RESIDUAL FINITENESS IN HOMOTOPY THEORY

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Dedicated to the memory of my mother, Rose Roitberg

1.

The concept of residual finiteness is familiar and important in combinatorial
group theory; see, for example, [4] or [5]. Recall that a group \( G \) is residually finite
if for every \( x \in G, x \neq 1 \), there exists a finite group \( \Gamma \) and a homomorphism \( \phi : G \to \Gamma \)
such that \( \phi x \neq 1 \).

Sullivan's introduction of a 'profinite completion' process into homotopy theory
([10]; see also [1]) suggests strongly that there ought to be a corresponding concept
of residual finiteness in 'the homotopy category'. (Throughout this paper, we work
exclusively in the pointed homotopy category \( \mathcal{H} \) of path-connected CW-spaces. For
any two objects \( A, B \) of \( \mathcal{H} \), we write the morphism set, as usual, as \([A, B]\).) This
paper represents a first step (aside from [10]) in developing such a concept and
features as its main results homotopy-theoretic analogs of the 'two remarkable
properties of finitely generated residually finite groups' described in [5; pp.
414–415], due respectively to A.I. Mal'cev (1940) and G. Baumslag (1963).

In order to define the notion of residually finite space (we actually give two,
possibly inequivalent, notions), we must first provide the homotopy-theoretic
analog of 'finite group'. We say that a space \( F \) is (homotopically) finite if the
homotopy groups \( \pi_n F, n \geq 1 \), are finite groups; if, in addition, there exists \( N \) such
that \( \pi_n F = 0, n > N \), we say that \( F \) is (homotopically) totally finite.

Definition 1. The space \( X \) is residually finite (sometimes abbreviated rf) if for any
finite CW-complex \( W \) and any pair of elements \( \alpha, \beta \in [W, X], \alpha \neq \beta \), there exists a
totally finite space \( Z \) and a map \( f : X \to Z \) such that \( f \circ \alpha \neq f \circ \beta \).

The space \( X \) is weakly residually finite (sometimes abbreviated wrf) if the
condition just stated holds in the special case where \( \beta = 0 \), the trivial element.

Recall [2; p. 83] that a map \( m : A \to B \) is an \( F \)-monomorphism (weak \( F \)-mono-
morphism) if for all finite CW-complexes \( W \), the induced \( m_* : [W, A] \to [W, B] \) is a
monomorphism (weak monomorphism, that is, $m^{-1}_{\mathcal{F}}(0) = 0$). Definition 1 may then be reformulated as follows.

**Definition 1'.** The space $X$ is residually finite (weakly residually finite) if there exists an $\mathcal{F}$-monomorphism (weak $\mathcal{F}$-monomorphism) $m : X \to Y$ with $Y$ a Cartesian product of totally finite spaces.

If $W$ is a cogroup-like space, for instance a suspension, then the two conditions in Definition 1 (or Definition 1') are, of course, equivalent. In general, they may differ; indeed, even if $X$ is an H-space, it is not clear that the notions of residual finiteness and weak residual finiteness agree since it is not postulated in that case that $Z$ be an H-space and $f : X \to Z$ an H-map.

The author cannot say with confidence which of the two notions is ‘better’. For the two principal results of this paper (Theorems 2 and 3 below), the first holds under the assumption of weak residual finiteness while the second seems to require the assumption of residual finiteness.

The stipulation of Definition 1 that $W$ be a finite CW-complex is important. If $W$ is taken to be arbitrary or even finite-dimensional, the resulting notions of residual finiteness and weak residual finiteness are rendered almost totally uninteresting; see Section 4.

Residual finiteness and weak residual finiteness behave well with respect to certain standard constructions in homotopy theory. The following simple proposition summarizes what is known in this direction.

**Proposition 1.** (a) If $\{X_{\lambda}\}, \lambda \in \Lambda$, is a family of $\mathcal{F}$ (wrf) spaces, then the Cartesian product $\prod_{\lambda \in \Lambda} X_{\lambda}$ is again $\mathcal{F}$ (wrf).

(b) If $X$ is $\mathcal{F}$ (wrf) and $V$ is a finite CW-complex, then the path-component of 0 of the pointed mapping space $X^V$ is again $\mathcal{F}$ (wrf).

(c) If $X$ is $\mathcal{F}$ (wrf) and $m : Y \to X$ is an $\mathcal{F}$-monomorphism (weak $\mathcal{F}$-monomorphism), then $Y$ is $\mathcal{F}$ (wrf).

