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Bijective linear rank preservers for spaces of matrices over antinegative semirings

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ABSTRACT

We classify the bijective linear operators on spaces of matrices over antinegative commutative semirings with no zero divisors which preserve certain rank functions such as the symmetric rank, the factor rank and the tropical rank. We also classify the bijective linear operators on spaces of matrices over the max-plus semiring which preserve the Gondran–Minoux row rank or the Gondran–Minoux column rank.

1. Semirings and rank functions

We begin with the definition of a semiring. Refs. [6,7] are good references for semiring theory.

Definition 1.1. A semiring is a set S together with two operations $\oplus$ and $\otimes$ and two distinguished elements $\mathbf{0}, \mathbf{1}$ in S with $\mathbf{0} \neq \mathbf{1}$ such that

(1) $(S, \oplus, \mathbf{0})$ is a commutative monoid,
(2) $(S, \otimes, \mathbf{1})$ is a monoid,
(3) $\otimes$ is both left and right distributive over $\oplus$.
(4) whose additive identity $\mathbf{0}$ satisfies the property $r \otimes \mathbf{0} = \mathbf{0} \otimes r = \mathbf{0}$ for all $r \in S$.

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Throughout this paper S will always be taken to be a commutative semiring (i.e. where $(S, \otimes, \mathbf{1})$ is a commutative monoid) unless otherwise noted.

Definition 1.2. A semiring S is said to be antinegative if the only element with an additive inverse is the additive identity $\mathbf{0}$.

Antinegative semirings are also sometimes called zerosumfree semirings or antirings.

Definition 1.3. Let S be a commutative semiring. A non-zero element $x \in S$ is called a zero divisor if there exists a non-zero $y \in S$ such that $x \otimes y = \mathbf{0}$.

A much studied example of a semiring is the max-plus semiring (sometimes also called the tropical semiring) where $\mathbb{R}_{max} = \mathbb{R} \cup \{-\infty\}$ with $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$. ($+$ here represents ordinary addition.) Note that in this case $\mathbf{0} = -\infty$ and $\mathbf{1} = 0$. The non-negative numbers or more generally the non-negative elements of a unital subring of $\mathbb{R}$ form a semiring under the usual addition and multiplication. A chain semiring is a totally ordered set S with greatest element $\mathbf{1}$ and least element $\mathbf{0}$ where $a \oplus b = \max\{a, b\}$ and $a \otimes b = \min\{a, b\}$. The chain semiring consisting entirely of two elements $\{\mathbf{0}, \mathbf{1}\}$ is called the Boolean semiring. All of these semirings are commutative antinegative and have no zero divisors.

We can consider matrices over semirings. $M_{m,n}(S)$ denotes the set of m by n matrices over S and $M_n(S)$ denotes the set of n by n matrices over S. Addition and multiplication of these matrices can be defined in the usual way. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be m by n matrices over a semiring $(S, \oplus, \otimes)$ and let $C = [c_{ij}]$ be an n by p matrix over the same semiring. Then $A + B = [a_{ij} \oplus b_{ij}]$ and $AC = [\bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj}]$. The set of n by n matrices over a semiring is itself a semiring. For any $k \in S$, the matrix $kA = [k \otimes a_{ij}]$.

There are many different yet equivalent ways of defining the rank of a matrix over a field. Many of these definitions may be generalized to matrices over semirings. With matrices over semirings, these different definitions are no longer equivalent and they yield different rank functions. We will introduce the rank functions used in our paper; these and many others are discussed extensively in [1,2,8].

Definition 1.4. Let $A \in M_{m,n}(S)$. Then the factor rank of A is the smallest integer r for which there exists an m by r matrix B and an r by n matrix C such that $A = BC$. The factor rank is denoted $f(A)$.

The factor rank is sometimes called the Schein rank or the semiring rank.

There is a close connection between ranks and determinants. Since subtraction is not generally allowed in semirings, we have to split the determinant into positive and negative parts. In what follows S_n is the symmetric group on n elements and A_n is the alternating group on n elements.

Definition 1.5. Given $A \in M_n(S)$ when $n \geqslant 2$, the positive determinant and the negative determinant are respectively the following sums: $|A|^+ = \bigoplus_{\sigma \in A_n} (\bigotimes_{i=1}^{n} a_{i\sigma(i)})$, and $|A|^- = \bigoplus_{\sigma \in S_n \setminus A_n} (\bigotimes_{i=1}^{n} a_{i\sigma(i)})$. If $a \in S$ is considered as a 1 by 1 matrix then we take $|a|^+ = a$ and $|a|^- = \mathbf{0}$.

