Poincaré inequality for linear SPDE driven by Lévy Noise

Yingchao Xie

School of Mathematical Sciences, Xuzhou Normal University, Xuzhou 221116, Jiangsu, PR China

Received 24 May 2009; received in revised form 22 November 2009; accepted 23 May 2010
Available online 1 June 2010

Abstract

In this paper, we prove the Poincaré inequality and the integration by parts formula for the invariant measure of the linear SPDE driven by Lévy Noise. The equation was researched in Dong and Xie [5], which has proved the existence and uniqueness of the weak solution and the ergodicity of the Markov semigroup associated with the solution.

MSC: primary 34D08, 34D25; secondary 60H20

Keywords: Poincaré inequality; Integration by parts formula; SPDE with Lévy Noise; Invariant measure

1. Introduction

The article is concerned with a linear PDE with a random perturbation of Lévy noise. The aim is to prove the Poincaré inequality and the integration by parts formula for the invariant measure with the solution of the equation. The equation was researched in [5]. We have proved the existence and uniqueness of the weak solution and the ergodicity of the Markov semigroup associated with the solution. We mainly use the method as [3] to prove our results in this paper.

Let $H$ be a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$ and the norm $|\cdot|$. Let the self-adjoint operator $A : \mathcal{D}(A) \subset H \rightarrow H$ be the infinitesimal generator of strongly continuous semigroup $e^{-tA}$ and it satisfies

$$\langle Ax, x \rangle \geq \lambda |x|^2, \quad x \in H \quad (1.1)$$

for some $\lambda > 0$. 

The work was supported by the National Science Foundations of China (10671168 and 10971180) and the Science Foundation of Jiangsu Province (07-333).

E-mail address: ycxie@xznu.edu.cn.
For any $\delta > 0$, we can introduce the fractional power $A^\delta$ of $A$ defined by

$$A^\delta = \frac{1}{\Gamma(\delta)} \int_0^\infty t^{\delta-1} e^{-tA} dt,$$

where $\Gamma(\cdot)$ is the Euler function. We have $\text{Range } (e^{-tA}) \subset \mathcal{D}(A)$ for $t > 0$.

Set $V = \mathcal{D}(A^{1/2})$. Let $V^*$ be the dual space of $V$ and identify $H$ with its dual $H^*$, then one has the Gelfand triple $V \subset H \subset V^*$. In this article, we suppose

The injection from $V$ to $H$ is compact.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with the filtration $\{\mathcal{F}_t, t \geq 0\}$, $N(ds, du)$ be the Poisson measure with $\sigma$-finite characteristic measure $\lambda(du)$ on measurable set $U$. $\tilde{N}(ds, du) = N(ds, du) - \lambda(du)ds$ is martingale measure. $W$ is a cylindrical Wiener process defined on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Suppose that $W$ and $N(dt, du)$ are independent. We will consider the following stochastic partial differential equation with jumps

$$\begin{cases}
    dX_t + AX_t dt = \int_U f(X_{t-}, u) \tilde{N}(dt, du) + BdW_t, & t > 0, \\
    X_0 = x,
\end{cases}$$

(1.2)

where $B$ is a continuous bounded linear operator on $H$.

In the past years, the SPDEs with white noise perturbation were an active research field, which accumulated a lot of works, e.g. [3,4,9–11] and references therein. In the recent years there were a number of very interesting results on SPDEs driven by Lévy noise, e.g. [1,2,6,5,7] and references therein. For the integration by parts formula, Da Prato researched this problem with respect to the invariant measure for SPDE driven by Wiener Noise in [3], Hariya proved this formula for Wiener measures restricted to subsets in $\mathbb{R}^d$ in [8]. We shall study this problem for SPDE driven by Lévy Noise.

The paper consists of three sections. In Section 1, we give some definitions, hypotheses and results which were proved in [5]. In Section 2, we research the transition semigroup in $L^p(H, \mu)$, where $\mu$ is the unique invariant measure of the solution of (1.2). We obtain the integration by parts formula with $\mu$ in Section 3. Finally, the Poincaré inequality is proved.

**Definition 1.1.** An $H$-valued càdlàg process $X$ is called the weak solution of (1.2), if for $T \geq 0$, $X \in L^\infty(0, T; H)$ and for $\xi \in \mathcal{D}(A)$,

$$\langle X_t, \xi \rangle = \langle x, \xi \rangle - \int_0^t \langle X_s, AX_s \rangle ds + \int_0^t \int_U \langle f(X_{s-}, u), \xi \rangle \tilde{N}(ds, du) + \langle \xi, BW_t \rangle,$$

$\mathbb{P}$-a.s.

Now, we introduce some hypotheses for the function $f$, which are needed in the following. Let $U_k$ be a measurable subset of $U$ with $U_k \uparrow U$ and $\lambda(U_k) < \infty$.

Suppose that the measurable function $f(\cdot, \cdot) : \mathbb{H} \times U \mapsto \mathbb{H}$ satisfies for some positive constants $C, K$

$$(H_1) \int_U |f(0, u)|^2 \lambda(du) = C < \infty.$$

$$(H_2) \int_U |f(x, u) - f(y, u)|^2 \lambda(du) \leq K|x - y|^2, \forall x, y \in \mathbb{H}.$$

$$(H_3) \sup_{|y| \leq k} \int_{U_m} |f(y, u)|^2 du \to 0 \text{ as } m \to \infty, \text{ for any fixed } k > 0.$$

$$(H_4) \text{ There exists } \epsilon \in (0, 1) \text{ such that } \text{Range } \left(A^{-\epsilon} \right) \subset \text{Range}(B).$$
In [5], we have proved the following results:

**Theorem 1.1.** Under the conditions \((H_1)\)–\((H_3)\), for initial value \(x \in \mathbb{H}\), the Eq. (1.2) has a unique global weak solution \(X \in L^\infty(0, T; \mathbb{H}) \cap L^1(0, T; \mathbb{V})\) for any \(T > 0\).

**Theorem 1.2.** Suppose that the conditions \((H_1)\)–\((H_4)\) hold, the function \(f(x, z)\) is differentiable with respect to \(x\), then the Markov semigroup associated with the solution \(X\) of (1.2) has strong Feller property and irreducibility.

**Remark 1.1.** Theorem 1.2 shows that there exists a unique invariant measure denoted by \(\mu\) for the transition semigroup \((P_t)_{t \geq 0}\) associated with the solution process \(X\) of (1.2).

### 2. The transition semigroup in \(L^p(\mathbb{H}, \mu)\)

In this section, we shall give some energy inequalities, which can be proved by using the Galerkin approximation. Since the method in infinite dimension is the same as the case in finite dimension, we prove them in infinite dimension case directly.

**Proposition 2.1.** Suppose that the conditions \((H_1)\)–\((H_4)\) hold, the function \(f(x, z)\) is differentiable with respect to \(x\). Then for any \(p > 1\), \((P_t)_{t \geq 0}\) has a unique extension to a strongly continuous semigroup of contractions in \(L^p(\mathbb{H}, \mu)\) (which is still denoted by \((P_t)_{t \geq 0}\)).

