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LINEAR ALGEBRA AND ITS APPLICATIONS

Unitary and Euclidean representations of a quiver

Vladimir V. Sergeichuk [1]

Institute of Mathematics, National Academy of Sciences, Tereshchenkivska 3, Kiev-4, Ukraine

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Abstract

A unitary (Euclidean) representation of a quiver is given by assigning to each vertex a unitary (Euclidean) vector space and to each arrow a linear mapping of the corresponding vector spaces. We recall an algorithm for reducing the matrices of a unitary representation to canonical form, give a certain description of the representations of canonical form, and reduce the problem of classifying Euclidean representations to the problem of classifying unitary representations. We also describe the set of dimensions of all indecomposable unitary (Euclidean) representations of a quiver and establish the number of parameters in an indecomposable unitary representation of a given dimension.

Keywords: Canonical matrix; Unitary transformation; Quiver

1. Introduction

Many problems of linear algebra can be formulated and studied in terms of quivers and their representations, which were proposed by Gabriel [1] (see also [2]). A *quiver* is a directed graph. Its *representation* $\mathscr{A}$ is given by assigning to each vertex i a vector space $\mathscr{A}_i$ and to each arrow $\alpha : i \to j$ a linear mapping $\mathscr{A}_\alpha : \mathscr{A}_i \to \mathscr{A}_j$. For example, the canonical form problems for representations of the quivers $\circlearrowleft\!\cdot$ and $\rightleftarrows\!\cdot$ correspond to the canonical form problems for

[1] E-mail: sergeich@ukrpack.net.

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linear operators (whose solution is the Jordan normal form) and for pairs of linear mappings from one space to another (the matrix pencil problem, solved by Kronecker).

In this article we study unitary and Euclidean representations of a quiver up to isometry. A *unitary* (*Euclidean*) *representation* $\mathscr{A}$ is given by assigning to each vertex i a finite-dimensional unitary (Euclidean) space $\mathscr{A}_i$ and to each arrow $\alpha : i \to j$ a linear mapping $\mathscr{A}_\alpha : \mathscr{A}_i \to \mathscr{A}_j$. We say that two unitary (Euclidean) representations $\mathscr{A}$ and $\mathscr{B}$ are *isometric* and write $\mathscr{A} \simeq \mathscr{B}$ if there exists a system of isometries $\Phi_i : \mathscr{A}_i \to \mathscr{B}_i$ such that $\Phi_j \mathscr{A}_\alpha = \mathscr{B}_\alpha \Phi_i$ for each $\alpha : i \to j$.

Our main tool is Littlewood's algorithm [3] for reducing matrices to triangular canonical form via unitary similarity. In [4] I rediscovered Littlewood's algorithm and applied it to the canonical form problem for unitary representations of a quiver. Various algorithms for reducing matrices to different canonical forms under unitary similarity were also proposed by Brenner, Mitchell, McRae, Radjavi, Benedetti and Gragnolini, and others; see Shapiro's survey [5].

In Section 2 we recall briefly Littlewood's algorithm and study the structure of canonical matrices much as it was made in [4] for the matrices of linear operators in a unitary space.

We say that a matrix problem is *unitarily wild* if it contains the problem of classifying linear operators in a unitary space. In Section 2.3 we show that the last problem contains the problem of classifying unitary representations of an arbitrary quiver (i.e., it is hopeless in a certain sense) and give examples of unitarily wild matrix problems.

The vector $\dim \mathscr{A} = (\dim \mathscr{A}_1, \dim \mathscr{A}_2, \ldots, \dim \mathscr{A}_p) \in \mathbb{N}_0^p$ is called the *dimension* of a representation $\mathscr{A}$ of a quiver Q with vertices $1, 2, \ldots, p$ (we denote $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$). In Section 3 we describe the set of dimensions of direct-sum-indecomposable unitary representations of a quiver, and establish the number of parameters in an indecomposable unitary representation of a given dimension. (Analogous, but much more fundamental and complicated, results for non-unitary representations of a quiver were obtained by Kac [6–8] (see also [2], Section 7.4). In [9] I extended his results to systems of linearly mappings and forms, assigning a bilinear form to an undirected edge of a partially directed graph.)

In particular, if $z \in \mathbb{N}^p$ and Q is a connected quiver other than $\bullet$ and $\bullet \to \bullet$, then there exists an indecomposable unitary representation of dimension z if and only if $zM_Q \geqslant z$, where $M_Q = (m_{ij})$ is the $p \times p$ matrix whose entry m_{ij} is the number of arrows of the form $i \to j$ and $i \leftarrow j$ ($(t_1, \ldots, t_p) \geqslant (z_1, \ldots, z_p)$ means $t_1 \geqslant z_1, \ldots, t_p \geqslant z_p$).

In Section 4 we study Euclidean representations of a quiver. Let $\mathscr{A}^{\mathbb{C}}$ denote the unitary representation obtained from a Euclidean representation $\mathscr{A}$ by complexification ($\mathscr{A}$ and $\mathscr{A}^{\mathbb{C}}$ are given by the same set of real matrices).

In Section 4.1 we prove intuitively obvious facts that (i) $\mathscr{A}^{\mathbb{C}} \simeq \mathscr{B}^{\mathbb{C}}$ implies $\mathscr{A} \simeq \mathscr{B}$, and (ii) if $\mathscr{A}$ is indecomposable and $\mathscr{A}^{\mathbb{C}}$ is decomposable, then $\mathscr{A}^{\mathbb{C}} \simeq \mathscr{U} \oplus \bar{\mathscr{U}}$, where $\mathscr{U}$ is an indecomposable unitary representation. This will imply that unitary and Euclidean representations have the coincident sets of dimensions of indecomposable representations.

In Section 4.2 we study, when a given unitary representation of a quiver can be obtained by complexification. In particular, let A be a complex matrix that is not unitarily similar to a direct sum of matrices, and let $S^{-1}AS = \bar{A}$ for a unitary matrix S (such S exists if A is unitarily similar to a real matrix). Then A is unitarily similar to a real matrix if and only if S is symmetric.

In Section 4.3 we construct an infinite set of real orthogonally non-similar pairs (A, B) of symmetric real matrices having the same characteristic polynomial $\chi(\lambda, x) = |\lambda I - A - Bx|$ (a counter example to Friedland's Problem 11.11 from [10]).

2. Unitary matrix problems

We assume the complex numbers to be lexicographically ordered:

$$a + bi \preceq a' + b'i \quad \text{if either } a = a' \text{ and } b \leqslant b', \text{ or } a < a' \tag{1}$$

and the set of blocks of a block matrix $A = [A_{ij}]$ to be linearly ordered:

$$A_{ij} \leqslant A_{i'j'} \quad \text{if either } i = i' \text{ and } j \leqslant j', \text{ or } i > i'. \tag{2}$$

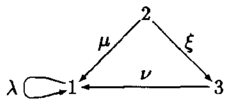
A block complex matrix with a given (perhaps empty) set of marked square blocks will be called a *marked block matrix*; a square block is marked by a line along its principal diagonal. By a *unitary matrix problem* we mean the classification problem for marked block matrices $A = [A_{ij}]$ $(1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant r)$ up to transformations

$$A \mapsto B := R^{-1}AS = [R_i^{-1}A_{ij}S_j], \tag{3}$$

where $R = R_1 \oplus \cdots \oplus R_l$ and $S = S_1 \oplus \cdots \oplus S_r$ are unitary matrices, and $R_i = S_j$ whenever the block A_{ij} is marked. The transformation (3) is called an *admissible transformation*; we say that these marked block matrices A and B (with the same disposition of marked blocks) are *equivalent* and write $A \sim B$ or

$$(R, S) : A \rightsquigarrow B. \tag{4}$$

Notice that a matrix consisting of a single block is reduced by transformations of unitary similarity if the block is marked, and by transformations of unitary equivalence otherwise. Moreover, the matrices of every unitary representation $\mathscr{A}$ of a quiver can be accommodated in a marked block matrix A such that the admissible transformations with A correspond to reselections of the orthogonal bases in the spaces of $\mathscr{A}$, for example,



$$\begin{array}{c|c|c|c|} & S_1 & S_2 & S_3 \\ \hline S_1^{-1} & A_\lambda & A_\mu & A_\nu \\ \hline S_3^{-1} & 0 & A_\xi & 0 \\ \hline \end{array} \tag{5}$$

2.1. An algorithm

Lemma 2.1. (a) *Each complex matrix A is unitarily equivalent to the matrix*

$$D = a_1 I \oplus \cdots \oplus a_{k-1} I \oplus 0, \quad a_i \in \mathbb{R}, \ a_1 > \cdots > a_{k-1} > 0. \tag{6}$$

(b) *If $R^{-1}DS = D'$, where R and S are unitary matrices and D, D' are of the form* (6), *then $D = D'$, $S = S_1 \oplus \cdots \oplus S_{k-1} \oplus S'$ and $R = S_1 \oplus \cdots \oplus S_{k-1} \oplus R'$, where each S_i has the same size as $a_i I$.*

Lemma 2.2. (a) *Each square complex matrix A is unitarily similar to the block-triangular matrix*

$$F = \begin{bmatrix} \lambda_1 I & F_{12} & \cdots & F_{1k} \\ & \lambda_2 I & \cdots & F_{2k} \\ & & \ddots & \vdots \\ 0 & & & \lambda_k I \end{bmatrix}, \quad \begin{array}{l} \lambda_1 \succeq \lambda_2 \succeq \cdots \succeq \lambda_k \text{ (see (1))}, \\ \text{the columns of } F_{i,i+1} \\ \text{are linearly independent} \\ \text{if } \lambda_i = \lambda_{i+1}. \end{array} \tag{7}$$

(b) *If $S^{-1}FS = F'$, where S is a unitary matrix and F and F' have the form* (7), *then $\lambda_i I = \lambda_i' I$ and $S = S_1 \oplus \cdots \oplus S_k$, where each S_i has the same size as $\lambda_i I$.*

Proof. These lemmas were proved in many articles, see, for example [3–5], so we give only an outline of their proofs. Part (a) of Lemma 2.1 is the singular value decomposition; part (b) follows from $D = D'$, $S^*D^*R^{*-1} = D^*$, $S^{-1}DR = D$, $D^2R = RD^2$, and $D^2S = SD^2$. The matrix (7) is the matrix of an arbitrary linear operator $A : \mathbb{C}^n \to \mathbb{C}^n$ in an orthogonal basis $f_1, \ldots, f_n$ such that $f_1, \ldots, f_{t_r}$ is a basis of $\operatorname{Ker}(A - \lambda_1 I) \cdots (A - \lambda_r I)$, $1 \leqslant r \leqslant k$, where $(x - \lambda_1) \cdots (x - \lambda_k)$, $\lambda_1 \succeq \cdots \succeq \lambda_k$, is the minimal polynomial of A; it proves part (a) of Lemma 2.2. Successively equating the blocks of $FS = SF'$ ordered with Eq. (2), we prove part (b). $\square$

By the *canonical part* of the matrix (6) or (7), we mean the matrix (6) or, respectively, the join of blocks F_{ij}, $i \geqslant j$. According to Lemmas 2.1 and 2.2, the canonical part is uniquely determined by the initial matrix A and does not change if A is replaced by a unitarily equivalent or, respectively, similar matrix.

