* $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$

$$(E \cap S) = \mathrm{GE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big)$$

Proof. Let us consider the first part of the proof, by supposing that $E = \mathrm{GE}(\mathrm{AF}, C)$. On the basis of Proposition 30, E is in particular a complete extension in C, i.e., $E \in \mathcal{CE}(\mathrm{AF}, C)$, therefore Proposition 39 entails that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$ $(E \cap S) \in \mathcal{CE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C)$. Taking into account Proposition 30, we have to prove that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$ $(E \cap S)$ is the least element (with respect to set inclusion) in $\mathcal{CE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C)$. We reason by contradiction, supposing that there is at least one strongly connected component where the thesis is not verified. In particular, since the strongly connected components of AF make up an acyclic graph, we can choose $\widehat{S} \in \mathrm{SCCS}_{\mathrm{AF}}$ such that

$$\forall S \in \mathrm{sccanc}_{\mathrm{AF}}(\widehat{S}), \quad (E \cap S) = \mathrm{GE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big) \tag{6}$$

and

$$\exists \widehat{E} \subsetneq (E \cap \widehat{S}), \quad \widehat{E} = \mathrm{GE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C\big) \tag{7}$$

Note that condition (6) is trivially verified if $\widehat{S}$ is initial.

Moreover, the second condition follows from the fact that, on the basis of Lemma 29, $\mathrm{GE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$ must exist, and according to Proposition 30 it is included in all the elements of $\mathcal{CE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$.

Now, taking again into account that the strongly connected components of AF make up an acyclic graph, it is easy to see that it is possible to construct a set E' such that:

- $\forall S \in \mathrm{sccanc}_{\mathrm{AF}}(\widehat{S}), (E' \cap S) = (E \cap S)$;
- $(E' \cap \widehat{S}) = \widehat{E}$;
- $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}, (E' \cap S) = \mathrm{GE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E')}, U_{\mathrm{AF}}(S, E') \cap C)$.

To this purpose, it is obviously possible to construct a set E'_* contained in the strongly connected components of $(\widehat{S} \cup \mathrm{sccanc}_{\mathrm{AF}}(\widehat{S}))$ which satisfies the first two conditions. Thus, it turns out that $\forall S \in (\widehat{S} \cup \mathrm{sccanc}_{\mathrm{AF}}(\widehat{S}))\ U_{\mathrm{AF}}(S, E'_*) = U_{\mathrm{AF}}(S, E)$ and $P_{\mathrm{AF}}(S, E'_*) = P_{\mathrm{AF}}(S, E)$. Therefore, taking into account (6) and (7), E'_* satisfies the third condition too for any such S, i.e.,

$$\forall S \in \big(\widehat{S} \cup \mathrm{sccanc}_{\mathrm{AF}}(\widehat{S})\big) \quad (E'_* \cap S) = \mathrm{GE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E'_*)}, U_{\mathrm{AF}}(S, E'_*) \cap C\big)$$

Now, E' can be obtained constructively from E'_* by proceeding along the other strongly connected components of the defeat graph: in fact $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}\ \mathrm{GE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E')}, U_{\mathrm{AF}}(S, E') \cap C)$ always exists by Lemma 29.

On the basis of Proposition 30, we have that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}\ (E' \cap S) \in \mathcal{CE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E')}, U_{\mathrm{AF}}(S, E') \cap C)$. As a consequence, by Proposition 39 it turns out that $E' \in \mathcal{CE}(\mathrm{AF}, C)$, while since $(E' \cap \widehat{S}) = \widehat{E} \subsetneq (E \cap \widehat{S})$ it is not true that $E \subseteq E'$. However, this contradicts the hypothesis that $E = \mathrm{GE}(\mathrm{AF}, C)$, which according to Proposition 30 is the least element of $\mathcal{CE}(\mathrm{AF}, C)$, i.e., the least complete extension in C.

Let us turn now to the other direction of the proof, by supposing that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}$, $(E \cap S) = \mathrm{GE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C)$. On the basis of Proposition 30, we have that $\forall S \in \mathrm{SCCS}_{\mathrm{AF}}, (E \cap S) \in \mathcal{CE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C)$, therefore Proposition 39 entails that $E \in \mathcal{CE}(\mathrm{AF}, C)$. As a consequence, taking into account Proposition 30 we have only to prove that E is the least element of $\mathcal{CE}(\mathrm{AF}, C)$. We reason by contradiction, assuming that the grounded extension $E' = \mathrm{GE}(\mathrm{AF}, C)$, which must exist by Lemma 29 and is a subset of E by Proposition 30, is strictly included in E. Thus, there must be at least a strongly connected component S such that $(E' \cap S) \subsetneq (E \cap S)$: since the strongly connected components form an acyclic graph, there is in particular a strongly connected component $\widehat{S}$ such that:

$$\forall S \in \mathrm{sccanc}_{\mathrm{AF}}(\widehat{S}), \quad (E' \cap S) = (E \cap S) \tag{8}$$

$$(E' \cap \widehat{S}) \subsetneq (E \cap \widehat{S}) \tag{9}$$

Moreover, since $E' = \mathrm{GE}(\mathrm{AF}, C) \in \mathcal{CE}(\mathrm{AF}, C)$, Proposition 39 applied to $\widehat{S}$ entails that $(E' \cap \widehat{S}) \in \mathcal{CE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E')}, U_{\mathrm{AF}}(\widehat{S}, E') \cap C)$. Taking into account (8), it is easy to see that $U_{\mathrm{AF}}(\widehat{S}, E') = U_{\mathrm{AF}}(\widehat{S}, E)$ and $P_{\mathrm{AF}}(\widehat{S}, E') = P_{\mathrm{AF}}(\widehat{S}, E)$, therefore $(E' \cap \widehat{S}) \in \mathcal{CE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$. However, according to (9) we have that $(E' \cap \widehat{S})$ is strictly included in $(E \cap \widehat{S})$, contradicting the hypothesis (referred to $\widehat{S}$) that $(E \cap \widehat{S}) = \mathrm{GE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$ and therefore, on the basis of Proposition 30, that $(E \cap \widehat{S})$ is the least element of $\mathcal{CE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(\widehat{S},E)}, U_{\mathrm{AF}}(\widehat{S}, E) \cap C)$. □

5.7. Traditional semantics as SCC-recursive semantics

As discussed in Section 4, each specific SCC-recursive semantics is identified by its own base function. On the basis of the results obtained in the previous sections, it is possible to identify the base functions corresponding to the traditional semantics introduced in [9] and, thus, to definitely prove that these traditional semantics fit the general SCC-recursive schema.

Theorem 43. *The stable, complete, preferred and grounded semantics are SCC-recursive, characterized by the following base functions* (*defined for generic argumentation frameworks* AF *such that* $|\mathrm{SCCS}_{\mathrm{AF}}| = 1$):

- $\mathcal{BF}_{\mathcal{ST}}(\mathrm{AF}, C) \equiv \mathcal{SE}(\mathrm{AF}, C)$;
- $\mathcal{BF}_{\mathcal{CO}}(\mathrm{AF}, C) \equiv \mathcal{CE}(\mathrm{AF}, C)$;
- $\mathcal{BF}_{\mathcal{PR}}(\mathrm{AF}, C) \equiv \mathcal{PE}(\mathrm{AF}, C)$;
- $\mathcal{BF}_{\mathcal{GR}}(\mathrm{AF}, C) \equiv \{\mathrm{GE}(\mathrm{AF}, C)\}$.

Proof. Let us prove the claim with reference to stable semantics. First, $E \in \mathcal{SE}(\mathrm{AF})$ if and only if $E \in \mathcal{SE}(\mathrm{AF}, \mathcal{A})$, since as noticed in Section 5.1 Dung's original definitions are recovered from the extended ones in case C coincides with the set of all arguments. Now, it is easy to see that $\mathcal{SE}(\mathrm{AF}, C)$ for generic AF and C adheres to Definition 20: if $|\mathrm{SCCS}_{\mathrm{AF}}| = 1$, then $\mathcal{SE}(\mathrm{AF}, C)$ coincides by definition with the base function $\mathcal{BF}_{\mathcal{ST}}(\mathrm{AF}, C)$, otherwise the decomposition schema along the strongly connected components follows from Proposition 34.

As far as complete, preferred and grounded semantics are concerned, proofs are similar and are based on Propositions 39, 41 and 42, respectively. □

For the grounded semantics, the base function admits a simple explicit formulation.