It is plain that if $X$ is $\mathcal{F}$, or even wrf, then $\pi_1 X$ is a $\mathcal{F}$ group. Specializing $V$, in Proposition 1(b), to be a sphere, we obtain more generally:

**Corollary 1.** If $X$ is wrf, then the homotopy groups $\pi_n X$, $n \geq 1$, are $\mathcal{F}$ groups.

A very weak converse of Corollary 1, whose proof is deferred to Section 4, is given by

**Proposition 2.** If $G$ is a $\mathcal{F}$ group, abelian if $n \geq 2$, then the Eilenberg–MacLane space $K(G, n)$ is $\mathcal{F}$.

Proposition 1(a) has an evident group-theoretic analog. The known behavior of residual finiteness in the category of groups with respect to the formation of free
products [4] suggests the following question, which we formulate in its simplest form.

**Question 1.** If $X$ and $X'$ are rf (wrf), is then the wedge $X \vee X'$ rf (wrf)?

A special case of Proposition 1(c) arises when $m: Y \to X$ is a covering map; indeed, in that case, $m$ is actually a monomorphism rather than merely an $f$-monomorphism. The group-theoretic analog of this special case asserts that subgroups of rf groups are themselves rf. Now if $H$ is a subgroup of finite index of a group $G$ and if $H$ is rf, then it is easy to show that $G$ is rf. This, in turn, suggests the following question.

**Question 2.** If $g: \hat{X} \to X$ is a finite-sheeted covering map and $\hat{X}$ is rf (wrf), is $X$ then rf (wrf)?

Some comments on Question 2 are given in Section 3.

To obtain an abundant supply of rf spaces, we may appeal to the work of Sullivan [10].

**Theorem 1.** (a) If $\hat{X}$ is the profinite completion of a space $X$, then $\hat{X}$ is rf; the same is true for the $P$-profinite completion $\hat{X}_P$ where $P$ is a nonempty family of primes.

(b) If $X$ is a nilpotent space of finite type, that is the homotopy groups $\pi_n X$, $n \geq 1$, are finitely generated, then $X$ is rf.

**Proof.** To prove (a), recall that for any $W$ (not necessarily a finite CW-complex), there is a bijection

$$[W, \hat{X}] \cong \lim_{\leftarrow} [W, Z],$$

the inverse limit being taken over the category whose objects are maps $\phi: X \to Z$ with $Z$ totally finite and whose morphisms are homotopy-commutative diagrams

$$\begin{array}{ccc}
X & \xrightarrow{\phi} & Z \\
\downarrow \phi' & & \downarrow \psi \\
Z' & \xrightarrow{\phi_0} & Z
\end{array}$$

Thus, if $\alpha, \beta \in [W, \hat{X}]$, $\alpha \neq \beta$, there exists $\phi_0: X \to Z_0$ with $Z_0$ totally finite such that $\alpha_{\phi_0} \neq \beta_{\phi_0}$, where $\alpha_{\phi_0}, \beta_{\phi_0} \in [W, Z_0]$ are the projections of $\alpha, \beta$. The evident natural transformation of functors

$$\lim_{\leftarrow} [-, Z] \to [-, Z_0]$$

gives rise to a map $f: \hat{X} \to Z_0$ and clearly $f_\ast \alpha = \alpha_{\phi_0}$, $f_\ast \beta = \beta_{\phi_0}$; thereby establishing (a) in case $P = \text{all primes}$. The general case is handled similarly.
To prove (b), we invoke the 'Hasse principle' [10; Theorem 3.2 and p. 53] (or 'fracture lemma' [1; pp. 192-194]), according to which the profinite completion map \( c : X \to \bar{X} \) is an \( \mathbb{F} \)-monomorphism. Since, by (a), \( \bar{X} \) is rf, (b) follows on appealing to Proposition 1(c).

**Addendum to Theorem 1.** If \( X \) is nilpotent of finite type, then, for any nonempty family of primes \( P \), there exists an \( \mathbb{F} \)-monomorphism \( m : X \to Y \times Y' \) with \( Y \), resp. \( Y' \), a Cartesian product of totally finite spaces, each of which is nilpotent and \( P \)-torsion, resp. \( P' \)-torsion. (Compare Definition 1'. A space is said to be \( P \)-torsion if its homotopy groups are \( P \)-torsion groups. Also \( P' \) denotes the family of all primes not in \( P \).) Further, \( m \) 'extends' uniquely to a map \( M : \hat{X}_P \to Y \), which is a monomorphism.