The algorithm for reducing a marked block matrix $A = [A_{ij}]$ to canonical form is as follows:

Let A_{pq} be the first (in the ordering (2)) block of A that changes under admissible transformations (3). Depending on the arrangement of the marked blocks, it is reduced by the transformations of unitary equivalence or similarity. Respectively, we reduce $A = [A_{ij}]$ to the matrix $\tilde{A} = [\tilde{A}_{ij}]$ with $\tilde{A}_{pq}$ of the form (6) or (7), and then restrict ourselves to those admissible transformations with $\tilde{A}$ that preserve the canonical part of $\tilde{A}_{pq}$. As follows from Lemmas 2.1(b) and 2.2(b), it is exactly the admissible transformations with the marked block matrix A' that is obtained in the following way: The block $\tilde{A}_{pq}$ of the form (6) or (7) consists of k horizontal and k vertical strips; we extend this partition to the whole pth horizontal and the whole qth vertical strips of $\tilde{A}$. If new k divisions pass through the marked block $\tilde{A}_{ij}$, we carry out k perpendicular divisions such that $\tilde{A}_{ij}$ is partitioned into $k \times k$ subblocks with square diagonal blocks (they are crossed by the marking line) and repeat this for all new divisions. We additionally mark the subblocks $a_1 I, \dots, a_{k-1} I$ of $\tilde{A}_{pq}$ if it has the form (6). The obtained marked block matrix A' will be called the *derived matrix* of A. Clearly, $A \sim B$ implies $A' \sim B'$.

Let us consider the sequence of derived matrices

$$A^{(0)} := A,\ A',\ A'', \dots, A^{(s)}. \tag{8}$$

This sequence ends with a certain matrix $A^{(s)}, s \geqslant 0$, for which the admissible transformations do not change any of its blocks, i.e, $A^{(s)}$ is equivalent only to itself. Then $A \sim B$ implies $A^{(s)} \sim B^{(s)}$, i.e., $A^{(s)} = B^{(s)}$. Remove from $A^{(s)}$ all additional divisions into subblocks and additional marking lines that have appeared during the reduction of A to $A^{(s)}$. The obtained marked block matrix will be called a *canonical matrix* or the *canonical form of A* and will be denoted by A^{∞}. We have the following theorem.

Theorem 2.3. *Each marked block matrix A is equivalent to the uniquely determined canonical matrix A^{∞}; moreover, $A \sim B$ if and only if $A^{\infty} = B^{\infty}$.*

We will take under consideration the null matrices 0_{0n}, 0_{m0}, and 0_{00} of size $0 \times n$, $m \times 0$, and 0×0, putting for a $p \times q$ matrix M

$$M \oplus 0_{0n} = [M \quad 0_{pn}], \quad M \oplus 0_{m0} = \begin{bmatrix} M \\ 0_{mq} \end{bmatrix}, \quad 0_{m0} \oplus 0_{0n} = 0_{mn}.$$

Respectively, we will consider block matrices with "empty" horizontal and/or vertical strips.

Let $A = [A_{ij}]$ and $B = [B_{ij}]$ $(1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant r)$ be marked block matrices with the same set of indices (i,j) of the marked blocks. By the *block direct sum* of A and B we mean the marked block matrix

$$A \uplus B := [A_{ij} \oplus B_{ij}]$$

with the same disposition of marked blocks. If $T_1 = (R_1, S_1) : A \rightsquigarrow C$ and $T_2 = (R_2, S_2) : B \rightsquigarrow D$ (see (4)), then R_1, R_2 and, respectively, S_1, S_2 are block diagonal matrices with l and, respectively, r diagonal square blocks, and

$$T_1 \uplus T_2 := (R_1 \uplus R_2, S_1 \uplus S_2) : A \uplus B \rightsquigarrow C \uplus D.$$

A marked block matrix A is said to be *indecomposable* if (i) its size other than 0×0, and (ii) $A \sim B \uplus C$ implies B or C has size 0×0.

For every matrices $M_1, \ldots, M_n, N$, we define

$$(M_1, \ldots, M_n) \otimes N := (M_1 \otimes N, \ldots, M_n \otimes N), \tag{9}$$

where $M_i \otimes N$ is obtained from M_i by replacing its entries a with aN.

Theorem 2.4. (a) *Each marked block matrix A is equivalent to a matrix of the form*

$$\begin{aligned} B &= (P_1 \otimes I_{m_1}) \uplus \cdots \uplus (P_t \otimes I_{m_t}) \\ &\sim \underbrace{P_1 \uplus \cdots \uplus P_1}_{m_1 \text{ copies}} \uplus \cdots \uplus \underbrace{P_t \uplus \cdots \uplus P_t}_{m_t \text{ copies}}, \end{aligned}$$

where $P_1, \ldots, P_t$ are non-equivalent indecomposable marked block matrices, uniquely determined up to equivalence (we may take $P_1 = P_1^{\infty}, \ldots, P_t = P_t^{\infty}$), and $m_1, \ldots, m_t$ are uniquely determined natural numbers. Every preserving B admissible transformation $T : B \rightsquigarrow B$ has the form

$$T = (\mathbf{1}_{P_1} \otimes U_1) \uplus \cdots \uplus (\mathbf{1}_{P_t} \otimes U_t),$$

where $\mathbf{1}_{P_i} = (I, I) : P_i \rightsquigarrow P_i$ is the identity transformation of P_i, and U_i is a unitary $m_i \times m_i$ matrix $(1 \leqslant i \leqslant t)$.

(b) *A marked block matrix A of size $\neq 0 \times 0$ is indecomposable if and only if every preserving A admissible transformation $T : A \rightsquigarrow A$ has the form $T = a\mathbf{1}_A$, $a \in \mathbb{C}$, $|a| = 1$.*

(c) *A canonical matrix can be reduced to an equivalent block direct sum of indecomposable canonical matrices using only admissible permutations of rows and columns.*

Proof. (a) We may take $A = A^{\infty}$. Since admissible transformations with $A^{(i)}$, $1 \leqslant i \leqslant s$, (see Eq. (8)) are exactly the admissible transformations with A that preserve the already reduced part of $A^{(i)}$ (preserve $A^{(s)}$ if $i = s$), the set of admissible transformations with $A^{(s)}$ consists of all (R,S) : $A \rightsquigarrow A$. By Eq. (3),

$R = R_1 \oplus \cdots \oplus R_l$ and $S = S_1 \oplus \cdots \oplus S_r$, where $l \times r$ is the number of blocks of A. Since $(R, S) : A^{(s)} \rightsquigarrow A^{(s)}$, we have

$$R_i = U_{f(i,1)} \oplus \cdots \oplus U_{f(i,l_i)} \quad \text{and} \quad S_j = U_{g(j,1)} \oplus \cdots \oplus U_{g(j,r_j)}, \tag{10}$$

where $f(i, \alpha), g(j, \beta) \in \{1, \ldots, t\}$ and $U_1, \ldots, U_t$ are arbitrary unitary matrices of fixed sizes. $A^{(s)}$ differs from A only by additional divisions of its strips into substrips (and by additional marking lines). We transpose substrips within each strip of A to obtain a matrix $B \sim A$ such that, for all $(R, S) : B \rightsquigarrow B$, we have Eq. (10) with $f(i, 1) \leqslant \cdots \leqslant f(i, l_i)$ and $g(j, 1) \leqslant \cdots \leqslant g(j, r_j)$. Clearly, B satisfies (a).

(b) and (c) are obvious. □

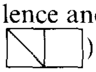
2.2. The structure of canonical matrices

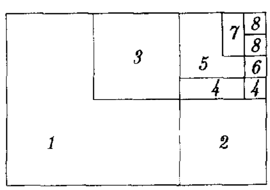
In this section we partition the set of canonical $m \times n$ matrices into disjoint subsets consisting of the canonical matrices with the same "scheme" (the number of such schemes is finite for each size $m \times n$), and show how to construct all the canonical matrices having a given scheme (for matrices under unitary similarity this was made briefly in [4]).

We partition a canonical matrix into zones, which illustrate the reduction process.

Let $A = A^{\infty}$ be a canonical matrix. Then all its derived matrices (8) differ from A only by additional divisions and marking lines. Denote by P_l $(0 \leqslant l < s)$ the first block of $A^{(l)}$ that changes under admissible transformations (it is reduced when we construct $A^{(l+1)}$).

Let $A_{ij}^{(l)}$ be a block of $A^{(l)}$ such that either $A_{ij}^{(l)} \leqslant P_l$ or $l = s$. The admissible transformations with $A^{(l)}$ induce the unitary equivalence or similarity transformations with $A_{ij}^{(l)}$. Respectively, $A_{ij}^{(l)}$ has the form (6) or (7); we denote by $Z(A_{ij}^{(l)})$ its canonical part (see Section 2.1). Defining by induction in l, we call $Z(A_{ij}^{(l)})$ by a *zone* and l by its *depth* if either $l = 0$ or $Z(A_{ij}^{(l)})$ is not contained in a zone of depth $< l$.

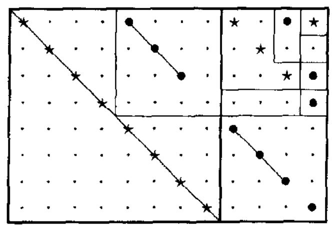
For each zone $Z = Z(A_{ij}^{(l)})$, we put $\mathrm{Bl}(Z) := A_{ij}^{(l)}$ and call Z by an *equivalence* (*similarity*) *zone* if $\mathrm{Bl}(Z)$ is transformed by unitary equivalence (similarity) transformations. Clearly, every canonical matrix A is partitioned into equivalence and similarity zones; for example (for a marked block matrix of the form )

$$A = \left[\begin{array}{cccc|cccc|cccc} i&0&0&0&2&0&0&0&3&0&2&i\\ 0&i&0&0&0&2&0&0&0&3&0&0\\ 0&0&i&0&0&0&2&0&0&0&i&4\\ 0&0&0&i&0&0&0&0&0&0&0&4\\ 0&0&0&0&0&0&0&0&5&0&0&0\\ 0&0&0&0&0&0&0&0&0&5&0&0\\ 0&0&0&0&0&0&0&0&0&0&5&0\\ 0&0&0&0&0&0&0&0&0&0&0&3 \end{array}\right] =$$  (11)

is partitioned into 10 zones, their depths are indicated on the right-hand side of Eq. (11).