Proposition 44. *For any argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ *such that* $|\mathrm{SCCS}_{\mathrm{AF}}| = 1$, *and for any* $C \subseteq \mathcal{A}$, *we have that*

$$\mathcal{BF}_{\mathcal{GR}}(\mathrm{AF}, C) = \big\{\mathrm{GE}(\mathrm{AF}, C)\big\} = \begin{cases} \{\{\alpha\}\}, & \textit{if } C = \mathcal{A} = \{\alpha\} \textit{ and } \rightarrow = \emptyset; \\ \{\emptyset\}, & \textit{otherwise.} \end{cases}$$

Proof. According to Definition 28, $\mathrm{GE}(\mathrm{AF}, C)$ is the least fixed point of $\mathrm{F}_{\mathrm{AF},C}$. Let us consider its computation in the following exhaustive cases.

First, if $C = \emptyset$ then $\forall E \subseteq \mathcal{A}$ $\mathrm{F}_{\mathrm{AF},C}(E) = \emptyset$ by definition, obviously entailing that the empty set is the least fixed point of $\mathrm{F}_{\mathrm{AF},C}$ and $\mathcal{BF}_{\mathcal{GR}}(\mathrm{AF}, C) = \{\emptyset\}$.

Second, if $|\mathcal{A}| > 1$ then it must be the case that $\forall \alpha \in \mathcal{A}$, $\mathrm{parents}_{\mathrm{AF}}(\alpha) \neq \emptyset$, since AF is made up of a unique strongly connected component, while an initial node would not be reachable from another one. As a consequence, taking into account the definition of $\mathrm{F}_{\mathrm{AF},C}$, it turns that, $\forall C \subseteq \mathcal{A}$,

$$\begin{aligned}\mathrm{F}_{\mathrm{AF},C}(\emptyset) &= \{\alpha \in C \mid \forall \beta \in \mathcal{A}\colon\ \beta \rightarrow \alpha,\ \emptyset \rightarrow \beta\}\\ &= \{\alpha \in C \mid \mathrm{parents}_{\mathrm{AF}}(\alpha) = \emptyset\}\\ &= \emptyset\end{aligned}$$

As a consequence, the least fixed point of $\mathrm{F}_{\mathrm{AF},C}$ is the empty set, therefore again $\mathcal{BF}_{\mathcal{GR}}(\mathrm{AF}, C) = \{\emptyset\}$.

The remaining case to be considered is $\mathcal{A} = C = \{\alpha\}$. There are only two possibilities for the attack relation, namely either $\rightarrow = \{(\alpha, \alpha)\}$ or $\rightarrow = \emptyset$. The first situation can be treated as in the previous case, since the unique node α is not initial, yielding again $\mathcal{BF}_{\mathcal{GR}}(\mathrm{AF}, C) = \{\emptyset\}$. In the other situation, i.e., $C = \mathcal{A} = \{\alpha\}$ and $\rightarrow = \emptyset$, it is easy to see that $\mathrm{F}_{\mathrm{AF},C}(\{\alpha\}) = \{\alpha\}$ and $\mathrm{F}_{\mathrm{AF},C}(\emptyset) = \{\alpha\}$, therefore the least fixed point of $\mathrm{F}_{\mathrm{AF},C}$ is made up of the unique node α and $\mathcal{BF}_{\mathcal{GR}}(\mathrm{AF}, C) = \{\{\alpha\}\}$. $\square$

6. General properties of SCC-recursive semantics

Having proved that SCC-recursiveness is general enough to include traditional semantics, we now show that it is also restrictive enough to ensure that the basic desirable properties of an extension are satisfied by any SCC-recursive semantics, provided that very simple and intuitive constraints on the base function are respected.

6.1. Conflict-free property

As stated in Section 2.2, a basic requirement of any extension is the absence of conflicts, as expressed by the following definition:

Definition 45. A semantics $\mathcal{S}$ satisfies the *conflict-free property* if and only if $\forall \mathrm{AF}, \forall E \in \mathcal{E}_{\mathcal{S}}(\mathrm{AF})$ E is conflict-free.

Clearly, a necessary condition for a given SCC-recursive semantics to be conflict-free is that its base function is conflict-free:

Definition 46. The base function $\mathcal{GF}^*_{\mathcal{S}}$ of a SCC-recursive semantics $\mathcal{S}$ is *conflict-free* if and only if $\forall \mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and $\forall C \subseteq \mathcal{A}$ each element of $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, C)$ is conflict-free.

We also prove that this is a sufficient condition.

Proposition 47. *Let $\mathcal{S}$ be a SCC-recursive semantics identified by the base function $\mathcal{GF}^*_{\mathcal{S}}$. If $\mathcal{GF}^*_{\mathcal{S}}$ is conflict-free, then $\forall \mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and $\forall C \subseteq \mathcal{A}$ the elements of the function $\mathcal{GF}(\mathrm{AF}, C)$ based on $\mathcal{GF}^*_{\mathcal{S}}$ are conflict-free as well.*

Proof. Let us consider a generic argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$ and a generic set $C \subseteq \mathcal{A}$: we have to prove that, given a generic $E \in \mathcal{GF}(\mathrm{AF}, C)$, E is conflict-free.

Given the recursive characterization of E as in Definition 20, we first prove the claim in the base case, namely $|\mathrm{SCCS}_{\mathrm{AF}}| = 1$, then we assume as an inductive hypothesis that the claim holds for any $S \in \mathrm{SCCS}_{\mathrm{AF}}$ at the level of $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$ and show that this hypothesis entails the claim at the level of AF.

If $|\mathrm{SCCS}_{\mathrm{AF}}| = 1$, then, by Definition 20, $E \in \mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, C)$, and therefore is conflict-free by the hypothesis.

In the other case, namely $|\mathrm{SCCS}_{\mathrm{AF}}| > 1$, the inductive hypothesis can be expressed as

$$\forall S \in \mathrm{SCCS}_{\mathrm{AF}}\ (E \cap S) \text{ is conflict-free in } \mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)} \tag{10}$$

Let us reason by contradiction, assuming the existence of two elements $\alpha, \beta \in E$ such that $\alpha \rightarrow \beta$, and let us indicate $\mathrm{SCC}_{\mathrm{AF}}(\beta)$ as S. Clearly, $\alpha \notin S$, otherwise $(E \cap S)$ would not be conflict-free in $\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$. As a consequence, $\alpha \in \mathrm{outparents}_{\mathrm{AF}}(S) \cap E$. However, this would entail that $\beta \in D_{\mathrm{AF}}(S, E)$ by definition of $D_{\mathrm{AF}}(S, E)$, while since $\beta \in E$ this would contradict the fact that $(E \cap S) \subseteq UP_{\mathrm{AF}}(S, E)$, prescribed by (10). □

Theorem 48. *Given a SCC-recursive semantics $\mathcal{S}$, if its base function $\mathcal{GF}^*_{\mathcal{S}}$ is conflict-free then $\mathcal{S}$ satisfies the conflict-free property.*

Proof. Since, according to Definition 20, any extension $E \in \mathcal{E}_{\mathcal{S}}(\mathrm{AF})$ belongs to $\mathcal{GF}(\mathrm{AF}, \mathcal{A})$, the claim easily follows from Proposition 47 applied with $C = \mathcal{A}$. □

6.2. Agreement with the grounded semantics

As discussed in Section 2.2, the grounded semantics represents a sort of lower bound among argumentation semantics, since in Dung's framework the grounded extension is the least among all conceivable extensions, namely complete extensions. More generally, the agreement with grounded semantics can be regarded as a fundamental requirement for any argumentation semantics, as it appears evident considering the constructive characterization of grounded semantics recalled in the same subsection. As a confirmation of the well-foundedness of the property of SCC-recursiveness, we prove in Theorem 52 that for any SCC-recursive semantics each of its extensions includes the grounded extension, provided that a very simple condition on the base function is satisfied.

First, we prove a general property of the SCC-recursive schema showing that inclusion between extensions entails inclusion relations between the elements of the partitions of strongly connected components introduced in Definition 18.

Lemma 49. *Given an argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$, let $E_1, E_2 \subseteq \mathcal{A}$ be two sets of arguments such that $E_1 \subseteq E_2$ and E_2 is conflict-free (and therefore also E_1). For any strongly connected component $S \subseteq \mathrm{SCCS}_{\mathrm{AF}}$, it holds that:*

- $D_{\mathrm{AF}}(S, E_1) \subseteq D_{\mathrm{AF}}(S, E_2)$;
- $UP_{\mathrm{AF}}(S, E_2) \subseteq UP_{\mathrm{AF}}(S, E_1)$;
- $U_{\mathrm{AF}}(S, E_1) \subseteq U_{\mathrm{AF}}(S, E_2)$.