**Proof.** The first statement follows from [10; p. 53], the second from [10; Corollary to Theorem 3.1].

Theorem 1 suggests the following question.

**Question 3.** If \( X \) is a nilpotent space of finite type and \( P \) is a nonempty family of primes, is then the \( P \)-localization \( X_P \) rf?

Notice that if \( P = \emptyset \), then \( X_P \), the rationalization of \( X \), is not rf (or wrf) unless, of course, it is contractible. This follows from Corollary 1.

We next turn to a connection between (w)rf spaces and Hopfian spaces ([3], [7]). In order to state our main theorem in this direction, we need to introduce certain finiteness conditions which come into play.

(i) We say that \( X \) satisfies (F1) if the homotopy groups \( \pi_n X, n \geq 2 \), are either finitely generated (Hft in the terminology of [3]) or the \( P \)-profinite completions of finitely generated abelian groups, \( P \) being a nonempty family of primes.

(ii) We say that \( X \) satisfies (F2) if the set \( [X, Z] \) is finite for any totally finite \( Z \).

(ii') We say that \( X \) satisfies (F2') if the condition (ii) is met for any totally finite, nilpotent \( Z \).

The following proposition is easily established with the aid of obstruction theory.

**Proposition 3.** (a) If \( X \) is geometrically of finite-finite type, that is the homotopy groups \( \pi_n X, n \geq 2 \), are finitely generated and \( X \) has the homotopy type of a CW-complex with finite skeleta (in particular, \( \pi_1 X \) is finitely presented), then \( X \) satisfies (F1) and (F2).

(b) If \( X \) is algebraically of finite-finite type, that is the homotopy groups \( \pi_n X, n \geq 1 \), and the integral homology groups \( H_n X, n \geq 1 \), are finitely generated, then \( X \) satisfies (F1) and (F2').

We now state our homotopy-theoretic analog of a theorem of Mal'cev [5; p. 415].
Theorem 2. If $X$ is a weakly residually finite space which satisfies $(F1)$ and $(F2)$, then $X$ is a Hopfian object of $\mathcal{X}$. The same conclusion holds if $(F2)$ is replaced by $(F2')$ provided there is a weak $\mathcal{F}$-monomorphism $X \to Y$ as in Definition 1' with each factor of $Y$ nilpotent.

The proof is given in Section 2.

Corollary 2. The following classes of spaces consist of Hopfian objects of $\mathcal{X}$:

(a) Wrf spaces of geometrically finite-finite type.
(b) Nilpotent spaces of finite type.
(c) $P$-profinite completions of nilpotent spaces of finite type.
(d) Finite spaces.

The proof for (a) is clear in view of Proposition 3(a). For (b) and (c), we use the fact [2; Theorem II.2.16, p. 69] that nilpotent spaces of finite type are algebraically of finite-finite type, together with Proposition 3(b), the Addendum to Theorem 1 and [10; the $P$-profinite analog of Theorem 3.1(ii)]. For (d) we need to observe that finite spaces satisfy $(F2)$. This is shown in Section 3, which, more generally, treats spaces having finite fundamental group.

Note that Corollary 2(b) coincides with [3; Corollary 4]. However, [7; Corollary 1.1] is not subsumed by Theorem 2. In fact, any noncontractible, rational nilpotent space whose integral homology groups $H_nX$, $n \geq 1$, are finitely generated $\mathbb{Q}$-modules (for instance $K(\mathbb{Q}, n)$, $n \geq 1$) is Hopfian but not wrf.

Finally, we come to a homotopy-theoretic analog of a theorem of G. Baumslag [5; p. 414].

Theorem 3. If $X$ is a finite CW-complex which is residually finite, then $\text{Aut}(X)$, the group of (pointed) homotopy classes of self-homotopy equivalences of $X$, is a residually finite group.

The proof is given in Section 2.

At this point, I would like to record my indebtedness to G. Baumslag for some very helpful discussions during the initial stage of this work.

2.