Let A be a canonical matrix partitioned into zones. For each similarity zone, we replace all its diagonal elements by stars. For each equivalence zone, we replace all its non-zero elements by circles, and join with a line its circles corresponding to equal elements (this line does not coincide with a marking line because the marking lines connect stars). The other elements of A are zeros, we replace theirs by points. The obtained picture will be called the *scheme* $\mathscr{S}(A)$ of A.

For example, the canonical matrix (11) has the scheme

$S(A) =$ 

Theorem 2.5. *Each canonical matrix $A = [a_{ij}]$ with a given scheme $\mathscr{S} = [s_{ij}]$ can be constructed by successive filling of its zones by numbers starting with the zones of greatest depth as follows: Let Z be a zone of depth $d(Z)$ and let all entries in zones of depth $>d(Z)$ be replaced by numbers. Then we replace all points, circles, and stars of Z, respectively, by zeros, positive real numbers, and complex numbers such that the following conditions hold:*

(1) *Let s_{ij} and $s_{i+1,j+1}$ be circles in Z. Then $a_{ij} = a_{i+1,j+1}$ if s_{ij} and $s_{i+1,j+1}$ are linked by a line, and $a_{ij} > a_{i+1,j+1}$ otherwise.*

(2) *Let $s_{\alpha,\beta}, \dots, s_{\alpha+k,\beta+k}$ be all stars of Z that lie under a certain stair of Z. Then $a_{\alpha,\beta} = \cdots = a_{\alpha+k,\beta+k}$. If $s_{\alpha+k+1,\beta+k+1}, \dots, s_{\alpha+t,\beta+t}$ are all stars of Z that lie under the next stair of Z, then $a_{\alpha,\beta} \succeq a_{\alpha+t,\beta+t}$; moreover, $a_{\alpha,\beta} \succ a_{\alpha+t,\beta+t}$ whenever the columns of the block $[a_{ij} | \alpha \leqslant i \leqslant \alpha + k, \beta + k + 1 \leqslant j \leqslant \beta + t]$ are linearly dependent (this block has been filled by numbers because all its entries are located in zones of depth $>d(Z)$).*

This theorem gives a convenient way to present solutions of unitary matrix problems in small sizes by their sets of schemes. Thus, the list of schemes of canonical 5×5 matrices under unitary similarity was obtained by Klimenko [11].

2.3. Unitarily wild matrix problems

The canonical form problem for pairs of $n \times n$ matrices under simultaneous similarity (i.e., for representations of the quiver $\circlearrowleft\!\bullet\!\circlearrowright$) plays a special role in the theory of (non-unitary) matrix problems. It may be proved that its solution implies the classification of representations of every quiver (and even representations of every finite-dimensional algebra). For this reason, the classification problem for pairs of matrices under simultaneous similarity is used as a yardstick of the complexity; Donovan and Freislich [12] (see also [2]) suggested to name a classification problem *wild* if it contains the problem of simultaneous similarity, and otherwise to name it *tame* (in accordance with the partition of animals into wild and tame ones).

The canonical form problem for an $n \times n$ matrix under unitary similarity (i.e., for unitary representations of the quiver $\circlearrowright$) plays the same role in the theory of unitary matrix problems: it contains the problem of classifying unitary representations of every quiver. For example, the problem of classifying unitary representations of the quiver (5) can be regarded (by Lemma 2.2) as the problem of classifying, up to unitary similarity, matrices of the form:

$$\left[\begin{array}{c|c|c|c|c} 5I & I & A_\lambda & A_\nu & A_\mu \\ \hline 0 & 4I & I & 0 & 0 \\ \hline 0 & 0 & 3I & 0 & 0 \\ \hline 0 & 0 & 0 & 2I & A_\xi \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right].$$

A matrix problem is called *unitarily wild* (or *-wild, see [13]) if it contains the problem of classifying matrices via unitary similarity, and *unitarily tame* otherwise.

For each unitary problem, one has an alternative: to solve it or to prove that it is unitarily wild (and hence is hopeless in a certain sense). In this section we give some examples of such alternatives.

Let us consider the problems of classifying nilpotent linear operators φ, $\varphi^n = 0$, in a unitary space. For $n = 2$ this problem is unitarily tame; the canonical matrix of φ (in the sense of contents in Section 2.1) is

$$\begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix},$$

where D is of the form (6) without zero columns. Indeed, a matrix $F = [F_{ij}]$ of the form Eq. (7) satisfies $F^2 = 0$ only if $k = 2$, $F_{11} = 0$, and $F_{22} = 0$; we can reduce F_{12} to the form (6). For $n > 2$ this problem is unitarily wild since the matrices

$$\begin{bmatrix} 0 & I & X \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & I & Y \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}$$

are unitarily similar if and only if X and Y are unitarily similar (see also [14]).

Let us consider the problem of classifying m-tuples $(p_1, \dots, p_m)$ of projectors $p_i^2 = p_i$ in a unitary space. For $m = 1$ this problem is unitarily time; the canonical matrix of a projector $p = p^2$ was obtained in [15,16]. Of course, it is

$$\begin{bmatrix} I & D \\ 0 & 0 \end{bmatrix},$$

where D is of the form (6), since a matrix $F = [F_{ij}]$ of the form (7) satisfies $F^2 = F$ only if $k = 2$, $F_{11} = I$, and $F_{22} = 0$. As was proved recently in [13], for $m \geqslant 2$ this problem is unitarily wild even if p_1 is an orthoprojector, i.e. $p_1 = p_1^2 = p_1^*$, (since the pairs of idempotent matrices

$$\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} X & I-X \\ X & I-X \end{bmatrix}\right) \quad \text{and} \quad \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} Y & I-Y \\ Y & I-Y \end{bmatrix}\right)$$

are unitarily similar iff X and Y are unitarily similar), or if $p_1 p_2 = p_2 p_1 = 0$.

The problems of classifying the following operators and systems of operators in unitary spaces are unitarily wild:

(a) Pairs of linear operators (φ, ψ) such that $\varphi^2 = \psi^2 = \varphi\psi = \psi\varphi = 0$ since

$$\left(\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\right) \quad \text{and} \quad \left(\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix}\right)$$

are unitarily similar if and only if X and Y are unitarily similar.

(b) Pairs of self-adjoint operators (φ, ψ) because $\varphi + \mathrm{i}\psi$ is an arbitrary operator. The tame–wild dichotomy for satisfying quadratic relation pairs of self-adjoint operators in a Hilbert space was studied in [17].

(c) Pairs of unitary operators (φ, ψ) since $(\mathrm{i}(\varphi + \mathbf{1})(\varphi - \mathbf{1})^{-1},\ \mathrm{i}(\psi + \mathbf{1})(\psi - \mathbf{1})^{-1})$ is a pair of self-adjoint operators (the Cayley transformation).

(d) Partial isometries (i.e., linear operators φ such that $(\varphi^*\varphi)^2 = \varphi^*\varphi$), it was proved in [18].

The problem of classifying unitary representations of a connected quiver Q is unitarily tame if $Q \in \{\bullet,\ \bullet \to \bullet\}$ and unitarily wild otherwise. Indeed, the classification of unitary representations of the quiver $\bullet \to \bullet$ is given by the singular value decomposition (Lemma 2.1). The problem of classifying unitary representations of the quiver $\bullet \to \bullet \leftarrow \bullet$ is unitarily wild because it reduces to the unitary matrix problem for marked block matrices of the form $\begin{array}{|c|c|}\hline \quad & \quad \\ \hline \end{array}$, and two block matrices

$$\begin{array}{|ccc|c:c|}\hline 2I & 0 & 0 & I & X \\ \hdashline 0 & I & 0 & I & I \\ \hdashline 0 & 0 & 0 & I & 0 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|ccc|c:c|}\hline 2I & 0 & 0 & I & Y \\ \hdashline 0 & I & 0 & I & I \\ \hdashline 0 & 0 & 0 & I & 0 \\ \hline \end{array}$$

are equivalent if and only if X and Y are unitarily similar. We can change the direction of an arrow in a quiver by replacing in each representation the corresponding linear mapping by the adjoint one.

Let us consider the problem of classifying n-tuples $(V_1, \dots, V_n)$ of subspaces of a unitary space U up to the following equivalence: $(V_1, \dots, V_n) \sim (V_1', \dots, V_n')$ if there exists an isometry $\varphi : U \to U$ such that $\varphi V_1 = V_1', \dots, \varphi V_n = V_n'$. Fixing an orthogonal basis in U and (non-orthogonal) bases in $V_1, \dots, V_n$, we reduce it to the canonical form problem for block matrices $A = [A_1 | \dots | A_n]$ (the columns of A_i are the basis vectors of V_i and hence are linearly independent) up to unitary transformations of rows of A and elementary (non-unitary) transformations of columns of A_i $(i = 1, \dots, n)$.

For $n = 1$, $A = [A_1]$ reduces to $I \oplus 0_{p0}$ (it follows from Lemma 2.1). For $n = 2$, $A = [A_1 | A_2]$ reduces to

$$\left[\begin{array}{c|cc} I & I & 0 \\ & 0 & D \\ \hline 0 & 0 & I \\ & 0 & 0 \end{array}\right]$$

where D is of the form (6). This block matrix reduces to a block direct sum of matrices $\left[\begin{array}{c|c} 1 & \alpha \\ 0 & 1 \end{array}\right]$ $(\alpha > 0)$,$[1|1]$, $[1|0_{10}]$, $[0_{10}|1]$, $[0_{10}|0_{10}]$. (The problem of classifying pairs of subspaces in a complex or real vector space with scalar product given by a symmetric, or skew-symmetric, or Hermitian form was solved in [19].)

For $n = 3$ this problem is unitarily wild even if we restrict our consideration to the triples (V_1, V_2, V_3) with $V_1 \perp V_2$ since

$$\left[\begin{array}{c|c|c} I & 0 & X \\ 0 & I & Y \\ 0 & 0 & I \end{array}\right] \quad \text{reduces to} \quad \left[\begin{array}{c|c|c} I & 0 & X' \\ 0 & I & Y' \\ 0 & 0 & I \end{array}\right]$$

if and only if (X, Y) and (X', Y') determine isometric unitary representations of the quiver $\bullet \leftarrow \bullet \rightarrow \bullet$. An analogous statement was proved in [20] and [13]: the problem of classifying triples (p_1, p_2, p_3) of orthoprojectors $p_i = p_i^2 = p_i^*$ in a unitary space is unitarily wild even if $p_1 p_2 = p_2 p_1 = 0$; such a triple determines, in one-to-one manner, a triple (V_1, V_2, V_3) with $V_1 \perp V_2$ by means of $V_i = \operatorname{Im} p_i$.