**Proof.** For \(\varphi \in C_b(\mathbb{H})\), we have

\[
|P_t \varphi(x)|^p \leq \int_{\mathbb{H}} |\varphi(y)|^p P_t(x, dy) = P_t(|\varphi(x)|^p), \quad t > 0, \ x \in \mathbb{H}.
\]

Integrating this inequality with respect to \(\mu\) over \(\mathbb{H}\), and taking into account the invariance of \(\mu\) yield

\[
\int_{\mathbb{H}} |P_t \varphi(x)|^p \mu(dx) \leq \int_{\mathbb{H}} P_t(|\varphi(x)|^p) \mu(dx) = \int_{\mathbb{H}} |\varphi(x)|^p \mu(dx).
\]

Since \(C_b(\mathbb{H})\) is dense in \(L^p(\mathbb{H}, \mu)\), \((P_t)_{t \geq 0}\) is uniquely extendible to a contraction semigroup in \(L^p(\mathbb{H}, \mu)\).

For \(0 \leq s < t < \infty\),

\[
\|P_t - P_s\|_{L^p(\mathbb{H}, \mu)} \leq \|P_{t-s} - I\|_{L^p(\mathbb{H}, \mu)}
\]

and the dominated convergence theorem imply the strong continuity of \((P_t)_{t \geq 0}\). \(\Box\)

For any \(p \geq 1\), we shall use \(\mathcal{L}_p\) to denote the infinitesimal generator of \(P_t\) in \(L^p(\mathbb{H}, \mu)\) and \(\mathcal{D}(\mathcal{L}_p)\) to denote its domain. The definition of the infinitesimal generator yields

\[
\varphi_h(x) \equiv e^{i(h, x)} \in \mathcal{D}(\mathcal{L}_p) \iff h \in \mathcal{D}(A).
\]

Hence, for \(h \in \mathcal{D}(A)\) and \(x \in \mathbb{H}\),

\[
\mathcal{L}_p \varphi_h(x) = \left( -\frac{1}{2} Q h, h \right) - i \langle A h, x \rangle \varphi_h(x) + \varphi_h(x) \int_U \left[ e^{i(h, f(x, u))} - 1 - i(h, f(x, u)) \right] \lambda(du)
\]
by

\eta

where

provided the limit exists. Successive derivatives are defined in a natural way. 

\[ D f \]

for \( x \) and \( H \) is stable for \( P_t \) and it is dense in \( L^p(\mathbb{H}, \mu) \) for all \( p \geq 1 \).

We shall prove that \( \mathcal{L}_2 \) is the closure of the Kolmogorov operator \( \mathcal{L}_0 \) defined by

\[ \mathcal{L}_0 \phi(x) = \frac{1}{2} \text{Tr} \left[ Q D^2 \phi(x) \right] - \langle Ax, D\phi(x) \rangle + \int_U [\phi(x + f(x, u)) - \phi(x) - \langle D\phi(x), f(x, u) \rangle] \lambda(du), \]

for \( x \in \mathbb{H} \) and \( \phi \in \mathcal{E}_A(\mathbb{H}) \).

For the follow proof, we need the hypothesis \( (H_5) \) below:

\( (H_5) \) Suppose that the function \( f(x, u) \) is second order differentiable with respect to the first variable \( x \) and

\[ C_f \triangleq \max \left\{ \sup_{x \in \mathbb{H}} \int_U |Df(x, u)|^4 \lambda(du), \sup_{x \in \mathbb{H}} \int_U |D^2 f(x, u)|^2 \lambda(du) \right\} < \infty, \]

where \( Df(x, u) = D_x f(x, u) \).

Let \( S \) be any Banach space and \( \phi : \mathcal{D}(A) \to S \). For any \( x, h \in \mathbb{H}, \) we set

\[ D\phi(x) \cdot h = \lim_{t \to 0} \frac{1}{t} \left( \phi(x + th) - \phi(x) \right), \]

provided the limit exists. Successive derivatives are defined in a natural way.

For \( \phi \in C^1(\mathbb{H}) \) we have

\[ D \mathbb{E} \phi(X_t(x)) \cdot h = \mathbb{E} \left[ D\phi(X_t(x)) \cdot \eta_t^h(x) \right], \]

where \( \eta_t^h(x) = DX_t(x) \cdot h \) is the solution of the equation:

\[ \begin{cases} 
\text{d} \eta_t^h(x) = -A \eta_t^h(x) \text{d}t + \int_U Df(X_t(x), u) \cdot \eta_t^h(x) \tilde{N}(dt, du), \\
\eta_0^h(x) = h(x).
\end{cases} \]  

Multiply scalarly the Eq. (2.3) by \( \eta_t^h(x) \), we have

\[ |\eta_t^h(x)|^2 + 2 \int_0^T \|\eta_t^h(x)\|^2 \text{d}s = \begin{aligned}
|h|^2 + 2 \int_0^T \int_U \left\{ Df(X_s(x), u) \cdot \eta_s^h(x), \eta_s^h(x) \right\} \tilde{N}(dt, du) \\
+ \int_0^T \int_U \left| Df(X_s(x), u) \cdot \eta_s^h(x) \right|^2 N(dt, du).
\end{aligned} \]
The Davis inequality and the Young inequality imply, for every $T > 0$,
\[
\mathbb{E} \sup_{t \leq T} |\eta^h_t(x)|^2 + 2 \mathbb{E} \int_0^T ||\eta^h_s(x)||^2 \, ds \\
\leq |h|^2 + 2 \mathbb{E} \sup_{t \leq T} \left| \int_0^t \int_U \left( Df(X_{s-}(x), u) \cdot \eta^h_s(x), \eta^h_{s-}(x) \right) \tilde{N}(dr, du) \right| \\
+ \mathbb{E} \int_0^T \int_U |Df(X_s(x), u) \cdot \eta^h_s(x)|^2 \lambda(du) \, ds \\
\leq |h|^2 + 2 \sqrt{6} \mathbb{E} \left[ \int_0^T \int_U |Df(X_s(x), u) \cdot \eta^h_s(x)|^2 |\eta^h_s(x)|^2 \lambda(du) \, ds \right]^{1/2} \\
+ \mathbb{E} \int_0^T \int_U |Df(X_s(x), u) \cdot \eta^h_s(x)|^2 \lambda(du) \, ds \\
\leq |h|^2 + 6 \mathbb{E} \sup_{t \leq T} |\eta^h_t(x)|^2 + \frac{(6 + \varepsilon)K}{\varepsilon} \mathbb{E} \int_0^T |\eta^h_s(x)|^2 \, ds,
\]
since $\sup_{x \in \mathbb{H}} \int_U |Df(x, z)|^2 \lambda(dz) \leq K$ by (H2), where $\varepsilon \in (0, 1)$. Hence, we have
\[
(1 - \varepsilon) \mathbb{E} \sup_{t \leq T} |\eta^h_t(x)|^2 + \mathbb{E} \int_0^T ||\eta^h_s(x)||^2 \, ds \leq |h|^2 + \frac{(6 + \varepsilon)K}{\varepsilon} \mathbb{E} \int_0^T |\eta^h_s(x)|^2 \, ds.
\]
The Gronwall inequality implies
\[
(1 - \varepsilon) \mathbb{E} \sup_{t \leq T} |\eta^h_t(x)|^2 + \mathbb{E} \int_0^T ||\eta^h_s(x)||^2 \, ds \\
\leq |h|^2 \exp \left\{ \frac{(6 + \varepsilon)K}{\varepsilon} T \right\} \leq |h|^2 \exp \left\{ \frac{7K}{\varepsilon} T \right\}.
\]
This yields
\[
(1 - \varepsilon) \mathbb{E} \sup_{t \leq T} |DX_t(x)|^2 \leq \exp \left\{ \frac{7K}{\varepsilon} T \right\}.
\] (2.5)