The proofs of both Theorems 2 and 3 depend on a key lemma, formulated below, on the existence of what we call distinguished maps. To explain, we consider spaces $X, Y_0$ such that $[X, Y_0]$ is finite and enumerate the elements of $[X, Y_0]$ as $\{f_1, \ldots, f_r\}$. Now let

$$Y = Y_0 \times \cdots \times Y_0$$

$r$ factors
and let \( f: X \to Y \) be the map whose \( j \)th component is \( f_j, 1 \leq j \leq r \). (Here, and subsequently, we sometimes 'confuse' a map with its homotopy class.) There is a natural action of the symmetric group \( \mathcal{S}_r \) on \( Y \) given by permuting coordinates and we may regard \( \mathcal{S}_r \) as a submonoid of \( \mathcal{H}(Y) \), the monoid of (pointed) self-homotopy equivalences of \( Y \).

**Lemma 1.** For any epimorphism \( e: X \to X \) (in \( \mathcal{H} \)), there is a unique \( \sigma \in \mathcal{S}_r \subseteq \mathcal{H}(Y) \) making the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{e} & X \\
\downarrow{f} & & \downarrow{f} \\
Y & \xrightarrow{\sigma} & Y
\end{array}
\]

homotopy-commutative. The assignment \( e \mapsto \sigma \) induces a map \( \Phi: \text{Epi}(X) \to \mathcal{S}_r \subseteq \mathcal{H}(Y) \) from the monoid of self-epimorphisms of \( X \) to \( \mathcal{S}_r \) which is a monoid homomorphism (not merely up to homotopy).

**Proof.** As \( e \) is an epimorphism, the induced map \( e^*: [X, Y_0] \to [X, Y_0] \) is a monomorphism. Moreover, as \([X, Y_0]\) is finite, \( e^* \) is a bijection. Thus, for each \( j \in \{1, \ldots, r\} \), there is a unique \( \sigma_j \in \{1, \ldots, r\} \) such that \( f_j \circ e = f_{\sigma_j} \). The assignment \( j \mapsto \sigma_j \) gives an element \( \sigma \in \mathcal{S}_r \) which plainly makes (2.1) homotopy-commutative.

The assertion about \( \Phi \) is evident.

The map \( f: X \to Y \) just described is said to be distinguished by analogy with the concept of distinguished subgroup [5; p. 114].

**Proof of Theorem 2.** We must show that any epimorphism \( e: X \to X \) is a homotopy equivalence and begin by showing that \( e \) is a weak \( \mathcal{F} \)-monomorphism.

Thus, let \( W \) be a finite CW-complex and \( \alpha \in [W, X], \alpha \neq 0 \). As \( X \) is wrf, there exists a totally finite space \( Z_0 \) and a map \( f_0: X \to Z_0 \) such that \((f_0)_*\alpha \neq 0\). Now either \( X \) satisfies (F2) or \( Z_0 \) may be chosen nilpotent and \( X \) satisfies (F2'); in any event, the set \([X, Z_0]\) is finite and we may apply Lemma 1 to obtain a (totally finite) space \( Z \) (nilpotent if \( Z_0 \) is nilpotent) and a distinguished map \( f: X \to Z \). Since one of the components of \( f \) is \( f_0 \), it is plain that \( f_*\alpha \neq 0 \). Appeal to diagram (2.1) (with \( Z \) in place of \( Y \)) shows that \( e_*\alpha \neq 0 \), that is \( e \) is a weak \( \mathcal{F} \)-monomorphism.

We wish next to invoke [3; Theorem 7], a special case of which asserts that any self-monomorphism \( m: X \to X \) with \( X \) wrf (see Section 1, (i) - the definition of (F1)) inducing an isomorphism \( m_*: \pi_1 X \cong \pi_1 X \) is a homotopy equivalence. Close examination of the proof of [3; Theorem 7], carried out below, reveals two things: first, the condition that \( m \) be a monomorphism may be relaxed - it suffices that \( m \) be a weak \( \mathcal{F} \)-monomorphism; second, the condition that the homotopy groups
\(\pi_n X, n \geq 2,\) be finitely generated may also be relaxed – it suffices that the \(\pi_n X, n \geq 2,\) be either finitely generated or \(P\)-profinite completions of finitely generated abelian groups for some nonempty family of primes \(P.\) Now our map \(e : X \to X\) induces an isomorphism \(e_* : \pi_1 X \cong \pi_1 X (e_* : \pi_1 X \to \pi_1 X\) since \(e\) is a weak \(f\)-monomorphism and \(e_* : \pi_1 X \to \pi_1 X\) since \(e\) is an epimorphism [3; Proposition 1]). Hence, in view of the observations above, [3; Theorem 7] may be applied to \(e\) to yield the desired result.