3. Unitary representations of a quiver

From now on, Q denotes a quiver with vertices $1, \dots, p$ and arrows $\alpha_1, \dots, \alpha_q$. A *unitary representation* of dimension $d = (d_1, \dots, d_p) \in \mathbb{N}_0^p$ (in short, a *unitary d representation*) will be given by assigning a matrix

$A_\alpha \in \mathbb{C}^{d_j \times d_i}$ to each arrow $\alpha : i \to j$, i.e., by the sequence $A = (A_{\alpha_1}, \dots, A_{\alpha_q})$ (assigning to each vertex i the unitary vector space $\mathbb{C}^{d_i}$ with scalar product $(x, y) = \bar{x}_1 y_1 + \dots + \bar{x}_{d_i} y_{d_i}$, we obtain a unitary representation in the sense of Section 1).

An *isometry* $A \tilde{\to} B$ of d representations A and B (an *autometry* if $A = B$) is given by a sequence $S = (S_1, \dots, S_p)$ of unitary $d_i \times d_i$ matrices S_i such that $S_j A_\alpha = B_\alpha S_i$ for each arrow $\alpha : i \to j$; we say also that B is obtained from A by *admissible transformations* and write $A \simeq B$. An autometry $S : A \tilde{\to} A$ is *scalar* if $S = a\mathbf{1}_A$, where $a \in \mathbb{C}$, $\mathbf{1}_A = (I_{d_1}, \dots, I_{d_p})$.

For two sequences of matrices $M = (M_1, \dots, M_t)$ and $N = (N_1, \dots, N_t)$, we denote $M \oplus N = (M_1 \oplus N_1, \dots, M_t \oplus N_t)$. A unitary d representation A of Q is *indecomposable* if (i) $d \neq (0, \dots, 0)$ and (ii) $A \simeq B \oplus C$ implies B or C has dimension $(0, \dots, 0)$.

3.1. Canonical representations

Let A be a unitary representation of Q. Using the algorithm (see Section 2.1), we reduce A_{α_1} to its canonical form $A^\infty_{\alpha_1}$, then restrict the set of admissible transformations with A to those that preserve $A^\infty_{\alpha_1}$ (it gives certain unitary matrix problems for $A_{\alpha_2}, \dots, A_{\alpha_q}$ with partitions them into blocks) and reduce A_{α_2} to its canonical form $A^\infty_{\alpha_2}$, and so on. The obtained representation $A^\infty = (A^\infty_{\alpha_1}, \dots, A^\infty_{\alpha_q})$ (we omit the marking lines) will be called a *canonical representation* of the quiver Q; the sequence of the schemes $\mathscr{S}(A^\infty) = (\mathscr{S}(A^\infty_{\alpha_1}), \dots, \mathscr{S}(A^\infty_{\alpha_q}))$ will be called the *scheme* of A^∞.

Clearly, $A \simeq A^\infty$ and $A \simeq B$ if and only if $A^\infty = B^\infty$.

Theorem 3.1. (a) *Every unitary representation is isometric to a representation of the form*

$$\begin{aligned} B &= (P_1 \otimes I_{m_1}) \oplus \dots \oplus (P_t \otimes I_{m_t}) \\ &\simeq \underbrace{P_1 \oplus \dots \oplus P_1}_{m_1 \text{ copies}} \oplus \dots \oplus \underbrace{P_t \oplus \dots \oplus P_t}_{m_t \text{ copies}} \end{aligned}$$

(*see Eq.* (9)), *where $P_1, \dots, P_t$ are non-isometric indecomposable representations, uniquely determined up to isometry, and $m_1, \dots, m_t$ are uniquely determined natural numbers. Every autometry $S : B \tilde{\to} B$ has the form*

$$S = (\mathbf{1}_{P_1} \otimes U_1) \oplus \dots \oplus (\mathbf{1}_{P_t} \otimes U_t),$$

where U_i is a unitary $m_i \times m_i$ matrix $(1 \leqslant i \leqslant t)$.

(b) *A unitary representation of dimension $\neq (0, \dots, 0)$ is indecomposable if and only if all its autometries are scalar.*

Proof. Analogously Eq. (5), the matrices of every unitary representation A of Q can be accommodated in a block diagonal matrix $A = \mathrm{diag}(A_{\alpha_q}, A_{\alpha_{q-1}}, \ldots, A_{\alpha_1}, 0, \ldots, 0)$ with a certain set of marked blocks such that the admissible transformations with A correspond to the admissible transformations with $M(A)$. Then $M(A^\infty) = M(A)^\infty$ and we can apply Theorem 2.4. □

3.2. The set of dimensions of indecomposable unitary representations

We will use the following designations: $M_Q = [m_{ij}]$ is the $p \times p$ matrix, in which m_{ij} is the number of arrows $i \to j$ and $i \leftarrow j$ of the quiver Q; $\mathrm{supp}(z)$ is the full subquiver of Q with the vertex set $\{i \mid z_i \neq 0\}$ for each $z \in \mathbb{N}_0^p$; $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{N}_0^p$ with 1 in the ith position. Denote by $D(Q)$ the subset of $\mathbb{N}_0^p$ consists of $e_1, \ldots, e_p$, all $e_i + e_j$ with $m_{ij} = 1$, and all non-zero z with connected $\mathrm{supp}(z) \notin \{\bullet,\ \bullet \to \bullet\}$ such that $zM_Q \geqslant z$. In this section we prove the following theorem.

Theorem 3.2. *$D(Q)$ is the set of dimensions of indecomposable unitary representations of a quiver Q.*

Put $\Delta_i(z) := \sum_j m_{ij} z_j$ for $z \in \mathbb{N}_0^p$ and $1 \leqslant i \leqslant p$. Then $zM_Q = (\Delta_1(z), \ldots, \Delta_p(z))$.

Lemma 3.3. *$D(Q)$ satisfies the following conditions:*

(i) *If $z \in D(Q)$ and* $\mathrm{supp}(z) \notin \{\bullet,\ \bullet \to \bullet\circlearrowright\}$ *then $zM_Q > z$.*

(ii) *If $z, u \in D(Q)$ and $z < u$, then there exists i such that $z + e_i \leqslant u$ and $z + e_i \in D(Q)$.*

Proof. (i) Let $z \in D(Q)$, $\mathrm{supp}(z) \notin \{\bullet,\ \bullet \to \bullet\circlearrowright\}$, and $zM_Q = z$. Fix i such that $z_i = \max\{z_1, \ldots, z_p\}$. Then $m_{ij} \neq 0$ for a certain $j \neq i$. Since $z_j = \Delta_j(z) \geqslant m_{ij} z_i \geqslant z_i$, $z_i = z_j$, $m_{ij} = 1$, and $m_{kj} z_k = 0$ for all $k \neq i$. Taking z_j and z_i instead of z_i and z_j, we have $m_{ki} z_k = 0$ for all $k \neq j$. Hence $supp(z) = \bullet \to \bullet$, a contradiction.

(ii) Let $z, u \in D(Q)$ and $z < u$. If $\mathrm{supp}(z) \neq \mathrm{supp}(u)$, then there exists a non-zero m_{ij} with $i \in \mathrm{supp}(u) \setminus \mathrm{supp}(z)$ and $j \in \mathrm{supp}(u) \cap \mathrm{supp}(z)$. The $z + e_i$ satisfies the requirements.

We may assume that $\mathrm{supp}(z) = \mathrm{supp}(u) = Q$. Then $Q \notin \{\bullet, \bullet \to \bullet\}$. Fix a vertex l such that $z_l < u_l$. We will suppose that $\Delta_l(z) = z_l$ and $m_{ll} = 0$ (otherwise $z + e_l$ satisfies the requirements).

Assume first that $z_l \leqslant z_j$ for some $m_{lj} \neq 0$. The condition $\Delta_l(z) = z_l$ implies $z_l = z_j$, $m_{lj} = 1$, and $m_{lk} = 0$ for all $k \neq j$. Hence $z_j = z_l < u_l \leqslant \Delta_l(u) = u_j$. Since $Q \neq \bullet \to \bullet$, $m_{jk} \neq 0$ for some $k \neq l$, and we can take $z + e_j$. Next, let $z_l > z_j$ (and hence $z + e_j \in D(Q)$) for all non-zero m_{lj}. If $z_j = u_j$ for all $m_{lj} \neq 0$, then $u_l \leqslant \Delta_l(u) = \Delta_l(z) = z_l$, a contradiction. Hence $z_j < u_j$ for a certain $m_{lj} \neq 0$, and we can take $z + e_j$. □

Lemma 3.4. *If A is a unitary d representation of Q and $d \notin D(Q)$, then A is decomposable.*

Proof. Assume to the contrary, that A is indecomposable. Then $\text{supp}(d)$ is connected; Lemma 2.1 and $d \notin D(Q)$ imply $\text{supp}(d) \notin \{\bullet, \bullet \to \bullet\}$ and $dM_Q \ngeqslant d$, i.e., there exists l such that $\Delta_l(d) < d_l$. Then $m_{ll} = 0$ and we can assume that there are no arrows starting from l (otherwise we replace each arrow $\alpha : l \to i$ by $\alpha^* : i \to l$, simultaneously replacing A_α by the adjoint matrix).

Let $\alpha, \beta, \ldots, \gamma$ be all arrows stopping at l; combine the corresponding them matrices of A into a single $d_l \times \Delta_l(d)$ matrix $[A_\alpha|A_\beta|\ldots|A_\gamma]$. The number of its rows is greater than the number of its columns; making a zero row by unitary transformations of rows, we obtain $A \simeq B \oplus P$, where P is the zero representation of dimension e_l, a contradiction. □

Lemma 3.5. *If there exists an indecomposable unitary z representation and $z_i < \Delta_i(z)$ for a certain vertex i, then there exists an indecomposable unitary $z + e_i$ representation.*

Proof. Let A be an indecomposable unitary z representation and $z_1 < \Delta_1(z)$. We can assume that each starting from the vertex 1 arrow is a loop (replacing each $\lambda : 1 \to j$, $j \neq 1$, by $\lambda^* : j \to 1$ and, respectively, A_λ by $A^*_{\lambda^*}$).

(1) Assume first that there is a loop $\alpha : 1 \to 1$ and define a unitary $z + e_1$ representation H in which H_α is the nilpotent Jordan cell of size $(z_1 + 1) \times (z_1 + 1)$, $H_\beta := A_\beta \oplus 0_{11}$ for each $\beta : 1 \to 1$, $\beta \neq \alpha$; $H_\gamma : = A_\gamma \oplus 0_{10}$ for each $\gamma : j \to 1$, $j \neq 1$; and $H_\delta := A_\delta$ for each $\delta : j \to k$, $k \neq 1$. The representation H is indecomposable.