The Itô formula and (2.4) give
\[
|\eta^h_t(x)|^4 \leq |h|^4 + 4 \int_0^T \int_U |\eta^h_{s-}(x)|^2 \left( Df(X_{s-}(x), u) \cdot \eta^h_{s-}(x), \eta^h_{s-}(x) \right) \tilde{N}(dr, du) \\
+ 2 \int_0^T \int_U |\eta^h_{s-}(x)|^2 \left| Df(X_{s-}(x), u) \cdot \eta^h_{s-}(x) \right|^2 N(dr, du) \\
+ 2 \int_0^T \int_U (\eta^h_{s-}(x), Df(X_{s-}(x), u) \cdot \eta^h_{s-})^2 N(dr, du) \\
+ 2 \int_0^T \int_U \left| Df(X_{s-}(x), u) \cdot \eta^h_{s-}(x) \right|^4 N(dr, du).
\]
This shows
\[
\mathbb{E} |\eta^h_t(x)|^4 \leq |h|^4 + 2 \mathbb{E} \int_0^T \int_U |\eta^h_s(x)|^2 \left| Df(X_s(x), u) \cdot \eta^h_s(x) \right|^2 \lambda(du) \, ds
\]
Hence, we have the inequality

\[ 2\mathbb{E} \int_0^T \int_U (\eta_s^h(x), Df(X_s(x), u) \cdot \eta_s^h(x))^2 \lambda(du)ds + 2\mathbb{E} \int_0^T \int_U |Df(X_s(x), u) \cdot \eta_s^h(x)|^4 \lambda(du)ds \leq |h|^4 + 6C_f \int_0^T \mathbb{E} |\eta_s^h(x)|^4 ds. \]

By the Gronwall inequality we have

\[ \mathbb{E} |\eta_s^h(x)|^4 \leq |h|^4 \exp\{6C_f T\}. \] (2.6)

For \( h, k \in H \), we can prove that the equation

\[
\begin{align*}
\{& d\xi_t^{h,k}(x) + A_s^{h,k}(x)dt = \int_U Df(X_{t-}, u)\xi_t^{h,k}(x)\tilde{N}(dt, du) \nonumber \\
& + \int_U D^2 f(X_{t-}, u) \left( \eta_t^{h}(x), \eta_t^{k}(x) \right) \tilde{N}(dt, du), \quad t > 0, \\
& \xi_0^{h,k}(x) = 0 \}
\end{align*}
\]

has a unique weak solution, where \( \xi_t^{h,k}(x) = D^2 X_t(x)(h, k) \). In fact, the Itô formula yields

\[
\begin{align*}
& \left| \xi_t^{h,k}(x) \right|^2 + 2\int_0^t \left\| \xi_s^{h,k}(x) \right\|^2 ds \leq \int_0^t \int_U \left( Df(X_{s-}, u)\xi_s^{h,k}(x), \xi_s^{h,k}(x) \right) \tilde{N}(ds, du) \\
& + \int_0^t \int_U \left( D^2 f(X_{s-}, u) \left( \eta_s^{h}(x), \eta_s^{k}(x) \right), \xi_s^{h,k}(x) \right) \tilde{N}(dt, du) \\
& + 2\int_0^t \int_U \left( |Df(X_{s-}, u)|^2 \left| \xi_s^{h,k}(x) \right|^2 + \left| D^2 f(X_{s-}, u) \right|^2 \left| \eta_s^{h}(x) \right|^2 \left| \eta_s^{k}(x) \right|^2 \right) N(dt, du).
\end{align*}
\]

This and (2.6) show

\[
\mathbb{E} \left| \xi_t^{h,k}(x) \right|^2 + 2\mathbb{E} \int_0^t \left\| \xi_s^{h,k}(x) \right\|^2 ds \leq 2\mathbb{E} \int_0^t \int_U \left( |Df(X_s, u)|^2 \left| \xi_s^{h,k}(x) \right|^2 + \left| D^2 f(X_s, u) \right|^2 \left| \eta_s^{h}(x) \right|^2 \left| \eta_s^{k}(x) \right|^2 \right) \lambda(du)ds \leq 2C_f \mathbb{E} \int_0^t \left[ \left| \xi_s^{h,k}(x) \right|^2 + \left| \eta_s^{h}(x) \right|^2 \left| \eta_s^{k}(x) \right|^2 \right] ds
\]

\[
\leq 2C_f \mathbb{E} \int_0^t \left[ \xi_t^{h,k}(x) \right] ds + 2C_f |h|^2 |k|^2 \exp\{6C_f t\}.
\]

Hence, we have the inequality

\[
\mathbb{E} \left| \xi_t^{h,k}(x) \right|^2 \leq 2C_f |h|^2 |k|^2 \exp\{6C_f t\} \exp\{2C_f t\} = 2C_f |h|^2 |k|^2 \exp\{8C_f t\}.
\]
This gives
\[ \mathbb{E} \left| D^2 X_t(x) \right|^2 \leq 2C_f \exp \left\{ 8C_f t \right\}. \]  
(2.7)

**Theorem 2.1.** Suppose that the conditions \((H_1)-(H_5)\) hold and \(K < \lambda\). Then \(\mathcal{L}_0\) is closable and \(\mathcal{L}_2\) is the closure of \(\mathcal{L}_0\) in \(L^2(\mathbb{H}, \mu)\).