$A \in M_n(S)$ is called balanced if $|A|^+ = |A|^-$ and is called unbalanced if $|A|^+ \neq |A|^-$.

Definition 1.6. The symmetric rank of an m by n matrix A is the largest natural number k for which there exists a k by k unbalanced submatrix of A. We will denote the symmetric rank of A as $rk_{sym}(A)$.

The symmetric rank is also sometimes called the determinantal rank.

Definition 1.7. A matrix $A \in M_n(S)$ with $n \geqslant 2$ is called tropically singular if there exists $\mathcal{T}$, a proper subset of S_n such that $\bigoplus_{\sigma \in \mathcal{T}} (\bigotimes_{i=1}^{n} a_{i\sigma(i)}) = \bigoplus_{\sigma \in S_n \setminus \mathcal{T}} (\bigotimes_{i=1}^{n} a_{i\sigma(i)})$. A matrix is called tropically non-singular if it is not tropically singular. Non-zero 1 by 1 matrices are defined to be tropically non-singular and the zero 1 by 1 matrix is defined to be tropically singular.

Definition 1.8. The tropical rank of m by n matrix A is the largest natural number k for which there exists a k by k tropically non-singular submatrix of A. We will denote the tropical rank of A as $trop(A)$.

Since any tropically non-singular matrix must be unbalanced, the tropical rank of a matrix is always less than or equal to its symmetric rank.

Definition 1.9. A finite subset $\{v_i\}_{i=1}^m$ in S^n is said to be linearly independent in the Gondran–Minoux sense if for any disjoint subsets I, J of $\{1, 2, \ldots, m\}$ with $I \cup J \neq \emptyset$ and any set $\{c_k\}_{k \in I \cup J}$ of elements of S, we have $\bigoplus_{i \in I} c_i \otimes v_i \neq \bigoplus_{j \in J} c_j \otimes v_j$ unless $c_k = \mathbf{0}$ for all $k \in I \cup J$.

Definition 1.10. Let $A \in M_{m,n}(S)$. The Gondran–Minoux row rank of A is the maximal integer r for which there exist r rows of A which are linearly independent in the Gondran–Minoux sense. The Gondran–Minoux column rank of A is the maximal integer s for which there exist s columns of A which are linearly independent in the Gondran–Minoux sense. The Gondran–Minoux row rank of A is denoted as $GM_r(A)$ and the Gondran–Minoux column rank of A is denoted as $GM_c(A)$.

Definition 1.11. Let $A \in M_{m,n}(S)$. Then the term rank of A is the smallest non-negative integer k for which there exists either a k by n or an m by k submatrix of A which contains every non-zero element of A. The term rank is denoted $t(A)$.

Unlike the other types of rank, the term rank and the tropical rank are not straightforward generalizations of one of the equivalent defintions of the rank of a matrix over a field. For matrices over fields, the factor rank, the symmetric rank, the Gondran–Minoux row rank and the Gondran–Minoux column rank are all equal while the term rank and the tropical rank may be different from these and each other. For matrices over semirings, all of these ranks may differ from one another.

We end this section by noting some inequalities between the various types of ranks from the literature which will be useful for us. The first result is part of Proposition 3.1 from [2].

Proposition 1.12 ([2]). *Let $A \in M_{m,n}(S)$, then $f(A) \leqslant t(A)$.*

The next result is a combination of parts of Lemmas 8.1 and 8.5 from [1].

Proposition 1.13 ([1]). *Let $A \in M_{m,n}(\mathbb{R}_{max})$, then $rk_{sym}(A) \leqslant GM_r(A), GM_c(A) \leqslant f(A)$.*

2. Rank preservers

In this section we characterize bijective linear operators on $M_{m,n}(S)$ which preserve any one of the factor, symmetric or tropical ranks when S is a commutative antinegative semiring with no zero divisors. In the special case where $S = \mathbb{R}_{max}$, we also characterize the bijective linear preservers of the Gondran–Minoux ranks. If $A, B \in M_{m,n}(S)$, we write $A \preceq B$ if there exists $C \in M_{m,n}(S)$ such that $A + C = B$. We note that $\preceq$ is a preorder (i.e. a reflexive, transitive relation) on $M_{m,n}(S)$ and any linear operator $T : M_n(S) \to M_n(S)$ will preserve this preorder. Further, if S is antinegative then the term rank is a monotone function in the sense that if $A \preceq B$ then $t(A) \leqslant t(B)$.