Indeed, let A^- and H^- denote the restrictions of A and H on the subquiver $Q^- := Q \setminus \alpha$. By Theorem 3.1(a), we may assume that

$$A^- = (P_1 \otimes I_{m_1}) \oplus \cdots \oplus (P_t \otimes I_{m_t}),$$

where $P_1, \ldots, P_t$ are non-isometric indecomposable representations of Q^-, P_1 is the zero representation of dimension e_1, and $m_1 \geqslant 0$, $m_2 > 0, \ldots, m_t > 0$. Clearly,

$$H^- = (P_1 \otimes I_{m_1+1}) \oplus (P_2 \otimes I_{m_2}) \oplus \cdots \oplus (P_t \otimes I_{m_t}).$$

Let $S = (S_1, S_2, \ldots) : H \tilde{\to} H$. Since $S : H^- \tilde{\to} H^-$, by Theorem 3.1(a)

$$S = (\mathbf{1}_{P_1} \otimes U_1) \oplus \cdots \oplus (\mathbf{1}_{P_t} \otimes U_t),$$
$$S_1 = U_1^{(d_{11})} \oplus \cdots \oplus U_t^{(d_{1t})}, \tag{12}$$

where $(d_{1j}, \ldots, d_{pj}) = \dim(P_j)$ and U_j is a unitary matrix $(1 \leqslant j \leqslant t)$. Since $S_1 H_\alpha = H_\alpha S_1$, H_α is a Jordan cell and S_1 is a unitary matrix, we have

$S_1 = aI$, $a \in \mathbb{C}$. The representation A is indecomposable, so that $d_{1j} \neq 0$ and by Eq. (12) $U_j = aI$ for all $1 \leqslant j \leqslant t$. Hence $S = a\mathbf{1}_H$ and H is indecomposable by Theorem 3.1(b).

(2) There remains the case $m_{11} = 0$. Let $\alpha_1 : j_1 \to 1, \ldots, \alpha_l : j_l \to 1$ be all the arrows stopping at 1. We denote by A^- the restriction of A on the subquiver $Q^- := Q \setminus \{1; \alpha_1, \ldots, \alpha_l\}$. By Theorem 3.1(a), we may assume that

$$A^- = (P_1 \otimes I_{m_1}) \oplus \cdots \oplus (P_t \otimes I_{m_t}),$$

where $P_1, \ldots, P_t$ are non-isometric indecomposable unitary representations of Q^-.

Let $(S_2, \ldots, S_p) : A^- \tilde{\to} A^-$. By Theorem 3.1(a)

$$S_i = (I_{d_{i1}} \otimes U_1) \oplus \cdots \oplus (I_{d_{it}} \otimes U_t),$$

where $(d_{2j}, \ldots, d_{pj}) = \dim P_j$. For an arbitrary unitary $z_1 \times z_1$ matrix S_1, we define $\tilde{A}$ by means of $S = (S_1, S_2, \ldots, S_p) : A \tilde{\to} \tilde{A}$ (then $\tilde{A}^- = A^-$). Taking into account that $\tilde{A}_{\alpha_\tau} = S_1^{-1} A_{\alpha_\tau} S_{j_\tau}$ and partitioning the sets of columns of every A_{α_τ} and $\tilde{A}_{\alpha_\tau}$ in the same manner as S_{j_τ}, we obtain $B := [A_{\alpha_1} | \ldots | A_{\alpha_l}] = [B_1 | \ldots | B_m]$ and $\tilde{B} := [\tilde{A}_{\alpha_1} | \ldots | \tilde{A}_{\alpha_l}] = [\tilde{B}_1 | \ldots | \tilde{B}_m]$, where $m = \sum_{\tau=1}^{l} (d_{j_\tau 1} + \cdots + d_{j_\tau t})$ and $\tilde{B}_i = S_1^{-1} B_i U_{f(i)}$ for a certain $f(i) \in \{1, \ldots, t\}$.

Let $z_1 \times u_i$ be the size of B_i and put $r_i = \operatorname{rank}[B_1 | \ldots | B_{i-1} | B_{i+1} | \ldots | B_m]$. B is a $z_1 \times \Delta_1(z)$ matrix and $z_1 < \Delta_1(z)$, so $z_1 - r_i < u_i$ for a certain i. Since S_1 and $U_{f(i)}$ are arbitrary unitary matrices, by Lemma 2.1(a) there exists S such that

$$\tilde{B} = \left[\begin{array}{c|c|c|c|c|c|c} C_1 & \cdots & C_{i-1} & C_i & C_{i+1} & \cdots & C_m \\ 0 & \cdots & 0 & D & 0 & \cdots & 0 \end{array}\right],$$

where the rows of $[C_1 | \ldots | C_{i-1} | C_{i+1} | \ldots | C_m]$ are linearly independent and D is a $(z_1 - r_i) \times u_i$ matrix of the form $\operatorname{diag}(a_1, \ldots, a_n) \oplus 0_{kh}$ with real $a_1 \geqslant \cdots \geqslant a_n > 0$. Since $\tilde{A}$ is indecomposable and $z_1 - r_i < u_i$, we have $k = 0$ and $h > 0$.

Let a_{n+1} be a real number such that $a_n > a_{n+1} > 0$. The replacement D by $D' = \operatorname{diag}(a_1, \ldots, a_n, a_{n+1}) \oplus 0_{0,h-1}$ changes $\tilde{B}$ to a new matrix $\tilde{B}'$ and $\tilde{A}$ to a new representation H of dimension $z + e_1$.

Let $R : H \tilde{\to} H$. Since H and A coincide on Q^- and $(R_2, \ldots, R_p) : A^- \tilde{\to} A^-$, by Theorem 3.1(a) the matrices $R_2, \ldots, R_p$ have the form $R_j = (I_{d_{j1}} \otimes V_1) \oplus \cdots \oplus (I_{d_{j\tau}} \otimes V_\tau)$ with unitary $V_1, \ldots, V_\tau$. By $R_1^{-1} \tilde{B}' (R_{j_1} \oplus \cdots \oplus R_{j_l}) = \tilde{B}'$, R_1 has the form $R_{11} \oplus R_{12}$, where $R_{12}^{-1} D' V_{f(i)} = D'$. Lemma 2.1 implies $R_{12} = R_{13} \oplus [c]$. Putting $\tilde{R}_1 = R_{11} \oplus R_{13}$ and $\tilde{R}_j = R_j$ $(j > 1)$, we have $\tilde{R} : \tilde{A} \tilde{\to} \tilde{A}$. By Theorem 3.1(b), $\tilde{R}_j = aI$ $(1 \leqslant j \leqslant p)$ for some $a \in \mathbb{C}$, so $V_j = aI$ $(1 \leqslant j \leqslant \tau)$. In particular, $V_{f(i)} = aI$ and, since $R_{12}^{-1} D' V_{f(i)} = D'$, $c = a$ and $R_1 = aI$. Therefore $R = a\mathbf{1}_H$ and H is indecomposable by Theorem 3.1(b). $\square$

Proof of Theorem 3.2. Let $U(Q)$ denote the set of dimensions of indecomposable unitary representations of Q. Lemma 3.4 implies $U(Q) \subset D(Q)$.

Let $u \in D(Q)$. Then $u_i \neq 0$ for a certain i. Using Lemma 3.3(ii), we select a sequence $u_1 := e_i, u_2, \ldots, u_t := u$ in $D(Q)$ such that $u_2 - u_1, \ldots, u_t - u_{t-1} \in \{e_1, \ldots, e_p\}$. By Lemma 3.5, $\{u_1, \ldots, u_t\} \subset U(Q)$ and $D(Q) \subset U(Q)$. $\square$

3.3. The number of parameters in an indecomposable unitary representation

By the *number of real (complex) parameters* of a unitary representation A we mean the number of circles (stars) in the scheme $\mathscr{S}(A^\infty)$. Recall that to circles correspond positive real numbers in A^∞, and to stars correspond complex numbers; the other entries in A^∞ are zeros.

Kac [7], Theorem C, proved that the maximal number of parameters in an indecomposable (non-unitary) representation of dimension d over an algebraically closed field is $1 - \varphi_Q(d)$, where

$$\varphi_Q(x) = x_1^2 + \cdots + x_p^2 - \sum_{i,j=1}^{p} m_{ij} x_i x_j$$

is a $\mathbb{Z}$-bilinear form called the *Tits form* of the quiver Q, and m_{ij} is the number of arrows $i \longrightarrow j$ and $i \longleftarrow j$.

We say that a zone (see Section 2.2) is in *general position* if all its diagonal entries are distinct and, if it is an equivalence zone, non-zero. A unitary representation A is said to be in *general position* if all zones in A^∞ are in general position.

Theorem 3.6. (a) *For every $d \in D(Q)$ (see Section* 3.2) *there exists an indecomposable canonical unitary d representation of general position, its scheme is uniquely determined by d.*

(b) *An indecomposable unitary d representation A has* $\sum d_i - 1$ *real parameters and at most*

$$1 - \varphi_Q(d) + \frac{1}{2}\sum d_i(d_i - 1)$$

complex parameters; this number is reached if and only if A is in general position.

Proof. We consider the set of zones of a canonical unitary representation $A^\infty = (A^\infty_{\alpha_1}, \ldots, A^\infty_{\alpha_q})$ as linearly ordered:

$$Z_1 < Z_2 \text{ if } i_1 < i_2; \quad \text{or } i_1 = i_2 \text{ and } l_1 < l_2; \quad \text{or } i_1 = i_2, \\ l_1 = l_2 \quad \text{and } \mathrm{Bl}(Z_1) < \mathrm{Bl}(Z_2) \tag{13}$$

(see Eq. (2) and contents of Section 2.2); where Z_k $(k = 1, 2)$ is a zone of depth l_k in $A^{\infty}_{\alpha_{i_k}}$.

(a) Let $d \in D(Q)$. By Theorem 3.2, there exists an indecomposable unitary d representation A. Let A be not in general position, and let Z be the first (in the sense of Eq. (13) zone of A^{∞} that is not in general position. Changing diagonal entries of Z, we transform it into a zone $\tilde{Z}$ of general position and A^{∞} into a new representation $\tilde{A}$. This exchange narrows down the set of admissible transformations that preserve all zones $\leqslant Z$, and, by Theorem 3.1(b), A^{∞} has only scalar autometries (as an indecomposable representation), therefore, $\tilde{A}$ has only scalar autometries and is indecomposable too.