Proof. As to the first point, we have by definition that $D_{\mathrm{AF}}(S, E_1) = \{\alpha \in S \mid (E_1 \cap \mathrm{outparents}_{\mathrm{AF}}(S)) \to \alpha\}$. Since $E_1 \subseteq E_2$, it must be the case that $D_{\mathrm{AF}}(S, E_1) \subseteq \{\alpha \in S \mid (E_2 \cap \mathrm{outparents}_{\mathrm{AF}}(S)) \to \alpha\} \equiv D_{\mathrm{AF}}(S, E_2)$.

The above result easily entails the second one, taking into account that $UP_{\mathrm{AF}}(S, E_2) = (S \setminus D_{\mathrm{AF}}(S, E_2))$: by the first point, $(S \setminus D_{\mathrm{AF}}(S, E_2)) \subseteq (S \setminus D_{\mathrm{AF}}(S, E_1)) = UP_{\mathrm{AF}}(S, E_1)$.

As to the third point, let us first prove that

$$\forall \alpha \in U_{\mathrm{AF}}(S, E_1), \quad \big(E_2 \cap \mathrm{outparents}_{\mathrm{AF}}(S)\big) \not\to \alpha \tag{11}$$

We reason by contradiction, assuming that $\exists \alpha \in U_{\mathrm{AF}}(S, E_1), \beta \in (E_2 \cap \mathrm{outparents}_{\mathrm{AF}}(S))$: $\beta \to \alpha$. In this case, we have in particular that $\beta \in (\mathrm{outparents}_{\mathrm{AF}}(S) \cap \mathrm{parents}_{\mathrm{AF}}(\alpha))$, therefore, according to the definition of $U_{\mathrm{AF}}(S, E_1)$ applied to α, $E_1 \to \beta$. However, since $E_1 \subseteq E_2$, also $E_2 \to \beta$ holds, with $\beta \in E_2$, contradicting the hypothesis that E_2 is conflict-free.

Now, it is easy to see that

$$\forall \alpha \in U_{\mathrm{AF}}(S, E_1), \forall \beta \in \big(\mathrm{outparents}_{\mathrm{AF}}(S) \cap \mathrm{parents}_{\mathrm{AF}}(\alpha)\big), \quad E_2 \to \beta \tag{12}$$

since by definition of $U_{\mathrm{AF}}(S, E_1)$ applied to α we have that $E_1 \to \beta$ and $E_1 \subseteq E_2$.

Finally, on the basis of (11) and (12) it turns out that $\forall \alpha \in U_{\mathrm{AF}}(S, E_1)$ $(E_2 \cap \mathrm{outparents}_{\mathrm{AF}}(S)) \not\to \alpha \wedge \forall \beta \in (\mathrm{outparents}_{\mathrm{AF}}(S) \cap \mathrm{parents}_{\mathrm{AF}}(\alpha))$ $E_2 \to \beta$, i.e., $\alpha \in U_{\mathrm{AF}}(S, E_2)$. $\square$

The following lemma shows that an inclusion relation between two sets C_1 and C_2 also holds between the grounded extensions in C_1 and C_2, even if the latter is referred to a restricted argumentation framework.

Lemma 50. *Let* $\mathrm{AF}_1 = \langle \mathcal{A}_1, \to_1 \rangle$ *and* $\mathrm{AF}_2 = \langle \mathcal{A}_2, \to_2 \rangle$ *be two argumentation frameworks such that* $\mathrm{AF}_2 = \mathrm{AF}_1{\downarrow_{\mathcal{A}_2}}$. *Given two sets* C_1 *and* C_2 *such that* $C_1 \subseteq C_2 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_1$, $\mathrm{GE}(\mathrm{AF}_1, C_1) \subseteq \mathrm{GE}(\mathrm{AF}_2, C_2)$.

Proof. First, let us consider the characteristic functions $\mathrm{F}_{\mathrm{AF}_1, C_1}$ and $\mathrm{F}_{\mathrm{AF}_2, C_2}$, respectively defined in 2^{C_1} and 2^{C_2} (with $2^{C_1} \subseteq 2^{C_2}$). We prove that, given two sets $E_1 \subseteq C_1$ and $E_2 \subseteq C_2$

$$\text{if } E_1 \subseteq E_2 \quad \text{then } \mathrm{F}_{\mathrm{AF}_1, C_1}(E_1) \subseteq \mathrm{F}_{\mathrm{AF}_2, C_2}(E_2) \tag{13}$$

Let us consider a generic element $\alpha \in \mathrm{F}_{\mathrm{AF}_1, C_1}(E_1)$: by definition, $\alpha \in C_1$ and α is acceptable with respect to E_1 in AF_1. Since $C_1 \subseteq C_2$, $\alpha \in C_2$, therefore to prove that $\alpha \in \mathrm{F}_{\mathrm{AF}_2, C_2}(E_2)$ we have only to show that α is acceptable with respect to E_2 in AF_2. To this purpose, let us consider a generic argument $\beta \in \mathcal{A}_2$ which attacks α in AF_2, i.e., $\beta \in \mathrm{parents}_{\mathrm{AF}_2}(\alpha)$. Clearly, since $\mathcal{A}_2 \subseteq \mathcal{A}_1$, β attacks α also in AF_1, i.e., $\beta \in \mathrm{parents}_{\mathrm{AF}_1}(\alpha)$. Since α is acceptable with respect to E_1 in AF_1, $E_1 \to \beta$ in AF_1, and taking into account that $E_1 \subseteq C_1 \subseteq \mathcal{A}_2$ it is easy to see that $E_1 \to \beta$ holds also in AF_2. Moreover, $E_1 \subseteq E_2$, therefore we get the desired conclusion that $E_2 \to \beta$ in AF_2.

Now, by extending an analogous result proved by Dung (see [9], pp. 342) in case AF_1 and AF_2 are finitary, i.e., such that every argument has a finite number of defeaters, $\mathrm{GE}(\mathrm{AF}_1, C_1)$ and $\mathrm{GE}(\mathrm{AF}_2, C_2)$ can be respectively expressed as $\bigcup_{i \geqslant 1} \mathrm{F}^i_{\mathrm{AF}_1, C_1}(\emptyset)$ and $\bigcup_{i \geqslant 1} \mathrm{F}^i_{\mathrm{AF}_2, C_2}(\emptyset)$: in order to prove the thesis, we show by induction on i that $\forall i \geqslant 1\ \mathrm{F}^i_{\mathrm{AF}_1, C_1}(\emptyset) \subseteq \mathrm{F}^i_{\mathrm{AF}_2, C_2}(\emptyset)$. The proof can be extended to the general case by using transfinite induction on ordinal numbers instead of ordinary induction, however we resort to the latter for simplicity.

As for the base case, according to (13) $\mathrm{F}_{\mathrm{AF}_1, C_1}(\emptyset) \subseteq \mathrm{F}_{\mathrm{AF}_2, C_2}(\emptyset)$, since obviously $\emptyset \subseteq \emptyset \subseteq C_1$. In the induction step, we assume that $\mathrm{F}^i_{\mathrm{AF}_1, C_1}(\emptyset) \subseteq \mathrm{F}^i_{\mathrm{AF}_2, C_2}(\emptyset)$: as a consequence, on the basis of (13) it turns out that $\mathrm{F}_{\mathrm{AF}_1, C_1}(\mathrm{F}^i_{\mathrm{AF}_1, C_1}(\emptyset)) \subseteq \mathrm{F}_{\mathrm{AF}_2, C_2}(\mathrm{F}^i_{\mathrm{AF}_2, C_2}(\emptyset))$, i.e., $\mathrm{F}^{i+1}_{\mathrm{AF}_1, C_1}(\emptyset) \subseteq \mathrm{F}^{i+1}_{\mathrm{AF}_2, C_2}(\emptyset)$, and the proof is complete. □

We now show that the agreement with grounded semantics is ensured if the base function properly deals with the simplest possible case of argumentation framework, i.e., a single node which does not attack itself and therefore should be justified.