**Proof.** Let \(X\) be the global weak solution of (1.2). The Itô formula implies
\[
\begin{align*}
|X_t(x)|^2 + 2 \int_0^t \|X_s(x)\|^2 \, ds \\
\leq |x|^2 + 2 \int_0^t \int_U \langle X_s-, f_s, X_s- \rangle \tilde{N}(ds, du) + 2t \text{ Tr } Q \\
+ 2 \int_0^t \langle X_s, dW(s) \rangle + \int_0^t \int_U |f(X_s, u)|^2 \tilde{N}(ds, du).
\end{align*}
\]
This yields
\[
\begin{align*}
\mathbb{E} |X_t(x)|^2 &\leq |x|^2 - 2 \int_0^t \mathbb{E} \|X_s(x)\|^2 \, ds + 2t \text{ Tr } Q + \int_0^t \int_U \mathbb{E} |f(X_s-, u)|^2 \lambda(du) \, ds \\
&\leq |x|^2 - 2 \lambda \int_0^t \mathbb{E} |X_s(x)|^2 \, ds + 2t \text{ Tr } Q + 2 \int_0^t \mathbb{E} |K|X_s(x)|^2 + C \, ds \\
&\leq \left( |x|^2 + 2t \text{ Tr } Q + 2Ct \right) + 2(K - \lambda) \int_0^t \mathbb{E} |X_s(x)|^2 \, ds.
\end{align*}
\]
This gives
\[
\begin{align*}
\mathbb{E} |X_t(x)|^2 - 2(K - \lambda) \int_0^t \mathbb{E} |X_s(x)|^2 \, ds \leq |x|^2 + 2 \left( \text{ Tr } Q + C \right) t,
\end{align*}
\]
that is,
\[
\frac{1}{t} \int_0^t \mathbb{E} |X_s(x)|^2 \, ds \leq \frac{|x|^2}{2(\lambda - K)t} + \frac{\text{ Tr } Q + C}{\lambda - K}.
\]
Hence, for any \(\theta > 0\),
\[
\frac{1}{t} \int_0^t \frac{\mathbb{E} |X_s(x)|^2}{1 + \theta |X_s(x)|^2} \, ds \leq \frac{1}{t} \int_0^t \mathbb{E} |X_s(x)|^2 \, ds \leq \frac{|x|^2}{2(\lambda - K)t} + \frac{\text{ Tr } Q + C}{\lambda - K}.
\]
Let \(t \to \infty\), the ergodicity of the solution \(X\) implies
\[
\int_{\mathbb{H}} \frac{|y|^2}{1 + \theta |y|^2} \mu(dy) \leq \frac{\text{ Tr } Q + C}{\lambda - K}.
\]
Put \(\theta \to 0\), by the Fatou lemma we have
\[
\int_{\mathbb{H}} |y|^2 \mu(dy) \leq \frac{\text{ Tr } Q + C}{\lambda - K}.
\]  
(2.8)
For $h \in \mathcal{D}(A)$ and $x \in \mathbb{H}$, there is some constant $\alpha > 0$ such that

$$
|\mathcal{L}_0 \varphi_h(x)| \leq \frac{1}{2} |(Qh, h)| + |Ah||x| + \left| \int_U \left[ e^{i(h, f(x, u))} - 1 - i(h, f(x, u)) \right] \lambda(du) \right|
$$

$$
\leq \frac{1}{2} |(Qh, h)| + |Ah||x| + 2\alpha|h|^2 \int_U |f(x, u) - f(0, u)|^2 \lambda(du)
$$

$$
+ 2\alpha|h|^2 \int_U |f(0, u)|^2 \lambda(du)
$$

$$
\leq \frac{1}{2} |(Qh, h)| + |Ah||x| + 2\alpha|h|^2 K|x|^2 + 2\alpha|h|^2 C. 
\tag{2.9}
$$

For any $\varphi \in \mathcal{E}_A(\mathbb{H})$, (2.1) yields that

$$
\mathcal{L}_2 \varphi(x) = \lim_{t \to 0} \frac{P_t \varphi(x) - \varphi(x)}{t} = \mathcal{L}_0 \varphi(x), \quad \forall x \in \mathbb{H}.
$$

Since

$$
\int_{\mathbb{H}} |\mathcal{L}_2 \varphi(x) - \mathcal{L}_0 \varphi(x)|^2 \mu(dx) \leq 2 \int_{\mathbb{H}} \left| \mathcal{L}_2 \varphi(x) - \frac{1}{t} [P_t \varphi(x) - \varphi(x)] \right|^2 \mu(dx)
$$

$$
+ 2 \int_{\mathbb{H}} \left| \mathcal{L}_0 \varphi(x) - \frac{1}{t} [P_t \varphi(x) - \varphi(x)] \right|^2 \mu(dx),
$$

let $t \to 0$, (2.1), (2.8), (2.9) and the dominated convergence theorem imply that

$$
\int_{\mathbb{H}} |\mathcal{L}_2 \varphi(x) - \mathcal{L}_0 \varphi(x)|^2 \mu(dx) = 0.
$$

Therefore, $\mathcal{L}_2$ extends $\mathcal{L}_0$. By the Phillips–Lumer theorem we have that $\mathcal{L}_2$ is dissipative, so is $\mathcal{L}_0$. Hence, $\mathcal{L}_0$ is closable. Let us use $\mathcal{L}_0^*$ denote its closure. In the following, we shall prove that $\mathcal{L}_2 = \mathcal{L}_0^*$.

For $\gamma > 0$, $\psi \in \mathcal{E}_A(\mathbb{H})$, put

$$
\varphi(x) = \int_0^\infty e^{-\gamma t} P_t \psi(x) dt, \quad x \in \mathbb{H},
$$

we are going to show that $\varphi$ belongs to the range of $\gamma - \mathcal{L}_0^*$.

(a) Suppose that the function $f(x, u)$ is third order differentiable with respect to the first variable $x$ and satisfies (H5). By (2.5) and (2.7), there exists a constant $C_0 > 0$ such that, for $\gamma > \left( \frac{7K}{\varepsilon} \right) \lor (8C_f)$ and $x \in \mathbb{H},$

$$
|D\varphi(x)| = \left| \int_0^\infty e^{-\gamma t} \mathbb{E}[(DX_t(x))^* D\psi(X_t(x))] dt \right|
$$

$$
\leq \int_0^\infty e^{-\gamma t} \left( \mathbb{E}|(DX_t(x))|^2 \right)^{\frac{1}{2}} \left( \mathbb{E}|D\psi(X_t(x))|^2 \right)^{\frac{1}{2}} dt
$$

$$
\leq C_0 \int_0^\infty e^{\left( \frac{7K}{\varepsilon} - \gamma \right) t} dt = \frac{\varepsilon C_0}{\gamma \varepsilon - 7K} < \infty, \tag{2.10}
$$
\begin{equation}
|D^2\varphi(x)| \leq \int_0^\infty e^{-\gamma t} \left[ \mathbb{E}\left| (D^2\psi(X_t(x)), (DX_t(x))^{\otimes 2}) \right| + \mathbb{E}\left| D^2X_t(x)D\psi(X_t(x)) \right| \right] dt \\
\leq C_0 \int_0^\infty e^{-\gamma t} \left[ \mathbb{E}|DX_t(x)|^2 + \mathbb{E}|D^2X_t(x)| \right] dt \\
\leq C_0 \int_0^\infty e^{-\gamma t} \left[ \frac{1}{1-\epsilon}e^{7Kt} + 2C_f e^{8C_f t} \right] dt \\
= C_0 \left( \frac{\epsilon}{(1-\epsilon)(\gamma\epsilon - 7K)} + \frac{2C_f}{\gamma - 8C_f} \right) < \infty. \tag{2.11}
\end{equation}

(2.10) and (2.11) show \( \varphi \in C^2_0(\mathbb{H}) \) for \( \gamma > \left( \frac{7K}{\epsilon} \right) \lor (8C_f) \).