If $\tilde{A}$ is not in general position, we repeat this process for it, and so on, until we obtain an indecomposable d representation B of general position. Its scheme is uniquely determined since, for each zone Z of B, the set of admissible transformations that preserve all zones $\leqslant Z$ (and hence the matrix problem for the remaining part of B) does not depend on diagonal entries of Z such that it is in general position.

(b) Let A be an indecomposable canonical d representation, and Z be its zone or the symbol ∞. Denote by $J(Z)$ the set of all isometries of the form $S : A \tilde{\to} \tilde{A}$ that preserve all zones $< Z$ (all zones if $Z = \infty$). As follows from the algorithms of Sections 2.1 and 3.1, $J(Z)$ consists of all sequences of the form $S = (S_1, \ldots, S_p)$, where

$$S_i = U_{\sigma(i1)} \oplus U_{\sigma(i2)} \oplus \cdots \oplus U_{\sigma(it_i)},$$

σ: $\{(ij) \mid 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant t_i\} \to \{1, \ldots, t\}$ is a fixed surjection, and $U_1, \ldots, U_t$ are arbitrary unitary matrices of fixed sizes $m_1 \times m_1, \ldots, m_t \times m_t$ (we will write $S = S(U_1, \ldots, U_t)$).

Put

$$\Delta_1(Z) = m_1 + \cdots + m_t, \qquad \Delta_2(Z) = m_1^2 + \cdots + m_t^2.$$

Let $Z \neq \infty$ and Z' be the first zone after Z ($Z' = \infty$ if Z is the last zone of A). We will prove that

$$\Delta_1(Z) - \Delta_1(Z') = n^{\bullet}(Z), \tag{14}$$

$$\Delta_2(Z) - \Delta_2(Z') \leqslant 2n(Z) - n^{\bullet}(Z) - 2n^{\star}(Z) \tag{15}$$

and that the equality in Eq. (15) holds if and only if Z is a zone of general position; where $n(Z)$ is the number of entries in Z, and $n^{\bullet}(Z)$ (resp., $n^{\star}(Z)$) is the number of circles (resp., stars) that correspond to the diagonal entries of Z.

As follows from the algorithms of Sections 2.1 and 3.1, the block $\mathrm{Bl}(Z)$ is reduced by transformations

$$\mathrm{Bl}(Z) \mapsto U_i^{-1} \mathrm{Bl}(Z) U_j, \tag{16}$$

where $S = S(U_1, \ldots, U_t) \in J(Z)$ and i and j are determined by Z; moreover, this S is contained in $J(Z')$ if and only if transformation (16) preserves Z.

(1) Let $i \neq j$, say, $i = 1$ and $j = 2$. Then, by Lemma 2.1, $Z = \mathrm{Bl}(Z) = a_1 I_{r_1} \oplus \cdots \oplus a_{k-1} I_{r_{k-1}} \oplus 0_{xy}$, $r_\alpha \geqslant 1$, $x \geqslant 0$ and $y \geqslant 0$. The transformation (16) preserves Z if and only if

$$U_1 = V_1 \oplus \cdots \oplus V_k \quad \text{and} \quad U_2 = V_1 \oplus \cdots \oplus V_{k-1} \oplus V_{k+1}, \tag{17}$$

where $V_1, \ldots, V_{k+1}$ are unitary matrices of size $r_1 \times r_1, \ldots, r_{k-1} \times r_{k-1}$, $x \times x$, $y \times y$. Hence, $J(Z')$ consists of all $S \in J(Z)$ with U_1 and U_2 of the form (17), that is, $S = S(V_1, \ldots V_{k+1}, U_3 \ldots U_t)$. Therefore, $\Delta_1(Z') = r_1 + \cdots + r_{k-1} + x + y + m_3 + \cdots + m_t$, $\Delta_2(Z') = r_1^2 + \cdots + r_{k-1}^2 + x^2 + y^2 + m_3^2 + \cdots + m_t^2$. By Eq. (16), $\mathrm{Bl}(Z)$ has size $m_1 \times m_2$, $m_1 = r_1 + \cdots + r_{k-1} + x$, $m_2 = r_1 + \cdots + r_{k-1} + y$, so $n(Z) = m_1 m_2$, $n^\bullet(Z) = r_1 + \cdots + r_{k-1}$, $n^\star(Z) = 0$. We have $\Delta_1(Z) - \Delta_1(Z') = r_1 + \cdots + r_{k-1} = n^\bullet(Z)$, $\Delta_2(Z) - \Delta_2(Z') = (r_1 + \cdots + r_{k-1} + x)^2 + (r_1 + \cdots + r_{k-1} + y)^2 - r_1^2 - \cdots - r_{k-1}^2 - x^2 - y^2 = [(r_1 + \cdots + r_{k-1} + x) - (r_1 + \cdots + r_{k-1} + y)]^2 + 2(r_1 + \cdots + r_{k-1} + x)(r_1 + \cdots + r_{k-1} + y) - r_1^2 - \cdots - r_{k-1}^2 - x^2 - y^2 = (x - y)^2 + 2n(Z) - r_1^2 - \cdots - r_{k-1}^2 - x^2 - y^2 = -2xy + 2n(Z) - r_1^2 - \cdots - r_{k-1}^2 \leqslant 2n(Z) - r_1 - \cdots - r_{k-1} = 2n(Z) - n^\bullet(Z)$. Moreover, we have the equality if and only if $r_1 = \cdots = r_{k-1} = 1$ and $xy = 0$, i.e, Z is in general position.

(2) Let $i = j$, say, $i = j = 1$. Then, by Lemma 2.1, $\mathrm{Bl}(Z) = [F_{\alpha\beta}]$, where $F_{\alpha\beta} = 0$ for $\alpha > \beta$, $F_{\alpha\alpha} = \lambda_\alpha I_{r_\alpha}$, $r_\alpha \geqslant 1$, and $r_1 + \cdots + r_k = m_1$. The transformation (16) preserves $Z = \{F_{\alpha\beta} \mid \alpha \leqslant \beta\}$ if and only if $U_1 = V_1 \oplus \cdots \oplus V_k$, where $V_1, \ldots, V_k$ are unitary matrices of size $r_1 \times r_1, \ldots, r_k \times r_k$. Hence, $J(Z')$ consists of all $S \in J(Z)$ with $U_1 = V_1 \oplus \cdots \oplus V_k$, i.e., $S = S(V_1, \ldots, V_k, U_2, \ldots, U_t)$. So $\Delta_1(Z) - \Delta_1(Z') = m_1 - r_1 - \cdots - r_k = 0 = n^\bullet(Z)$, $\Delta_2(Z) - \Delta_2(Z') = (r_1 + \cdots + r_k)^2 - r_1^2 - \cdots - r_k^2 = 2\sum_{\alpha \leqslant \beta} r_\alpha r_\beta - 2(r_1^2 + \cdots + r_k^2) = 2n(Z) - 2(r_1^2 + \cdots + r_k^2) \leqslant 2n(Z) - 2(r_1 + \cdots + r_k) = 2n(Z) - 2n^\star(Z)$. Moreover, we have the equality if and only if $r_1 = \cdots = r_k = 1$, i.e., Z is in general position.

Hence, relations (14) and (15) hold.

Let $Z_1 < \cdots < Z_r$ be all zones of A ordered by Eq. (13) and let $\dim A = (d_1, \ldots, d_p)$. Then $Z_i' = Z_{i+1}$ for $i < r$, and $Z_r' = \infty$. Since $J(Z_1)$ consists of all sequences $S = (S_1, \ldots, S_p)$ of unitary $d_1 \times d_1, \ldots, d_p \times d_p$ matrices, $\Delta_1(Z_1) = d_1 + \cdots + d_p$, $\Delta_2(Z_1) = d_1^2 + \cdots + d_p^2$. Since A is indecomposable, by Theorem 3.1(b) $J(\infty)$ consists of all sequences $S = \lambda(I_{d_1}, \ldots, I_{d_p})$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$, so $S = S([\lambda])$ and $\Delta_1(\infty) = \Delta_2(\infty) = 1$. By Eq. (14), $d_1 + \cdots + d_p - 1 = \Delta_1(Z_1) - \Delta_1(\infty) = \sum_{i=1}^r (\Delta_1(Z_i) - \Delta_1(Z_i')) = \sum_{i=1}^r n^\bullet(Z_i)$ is the number of circles in $S(A^\infty)$, i.e., the number of real parameters in A.

By Eq. (15), $d_1^2 + \cdots + d_p^2 - 1 = \Delta_2(Z_1) - \Delta_2(\infty) = \sum_{i=1}^r (\Delta_2(Z_i) - \Delta_2(Z_i')) \leqslant 2\sum_{i=1}^r n(Z_i) - \sum_{i=1}^r n^\bullet(Z_i) - 2\sum_{i=1}^r n^\star(Z_i)$. But $\sum_{i=1}^r n(Z_i)$ is the number of entries in $A_{\alpha_1}, \ldots, A_{\alpha_q}$, hence, is equal to $\sum_{i,j=1}^p m_{ij} d_i d_j$, where m_{ij} is the number of arrows $i \to j$ and $i \leftarrow j$; $n^\star(A) := \sum_{i=1}^r n^\star(Z_i)$ is the number of complex parameters in A. Therefore, $n^\star(A) \leqslant \sum m_{ij} d_i d_j - \frac{1}{2}(\sum d_i^2 - 1) - \frac{1}{2}(\sum d_i - 1) =$

$1-[\sum d_i^2-\sum m_{ij}\ d_id_j]+\frac{1}{2}\sum(d_i^2-d_i)=1-\varphi_Q(d)+\frac{1}{2}\sum d_i(d_i-1)$. We have the equality if and only if all Z_i are in general position, i.e., A is in general position. □

The proof implies the following corollary.

Corollary 3.7. (a) *Let $d\in D(Q)$ and $m=\max\{d_1,\dots,d_p\}$. Then there exists an indecomposable canonical d representation of general position with entries in $\{0,\ 1,\dots,m\}$.*

(b) *A decomposable unitary d representation has less than $\sum d_i-1$ real parameters and less than $1-\varphi_Q(d)+\frac{1}{2}\sum d_i(d_i-1)$ complex parameters.*

Proof. (a) follows from Theorems 3.6(a) and 2.5.