Proposition 51. *Let $\mathcal{S}$ be a SCC-recursive semantics identified by a conflict-free base function $\mathcal{GF}^*_{\mathcal{S}}$ such that*

$$\mathcal{BF}_{\mathcal{S}}\big(\langle\{\alpha\}, \emptyset\rangle, \{\alpha\}\big) = \big\{\{\alpha\}\big\}$$

For any argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \to \rangle$ and for any set $C \subseteq \mathcal{A}$, it holds that

$$\forall E \in \mathcal{GF}(\mathrm{AF}, C), \quad \mathrm{GE}(\mathrm{AF}, C) \subseteq E$$

where $\mathcal{GF}(\mathrm{AF}, C)$ is the recursive function of the SCC-recursive schema characterized by the base function $\mathcal{GF}^*_{\mathcal{S}}$.

Proof. Given a generic $\mathrm{AF} = \langle \mathcal{A}, \to \rangle$ and a set $C \subseteq \mathcal{A}$, let us consider a generic set $E \in \mathcal{GF}(\mathrm{AF}, C)$: we have to prove that $\mathrm{GE}(\mathrm{AF}, C) \subseteq E$. Let us distinguish two cases for the argumentation framework AF.

First, if $\mathrm{SCCS}_{\mathrm{AF}} = 1$, then the base case of the recursive definition applies to E, i.e., according to Definition 20 $E \in \mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, C)$. Of course, if $\mathrm{GE}(\mathrm{AF}, C)$ is the empty set then the thesis trivially holds for E. On the basis of Proposition 44, $\mathrm{GE}(\mathrm{AF}, C)$ is non empty only if $\mathrm{AF} = \langle\{\alpha\}, \emptyset\rangle$ and $C = \{\alpha\}$, and it turns out that $\mathrm{GE}(\mathrm{AF}, C) = \{\alpha\}$. By the hypothesis, in this case $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, C) = \{\{\alpha\}\}$, thus the only possible case for E is $E = \{\alpha\}$, obviously entailing that $\mathrm{GE}(\mathrm{AF}, C) \subseteq E$.

Let us now consider the case where $\mathrm{SCCS}_{\mathrm{AF}} > 1$. Taking into account the recursive definition of E according to Definition 20 and the proof for the case $|\mathrm{SCCS}_{\mathrm{AF}}| = 1$, we can recursively assume that, if $|\mathrm{SCCS}_{\mathrm{AF}}| > 1$, the thesis holds at the level of restricted argumentation frameworks:

$$\forall S \in \mathrm{SCCS}_{\mathrm{AF}}, \quad \mathrm{GE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big) \subseteq (E \cap S) \tag{14}$$

In order to simplify the notation, let us indicate $\mathrm{GE}(\mathrm{AF}, C)$ as E^*: we proceed inductively along the strongly connected components of the argumentation framework, by proving that

$$\forall S \in \mathrm{SCCS}_{\mathrm{AF}}, \quad (E^* \cap S) \subseteq (E \cap S)$$

which obviously entails the claim. Let us first notice that, according to Proposition 42

$$\forall S \in \mathrm{SCCS}_{\mathrm{AF}}, \quad (E^* \cap S) = \mathrm{GE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E^*)}, U_{\mathrm{AF}}(S, E^*) \cap C\big) \tag{15}$$

In the basis case, we consider the initial strongly connected components, namely we refer to a generic $S \in \mathrm{SCCS}_{\mathrm{AF}}$ such that $\mathrm{sccparents}_{\mathrm{AF}}(S) = \emptyset$, entailing that $\mathrm{outparents}_{\mathrm{AF}}(S) = \emptyset$. This, in turn, entails that, for any set E, $D_{\mathrm{AF}}(S, E) = P_{\mathrm{AF}}(S, E) = \emptyset$ and $U_{\mathrm{AF}}(S, E) = S$. As a consequence, according to (15) we have that $(E^* \cap S) = \mathrm{GE}(\mathrm{AF}{\downarrow}_S, S \cap C)$, which is in turn equal to $\mathrm{GE}(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C)$. Since by (14) the latter is contained in $(E \cap S)$, we get the desired result that $(E^* \cap S) \subseteq (E \cap S)$.

In the inductive step, we can assume that

$$\forall P \in \mathrm{sccanc}_{\mathrm{AF}}(S), \quad (E^* \cap P) \subseteq (E \cap P)$$

Now, since E is conflict-free by Proposition 47 and the definitions of $U_{\mathrm{AF}}(S, E)$ and $P_{\mathrm{AF}}(S, E)$ only depend on the subset of E contained in the strongly connected components of $\mathrm{sccanc}_{\mathrm{AF}}(S)$, Lemma 49 can be applied obtaining

$$U_{\mathrm{AF}}(S, E^*) \subseteq U_{\mathrm{AF}}(S, E)$$

and

$$UP_{\mathrm{AF}}(S, E) \subseteq UP_{\mathrm{AF}}(S, E^*)$$

On the basis of these results, it is possible to apply Lemma 50 with

- $\mathrm{AF}_1 = \mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E^*)}$,
- $\mathrm{AF}_2 = \mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)} = \mathrm{AF}_1{\downarrow}_{UP_{\mathrm{AF}}(S,E)}$,
- $C_1 = U_{\mathrm{AF}}(S, E^*) \cap C$,
- $C_2 = U_{\mathrm{AF}}(S, E) \cap C$

obtaining

$$\mathrm{GE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E^*)}, U_{\mathrm{AF}}(S, E^*) \cap C\big) \subseteq \mathrm{GE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big)$$

Now, on the basis of (15) we have that

$$(E^* \cap S) = \mathrm{GE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E^*)}, U_{\mathrm{AF}}(S, E^*) \cap C\big)$$

while by (14)

$$\mathrm{GE}\big(\mathrm{AF}{\downarrow}_{UP_{\mathrm{AF}}(S,E)}, U_{\mathrm{AF}}(S, E) \cap C\big) \subseteq (E \cap S)$$

entailing that $(E^* \cap S) \subseteq (E \cap S)$. □

Theorem 52. *Let $\mathcal{S}$ be a SCC-recursive semantics identified by a conflict-free base function $\mathcal{GF}^*_{\mathcal{S}}$ such that*

$$\mathcal{BF}_{\mathcal{S}}\big(\langle\{\alpha\}, \emptyset\rangle, \{\alpha\}\big) = \big\{\{\alpha\}\big\}$$

For any argumentation framework $\mathrm{AF} = \langle \mathcal{A}, \rightarrow\rangle$, $\forall E \in \mathcal{E}_{\mathcal{S}}(\mathrm{AF}), \mathrm{GE}(\mathrm{AF}) \subseteq E$.

Proof. The theorem directly follows from Proposition 51 applied in the case $C = \mathcal{A}$, taking into account that $\mathcal{E}_{\mathcal{S}}(\mathrm{AF}) = \mathcal{GF}(\mathrm{AF}, \mathcal{A})$ and $\mathrm{GE}(\mathrm{AF}) = \mathrm{GE}(\mathrm{AF}, \mathcal{A})$. □

Thanks to the above properties, defining an SCC-recursive semantics which is sound, i.e., conflict-free and in agreement with the grounded semantics, turns out to be a relatively easy task. In fact, in order for these properties to be satisfied it is sufficient that the base function is conflict-free and correctly treats the case of a single node without defeaters. In the following section we exploit the SCC-recursive schema to introduce four novel semantics which can cope in different ways with the problematic examples affecting preferred semantics. These semantics are introduced mainly for the purpose of demonstrating the potential of SCC-recursiveness: the possibility of defining novel semantics in a rather straightforward way confirms that our formalism is a sound basis for further studies in the spirit of the analysis of Section 3, where the need of a framework encompassing a large variety of semantics has been pointed out. In this sense, the semantics presented in the following section can be regarded more as illustrative than as definitive achievements: they represent an initial excursion in the space of SCC-recursive semantics, whose deeper and more complete exploration is indeed an interesting subject for future work. The definition of the semantics is deliberately example-driven; however, at a more general level, the four semantics are related by an underlying conceptual analysis, whose outcome seems to suggest that, in the SCC-recursive context, simpler base functions are better.

7. Supporting the definition of novel argumentation semantics

The results of the previous section suggest that relatively simple base functions are appropriate in the context of our SCC-recursive schema. Therefore, our search of novel interesting SCC-recursive semantics is driven by the idea of defining base functions based on progressively simpler concepts. Accordingly, we adopt preferred semantics as a starting point, and we use the focused survey presented in Section 2.2 as a guideline, by following it backward from more articulated concepts to the basic ones.

7.1. Beyond preferred semantics

In this subsection, we explore solutions which preserve the fundamental notion of defense, formally represented by the property of admissibility in Dung's framework; complete admissible sets, i.e., complete extensions, are regarded as the most general family of conceivable extensions in this context. As shown by Theorem 43, the recursive schema turns out to completely include this framework, since all complete extensions are recursively characterized. Therefore, considering the recursive schema does not introduce any limitation in this respect.