It remains then to refine the argument in [3; Theorem 7] in order to accommodate the relaxed assumptions. What we need to know first is that the map \(m : X \to X\) induces weak monomorphisms

\[
m_* : \pi_n (X; \mathbb{Z}/r) \to \pi_n (X; \mathbb{Z}/r), \quad n \geq 2, \quad r > 0,
\]

\(\pi_n (X; \mathbb{Z}/r)\) denoting \([S^{n-1} \cup e^n, X].\) (The 'w', indicating weak monomorphism, may be removed if \(n \geq 3\) but \(\pi_2 (X; \mathbb{Z}/r)\) need not admit a group structure.) For this it plainly suffices that \(m\) be a weak \(f\)-monomorphism. We then wish to conclude from (2.2) that 

\[
m_* : \pi_n X \cong \pi_n X.
\]

From (2.2) and the appropriate Universal Coefficient Theorem, we immediately infer

\[
m_* \otimes 1 : \pi_n X \otimes \mathbb{Z}/r \to \pi_n X \otimes \mathbb{Z}/r, \quad n \geq 2, \quad r > 0.\]

[3; Theorem 7] handles the case where \(\pi_n X\) is finitely generated, so we concentrate attention on the case where \(\pi_n X\) is the \(P\)-profinite completion of a finitely generated abelian group, \(P\) a nonempty family of primes. To simplify notation, write \(A = \pi_n X\) and \(\mu = m_*;\) further, write \(A = \hat{B}_p\) with \(B\) finitely generated abelian. Now \(\hat{B}_p\) factors as a Cartesian product \(\prod_{p \in P} \hat{B}_p\) and \(\mu\) respects this factorization, so we may as well assume \(P\) is a singleton set \(\{p\}.\) Identifying \(A \otimes \mathbb{Z}/p^i\) with \(A/p^i A, i \geq 1,\) we find from (2.3),

\[
\mu \otimes 1 : A/p^i A \to A/p^i A, \quad i \geq 1,
\]

and hence, since \(A/p^i A \cong B/p^i B\) is finite

\[
\mu \otimes 1 : A/p^i A \cong A/p^i A, \quad i \geq 1.
\]

Consider the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\mu} & A \\
\downarrow c_p & & \downarrow c_p \\
\hat{A}_p = \lim_{\leftarrow} A/p^i A & \xrightarrow{\beta_p} & \lim_{\leftarrow} A/p^i A = \hat{A}_p
\end{array}
\]

with \(c_p\) denoting \(p\)-profinite completion. By (2.4), \(\beta_p\) is an isomorphism and since \(A\) is \(p\)-complete, \(c_p\) is an isomorphism. It follows that \(\mu : A \cong A\) and the proof of Theorem 2 is completed.
Proof of Theorem 3. Let $\theta \in \text{Aut}(X)$ be different from the identity element $i$. We must find a finite group $\Gamma$ and a homomorphism $\phi : \text{Aut}(X) \to \Gamma$ such that $\phi \theta \neq \phi i$. Viewing $\theta$ and $i$ as elements of $[X, X]$ and using the assumptions on $X$, we find a totally finite space $Z_0$ and a map $f_0 : X \to Z_0$ such that $(f_0)_* \theta \neq (f_0)_* i$. As $[X, Z_0]$ is clearly finite (compare Proposition 3(a)), Lemma 1 guarantees the existence of a homotopy-commutative diagram

$$
\begin{array}{c}
x \\ f \\ Z
\end{array} \xymatrix{ & \mathcal{H}(Z) \\ X \\ f \\ Z}
$$

with $\sigma \in \mathcal{H}_r \subset \mathcal{H}(Z)$ for some $r > 0$ and $f_* \theta \neq f_* i$; thus $\sigma \neq 1$. Now the monoid homomorphism $\Phi : \text{Epi}(X) \to \mathcal{H}$ of Lemma 1 restricted to $\mathcal{H}(X) \subset \text{Epi}(X)$ induces a group homomorphism $\phi : \text{Aut}(X) \to \mathcal{H}$, and the above discussion shows that $\phi \theta = \sigma \neq 1$, thereby concluding the proof of Theorem 3.

3.