(b) This statement is proved as Theorem 3.6(b), but, in the last two paragraphs of its proof, we must use $\Delta_1(\infty)>1$ and $\Delta_2(\infty)>1$ instead of $\Delta_1(\infty)=\Delta_2(\infty)=1$ since $J(\infty)$ contains a non-scalar authometry in the case of a decomposable representation A. □

4. Euclidean representations of a quiver

Let Q be a quiver with vertices $1,\dots,p$ and arrows $\alpha_1,\dots,\alpha_q$. A *Euclidean representation* A of dimension $d=(d_1,\dots,d_p)\in\mathbb{N}_0^p$ will be given by assigning a matrix $A_\alpha\in\mathbb{R}^{d_j\times d_i}$ to each arrow $\alpha:i\to j$; i.e., by the sequence $A=(A_{\alpha_1},\dots,A_{\alpha_q})$. An $\mathbb{R}$-*isometry* $A\tilde{\to}_{\mathbb{R}}B$ of Euclidean representations A and B will be given by a sequence $S=(S_1,\dots,S_p)$ of real orthogonal matrices such that $S_jA_\alpha=B_\alpha S_i$ for each arrow $\alpha:i\to j$ (analogously, $R:A\tilde{\to}_{\mathbb{C}}B$ denotes an isometry in the sense of Section 3). A Euclidean d representation A is said to be $\mathbb{R}$-*indecomposable* if (i) $d\neq(0,\dots,0)$ and (ii) $A\simeq_{\mathbb{R}}B\oplus C$ implies B or C has dimension $(0,\dots,0)$.

For a sequence of complex matrices $M=(M_1,\dots,M_n)$, we define the *conjugate sequence* $\bar{M}=(\bar{M}_1,\dots,\bar{M}_n)$, the *transposed sequence* $M^{\mathrm{T}}=(M_1^{\mathrm{T}},\dots,M_n^{\mathrm{T}})$, and the *adjoint sequence* $M^*=\bar{M}^{\mathrm{T}}$. Clearly, the Euclidean representations are the selfconjugate unitary representations.

4.1. A reduction to unitary representations

We give a standard reduction of the problem of classifying Euclidean representations to the problem of classifying unitary representations.

Let $\mathrm{ind}(Q)$ and $\mathrm{ind}_{\mathbb{R}}(Q)$ denote complete systems of non-isometric indecomposable unitary representations and non-$\mathbb{R}$-isometric $\mathbb{R}$-indecomposable Euclidean representations, respectively. Let us replace each representation in $\mathrm{ind}(Q)$ that is isometric to a Euclidean representation by a Euclidean one, and denote the set of such by $\text{ind}_0(Q)$ (if $A \in \text{ind}(Q)$ and $S : A \tilde{\rightarrow}_{\mathbb{C}} \bar{A}$, then A is isometric to a Euclidean representation if and only if $S^T = S$; see Theorem 4.4). Denote by $\text{ind}_1(Q)$ the set consisting of all representations from $\text{ind}(Q)$ that are isometric to their conjugates, but not to a self-conjugate, together with one representation from each pair $\{A, B\} \subset \text{ind}(Q)$ such that $A \not\simeq_{\mathbb{C}} \bar{A} \simeq_{\mathbb{C}} B$.

For a unitary d representation $A = (A_{\alpha_1}, \ldots, A_{\alpha_q})$, we define the Euclidean $2d$ representation $A^{\mathbb{R}} = (A^{\mathbb{R}}_{\alpha_1}, \ldots, A^{\mathbb{R}}_{\alpha_q})$, where $A^{\mathbb{R}}_{\alpha}$, is obtained from A_α by replacing each entry $a + bi$ $(a, b \in \mathbb{R})$ with

$$\begin{matrix} a & b \\ -b & a \end{matrix}.$$

Since

$$U^{-1}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad U = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}$$

with the unitary

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix},$$

we have

$$A^{\mathbb{R}} \simeq_{\mathbb{C}} A \oplus \bar{A}. \tag{18}$$

Theorem 4.1. (a) *Let A and B be Euclidean representations of a quiver Q. Then $A \simeq_{\mathbb{R}} B$ if and only if $A \simeq_{\mathbb{C}} B$.*

(b) *Every Euclidean representation is $\mathbb{R}$-isometric to a direct sum of indecomposable Euclidean representations, uniquely determined up to $\mathbb{R}$-isometry of summands. Moreover,*

$$\text{ind}_{\mathbb{R}}(Q) = \text{ind}_0(Q) \cup \{A^{\mathbb{R}} \mid A \in \text{ind}_1(Q)\}. \tag{19}$$

(c) *The set of dimensions of $\mathbb{R}$-indecomposable Euclidean representations of Q coincides with the set of dimensions of indecomposable unitary representations and is equal to $D(Q)$ (see Section* 3.2).

A homomorphism ($\mathbb{R}$-*homomorphism*) $S : A \to B$ of representations A and B of Q is a sequence of complex (real) matrices $S = (S_1, \ldots, S_p)$ such that $S_j A_\alpha = B_\alpha S_i$ for each arrow $\alpha : i \to j$. Clearly, an isomorphism S is an isometry if and only if $S^* = S^{-1}$.

Lemma 4.2. *The following properties are equivalent for a unitary (Euclidean) representation A:*

(i) *A is decomposable ($\mathbb{R}$-decomposable).*

(ii) *There exists an endomorphism ($\mathbb{R}$-endomorphism) $F : A \to A$ such that $F = F^* = F^2 \notin \{\mathbf{0}_A, \mathbf{1}_A\}$.*

(iii) *There exists a non-scalar self-adjoint endomorphism ($\mathbb{R}$-endomorphism) $S = S^* : A \to A$.*

Proof. (i) $\Rightarrow$ (ii) Let $S : A \tilde{\to} B \oplus C$ be an isometry of unitary representations ($\mathbb{R}$-isometry of Euclidean representations) and $B \neq 0 \neq C$. Then $F := S^{-1}(\mathbf{1}_B \oplus \mathbf{0}_C)S : A \to A$ satisfies (ii).

(ii) $\Rightarrow$ (iii) Put $S := F$.

(iii) $\Rightarrow$ (i) Since every S_i in S is a Hermitian (resp., real symmetric) matrix, there exists a unitary (real orthogonal) matrix U_i such that $R_i := U_i S_i U_i^{-1} = \operatorname{diag}(a_{i1}, \dots, a_{it_i})$, where $a_{ij} \in \mathbb{R}$ and $a_{i1} \geqslant \cdots \geqslant a_{it_i}$. Define the unitary (Euclidean) representation B by means of the isometry $U := (U_1, \dots, U_p) : A \tilde{\to} B$. Then $R := USU^{-1} : B \tilde{\to} B$ is an isometry and $R = a_1 \mathbb{I}_1 \oplus \cdots \oplus a_t \mathbb{I}_t$, where $a_1 > \cdots > a_t$, $\mathbb{I}_i = (I_{n_{i1}}, \dots, I_{n_{ip}})$, $n_{ij} \geqslant 0$. Clearly, $B = B_1 \oplus \cdots \oplus B_t$, where $\dim(B_i) = (n_{i1}, \dots, n_{ip})$. $\square$

Proof of Theorem 4.1. (A) We first prove the statement (a) for an $\mathbb{R}$-*indecomposable* Euclidean representation A. Let $S = \Phi + i\Psi : A \tilde{\to}_{\mathbb{C}} B$, where Φ and Ψ are real matrices and B is a Euclidean representation. Then Φ and Ψ are $\mathbb{R}$-homomorphisms $A \to B$. Since $\mathbf{1}_A = S^*S = (\Phi^{\mathrm{T}} - \mathrm{i}\Psi^{\mathrm{T}})(\Phi + \mathrm{i}\Psi) = (\Phi^{\mathrm{T}}\Phi + \Psi^{\mathrm{T}}\Psi) + \mathrm{i}(\Phi^{\mathrm{T}}\Psi - \Psi^{\mathrm{T}}\Phi)$, $\Phi^{\mathrm{T}}\Phi + \Psi^{\mathrm{T}}\Psi = \mathbf{1}_A$. By Lemma 4.2, the self-adjoint $\mathbb{R}$-endomorphisms $\Phi^{\mathrm{T}}\Phi$ and $\Psi^{\mathrm{T}}\Psi$ are scalar, i.e., $\Phi^{\mathrm{T}}\Phi = \lambda\mathbf{1}_A$, $\Psi^{\mathrm{T}}\Psi = \mu\mathbf{1}_A$, and $\lambda + \mu = 1$. Obviously, λ and μ are non-negative real numbers. For definiteness, $\lambda > 0$, then $\lambda^{-1/2}\Phi : A \tilde{\to}_{\mathbb{R}} B$.

(B) Let A be an $\mathbb{R}$-indecomposable Euclidean representation that is decomposable as a unitary representation. We prove that $A \simeq_{\mathbb{R}} B^{\mathbb{R}} \simeq_{\mathbb{C}} B \oplus \bar{B}$, where B is an indecomposable unitary representation that is not isometric to a Euclidean representation.

Indeed, by Lemma 4.2 there exists an endomorphism $F : A \to A$ such that $F = F^* = F^2 \notin \{\mathbf{0}_A, \mathbf{1}_A\}$. Let $F = \Phi + \mathrm{i}\Psi$, where Φ and Ψ are sequences of real matrices. Since $F = F^* = \Phi^{\mathrm{T}} - \mathrm{i}\Psi^{\mathrm{T}}$, it follows that $\Phi = \Phi^{\mathrm{T}}$ and $\Psi = -\Psi^{\mathrm{T}}$. By Lemma 4.2, the endomorphism Φ is scalar, i.e., $\Phi = \lambda\mathbf{1}_A$, $\lambda \in \mathbb{R}$. If $\lambda = 0$, then $\mathrm{i}\Psi = F = F^2 = -\Psi^2$ and $\Psi = \mathbf{0}_A$, a contradiction.

Hence $\lambda \neq 0$. Since $F = F^2 = (\lambda\mathbf{1}_A + \mathrm{i}\Psi)^2 = (\lambda^2\mathbf{1}_A - \Psi^2) + 2\lambda\mathrm{i}\Psi$, we have $\lambda^2\mathbf{1}_A - \Psi^2 = \lambda\mathbf{1}_A$ and $2\lambda\Psi = \Psi$. The condition $F \neq \mathbf{1}_A$ implies $\Psi \neq \mathbf{0}_A$, $\lambda = \frac{1}{2}$, and $\Psi^2 = -\frac{1}{4}\mathbf{1}_A$.