The Itô formula yields

\[
\psi(X_t(x)) = \psi(x) + \int_0^t \mathcal{L}_0 \psi(X_s(x)) ds + M_t,
\]

where \( M \) is a martingale, we have

\[
\varphi(x) = \frac{1}{\gamma}\psi(x) + \frac{1}{\gamma} \int_0^\infty e^{-\gamma t} P_t \mathcal{L}_0 \psi(X_t(x)) dt \\
= \frac{1}{\gamma}\psi(x) + \frac{1}{\gamma} \mathcal{L}_0 \int_0^\infty e^{-\gamma t} P_t \psi(X_t(x)) dt \\
= \frac{1}{\gamma}\psi(x) + \frac{1}{\gamma} \mathcal{L}_0 \varphi(x),
\]

that is, \( \mathcal{L}_0 \varphi = \gamma \varphi - \psi \). This shows

\[
L\varphi = \gamma \varphi - \psi - \left\{ D\varphi, \int_U [\varphi(\cdot, + f(\cdot, u)) - \varphi(\cdot) - \langle D\varphi, f(\cdot, u) \rangle] \lambda(du) \right\}, \tag{2.12}
\]

where \( L \) is the infinitesimal generator of the semigroup defined by \( R_t \varphi(x) = \mathbb{E}(\varphi(Z_t(x))) \) and

\[
Z_t(x) = e^{-tA}x + \int_0^t e^{(t-s)A} B dW_s.
\]

From the above proof, we have that \( \varphi \) satisfies the conditions of Proposition 2.68 of [3]. Hence, there exists a three-index sequence \( \{\varphi_{kmn}, k, m, n \geq 1\} \subset \mathcal{D}^r_A(\mathbb{H}) \) such that, for any \( x \in \mathbb{H}, \)

\[
\lim_{k \to \infty} \lim_{m \to \infty} \lim_{n \to \infty} \varphi_{kmn} = \varphi, \\
\lim_{k \to \infty} \lim_{m \to \infty} \lim_{n \to \infty} D\varphi_{kmn} = D\varphi
\]

and

\[
\lim_{k \to \infty} \lim_{m \to \infty} \lim_{n \to \infty} L\varphi_{kmn} = L\varphi.
\]

By the dominated convergence theorem and (2.12) we have

\[
\lim_{k \to \infty} \lim_{m \to \infty} \lim_{n \to \infty} \mathcal{L}_0 \varphi_{kmn} = \mathcal{L}_0 \varphi \\
= L\varphi + \left\{ D\varphi, \int_U [\varphi(\cdot, + f(\cdot, u)) - \varphi(\cdot) - \langle D\varphi, f(\cdot, u) \rangle] \lambda(du) \right\} \\
= \gamma \varphi - \psi,
\]
that is,
\[ \gamma \varphi - \mathcal{L}_0 \varphi = \psi. \]

(b) Suppose that \( S : \mathcal{D}(S) \subset \mathbb{H} \rightarrow \mathbb{H} \) is a given self-adjoint negative definite operator such that \( S^{-1} \) is of trace class, \( I \) is the identity operator on \( \mathbb{H} \) and the measure \( N_Q \) is defined on the product space \( \mathbb{R}^\infty \) as follows:

\[ N_S(dx) = \prod_{k=1}^{\infty} N_{0, \alpha_k}, \quad Se_k = \alpha_k e_k, \quad \text{Tr} \ S = \sum_{k=1}^{\infty} \alpha_k < \infty, \]

that is, if \( \mathbb{H} \) is \( n \)-dimensional, we have

\[ N_S(dx) = (2\pi)^{-\frac{n}{2}} (\det S)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \langle S^{-1}x, x \rangle \right\} \ dx. \]

Let us introduce a regularization of \( f(x, u) \) with respect to the variable \( x \):

\[ \langle f_\beta(x, u), z \rangle = \int_{\mathbb{H}} \langle f(e^{\beta S}x + y, u), e^{\beta S}z \rangle N_{\frac{1}{2} S^{-1}(e^{2\beta S-1})(dy), \beta > 0}, \]

then \( f_\beta(x, u) \) is Lipschitz continuous of class \( C^\infty \) with respect to the first variable \( x \) and

\[ \int_U |D^k f_\beta(x, u)|^2 \lambda(du) < \infty, \quad \forall k \geq 1 \]

by the conditions \((H_1), (H_2)\) and \((H_5)\).

For the equation

\[ \gamma \varphi_\beta - L \varphi_\beta - \left( D \varphi_\beta, \int_U \varphi_\beta(\cdot + f_\beta(\cdot, u)) - \varphi_\beta(\cdot) - \langle D \varphi_\beta, f_\beta(\cdot, u) \rangle \lambda(du) \right) = \psi, \]

the step (a) yields that \( \varphi_\beta \in \mathcal{D}(\mathcal{L}_0) \) and

\[ \gamma \varphi_\beta - \mathcal{L}_0 \varphi_\beta = \psi + \left( D \varphi_\beta, \int_U \varphi_\beta(\cdot + f_\beta(\cdot, u)) - \varphi_\beta(\cdot) - \langle D \varphi_\beta, f_\beta(\cdot, u) \rangle \lambda(du) \right. \]

\[ \left. - \int_U \varphi_\beta(\cdot + f(\cdot, u)) - \varphi_\beta(\cdot) - \langle D \varphi_\beta, f(\cdot, u) \rangle \lambda(du) \right). \tag{2.13} \]

Since the condition \((H_2)\) implies

\[ \left| \int_U \varphi(\cdot + f(\cdot, u)) - \varphi(\cdot) - \langle D \varphi, f(\cdot, u) \rangle \lambda(du) \right| \]

\[ \leq C_1 \int_U |D^2 \varphi(\xi_f)| |f(\cdot, u)|^2 \lambda(du) \leq C_1 \int_U |f(\cdot, u)|^2 \lambda(du) < \infty, \tag{2.14} \]

the conditions \((H_2)\) and \((H_5)\) yield

\[ \left| \int_U \varphi_\beta(\cdot + f_\beta(\cdot, u)) - \varphi_\beta(\cdot) - \langle D \varphi_\beta, f_\beta(\cdot, u) \rangle \lambda(du) \right| \]

\[ \leq C_1 \int_U |D^2 \varphi_\beta(\delta \cdot +(1-\delta) f_\beta(\cdot, u))| |f_\beta(\cdot, u)|^2 \lambda(du) \]

\[ \leq C_1 \int_U |f(\cdot, u)|^2 \lambda(du) < \infty \tag{2.15} \]
for some constant $C_1 > 0$, where $\delta \in (0, 1)$.

$$
\lim_{\beta \to 0} \int_U f_\beta(x, u)\lambda(du) = \int_U f(x, u)\lambda(du), \quad x \in \mathbb{H},
$$