Throughout this section, $X$ is assumed to be Hft. The following proposition, which allows us to complete the proof of Corollary 2(d), is probably well known, but we include a proof for the sake of completeness.

Proposition 4. If the Hft space $X$ has a finite fundamental group $\pi$, and if $Z$ is totally finite, then $[X, Z]$ is finite.

Proof. It suffices, by obstruction theory, to establish the finiteness of the cohomology sets $H^n(X; \pi_n Z)$, $n \geq 1$; $H^1(X; \pi_1 Z)$ is understood to mean the set $\text{Hom}(\pi, \pi_1 Z)$ while the cohomology groups $H^n(X; \pi_n Z)$, $n \geq 2$, are understood to be taken with twisted coefficients.

The finiteness of $\text{Hom}(\pi, \pi_1 Z)$ is obvious. For $n \geq 2$, consider the fibration

$$\tilde{X} \to X \to K(\pi, 1)$$

with $\tilde{X}$ the universal cover of $X$. Writing $\Gamma = \pi_n Z$, there is an associated Serre spectral sequence (with twisted coefficients; see [9; §2]) with

$$E_2^{rs} = H^r(\pi; H^s(\tilde{X}; \Gamma)), \quad r, s \geq 0.$$

Since $\tilde{X}$ is 1-connected, the cohomology groups $H^s(\tilde{X}; \Gamma)$, $s \geq 0$, are with constant coefficients; moreover, since $X$ (and hence also $\tilde{X}$) is Hft, the integral homology groups of $\tilde{X}$ are finitely generated. It follows then from the Universal Coefficient Theorem and the finiteness of $\Gamma$ that $H^s(\tilde{X}; \Gamma)$ is finite, $s \geq 0$. Appealing to a
classical result on the cohomology of finite groups, we infer that $E_2^{r,s}$ is finite, $r,s \geq 0$. Passing through the spectral sequence, we conclude finally that $E_\infty^{r,s}$, and therefore also $H^{r+s}(X; \Gamma)$ is finite, $r,s \geq 0$.

The type of space considered in Proposition 4 is interesting in connection with Question 2 (Section 1). Indeed, the universal cover $\tilde{X}$ of such a space $X$ is wrf by Theorem 1(b) so a positive reply to Question 2 would imply that $X$ itself is wrf.

There is a possible attack on Question 2, which we would like to outline, at least in case $X$ has finite skeleta. It should be emphasized, though, that the line of argument presented falls far short of providing a proof of an affirmative answer to the question.

To begin, we may assume $g : \tilde{X} \to X$ is a regular covering map. (For certainly there is a finite-sheeted covering map $\tilde{X}_0 \to \tilde{X}$ such that the composite $\tilde{X}_0 \to \tilde{X} \to X$ is a regular covering map. The weak residual finiteness of $X$ implies that of $\tilde{X}_0$; see the paragraph preceding Question 2.) We then have a fibration

$$\tilde{X} \xrightarrow{g} X \xrightarrow{k} K(\pi, 1), \tag{3.1}$$

where now $\pi = \pi_1 X / g_* \pi_1 \tilde{X}$. If $W$ is a finite CW-complex and $0 \neq \alpha \in \pi [W, X]$, there are two possibilities. If $k_\ast \alpha \neq 0$, there is nothing more to do. If $k_\ast \alpha = 0$, then $\alpha$ lifts to a (unique) $a \in [W, \tilde{X}]$. Since $\tilde{X}$ is wrf, there is a totally finite space $\tilde{Z}$ and a map $f : \tilde{X} \to \tilde{Z}$ such that $f_* \alpha \neq 0$. Furthermore, since $X$ (and hence also $\tilde{X}$) has finite skeleta, Proposition 3(a) in conjunction with Lemma 1 insures that $f$ may be taken to be a distinguished map.

Now the fibration (3.1) is classified, up to (pointed) fiber homotopy type, by a monoid homomorphism $\kappa : \pi \to \mathcal{H}(\tilde{X})$. Composing $\kappa$ with the monoid homomorphism $\phi : \mathcal{H}(\tilde{X}) \to \mathcal{H}_Z \subseteq \mathcal{H}(\tilde{Z})$ provided by Lemma 1 yields a monoid homomorphism $\kappa' : \pi \to \mathcal{H}(\tilde{Z})$ which, in turn, gives rise to a fibration

$$\tilde{Z} \xrightarrow{g'} \tilde{Z} \xrightarrow{k'} K(\pi, 1), \tag{3.2}$$

unique up to (pointed) fiber homotopy type. (To be precise, it is necessary to deviate from our earlier dictum and interpret $\mathcal{H}(\tilde{X})$ and $\mathcal{H}(\tilde{Z})$ as the space of free (unpointed) homotopy equivalences. Lemma 1 certainly has a valid free analogue.)