Since preferred extensions are maximal admissible sets, in order to devise an alternative admissibility-based proposal the only possible way is giving up the requirement of maximality. Let us start our analysis by recalling the argumentation framework of Fig. 7(a), where the argument $\neg$*rain* turns out to be questionably justified according to preferred semantics. First of all, note that in any semantics based on the concept of admissibility, none of the arguments S, R, or J can be included in any extension, since any of them attacks its only possible defender. As a consequence, also *rain* cannot be included in any extension, since it cannot be defended against its defeater J. Therefore, the only complete extensions of the argumentation framework AF of Fig. 7(a) are $\{\neg rain\}$ and $\emptyset$. While preferred semantics prescribes $\{\neg rain\}$ as the only extension, we aim at finding a semantics $\mathcal{S}$ whose definition admits the empty set as extension. Adopting the general SCC-recursive schema to identify such definition, we are led to investigate $E \cap S$ for all $S \in \mathrm{SCCS}_{\mathrm{AF}} = \{S_1, S_2\}$, where $S_1 = \{R, S, J\}$ and $S_2 = \{rain, \neg rain\}$, as explained in Section 4. In particular, since for any complete extension E we have that $(E \cap S_1) = \emptyset$, it is necessarily the case that $P_{\mathrm{AF}}(S_2, E) = \{rain\}$ and $U_{\mathrm{AF}}(S_2, E) = \{\neg rain\}$. As a consequence, Definition 20 yields $(E \cap S_2) \in \mathcal{BF}_{\mathcal{S}}(\mathrm{AF}\downarrow_{S_2}, \{\neg rain\})$, and, therefore, we must look for a base function such that $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}\downarrow_{S_2}, \{\neg rain\}) = \{\emptyset\}$.

A hint to obtain this property in general comes from the local correction to stable semantics sketched in Section 2.2, where the empty set is selected as an extension when there is no set of nodes able to attack all the others. Similarly, we might select the empty set as an extension when the nodes of a restricted argumentation framework are not all defended, i.e., do not all belong to the set C. Formally, we impose to the base function that if $C \subsetneq \mathcal{A}$ then $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, C) = \{\emptyset\}$. Taking this for granted, to complete the definition of the base function only the case where $C = \mathcal{A}$ remains to be specified. To this purpose, the more direct approach is to consider the set of preferred extensions, thus obtaining the following base function (we denote the corresponding semantics as *AD*1):

$$\mathcal{BF}_{AD1}(\mathrm{AF}, C) = \begin{cases} \mathcal{PE}(\mathrm{AF}, C) & \text{if } C = \mathcal{A} \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

It is easy to see that *AD*1 semantics solves the problem related to the argumentation framework of Fig. 7(a). In fact, on the first strongly connected component, $\mathcal{BF}_{AD1}(\mathrm{AF}\downarrow_{S_1}, S_1)$ is invoked, which returns the set of preferred extensions of $\mathrm{AF}\downarrow_{S_1}$: $\mathcal{PE}(\mathrm{AF}, S_1) = \{\emptyset\}$. Then, as explained above, $\mathcal{BF}_{AD1}(\mathrm{AF}\downarrow_{S_2}, \{\neg rain\})$ is invoked, which also returns $\{\emptyset\}$ as result. Summarizing we have that, for any possible extension E, $E \cap S_1 = \emptyset$ and $E \cap S_2 = \emptyset$, yielding the empty set as the unique extension of the whole argumentation framework.

However, *AD*1 fails with the argumentation framework presented in Fig. 8(a). In fact, in this case we have a single strongly connected component, and, therefore, *AD*1 inherits from preferred semantics the counter-intuitive behavior discussed in Section 3, identifying as the unique extension the set $\{\alpha, \phi\}$, which is also a stable extension. To overcome this problem, we need to rule out $\{\alpha, \phi\}$ as an extension. Note that the only defeater of both α and ϕ is γ, therefore the node γ should retain the capability of preventing α and ϕ to be justified. Noting that one of the defeaters of γ, actually β, is not included in the extension, to obtain the desired behavior we take a further step going back to a notion of more "aggressive" behavior, which strengthens the requirement of attacking all external nodes, typical of stable semantics. More precisely, we require that an extension fully attacks its defeaters, i.e., it includes all the defeaters of its defeaters. In particular, $\{\alpha, \phi\}$ does not satisfy this condition since it does not include β, while $\emptyset$ trivially satisfies it. These considerations lead to define the following base function (we denote the corresponding semantics as *AD*2):

$$\mathcal{BF}_{AD2}(\mathrm{AF}, C) = \begin{cases} \{E \mid E \text{ maximal in } \mathcal{AS}^*_{\mathrm{AF}}\} & \text{if } C = \mathcal{A} \\ \{\emptyset\} & \text{otherwise} \end{cases}$$

where

$$\mathcal{AS}^*_{\mathrm{AF}} = \left\{F \in \mathcal{AS}(\mathrm{AF}) \mid \forall \alpha \in \mathcal{A}\colon\ \alpha \to F, \text{parents}_{\mathrm{AF}}(\alpha) \subseteq F\right\}$$

It can be verified that *AD*2 preserves the desired behavior in all cases presented in Figs. 1–8.

Considering first the problematic example for *AD*1 of Fig. 8(a), the argumentation framework consists of a unique strongly connected component, and $\mathcal{BF}_{AD2}(\mathrm{AF}, \mathcal{A})$ is therefore invoked. Now $\mathcal{AS}^*_{\mathrm{AF}} = \{\emptyset\}$, as explained above, and therefore its only maximal set is $\emptyset$ which coincides with the unique extension.

On the other hand, the behavior of *AD*1 is preserved in the case of the argumentation framework of Fig. 7(a). After recalling that it consists of two strongly connected components $S_1 = \{R, S, J\}$ and $S_2 = \{rain, \neg rain\}$, it can be seen that the only admissible set in $\mathrm{AF}{\downarrow_{S_1}}$ is the empty set, which obviously belongs to $\mathcal{AS}^*_{\mathrm{AF}{\downarrow_{S_1}}}$. Then $U_{\mathrm{AF}}(S_2, \emptyset) = \{\neg rain\}$, $P_{\mathrm{AF}}(S_2, \emptyset) = \{rain\}$, and, as for *AD*1, $(E \cap S_2)$ is determined by $\mathcal{BF}_{AD2}(\mathrm{AF}{\downarrow_{S_2}}, \{\neg rain\})$, which yields the empty set since $C \neq \mathcal{A}$.

In the other cases, it turns out that *AD*1 and *AD*2 prescribe the same extensions as preferred semantics, whose behavior is fully justified from an intuitive point of view. We describe in the following the treatment of these cases in order to provide other examples of application of the SCC-recursive schema: since in all these cases $\mathcal{BF}_{AD2}(\mathrm{AF}, C)$ coincides with $\mathcal{BF}_{AD1}(\mathrm{AF}, C)$, we will refer to *AD*2 only.

The treatment of the argumentation framework of Fig. 3(a) is similar to that of Fig. 7(a), since, again, $E \cap S_1 = \emptyset$ with $S_1 = \{R, S, J\}$, and, letting $S_2 = \{rain\}$, then $P_{\mathrm{AF}}(S_2, \emptyset) = S_2$ and $(E \cap S_2)$ is determined by $\mathcal{BF}_{AD2}(\mathrm{AF}{\downarrow_{S_2}}, \emptyset)$, yielding the empty set as the unique extension.

As for the argumentation framework presented in Fig. 1, recall that its strongly connected components coincide with all the singletons. First, the base function will be invoked on the initial strongly connected component, obtaining $\mathcal{BF}_{AD2}(\mathrm{AF}{\downarrow_{\{\alpha\}}}, \{\alpha\}) = \{\{\alpha\}\}$. Then, $D_{\mathrm{AF}}(\{\beta\}, \{\alpha\}) = \{\beta\}$ and the base function will be invoked in the trivial case of an empty argumentation framework: $\mathcal{BF}_{AD2}(\mathrm{AF}{\downarrow_{\emptyset}}, \emptyset) = \{\emptyset\}$, thus excluding β from any extension. Then, for any E, $U_{\mathrm{AF}}(\{\gamma\}, E) = \{\gamma\}$, and iterating the same reasoning as above we obtain the inclusion of γ and then the exclusion of δ from the (unique) extension of AF.