(2.5), (2.14), (2.15) and the dominated convergence theorem imply that, in $L^2(\mathbb{H}, \mu)$,

$$
\lim_{\beta \to 0} \left[ \left( D\varphi_\beta, \int_U [\varphi_\beta(\cdot + f_\beta(\cdot, u)) - \varphi_\beta(\cdot) - \langle D\varphi_\beta, f_\beta(\cdot, u) \rangle] \lambda(du) \right) - \int_U [\varphi_\beta(\cdot + f(\cdot, u)) - \varphi_\beta(\cdot) - \langle D\varphi_\beta, f(\cdot, u) \rangle] \lambda(du) \right] = 0. \tag{2.16}
$$

From (2.13) and (2.15) we have that the closure of the range of $\mathcal{L}_0$ includes $\mathcal{E}_A$, so that it is dense in $L^2(\mathbb{H}, \mu)$, and by the Lumer–Phillips theorem (Theorem 3.20 in [3]), $\mathcal{L}_0$ is $m$-dissipative. Since $\mathcal{L}_2$ is the infinitesimal generator of strongly continuous semigroup of contractions, we know that it is $m$-dissipative. Again, $\mathcal{L}_2$ extends $\mathcal{L}_0$, hence, $\mathcal{L}_2$ must coincide with $\mathcal{L}_0$. \hfill \Box

3. The integration by parts formula

**Theorem 3.1.** The operator $Q^{\frac{1}{2}}D : \mathcal{E}_A(\mathbb{H}) \subset L^2(\mathbb{H}, \mu) \to L^2(\mathbb{H}, \mu; \mathbb{H})$ is uniquely extendible to a bounded operator denoted by $D_0$ from $L^2(\mathbb{H}, \mu)$ into $L^2(\mathbb{H}, \mu; \mathbb{H})$. Moreover, the following identity holds:

$$
\int_{\mathbb{H}} \varphi(x)\mathcal{L}_2\varphi(x)\mu(dx) = -\frac{1}{2} \int_{\mathbb{H}} |D_0\varphi(x)|^2 \mu(dx)
$$

$$
+ \frac{1}{2} \int_{\mathbb{H}} \int_U [\varphi(x + f(x, u)) - \varphi(x)]^2 \lambda(du) \mu(dx). \tag{3.1}
$$

**Proof.** Let $\varphi \in \mathcal{E}_A(\mathbb{H})$. By a straightforward computation, we have

$$
\mathcal{L}_2\varphi^2(x) = 2\varphi(x)\mathcal{L}_2\varphi(x) + \left| Q^{\frac{1}{2}}D\varphi(x) \right|^2 - \int_U [\varphi(x + f(x, u)) - \varphi(x)]^2 \lambda(du).
$$

Integrating this identity over $\mathbb{H}$ with respect to $\mu$ and taking into account that

$$
\int_{\mathbb{H}} \mathcal{L}_2\varphi^2(x)\mu(dx) = 0
$$

by the invariance of $\mu$, yield that

$$
\int_{\mathbb{H}} \varphi(x)\mathcal{L}_2\varphi(x)\mu(dx) = -\frac{1}{2} \int_{\mathbb{H}} |D_0\varphi(x)|^2 \mu(dx)
$$

$$
+ \frac{1}{2} \int_{\mathbb{H}} \int_U [\varphi(x + f(x, u)) - \varphi(x)]^2 \lambda(du) \mu(dx) \tag{3.2}
$$

for $\varphi \in \mathcal{E}_A(\mathbb{H})$.

Now, let $\varphi \in \mathcal{D}(\mathcal{L}_2)$. Since $\mathcal{E}_A(\mathbb{H})$ is a core for $\mathcal{L}_2$, there exists a sequence $\{\varphi_n\}_{n \geq 1} \subset \mathcal{E}_A(\mathbb{H})$ such that

$$
\varphi_n \to \varphi, \quad \mathcal{L}_2\varphi_n \to \mathcal{L}_2\varphi \quad \text{in} \ L^2(\mathbb{H}, \mu).$$
Hence, by (3.2) it follows that
\[
\int_{\mathbb{H}} |D_Q(\varphi_n(x) - \varphi_m(x))|^2 \mu(dx)
+ \int_{\mathbb{H}} \int_U [(\varphi_n(x + f(x, u)) - \varphi_n(x)) - (\varphi_m(x + f(x, u)) - \varphi_m(x))]^2 \lambda(du) \mu(dx)
\leq 2 \int_{\mathbb{H}} |\varphi_n(x) - \varphi_m(x)|^2 \mu(dx) + \int_{\mathbb{H}} |\mathcal{L}_2(\varphi_n(x) - \varphi_m(x))|^2 \mu(dx).
\]
This yields that \( \{D_Q \varphi_n\}_{n \geq 1} \) and \( \{\varphi_n(x + f(x, u)) - \varphi_n(x)\}_{n \geq 1} \) are Cauchy sequences in \( L^2(\mathbb{H}, \mu; \mathbb{H}) \) and \( L^2(\mathbb{H} \times U, \lambda \times \mu; \mathbb{H}) \), respectively. Hence, the conclusion follows. \( \square \)

**Theorem 3.2.** For any \( \varphi \in L^2(\mathbb{H}, \mu) \), the following identities hold:
\[
\int_{\mathbb{H}} \varphi^2(x) \mu(dx) = \int_{\mathbb{H}} (P_t \varphi(x))^2 \mu(dx) + \int_0^t \int_{\mathbb{H}} |Q^{1/2} D_P \varphi(x)|^2 \mu(dx) ds
- \int_0^t \int_{\mathbb{H}} \int_U \{P_s[\varphi(x + f(x, u)) - \varphi(x)]\}^2 \lambda(du) \mu(dx) ds
\]
and
\[
\int_{\mathbb{H}} |\varphi(x) - \bar{\varphi}|^2 \mu(dx) = \int_0^\infty \int_{\mathbb{H}} |Q^{1/2} D_P \varphi(x)|^2 \mu(dx) ds
- \int_0^\infty \int_{\mathbb{H}} \int_U \{P_s[\varphi(x + f(x, u)) - \varphi(x)]\}^2 \lambda(du) \mu(dx) ds,
\]
where \( \bar{\varphi} = \int_{\mathbb{H}} \varphi(x) \mu(dx) \).