All that is missing is a map $f$ making the diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{g} & X \\
\downarrow f & & \downarrow f \\
\tilde{Z} & \xrightarrow{g'} & \tilde{Z}
\end{array}$$
of Lemma 1 were strictly commutative, then there would actually be a morphism of the fibration (3.1) to the fibration (3.2), mapping $K(\pi, 1)$ identically. Unfortunately, it appears that there are obstructions to making (3.3) strictly commutative for all $h \in \mathcal{X}(X)$.

An additional shred of evidence in favor of an affirmative answer to Question 2 is provided by a study of a familiar family of Hft spaces with finite fundamental group, namely the real projective spaces $\mathbb{R}P^n, n \geq 1$.

If $n$ is odd, then $\mathbb{R}P^n$ is nilpotent [2; II (2.19), p. 71), so the residual finiteness of $\mathbb{R}P^n$ in this case results from Theorem 1(b). If $n$ is even, however, $\mathbb{R}P^n$ is not nilpotent [2; Lemma II.2.18, p. 70]. We are unable to establish the weak residual finiteness of $\mathbb{R}P^n$ in this case but we do have the following

**Proposition 5.** The real projective space $\mathbb{R}P^n, n \geq 1$, is a Hopfian object of $\mathcal{X}$.

**Proof.** For $n$ odd, this follows at once from Corollary 2(b).

For $n$ even, we base ourselves on the work of Olum [6]. Thus, let $e : \mathbb{R}P^n \to \mathbb{R}P^n$ be an epimorphism. By [3; Proposition 1], $e_* : \pi_1(\mathbb{R}P^n) \to \pi_1(\mathbb{R}P^n)$, hence $e_* : \pi_1(\mathbb{R}P^n) \equiv \pi_1(\mathbb{R}P^n)$. Consider the cohomology group $H^n(\mathbb{R}P^n; \mathbb{Z}^i) \equiv \mathbb{Z}$, where $\mathbb{Z}^i$ denotes the integers twisted by the $\mathbb{Z}/2$-action $l \mapsto -l, l \in \mathbb{Z}$. According to [6; §5], there is a 1-1 correspondence from the subset $[\mathbb{R}P^n, \mathbb{R}P^n] \subseteq [\mathbb{R}P^n, \mathbb{R}P^n]$ consisting of those $\phi$ for which $\phi_* : \pi_1(\mathbb{R}P^n) \equiv \pi_1(\mathbb{R}P^n)$ to the set of all odd integers given by associating to $\phi$ the 'twisted degree' of $\phi^* : H^n(\mathbb{R}P^n; \mathbb{Z}) \to H^n(\mathbb{R}P^n; \mathbb{Z})$.

Write $d$ for the twisted degree of $e$ (or $e^*$) and form the 2-stage Postnikov system

$$K(\mathbb{Z}/d, n) \to F \to K(\mathbb{Z}/2, 1)$$

associated to the $\mathbb{Z}/2$-action $l \mapsto -l, l \in \mathbb{Z}/d$. If $(\mathbb{Z}/d)^i$ denotes the integers mod $d$ twisted by this action, then any element of $H^n(\mathbb{R}P^n; (\mathbb{Z}/d)^i) \equiv \mathbb{Z}/d$ is 'represented' by a map $\mathbb{R}P^n \to F$ 'over $K(\mathbb{Z}/2, 1)$'; see [8; Theorem 3.6]. Let $f, g : \mathbb{R}P^n \to F$ represent any two elements of $H^n(\mathbb{R}P^n; (\mathbb{Z}/d)^i)$. It is easy to see that, viewed as elements of $[\mathbb{R}P^n, F]$,

$$f \circ e = g \circ e.$$

As $e$ is an epimorphism, this can only occur if $f = g$, that is if $d = \pm 1$. But then $e$ is a homotopy equivalence and we have succeeded in proving that $\mathbb{R}P^n$ is Hopfian.
In this section, we supply the proofs for two assertions made in Section 1, the common feature of these assertions being that their proofs both depend on the behavior of certain Ext groups.