Definition 2.1. Let S be a commutative semiring. An element of S is called a unit if it has a multiplicative inverse.

We note that the set of all units of S form a group under $\otimes$. We also need the definitions of the Schur product and a submonomial matrix.

Definition 2.2. Let $A, B \in M_{m,n}(S)$, then $A \circ B$ is the m by n matrix whose (i, j)th entry is $a_{ij} \otimes b_{ij}$.

Definition 2.3. A matrix $A \in M_{m,n}(S)$ is called a submonomial matrix if every line (row or column) of A contains at most one non-zero entry. A matrix $A \in M_n(S)$ is called a monomial matrix if every line (row or column) of A contains exactly one non-zero entry.

For submonomial matrices, we have the following observation.

Proposition 2.4. *The term rank of a submonomial matrix $A \in M_{m,n}(S)$ is equal to the number of non-zero entries of A. If S is a semiring with no zero divisors, then the factor rank, symmetric rank, tropical rank, Gondran–Minoux row rank and Gondran–Minoux column rank of a submonomial matrix A are all equal to the term rank of A.*

We also need Lemma 2 from [4].

Lemma 2.5 ([4, Lemma 2])**.** *Let S be an antinegative semiring and $A \in M_n(S)$, then there exists a submonomial matrix $M \preceq A$ such that $t(M) = t(A)$.*

The following characterization of invertible matrices over an antinegative commutative semiring with no zero divisors is due to Skornyakov [10].

Proposition 2.6 ([10, Theorem 1])**.** *Let S be an antinegative commutative semiring with no zero divisors, then $A \in M_n(S)$ is invertible if and only if it is a monomial matrix all of whose non-zero entries are units.*

This result can also be viewed as a special case of Dolžan and Oblak's [5, Theorem 1] characterization of invertible matrices over an antinegative semiring.

The concept of a (P, Q, B) operator is a fundamental concept in the theory of linear preservers over semirings. This concept has been defined in slightly different ways. We will use weak (P, Q, B) operator and strong (P, Q, B) operator to distinguish between the two different definitions in the literature.

Definition 2.7 ([4, p. 37])**.** Let T be a linear operator from $M_{m,n}(S)$ to itself. Then we say that T is a weak (P, Q, B) operator if there exists $P \in M_m(S)$, $Q \in M_n(S)$, and $B \in M_{m,n}(S)$ such that P and Q are permutation matrices, B has no entries which are zeros or zero divisors and either $T(X) = P(X \circ B)Q$ or $m = n$ and $T(X) = P(X^t \circ B)Q$.

Definition 2.8 ([2, Definition 2.11])**.** Let T be a linear operator from $M_{m,n}(S)$ to itself. Then we say that T is a strong (P, Q, B) operator if there exists $P \in M_m(S)$, $Q \in M_n(S)$, and $B \in M_{m,n}(S)$ such that P and Q are permutation matrices, all of the entries of B are units and either $T(X) = P(X \circ B)Q$ or $m = n$ and $T(X) = P(X^t \circ B)Q$.

In [4], Beasley and Pullman have given the following characterization of linear term rank preservers.

Lemma 2.9 ([4, Theorem 1])**.** *Let S be a commutative semiring and let T be a linear operator which maps $M_n(S)$ to itself. Then T preserves term rank if and only if it is a weak (P, Q, B) operator.*

We will also need the following lemma.

Lemma 2.10. *Let S be a commutative semiring and suppose $A \in M_{m,n}(S)$ is matrix at least one of whose entries is a unit having symmetric rank one (or equivalently tropical rank one). Then the factor rank of A is also one. If $A \in M_{m,n}(\mathbb{R}_{max})$ is matrix which has either Gondran–Minoux row rank one or Gondran–Minoux column rank one it must also have factor rank one.*

Proof. Let $a_{r,s}$ be an entry of A which is a unit. For any i, j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, we have $a_{ij} \otimes a_{rs} = a_{is} \otimes a_{rj}$. (If one of $i = r$ or $j = s$, this follows immediately. Otherwise, the sides of the equation are the positive and negative determinants of the two by two submatrix $A[i, r|j, s]$ of A and hence are equal.) Then $A = de^T$ where $d = (a_{rs}^{-1} \otimes a_{1s}, a_{rs}^{-1} \otimes a_{2s}, \ldots, a_{rs}^{-1} \otimes a_{ms})^T$ and $e = (a_{r1}, a_{r2}, \ldots, a_{rn})^T$ and hence has factor rank one. Now suppose $A \in M_{m,n}(\mathbb{R}_{max})$ is matrix which has either Gondran–Minoux row rank one or Gondran–Minoux column rank one. This matrix has symmetric rank one by proposition 1.13 and since every element of $\mathbb{R}_{max}$ except the additive identity is a unit, A has factor rank one.