**Proof.** For \( \varphi \in \mathcal{D}(\mathcal{L}_2) \), the Hille–Yosida theorem yields that \( P_t \in \mathcal{D}(\mathcal{L}_2) \) for all \( t \geq 0 \) and
\[
\frac{d}{dt} P_t \varphi = \mathcal{L}_2 P_t \varphi.
\]
Multiplying both sides of this identity by \( P_t \varphi \) and integrating with respect to \( x \) over \( \mathbb{H} \), (3.1) implies
\[
\frac{d}{dt} \int_{\mathbb{H}} [P_t \varphi(x)]^2 \mu(dx)
= -\int_{\mathbb{H}} |Q^{1/2} D_P \varphi(x)|^2 \mu(dx) + \int_{\mathbb{H}} \int_U [P_t \varphi(x + f(x, u)) - P_t \varphi(x)]^2 \lambda(du) \mu(dx).
\]
Integrating with respect to \( t \) yields (3.3). Let \( t \to +\infty \). We have
\[
\int_{\mathbb{H}} \varphi^2(x) \mu(dx) = \lim_{t \to \infty} \int_{\mathbb{H}} (P_t \varphi(x))^2 \mu(dx) + \int_0^\infty \int_{\mathbb{H}} |Q^{1/2} D_P \varphi(x)|^2 \mu(dx) ds
- \int_0^\infty \int_{\mathbb{H}} \int_U \{P_s[\varphi(x + f(x, u)) - \varphi(x)]\}^2 \lambda(du) \mu(dx) ds.
\]
On the other hand, by the invariance of \( \mu \) we have
\[
\lim_{t \to \infty} \int_{\mathbb{H}} (P_t \varphi(x))^2 \mu(dx) = \int_{\mathbb{H}} \left( \int_{\mathbb{H}} \varphi(x) \mu(dx) \right)^2 \mu(dy) = (\bar{\varphi})^2.
\]
Hence, the conclusion follows. □

**Lemma 3.1.** Let \( \{\varphi_n\}_{n \geq 1} \subset \mathcal{E}_A(\mathbb{H}) \) and \( G \in L^2(\mathbb{H}, \mu, \mathbb{H}) \) be such that

\[
\lim_{n \to \infty} \int_{\mathbb{H}} |D\varphi_n(x) - G(x)|^2 \mu(dx) = 0.
\]

Then, for any \( t \geq 0 \) we have

\[
\lim_{n \to \infty} \int_{\mathbb{H}} |DP_t\varphi_n(x) - \mathbb{E}[\{DX_t(x)\}^* G(X_t(x))]|^2 \mu(dx) = 0.
\]

In particular, if \( D\varphi_n \to 0 \) in \( L^2(\mathbb{H}, \mu, \mathbb{H}) \), we have \( DP_t\varphi_n \to 0 \) in \( L^2(\mathbb{H}, \mu, \mathbb{H}) \) for all \( t \geq 0 \).

**Proof.** Since, for \( t \geq 0 \) and \( x \in \mathbb{H} \),

\[
DP_t\varphi_n(x) = \mathbb{E}[\{DX_t(x)\}^* D\varphi_n(X_t(x))],
\]

taking into account the invariance of \( \mu \), by (2.5), we have

\[
\int_{\mathbb{H}} |DP_t\varphi_n(x) - \mathbb{E}[\{DX_t(x)\}^* G(X_t(x))]|^2 \mu(dx)
\]

\[
= \int_{\mathbb{H}} \left| \mathbb{E}[\{DX_t(x)\}^* (D\varphi_n(X_t(x)) - G(X_t(x)))] \right|^2 \mu(dx)
\]

\[
\leq \int_{\mathbb{H}} |DX_t(x)|^2 \mathbb{E}|D\varphi_n(X_t(x)) - G(X_t(x))|^2 \mu(dx)
\]

\[
\leq \frac{1}{1 - \varepsilon} \exp\left\{ \frac{7K_t}{\varepsilon} \right\} \int_{\mathbb{H}} \mathbb{E}|D\varphi_n(X_t(x)) - G(X_t(x))|^2 \mu(dx)
\]

\[
= \frac{1}{1 - \varepsilon} \exp\left\{ \frac{7K_t}{\varepsilon} \right\} \int_{\mathbb{H}} P_t|D\varphi_n(x) - G(x)|^2 \mu(dx)
\]

\[
= \frac{1}{1 - \varepsilon} \exp\left\{ \frac{7K_t}{\varepsilon} \right\} \int_{\mathbb{H}} |D\varphi_n(x) - G(x)|^2 \mu(dx).
\]

The conclusion of the lemma follows. □

**Theorem 3.3.** Suppose that the conditions (H1)–(H5) hold. Then the operator \( Q^{1/2}D \) is closable. Moreover, if \( \varphi \) belongs to the domain \( \overline{D}_Q \) of the closure of \( Q^{1/2}D \) and \( \overline{D}_Q \varphi = 0 \), we have \( \overline{D}_Q P_t \varphi = 0 \) for all \( t > 0 \).

**Proof.** The last statement follows from Lemma 3.1. We only prove that the operator \( Q^{1/2}D \) is closable.

Let \( \{\varphi_n\}_{n \geq 1} \subset \mathcal{E}_A(\mathbb{H}) \) and \( G \in L^2(\mathbb{H}, \mu, \mathbb{H}) \) be such that

\[
\int_{\mathbb{H}} |\varphi_n(x)|^2 \mu(dx) \to 0 \quad \text{and} \quad \int_{\mathbb{H}} |D\varphi_n(x) - G(x)|^2 \mu(dx) \to 0.
\]

By (3.3) and the Fubini theorem we have

\[
\int_{\mathbb{H}} \varphi_n^2(x) \mu(dx) = \int_{\mathbb{H}} (P_1\varphi_n(x))^2 \mu(dx) + \int_0^t \int_{\mathbb{H}} \left| Q^{1/2}DP_s\varphi_n(x) \right|^2 \mu(dx) ds
\]

\[
- \int_0^t \int_{\mathbb{H}} \int_{U} \left| P_s[\varphi_n(x + f(x, u)) - \varphi_n(x)] \right|^2 \lambda(du) \mu(dx) ds
\]
implies shows that the mapping is closable. We denote the closure by $h$. The arbitrariness of $\lambda$ yields that

$$\lim_{n \to \infty} \int_0^t \int_{\mathbb{H}} |Q^{1/2} DP_s \varphi_n(x)|^2 \mu(dx) ds = 0.$$ 

Since $Q^{1/2}$ is a bounded operator, the dominated convergence theorem yields

$$\int_0^t \int_{\mathbb{H}} |Q^{1/2} E [(D_x X_s(x))^* G(X_t(x))]|^2 \mu(dx) ds = 0.$$ 

This means that, for all $t > 0$,

$$Q^{1/2} E [(D_x X_t(x))^* G(X_t(x))] = 0, \quad \mu \times dt \text{-a.s.}$$

By the condition (H4) we have that $Q$ is invertible, which shows that, for all $t > 0$,

$$\mathbb{E} [(D_x X_t(x))^* G(X_t(x))] = 0, \quad \mu \times dt \text{-a.s.} \quad (3.5)$$