The argumentation frameworks presented in Figs. 2, 5, and 8(b) consist of a unique strongly connected component, therefore in all these cases the following invocation of the base function applies: $\mathcal{BF}_{AD2}(\mathrm{AF}, \mathcal{A})$. It is easy to see that, in all cases, the set of maximal elements of $\mathcal{AS}^*_{\mathrm{AF}}$ coincide with the set of preferred extensions.

A very similar reasoning applies to the argumentation framework of Fig. 4, whose two strongly connected components are both initial. Letting $S_1 = \{\alpha, \beta, \gamma\}$ and $S_2 = \{\delta\}$ and applying the base function to both of them we obtain, for any possible E, $E \cap S_1 = \emptyset$ and $E \cap S_2 = \{\delta\}$, therefore $\{\delta\}$ is the unique resulting extension.

As to the argumentation framework of Fig. 3(b), first the base function is applied to the initial strongly connected component $S_1 = \{P, R, J, S\}$. Also in this case it is easy to see that the sets $\{R, S\}$ and $\{P, J\}$, besides being the preferred extensions of the restricted argumentation framework $\mathrm{AF}{\downarrow_{S_1}}$, are the maximal sets of $\mathcal{AS}^*_{\mathrm{AF}{\downarrow_{S_1}}}$. Considering the propagation of the two possible choices to the second strongly connected component $S_2 = \{rain\}$, we have in the former case $U_{\mathrm{AF}}(S_2, \{R, S\}) = S_2$ and therefore the invocation $\mathcal{BF}_{AD2}(\mathrm{AF}{\downarrow_{S_2}}, S_2)$ returning $\{rain\}$, while in the latter case $D_{\mathrm{AF}}(S_2, \{P, J\}) = S_2$ and therefore the invocation $\mathcal{BF}_{AD2}(\mathrm{AF}{\downarrow_{\emptyset}}, \emptyset)$ returning $\{\emptyset\}$. Thus the same extensions $\{R, S, rain\}$ and $\{P, J\}$ are obtained as in preferred semantics.

A similar reasoning applies to the example of Fig. 7(b). Given the partition $S_1 = \{P, R, J, S\}$ and $S_2 = \{rain, \neg rain\}$, the possible choices for $E \cap S_1$ are $\{R, S\}$ and $\{P, J\}$. In the former case, $U_{\mathrm{AF}}(S_2, \{R, S\}) = S_2$ and since S_2 is a Nixon diamond, analogously to the case of Fig. 5, the possible choices for $E \cap S_2$ are $\{rain\}$ and $\{\neg rain\}$. In the latter case $D_{\mathrm{AF}}(S_2, \{P, J\}) = \{rain\}$ and $U_{\mathrm{AF}}(S_2, \{P, J\}) = \{\neg rain\}$, then $\mathcal{BF}_{AD2}(\mathrm{AF}{\downarrow_{\{\neg rain\}}}, \{\neg rain\})$ is invoked, obviously giving $\{\neg rain\}$ as result. Summing up, again the same extension as in preferred semantics are obtained, namely: $\{R, S, rain\}$, $\{R, S, \neg rain\}$, and $\{P, J, \neg rain\}$.

In the argumentation framework of Fig. 6 the initial strongly connected component $S_1 = \{\alpha, \beta\}$ is a Nixon diamond, thus, analogously to the case of Fig. 5, the possible choices for $E \cap S_1$ are $\{\alpha\}$ and $\{\beta\}$. Independently of this choice, considering the second strongly connected component $S_2 = \{\gamma\}$, it holds that $D_{\mathrm{AF}}(S_2, E) = S_2$ therefore $E \cap S_2 = \emptyset$. Consequently, for the last strongly connected component $S_3 = \{\delta\}$, $U_{\mathrm{AF}}(S_3, E) = S_3$ and $E \cap S_3 = S_3$. Therefore also in this case the same extensions $\{\alpha, \delta\}$ and $\{\beta, \delta\}$ are obtained as in preferred semantics.

It is interesting to note that the requirement of including in an extension all of the defeaters of a node which attacks the extension, imposed in the base function of *AD*2, would be harmful rather than useful outside the SCC-recursive schema: examples can easily be found where even initial nodes would not be justified. However, the SCC-recursive definition rules the behavior of the semantics in such a way as to exploit this constraint correctly.

Finally, we show in Proposition 53 that both *AD*1 and *AD*2 fit in Dung's framework, as all their extensions are actually complete extensions.

Proposition 53. *For any argumentation framework* $\mathrm{AF} = \langle \mathcal{A}, \rightarrow \rangle$*, the extensions prescribed by AD*1 *and AD*2 *are complete extensions.*

Proof. According to the characterization of complete semantics as SCC-recursive given in Theorem 43, any set E which is decomposable according to Definition 20 with a base function whose elements belong to $\mathcal{CE}(\mathrm{AF}, C)$ is a complete extension. Therefore, to prove the claim it is sufficient to show that the base functions introduced for *AD*1 and *AD*2 give elements belonging to $\mathcal{CE}(\mathrm{AF}, C)$ for any AF (consisting of a unique strongly connected component) and C.

Considering first *AD*1, we distinguish two cases for the base function. If $C = \mathcal{A}$, then $\mathcal{BF}_{AD1}(\mathrm{AF}, C) = \mathcal{PE}(\mathrm{AF}, C)$, and the conclusion directly follows from Proposition 31. In the other case, i.e., $C \subsetneq \mathcal{A}$, $\mathcal{BF}_{AD1}(\mathrm{AF}, C) = \{\emptyset\}$, therefore we have to prove that $\emptyset$ is a complete extension in C: since $\emptyset$ is obviously admissible in C, the only thing to show is that any $\alpha \in C$ is not acceptable with respect to $\emptyset$. This trivially holds if $C = \emptyset$. Otherwise, a generic $\alpha \in C$ could only be acceptable with respect to $\emptyset$ if $\text{parents}_{\text{AF}}(\alpha) = \emptyset$. However, this is impossible, since $\alpha \in C$ and $C \subsetneq \mathcal{A}$: taking into account that the argumentation framework admits a unique strongly connected component, there must be an argument $\beta \in \mathcal{A}, \beta \neq \alpha$ such that $\beta \to \alpha$.

Let us turn now to $AD2$-semantics, and let us notice that, if $C \subsetneq \mathcal{A}$, the relevant base function coincides with that of $AD1$-semantics, therefore the proof proceeds as in the previous case. If instead $C = \mathcal{A}$, then by definition any $E \in \mathcal{BF}_{AD2}(\text{AF}, C)$ is a maximal element of $\mathcal{AS}^*_{\text{AF}}$, i.e., it is a maximal set such that it is admissible in AF and the following property holds:

$$\forall \beta \in \mathcal{A}\colon\ \beta \to E,\ \text{parents}_{\text{AF}}(\beta) \subseteq E \tag{16}$$

Since $C = \mathcal{A}$ and E is admissible, we have only to prove that $\forall \alpha \in \mathcal{A}$ which is acceptable with respect to E, $\alpha \in E$. We reason by contradiction, assuming that $\exists \hat{\alpha} \in \mathcal{A}\colon\ \hat{\alpha} \notin E$ and $\hat{\alpha}$ is acceptable with respect to E. In case $E = \emptyset$, the acceptability of $\hat{\alpha}$ would entail that $\text{parents}_{\text{AF}}(\hat{\alpha}) = \emptyset$, and therefore the set $\{\hat{\alpha}\}$ would be admissible. However, $\{\hat{\alpha}\}$ would clearly belong to $\mathcal{AS}^*_{\text{AF}}$, contradicting the maximality of $E = \emptyset$. Therefore, let us now assume that $E \neq \emptyset$. Recalling that $\hat{\alpha} \notin E$ and that $|\text{SCCS}_{\text{AF}}| = 1$, there must exist a path between $\hat{\alpha}$ and any element of E. In particular, there must be an element $\gamma \in E$ such that the path from α to γ does not include other elements of E. In fact, since $\hat{\alpha}$ is acceptable with respect to E, Lemma 35 entails that $\hat{\alpha} \not\to E$, and the above mentioned path must have the following structure: $\hat{\alpha} \to \beta_1 \to \cdots \to \beta_n \to \gamma$, with $n \geqslant 1$, such that $\gamma \in E$ and $\beta_i \notin E$ for all $i\colon\ 1 \leqslant i \leqslant n$. In particular, this entails that $\beta_n \to E$ and $\text{parents}_{\text{AF}}(\beta_n) \nsubseteq E$, contradicting (16). $\square$

7.2. Beyond admissibility

In the previous subsection, we have identified the suppression of the requirement of maximality among complete extensions as a possible way to solve the problems affecting preferred semantics, while preserving the notion of admissibility. As already explained, in any admissibility-based semantics odd-length cycles admit only the empty extension, while handling floating defeat as in Fig. 6 requires multiple extensions for even-length cycles. As a consequence, both $AD1$ and $AD2$, as well as other possible proposals inspired to the same ideas, feature an asymmetry in the treatment of cycles. In this subsection, we follow the idea that this asymmetry is the primary cause of questionable behaviors of preferred semantics and should be avoided to allow alternative treatments of cycles.