We note that this lemma is no longer true if we remove the condition that at least one of the entries of the matrix be a unit as the following example shows. □

Example 2.11. Let $S = \{0\} \cup \{1\} \cup [2, \infty)$ with the usual real addition and real multiplication and let $B = \begin{pmatrix} 2 & 2\sqrt{2} \\ 2\sqrt{2} & 4 \end{pmatrix}$. Then B has symmetric rank one and factor rank two.

We are now ready to prove our main results. We begin with the following necessary condition for a bijective linear operator to preserve certain ranks. In the following proofs, we let $J_{m,n}$ denote the m by n matrix all of whose entries are $\mathbf{1}$.

Theorem 2.12. *Let S be a commutative antinegative semiring and let rank: $M_{m,n}(S) \to \mathbb{Z}^+$ be a function which satisfies $rank(A) \leqslant t(A)$ for all $A \in M_{m,n}(S)$ with equality whenever A is a submonomial matrix. Then any bijective linear operator on $M_{m,n}(S)$ which preserves this rank function must be a strong (P, Q, B) operator.*

Proof. Suppose T is a bijective linear operator with $rank(T(A)) = rank(A)$ for all $A \in M_{m,n}(S)$. Then for any $A \in M_{m,n}(S)$, there exists a submonomial matrix $B \preceq A$ such that $rank(B) = t(B) = t(A)$. Therefore $t(A) = rank(T(B)) \leqslant t(T(B)) \leqslant t(T(A))$. Now there exists a submonomial matrix $C \preceq T(A)$ such that $rank(C) = t(C) = t(T(A))$. Let $D \in M_n(S)$ be such that $C + D = T(A)$. Since T is surjective there exists $E, F \in M_n(S)$ such that $T(E) = C$ and $T(F) = D$. Since T is injective, $E + F = A$ and $t(T(A)) = rank(C) = rank(E) \leqslant t(E) \leqslant t(A)$. Therefore we have $t(T(A)) = t(A)$ for all $A \in M_n(S)$, which by Lemma 2.9 means that T is a weak (P, Q, B) operator. Since T is surjective there must be $G \in M_{m,n}(S)$ such that $T(G) = J_{m,n}$ which means that all the entries of B must be units and hence T is a strong (P, Q, B) operator.

We can now characterize the bijective linear operators which preserve certain rank functions. □

Theorem 2.13. *Let S be a commutative antinegative semiring with no zero divisors and let T be a linear operator on $M_{m,n}(S)$. Then the following are equivalent.*

(1) *T is bijective and preserves the tropical rank.*
(2) *T is bijective and preserves the symmetric rank.*
(3) *T is bijective and preserves the factor rank.*
(4) *There exist invertible monomial matrices $U \in M_m(S)$ and $V \in M_n(S)$ such that either $T(X) = UXV$ or $m = n$ and $T(X) = UX^tV$.*

Proof. In what follows fix rank to be either the tropical rank, symmetric rank or the factor rank. It is easy to verify that when U and V are invertible then $rank(UXV) = rank(X)$ for all $X \in M_{m,n}(S)$ and if further $m = n$, $rank(X^t) = rank(X)$. Therefore condition four implies conditions one, two and three. Now suppose T is a bijective linear operator on $M_{m,n}(S)$ such that $rank(T(X)) = rank(X)$ for all $X \in M_{m,n}(S)$. Since all three of our rank functions satisfy the hypotheses of Theorem 2.12 when S has no zero divisors, T must be a strong (P, Q, B)-operator and $rank(B) = rank(PBQ) = rank(T(J_{m,n})) = 1$. Hence by Lemma 2.10, B must have factor rank one and there exist $d \in S^m$ and $e \in S^n$ such that $B = de^t$. Let $D(d) \in M_m(S)$ and $D(e) \in M_n(S)$ be the diagonal matrices whose main diagonal entries are the elements of d and e

in the same order. Now let $U = PD(d)$ and $V = D(e)Q$ and hence either $T(X) = P(X \circ B)Q = UXV$ or $T(X) = P(X^t \circ B)Q = UX^tV$. Hence any one of conditions one, two or three implies condition four. □

For certain semirings there are non-surjective tropical and symmetric rank preservers. If we let S be the semiring and B be the matrix from Example 2.11, then $T(X) = X \circ B$ is a non-surjective tropical and symmetric rank preserver. If S is the non-negative reals, then $T(X) = MXN$ where M and N are square matrices which are invertible over $\mathbb{R}$ but not S is a non-surjective symmetric rank preserver. In contrast, the situation with the factor rank for certain semirings is different. If S is the semiring of non-negative elements of a unique factorization domain in $\mathbb{R}$ or if S is a chain semiring, it is known that all linear factor rank preservers on $M_{m,n}(S)$ with $m, n \geqslant 2$ are bijective and in these cases the characterization of factor rank preservers given in Theorem 2.13 is known (see [3] for details).