Hence, for any fixed $h \in \mathbb{H}$, (3.5) implies

$$\left| \mathbb{E} \langle G(X_t(x)), h \rangle \right| \leq |\mathbb{E} \langle G(X_t(x)), D_x X_t(x) \cdot h \rangle| + |\mathbb{E} \langle G(X_t(x)), h - D_x X_t(x) \cdot h \rangle|$$

$$= |\mathbb{E} \langle (D_x X_t(x))^* G(X_t(x)), h \rangle| + |\mathbb{E} \langle G(X_t(x)), h - D_x X_t(x) \cdot h \rangle|$$

$$= |\mathbb{E} \langle G(X_t(x)), h - D_x X_t(x) \cdot h \rangle|.$$ 

Taking into account the invariance of $\mu$ and (3.5), we get

$$\int_{\mathbb{H}} |P_t(G(x), h)|^2 \mu(dx) = \int_{\mathbb{H}} |\mathbb{E} \langle G(X_t(x)), h \rangle|^2 \mu(dx)$$

$$= \int_{\mathbb{H}} |\mathbb{E} \langle G(X_t(x)), h - D_x X_t(x) \cdot h \rangle|^2 \mu(dx)$$

$$\leq \left( \int_{\mathbb{H}} \mathbb{E} |G(X_t(x))|^2 \mu(dx) \right)^{1/2} \left( \int_{\mathbb{H}} \mathbb{E} |h - D_x X_t(x) \cdot h|^2 \mu(dx) \right)^{1/2}.$$ 

By the strong continuity of $P_t$ in $L^1(\mathbb{H}, \mu)$, letting $t \to 0$ we have

$$\int_{\mathbb{H}} |\langle G(x), h \rangle|^2 \mu(dx) = 0.$$ 

The arbitrariness of $h$ yields that $G = 0$ as required. \quad \Box

**Theorem 3.3** shows that the mapping

$$Q^{1/2} D : \mathcal{C}_A(\mathbb{H}) \subset L^2(\mathbb{H}, \mu) \to L^2(\mathbb{H}, \mu, \mathbb{H}), \quad \varphi \mapsto Q^{1/2} D \varphi$$

is closable. We denote the closure by $\overline{D_Q}$. 

Theorem 3.4. \( D(L_2) \subset D(D_Q) \) with continuous embedding and the following identity holds, for any \( \varphi \in D(L_2) \),
\[
\int_{\mathbb{H}} \varphi(x) \cdot \mathcal{L}_2 \varphi(x) \mu(dx) = -\frac{1}{2} \int_{\mathbb{H}} |D_Q \varphi(x)|^2 \mu(dx) + \frac{1}{2} \int_{\mathbb{H}} \int_{U} [\varphi(x + f(x, u)) - \varphi(x)]^2 \lambda(du) \mu(dx).
\]

Proof. Let \( \varphi \in D(L_2) \). Since \( \mathcal{E}_A(\mathbb{H}) \) is a core for \( L_2 \), there exists a sequence \( \{\varphi_n\}_{n \geq 1} \subset \mathcal{E}_A(\mathbb{H}) \) such that
\[
\int_{\mathbb{H}} |\varphi_n(x) - \varphi(x)|^2 \mu(dx) \to 0 \quad \text{and} \quad \int_{\mathbb{H}} |\mathcal{L}_0 \varphi_n(x) - \mathcal{L}_2 \varphi(x)|^2 \mu(dx) \to 0.
\]
By (3.1) it follows that
\[
\int_{\mathbb{H}} \left| Q^{\frac{1}{2}} D(\varphi_n(x) - \varphi_m(x)) \right|^2 \mu(dx) + \int_{\mathbb{H}} \int_{U} |\varphi_n(x + f(x, u)) - \varphi_m(x + f(x, u)) + \varphi_m(x)|^2 \lambda(du) \mu(dx) \leq 2 \int_{\mathbb{H}} |\mathcal{L}_0(\varphi_n(x) - \varphi_m(x))| |\varphi_n(x) - \varphi_m(x)| \mu(dx)
\]
\[
\leq \int_{\mathbb{H}} |\mathcal{L}_0(\varphi_n(x) - \varphi_m(x))|^2 \mu(dx) + \int_{\mathbb{H}} |\varphi_n(x) - \varphi_m(x)|^2 \mu(dx).
\]
This implies that the sequence \( \{Q^{\frac{1}{2}} D \varphi_n\}_{n \geq 1} \) is a Cauchy one in \( L^2(\mathbb{H}, \mu, \mathbb{H}) \). Since \( Q^{\frac{1}{2}} D \) is closable, it follows that \( \varphi \in W^{1,2}(\mathbb{H}, \mu) \) as required. \( \square \)

4. Poincaré inequality

Theorem 4.1. Suppose that \( K < 2\lambda \) and \( Q = 1 \). For any \( \varphi \in D(D_Q) \cong D(D) \), we have
\[
\int_{\mathbb{H}} |\varphi - \bar{\varphi}|^2 \mu(dx) \leq \frac{1}{2\lambda - K} \int_{\mathbb{H}} |D\varphi|^2 \mu(dx).
\]

Proof. By (2.4) we have
\[
\mathbb{E}|\eta^h_t(x)|^2 + 2\mathbb{E} \int_0^t \|\eta^h_s(x)\|^2 ds \leq |h|^2 + \mathbb{E} \int_0^t \int_U |Df(X_s(x), u) \cdot \eta^h_s(x)|^2 \lambda(du) ds \leq |h|^2 + K \mathbb{E} \int_0^T |\eta^h_s(x)|^2 ds.
\]
(1.1) yields that
\[
\mathbb{E}|\eta^h_t(x)|^2 \leq |h|^2 + (K - 2\lambda) \mathbb{E} \int_0^T |\eta^h_s(x)|^2 ds.
\]
The Gronwall inequality implies
\[
\mathbb{E}|\eta^h_t(x)|^2 \leq |h|^2 e^{(K - 2\lambda)t}.
\]
For $\varphi \in \mathcal{E}_A(H)$, (4.1) yields that, for $h \in H$ and $t \geq 0$,
\[
|\langle DP_t \varphi(x), h \rangle|^2 \leq \mathbb{E} \left( |D\varphi(X_t(x))|^2 \right) \mathbb{E} \left( |\eta_t^h(x)|^2 \right) \leq \mathbb{E} \left( |D\varphi(X_t(x))|^2 \right) \exp \{ (K - 2\lambda)t \} |h|^2 \leq e^{(K - 2\lambda)t} P_t \left( |D\varphi|^2 \right)(x) |h|^2.
\]
This shows
\[
|DP_t \varphi(x)|^2 \leq e^{(K - 2\lambda)t} P_t \left( |D\varphi|^2 \right)(x).
\]
From (3.4) and (4.2), taking into account the invariance of $\mu$, we get
\[
\int_H |\varphi(x) - \bar{\varphi}|^2 \mu(dx) \leq \int_0^\infty \int_H |DP_s \varphi(x)|^2 \mu(dx) ds \leq \int_0^\infty \int_H e^{(K - 2\lambda)s} P_s \left( |D\varphi|^2 \right)(x) \mu(dx) ds \leq \int_0^\infty e^{(K - 2\lambda)s} ds \int_H |D\varphi|^2(x) \mu(dx) \leq \frac{1}{2\lambda - K} \int_H |D\varphi|^2(x) \mu(dx).
\]
Hence, the conclusion holds for $\varphi \in \mathcal{E}_A(H)$. For $\varphi \in \mathcal{D}(\overline{D})$, we can prove the conclusion using the density of $\mathcal{E}_A(H)$ in $\mathcal{D}(\overline{D})$. □

References