By [21], Section 4.4, Exercise 25, every non-singular skew-symmetric real matrix is real orthogonally similar to a direct sum of matrices of the form

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}, \quad a > 0.$$

Since $\Psi^{\mathrm{T}} = -\Psi$ and $\Psi^2 = -\frac{1}{4}\mathbf{1}_A$, there exists a sequence S of real orthogonal matrices such that $S\Psi S^{-1} = \frac{1}{2}\mathbb{I} \otimes \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (see Eq. (9)), where $\mathbb{I} = (I, \dots, I)$. Put

$$G := SFS^{-1} = \frac{1}{2}\mathbb{I} \otimes \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

Define the Euclidean representation C by means of $S: A \tilde{\to}_{\mathbb{R}} C$. Then $G: C \to C$ is an $\mathbb{R}$-endomorphism. It follows from the form of G and the definition of homomorphisms, that $C = B^{\mathbb{R}}$ for a certain B. If B is a decomposable unitary representation, say, $B \simeq_{\mathbb{C}} X \oplus Y$, then by Eq. (18) $A \simeq_{\mathbb{R}} B^{\mathbb{R}} \simeq_{\mathbb{C}} B \oplus \bar{B} \simeq_{\mathbb{C}} X \oplus Y \oplus \bar{X} \oplus \bar{Y} \simeq_{\mathbb{C}} X^{\mathbb{R}} \oplus Y^{\mathbb{R}}$, by (A) $A \simeq_{\mathbb{R}} X^{\mathbb{R}} \oplus Y^{\mathbb{R}}$, a contradiction. If B is isometric to a Euclidean representation, say, $B \simeq_{\mathbb{C}} D = \bar{D}$, then $A \simeq_{\mathbb{R}} B^{\mathbb{R}} \simeq_{\mathbb{C}} B \oplus \bar{B} \simeq_{\mathbb{C}} D \oplus D$, by (A) $A \simeq_{\mathbb{R}} D \oplus D$, a contradiction. It proves (B).

(a)–(b). Let A and B be Euclidean representations, $A \simeq_{\mathbb{R}} B$, $A \simeq_{\mathbb{R}} A_1 \oplus \cdots \oplus A_l$ and $B \simeq_{\mathbb{R}} B_1 \oplus \cdots \oplus B_r$, where A_i and B_j are $\mathbb{R}$-indecomposable. From (B) and Theorem 3.1(a), $l = r$ and, after a permutation of summands, $A_i \simeq_{\mathbb{C}} B_i$. By (A), $A_i \simeq_{\mathbb{R}} B_i$. The equality (19) is obvious.

(c). By Corollary 3.7(a), there exists an $\mathbb{R}$-indecomposable Euclidean representation (with entries in $\mathbb{N}_0$) of dimension z for every $z \in D(Q)$. Conversely, let A be an $\mathbb{R}$-indecomposable Euclidean representation. If A is indecomposable as a unitary representation, then by Theorem 3.2 $\dim(A) \in D(Q)$. Otherwise by (B) $A \simeq_{\mathbb{C}} B \oplus \bar{B}$, where B is an indecomposable unitary representation, i.e., $d := \dim(B) \in D(Q)$. Since B is not isometric to a Euclidean representation, $\mathrm{supp}(d) \notin \{\bullet,\ \bullet \to \bullet\}$. Applying twice the definition of $D(Q)$ (see Section 3.2), we have $dM_Q \geqslant d$, $2dM_Q \geqslant 2d$, and $\dim(A) = 2d \in D(Q)$. $\square$

4.2. *Unitary representations that are isometric to Euclidean representations*

Theorem 4.1(b) reduces the problem of classifying Euclidean representations of a quiver Q to the following two problems:
1. to classify unitary representations of Q (i.e., to construct the set $\mathrm{ind}(Q)$);
2. to determine for each $A \in \mathrm{ind}(Q)$ whether it is isometric to a Euclidean representation and to construct that representation.

In this section we consider the second problem.

Lemma 4.3. (a) *If S is a symmetric unitary matrix, then there exists a unitary matrix U such that $S = U^{\mathrm{T}}U$.*

(b) *If S is a skew-symmetric unitary matrix, then there exists a unitary matrix U such that*

$$S = U^{\mathrm{T}}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)U.$$

Proof. Analogous statement for a non-unitary matrix S is in [21], Section 4.4, Corollary 4.4.4 and Exercise 26. The condition of unitarity makes its proof much more easy. We give it sketchily since an explicit form of U is needed for the applications of the next theorem.

Given a symmetric (skew-symmetric) unitary matrix S_n with rows $s_1, \dots, s_n$. If $s_1 \neq e_1 := (1, 0, \dots, 0)$, we take a unitary matrix U_n with rows $u_1, \dots, u_n$ such that $u_1 = \alpha(e_1 + s_1)$, $\alpha \in \mathbb{C}$, (resp., $\mathbb{C}u_1 + \mathbb{C}u_2 = \mathbb{C}e_1 + \mathbb{C}s_1$). Then $\bar{u}_1 S_n = \alpha(e_1 + \bar{s}_1)S_n = \alpha(e_1 S_n + \bar{s}_1 S_n) = \alpha(s_1 + e_1) = u_1 = e_1 U_n$, hence $(U_n^{-1})^{\mathrm{T}} S_n U_n^{-1} = \bar{U}_n S_n U_n^{-1} = [1] \oplus S_{n-1}$ (resp., then

$$\bar{U}_n S_n U_n^{-1} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} \oplus S_{n-2}, \quad |\beta| = 1;$$

replacing u_2 by βu_2, we make $\beta = 1$). If $s_1 = e_1$, we have $S_n = [1] \oplus S_{n-1}$, $U_n := I_n$. We repeat this procedure until we obtain the required $U := U_n (I_1 \oplus U_{n-1})(I_2 \oplus U_{n-2}) \cdots (I_{n-1} \oplus U_1)$ (resp., $U := U_n(I_2 \oplus U_{n-2}) \cdots$). $\square$

Theorem 4.4. (1) *Let A be a unitary representation and $A \not\simeq_{\mathbb{C}} \bar{A}$. Then A is not isometric to a Euclidean representation.*

(2) *Let A be an indecomposable unitary representation and $S : A \tilde{\to}_{\mathbb{C}} \bar{A}$.*

(a) *If $S = S^{\mathrm{T}}$, then A is isometric to a Euclidean representation B given by $U : A \tilde{\to}_{\mathbb{C}} B$, where $U_1, \dots, U_p$ are arbitrary unitary matrices such that $U_i^{\mathrm{T}} U_i = S_i$ (they exist by Lemma* 4.3(a)).

(b) *If $S \neq S^{\mathrm{T}}$, then $S = -S^{\mathrm{T}}$ and A is not isometric to a Euclidean representation but is isometric to a unitary representation C of the form*

$$\begin{bmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{bmatrix}$$

given by $V : A \tilde{\to}_{\mathbb{C}} C$, where $V_1, \dots, V_p$ are arbitrary unitary matrices such that

$$V_i^{\mathrm{T}} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} V_i = S_i$$

(*they exist by Lemma* 4.3(b)).

Proof. (1) Let $R : A \tilde{\to}_{\mathbb{C}} B$, where B is a Euclidean representation. Then $R^{\mathrm{T}} = \bar{R}^{-1} : \bar{B} \tilde{\to}_{\mathbb{C}} \bar{A}$ and $H := R^{\mathrm{T}} R : A \tilde{\to}_{\mathbb{C}} \bar{A}$ (observe that $H = H^{\mathrm{T}}$).

(2) Let A be an indecomposable unitary representation and $S : A \tilde{\to}_{\mathbb{C}} \bar{A}$. Then $\bar{S}S : A \tilde{\to}_{\mathbb{C}} A$, by Theorem 3.1(b) $\bar{S}S = \lambda \mathbf{1}_A$, $S = \lambda \bar{S}^{-1} = \lambda S^{\mathrm{T}} = \lambda(\lambda S^{\mathrm{T}}) = \lambda^2 S$, and $\lambda \in \{1, -1\}$.

(a) Let $\lambda = 1$, $U : A \tilde{\to}_{\mathbb{C}} B$ and $U^{\mathrm{T}} U = S$. Then $U = (U^{\mathrm{T}})^{-1} S = \bar{U} S : A \tilde{\to}_{\mathbb{C}} \bar{B}$ and $B = \bar{B}$.

(b) Let $\lambda = -1$. Then A is not isometric to a Euclidean representation (otherwise, by (1) there exists $H = H^{\mathrm{T}} : A \tilde{\to}_{\mathbb{C}} \bar{A}$; by Theorem 3.1(b) $S^{-1} H = \mu \mathbf{1}_A$ and $H^{\mathrm{T}} = \mu S^{\mathrm{T}} = -\mu S = -H$, a contradiction). Let $V : A \tilde{\to}_{\mathbb{C}} C$, where

$$V_i^{\mathrm{T}} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} V_i = S_i.$$

Then $\bar{V} S V^{-1} : C \tilde{\to}_{\mathbb{C}} \bar{C}$. If α is an arrow of Q, then

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} C_\alpha = \bar{C}_\alpha \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

and C_α is of the form $\begin{bmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{bmatrix}$. $\square$

Applying this theorem to unitary representations of the quiver $\circlearrowright$, we obtain the following corollary.

Corollary 4.5. *Let A be a complex matrix that is not unitarily similar to a direct sum of matrices, and let $S^{-1} A S = \bar{A}$ for a unitary matrix S (such S exists if A is unitarily similar to a real matrix). Then A is unitarily similar to a real matrix if and only if S is symmetric.*

4.3. Friedland's problem

Friedland [10] proved that the *characteristic polynomial*

$$\chi_A(\lambda, x) = |\lambda I - (A_1 + A_2 x)|$$

determines a finite number of similarity classes for almost all pairs $A = (A_1, A_2)$ of complex symmetric $n \times n$ matrices under complex orthogonal similarity. In Problem 11.11 he asked if the characteristic polynomial determines a finite number of similarity classes for pairs of real symmetric $n \times n$ matrices under real orthogonal similarity. The next theorem gives a negative solution of this problem.

Theorem 4.6. *There exists an infinite set of real orthogonally non-similar pairs of real symmetric 6×6 matrices having the same characteristic polynomial.*

Proof. Pairs of Hermitian matrices

$$A(a) = \left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\mathrm{i}a & \sqrt{1-2a^2} \\ \mathrm{i}a & 0 & a \\ \sqrt{1-2a^2} & a & 0 \end{bmatrix} \right)$$

have the same characteristic polynomial $\chi_{A(a)}(\lambda, x) = \lambda^3 - 3\lambda^2 + (2 - x^2)\lambda + x^2$ and are not unitarily similar for all $a \in \mathbb{R}$, $0 < a < 1/\sqrt{2}$, since they determine indecomposable canonical unitary representations of the quiver $\circlearrowleft\bullet\circlearrowright$. The corresponding pairs of real symmetric 6×6 matrices $A(a)^{\mathbb{R}}$, $0 < a < 1/\sqrt{2}$, have the same characteristic polynomial $\chi_{A(a)^{\mathbb{R}}}(\lambda, x) = \chi_{A(a)}(\lambda, x)^2$ and are not real orthogonally similar by Theorem 4.1(a). $\square$

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