Using similar methods, we can characterize the bijective linear operators on $M_{m,n}(\mathbb{R}_{max})$ which preserve the Gondran–Minoux ranks.

Theorem 2.14. *Let T be a linear operator on $M_{m,n}(\mathbb{R}_{max})$. Then the following are equivalent.*

(1) *T is bijective and preserves the Gondran–Minoux row rank.*
(2) *T is bijective and preserves the Gondran–Minoux column rank.*
(3) *There exist invertible matrices $U \in M_m(\mathbb{R}_{max})$ and $V \in M_n(\mathbb{R}_{max})$ such that $T(X) = UXV$ or $m = n \leqslant 5$ and $T(X) = UX^tV$.*

A result of Shitov [9] states that the $GM_r(A) = GM_c(A)$ for all $A \in M_n(\mathbb{R}_{max})$ if and only if $n \leqslant 5$. Hence the transpose operator preserves the Gondran–Minoux ranks on $M_n(\mathbb{R}_{max})$ if and only if $n \leqslant 5$. It is clear that if U and V are invertible then the Gondran–Minoux ranks of UAV and A are equal. Therefore condition three implies both one and two. Now fix GM to be either the Gondran–Minoux row rank or the Gondran–Minoux column rank. Using Proposition 1.13, we note that GM satisfies the hypotheses of Theorem 2.12. Hence if T is a bijective linear operator on $M_{m,n}(\mathbb{R}_{max})$ such that $GM(T(X)) = GM(X)$ for all $X \in M_{m,n}(\mathbb{R}_{max})$, T must be a strong (P, Q, B)-operator and $GM(B) = GM(PBQ) = GM(T(J)) = 1$. Therefore, by Lemma 2.10, B must have factor rank one and there exist $d \in S^m$ and $e \in S^n$ such that $B = de^t$. Let $D(d) \in M_m(\mathbb{R}_{max})$ and $D(e) \in M_n(\mathbb{R}_{max})$ be the diagonal matrices whose main diagonal entries are the elements of d and e in the same order. Now let $U = PD(d)$ and $V = D(e)Q$ and hence either $T(X) = P(X \circ B)Q = UXV$ or $T(X) = P(X^t \circ B)Q = UX^tV$. Therefore either of conditions one or two implies condition three.

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References

[1] M. Akian, S. Gaubert, A. Guterman, Linear independence over tropical semirings and beyond, Contemp. Math. 495 (2009) 1–38.
[2] L.B. Beasley, A.E. Guterman, Rank inequalities over semirings, J. Korean Math. Soc. 42 (2) (2005) 223–241.
[3] L.B. Beasley, C.K. Li, S. Pierce, Miscellaneous preserver problems, Linear and Multilinear Algebra 33 (1992) 109–119.
[4] L.B. Beasley, N.J. Pullman, Term rank, permanent and rook-polynomial preservers, Linear Algebra Appl. 90 (1987) 33–46.
[5] D. Dolžan, P. Oblak, Invertible and nilpotent matrices over antirings, Linear Algebra Appl. 430 (2009) 271–278.
[6] J. Golan, Semirings and Affine Equations Over them: Theory and Applications, Mathematics and its Applications, vol. 556, Kluwer Academic Publishers Group, Dordrecht, 2003.
[7] J.S. Golan, Semirings for the ring theorist, Rev. Roumaine Math. Pures Appl. 35 (6) (1990) 531–540.
[8] A.E. Guterman, Rank and determinant functions for matrices over semirings, in: Surveys in Contemporary Mathematics, 1–33, London Mathematical Society Lecture Note Series, vol. 347, Cambridge University Press, Cambridge, 2008.
[9] Ya.N. Shitov, Matrices with different Gondran–Minoux and determinantal ranks over max-algebras, J. Math. Sci. (New York) 163 (2009) 598–624.
[10] L.A. Skornyakov, Invertible matrices over distributive lattices, Siberian Math. J. 27 (1986) 289–292.