Proof of Proposition 1. For a finite CW-complex $W$ and elements $\alpha, \beta \in [W, K(G, n)]$, $\alpha \neq \beta$, we seek a finite group $\Gamma$, abelian if $n \geq 2$, and a homomorphism $\phi : G \to \Gamma$ such that the induced map

$$K(\phi, n)_* : [W, K(G, n)] \to [W, K(\Gamma, n)]$$

carries $\alpha, \beta$ into distinct elements. (Thus, for $n \geq 2$, $K(G, n)$ is rf in a strong sense; see the discussion following Definition 1'.)

If $n = 1$, we simply identify $[W, K(G, 1)]$ with Hom($\pi_1 W, G$) and use the residual finiteness of $G$ in an evident fashion to produce $\Gamma$ and $\phi : G \to \Gamma$.

If $n > 1$, we identify $[W, K(G, n)]$ with $H^n(W; G)$ and consider the Universal Coefficient Theorem

$$\text{Ext}(H_{n-1} W, G) \xrightarrow{\mu} H^n(W; G) \xrightarrow{\epsilon} \text{Hom}(H_n W, G)$$

If $c\alpha \neq c\beta$, we argue as in the case $n = 1$. If $c\alpha = c\beta$, we write $\alpha - \beta = \mu \xi$, $\xi \neq 0$, and attempt to find suitable $\Gamma$ and $\phi : G \to \Gamma$ so that the induced map $\phi_* : \text{Ext}(H_{n-1} W, G) \to \text{Ext}(H_{n-1} W, \Gamma)$ carries $\xi$ to a non-0 element. At this point, we no longer require the residual finiteness of $G$. In view of the finite generation of $H_{n-1} W$ and elementary facts about Ext, we may clearly reduce to the case $H_{n-1} W \cong \mathbb{Z}/r$, $r > 1$. Then $\text{Ext}(H_{n-1} W, G) \cong G/rG$; as a group of bounded order, the latter is, by standard abelian group theory, a direct sum of cyclic groups, each of order dividing $r$. Viewing $\xi$ as an element of $G/rG$, we thus clearly have a homomorphism $\psi : G/rG \to \Gamma$, with $\Gamma$ cyclic of order dividing $r$, such that $\psi \xi \neq 0$. If $\phi$ is the composite $G \to G/rG \xrightarrow{\psi} \Gamma$ of the canonical map with $\psi$, then $\phi_* : \text{Ext}(H_{n-1} W, G) \to \text{Ext}(H_{n-1} W, \Gamma)$ may be identified with $\psi$, thus establishing our contention.

We next prove a result which shows that if $W$, in Definition 1, is only required to be finite-dimensional, such simple spaces as $K(\mathbb{Z}, n)$ and $S^n, n \geq 2$, would fail to be wrf.

Proposition 6. Let $W$ be the Moore space $M(\mathbb{Q}, n-1)$, $n \geq 2$, and let $X$ be an $(n-1)$-connected space with $\pi_n X \cong \mathbb{Z}$. Then $[W, X]$ is uncountable but, for any finite space $F$, $[W, F]$ consists only of the trivial element.

Proof. The first part follows from the isomorphisms

$$[W, X] \cong H^n(W; \mathbb{Z}) \cong \text{Ext}(H_{n-1} W, \mathbb{Z}) \cong \text{Ext}(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{R}.$$
For the second part, consider the fibration
\[ \tilde{F} \to F \to K(\pi_1 F, 1), \]
with \( \tilde{F} \) the universal cover of \( F \). Now \([W, K(\pi_1 F, 1)] = \text{Hom}(\pi_1 W, \pi_1 F) = 0\), since \( \pi_1 W \) is divisible and \( \pi_1 F \) is finite. Thus \([W, F] = [W, \tilde{F}]\). To see that the latter set consists only of the trivial element, it suffices, by obstruction theory, to show that the groups \( H^m(W; \pi_m \tilde{F}) \), \( m \geq 2 \), vanish. But this follows from the Universal Coefficient Theorem since \( \text{Hom}(H_m W, \pi_m \tilde{F}) = 0 \) (\( H_m W \) is divisible and \( \pi_m \tilde{F} \) is finite) and \( \text{Ext}(H_{m-1} W, \pi_m \tilde{F}) = 0 \) (\( H_{m-1} W \) is torsion-free and \( \pi_m \tilde{F} \) is finite).

References