Let us investigate the definition of a semantics driven by this perspective. Consider again the simple case of a three length-cycle shown in Fig. 2: in order to enforce a symmetric treatment of cycles, we need to look for a possible set of non-empty extensions for this argumentation framework. To this purpose, note that, in order to preserve the conflict-free property, each extension has to include exactly one node. Moreover, obvious symmetry reasons entail that all nodes should be treated equally, therefore the only possibility is to identify as extensions the three sets $\{\alpha\}$, $\{\beta\}$ and $\{\gamma\}$.

Two general hints can be drawn from this simple example. First, it appears that our direction of investigation cannot be constrained by the admissibility requirement, since, in the considered example, the only sets that can be identified as extensions are not admissible. Second, it turns out that these extensions coincide with the maximal conflict-free sets of the argumentation framework (see Definition 6), and the same holds in the even-length cycle case. This suggests to exploit the notion of maximal conflict-free set as a basis for a new definition of extension. In a sense, this corresponds to the final step in our route from articulated concepts to basic ones, since the notion of conflict-free set has been the starting point of the survey in Section 2.2 and the conflict-free property is, actually, the most fundamental assumption underlying any definition of extension, that could never be removed.

The above intuition is confirmed by the fact that, by identifying the extensions as the maximal conflict-free sets, the problematic cases shown in Fig. 7 and Fig. 8 are handled correctly, since in all cases the intersection of maximal conflict-free sets is empty yielding all arguments not justified. However, as already discussed in Section 2.2, this intuition alone would not represent a satisfactory solution, since, due to the increased number of extensions prescribed, it would tend to assign the status of provisionally defeated to a large number of arguments, often to all of them. In other words, replacing the admissibility requirement with the less demanding notion of maximal conflict-free set requires some further condition in order to properly constrain the set of extensions prescribed by the semantics, i.e., to capture only a subset of the maximal conflict-free sets.

As a matter of fact, the SCC-recursive schema directly offers the solution to this difficulty: as it will be shown in the following, simply requiring that the semantics $\mathcal{S}$ is SCC-recursive represents an appropriate additional condition. To this purpose, let us consider again the example of Fig. 2 involving the three-length cycle. In this case, $|\mathrm{SCCS}_{\mathrm{AF}}| = 1$, therefore, according to Definition 20, the base function, applied to the set of all arguments, directly returns the set of the extensions. As a consequence, identifying extensions as the maximal conflict-free sets yields $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, \mathcal{A}) = \mathcal{MCF}_{\mathrm{AF}}$. Generalizing this idea, we obtain the specification of the base function for the case where $C = \mathcal{A}$, i.e., $\mathcal{BF}_{\mathcal{S}}(\mathrm{AF}, C) = \mathcal{MCF}_{\mathrm{AF}}$. Taking this for granted, to complete the definition of the base function only the case where $C \subsetneq \mathcal{A}$ remains to be specified. In this respect, it seems reasonable to exploit again the notion of maximal conflict-free set to provide a uniform conceptual basis to the approach. Two ways of applying this notion can be envisaged, depending on the role ascribed to the set C of defended nodes.

On the one hand, one may regard the inclusion of all the arguments in C as a requirement, preserving, as far as possible, the role of defense in the base function. This gives rise to the following definition (we denote the corresponding semantics as $CF1$):

$$\mathcal{BF}_{CF1}(\mathrm{AF}, C) = \mathcal{MCF}_{\mathrm{AF}\downarrow_C}$$

On the other hand, one may note that the perspective we are following is based on the assumption that the concept of defense is unnecessary in some cases, since the nodes of a three-length cycle are admitted as extensions though not being able to defend themselves. Generalizing from this remark, one may regard the notion of defense as definitely unnecessary. This leads to consider the following base function, where the requirement of inclusion of the arguments in C is overlooked (we denote as $CF2$ the corresponding semantics, first proposed in our [4]):

$$\mathcal{BF}_{CF2}(\mathrm{AF}, C) = \mathcal{MCF}_{\mathrm{AF}}$$

It can be seen that both $CF1$ and $CF2$-semantics are able to treat appropriately all argumentation frameworks presented in Figs. 1–8, as it was the case for $AD2$. To give a detailed account of the proposed semantics, we analyze their treatment of the argumentation framework of Fig. 7(a).[6]

Recall that $\mathrm{SCCS}_{\mathrm{AF}} = \{S_1, S_2\}$ with $S_1 = \{R, S, J\}$ and $S_2 = \{rain, \neg rain\}$. As for S_1, which is an initial strongly connected component, the base case of Definition 20 applies, yielding $(E \cap S_1) \in \mathcal{BF}_{\mathcal{S}}(\mathrm{AF}{\downarrow}_{S_1}, S_1)$. Therefore, according to $CF1$-semantics $(E \cap S_1)$ must belong to $\mathcal{MCF}_{\mathrm{AF}{\downarrow}_C}$ with $\mathrm{AF} = \mathrm{AF}{\downarrow}_{S_1}$ and $C = S_1$, which is of course equal to $\mathcal{MCF}_{\mathrm{AF}{\downarrow}_{S_1}}$. This is the same set prescribed by $CF2$-semantics, therefore the two semantics agree in this case. Now, $\mathcal{MCF}_{\mathrm{AF}{\downarrow}_{S_1}} = \{\{S\}, \{R\}, \{J\}\}$, leading to three alternative possibilities for $(E \cap S_1)$. Assuming that $(E \cap S_1) = \{J\}$, we have that $D_{\mathrm{AF}}(S_2, E) = \{rain\}$, $U_{\mathrm{AF}}(S_2, E) = \{\neg rain\}$ and $P_{\mathrm{AF}}(S_2, E) = \emptyset$. Therefore, the base function is applied to the argumentation framework $\mathrm{AF}{\downarrow}_{\{\neg rain\}}$ with $C = \{\neg rain\}$, yielding for both semantics $(E \cap S_2) = \{\neg rain\}$. Therefore, we have an extension $E_1 = \{J, \neg rain\}$ for both semantics. Let us turn now to the alternative $(E \cap S_1) = \{R\}$, which entails that $U_{\mathrm{AF}}(S_2, E) = S_2$ (as J is attacked by $R \in E$). Then, the base function is applied to the argumentation framework $\mathrm{AF}{\downarrow}_{S_2}$ with $C = S_2$: it is easy to see that both semantics give two alternatives for $(E \cap S_2)$, namely $\{rain\}$ and $\{\neg rain\}$. Therefore, we have for both $CF1$ and $CF2$ the additional extensions $E_2 = \{R, rain\}$ and $E_3 = \{R, \neg rain\}$. Finally, if $(E \cap S_1) = \{S\}$, we have that $P_{\mathrm{AF}}(S_2, E) = \{rain\}$, $U_{\mathrm{AF}}(S_2, E) = \{\neg rain\}$ and $D_{\mathrm{AF}}(S_2, E) = \emptyset$. Then, the base function is applied to the argumentation framework $\mathrm{AF}{\downarrow}_{S_2}$ with $C = \{\neg rain\}$. In this case, $CF1$ and $CF2$-semantics differ. In fact, $CF1$ gives $\mathcal{MCF}_{\mathrm{AF}{\downarrow}_C}$ which is equal to $\mathcal{MCF}_{\mathrm{AF}{\downarrow}_{\{\neg rain\}}}$, therefore $(E \cap S_2) = \{\neg rain\}$, giving $E_4 = \{S, \neg rain\}$. On the other hand, $CF2$ gives $\mathcal{MCF}_{\mathrm{AF}{\downarrow}_{S_2}} = \{\{rain\}, \{\neg rain\}\}$, yielding therefore the same extension E_4 as $CF1$ and an additional extension $E_5 = \{S, rain\}$.

In spite of this difference, both semantics provide the intuitively desirable result: no argument is justified, since the intersection of all extensions is empty. A similar treatment is provided by $CF1$ and $CF2$ to the argumentation framework of Fig. 7(b), where they identify the same extensions as preferred semantics, yielding again an empty intersection.

7.3. Comparing SCC-recursive semantics

Having provided the definition of four novel semantics, we now need to discuss their placement with respect to the traditional semantics of Dung's framework, in order to carry out a comparative analysis.

First of all, the basic requirement of agreement with the grounded semantics is satisfied by all the introduced proposals. In fact, it is easy to see that the sufficient conditions for agreement stated in Theorem 52 are respected by their base functions and, therefore, all the extensions of the proposed semantics contain the grounded extension. All semantics, in turn, differ from preferred semantics in the treatment of the problematic cases which have been the starting point of our investigation. In this respect, $AD1$ and $AD2$ are

[6] The treatment of other examples of Section 2.2 can be easily derived along the same lines, as well as that of the examples of Fig. 8, where there is only one strongly connected component and the extensions, coinciding with maximal conflict-free sets, have empty intersection.

closer by construction to preferred semantics, since, by Proposition 53, they select their extensions among complete extensions and, therefore, each one of their extensions is contained in a maximal complete extension, i.e., in a preferred extension. According to the relation of inclusion between extensions, $AD1$ and $AD2$ can be regarded as intermediate approaches lying between grounded and preferred semantics. On the other hand, $CF1$ and $CF2$ radically depart from preferred semantics, relying only on the conflict-free property and SCC-recursiveness, while relaxing the admissibility requirement and thus admitting extensions which are not complete extensions.

As remarked in the previous subsection, this departure supports a symmetric treatment of odd-length cycles with respect to even-length ones, which represents a distinguishing property of $CF1$ and $CF2$ with respect to $AD1$ and $AD2$. As far as the examples considered in previous subsections are concerned, this difference, though conceptually remarkable, does not give rise to significantly different results in the assignment of the justification status. In particular, only an example where $AD1$ fails to provide an intuitively adequate treatment has been pointed out, while $AD2$, $CF1$ and $CF2$ substantially agree in all examples, though sometimes achieving the desired result in different ways. Actually, this is not always the case: different behaviors are possible in other examples, which indeed show that the symmetric treatment of cycles can be a real advantage and make the difference in some cases.

To substantiate this statement, let us consider the argumentation framework shown in Fig. 9, corresponding to a floating defeat against argument γ by the nodes in the three-length cycle. In this case, any admissibility-based semantics admits the empty set as the unique extension, and therefore this is the case for $AD1$ and $AD2$ as well, which do not regard as justified any argument. On the contrary, both $CF1$ and $CF2$ admit as extensions the sets $\{\alpha, \delta\}$, $\{\beta, \delta\}$ and $\{\phi, \delta\}$. In fact, in the initial strongly connected component $S_1 = \{\alpha, \beta, \phi\}$ the maximal conflict-free sets, namely the singletons $\{\alpha\}$, $\{\beta\}$, and $\{\phi\}$, are selected as possible choices for $E \cap S_1$. Then $S_2 = \{\gamma\}$ is ruled out from possible selections, since γ is attacked in any case by E and, therefore, $D_{\mathrm{AF}}(S_2, E) = S_2$. As a consequence, δ is always selected, since, letting $S_3 = \{\delta\}$, $U_{\mathrm{AF}}(S_3, E) = S_3$ and $\{\delta\}$ is of course the only maximal conflict free set of S_3. As a result, α, β and ϕ are provisionally defeated, γ is defeated and δ is undefeated. This behavior is coherent with that obtained in the case of even-length cycles, like that shown in Fig. 6. As a consequence, this kind of examples discriminates admissibility-based semantics from the proposed novel semantics based on maximal conflict-free sets.

Considering now the case of a self-defeating argument shown in Fig. 10, it is easy to see that the only admissible set here is empty, therefore any admissibility-based semantics, including $AD1$ and $AD2$, is doomed to show the problematic behavior discussed in Section 3 and can not allow an alternative treatment where the self-defeating argument is ruled out. Let us now consider how this case is handled by $CF1$ and $CF2$. First, note that the argumentation framework AF is composed of two strongly connected components, namely $S_1 = \{\alpha\}$ and $S_2 = \{\beta\}$. Starting from the initial strongly connected component S_1, the base function applies to $\mathrm{AF}{\downarrow_{\{\alpha\}}}$ with $C = \{\alpha\}$: both $CF1$ and $CF2$ prescribe that $(E \cap S_1)$ is a maximal conflict-free set of S_1. Since S_1 consists just of a self-defeating argument, the only conflict-free set is the empty set, therefore both $CF1$ and $CF2$ exclude α from any extension. This in turn entails that, for any extension E,

$P_{\mathrm{AF}}(S_2, E) = \{\beta\}$ while $U_{\mathrm{AF}}(S_2, E) = D_{\mathrm{AF}}(S_2, E) = \emptyset$. Then, $E \cap S_2 \in \mathcal{GF}(\mathrm{AF}{\downarrow_{\{\beta\}}}, \emptyset)$, and, since $\mathrm{AF}{\downarrow_{\{\beta\}}}$ has obviously only one strongly connected component, this entails that $E \cap S_2 \in \mathcal{BF}_{CF}(\mathrm{AF}{\downarrow_{\{\beta\}}}, \emptyset)$, where *CF* stands for either *CF*1 or *CF*2. *CF*1 and *CF*2 behave differently in this case, due to the role of the parameter C in the definition of the relevant base functions. In particular, according to *CF*1 $(E \cap S_2) \in \mathcal{MCF}_{(\mathrm{AF}{\downarrow_{\{\beta\}}}){\downarrow_{\emptyset}}}$, i.e., $(E \cap S_2)$ must be the empty set. On the other hand, C plays no role in the base function of *CF*2, yielding $(E \cap S_2) \in \mathcal{MCF}_{\mathrm{AF}{\downarrow_{\{\beta\}}}}$, i.e., $(E \cap S_2) = \{\beta\}$. As a consequence, *CF*1 admits the empty set as the unique extension, thus sharing with admissibility-based semantics the inability to rule out self-defeating arguments. On the contrary, *CF*2 is able to select the only desired extension $\{\beta\}$, thus preventing the self-defeating argument α to exert a (possibly undesired) effect on the rest of the graph.

8. Conclusions

In this paper, we have proposed a general recursive schema that, while including Dung's framework as a special case, can be regarded as an alternative foundation of argumentation semantics. In particular, the property of SCC-recursiveness has a basic unifying role in argumentation semantics, in a similar way as admissibility in Dung's framework, and can ensure the correct propagation of the fundamental semantic principles coded in the base function to a generic argumentation framework.

In fact, SCC-recursiveness turns out to be an effective tool for supporting the development of new semantics. On the one hand, in order to define a particular SCC-recursive semantics it is sufficient to identify a suitable base function defined over the argumentation frameworks consisting of a single strongly connected component. On the other hand, the fundamental requirements that all extensions are conflict-free and contain the grounded extension come almost for free, since it is sufficient that the base function is conflict-free and correctly treats single nodes without defeaters.

To exemplify the potential of the recursive schema, we have introduced four novel semantics all providing a different treatment with respect to preferred semantics in cases where its behavior can be regarded as problematic. Our investigation has been developed in two directions: semantics *AD*1 and *AD*2 remain in the area of Dung's admissibility-based framework, while *CF*1 and *CF*2 more radically depart from traditional semantics, giving up the requirement of admissibility and resorting to simpler concepts. The most satisfactory behavior is achieved by *CF*2, which, in particular, is the only one able to deal with a thorny example concerning self-defeating arguments. It is worth noting that such semantics corresponds to the last step in our exploration of the use of progressively simpler concepts within the SCC-recursive schema, since it only relies on the notion of maximal conflict free set and completely overlooks the notion of defense in the definition of the base function.

As for future work, three main directions of investigation are worth considering. First, a further analysis of the general properties of the SCC-recursive schema will be carried out. In particular, in order to compare different proposals, it would be interesting to characterize the relationships between SCC-recursive semantics in terms of the notion of *skepticism*, i.e., on the basis of the level of commitment concerning the choices about the justification status assigned to the arguments [8]. Second, the space of SCC-recursive semantics will be extensively explored in order to identify further semantics, whose properties will be analyzed both from a theoretical point of view and with respect to their meaning and use in different application contexts. Third, since, as shown in the examples, the definition of SCC-recursiveness has a direct constructive interpretation, it is worth investigating the development of efficient and incremental algorithms based on local computation at the level of strongly connected components.

Acknowledgements

We are indebted to the anonymous referees for their helpful comments and suggestions.

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