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On the combinatorial invariance of Kazhdan–Lusztig polynomials

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Abstract

In this paper, we solve the conjecture about the combinatorial invariance of Kazhdan–Lusztig polynomials for the first open cases, showing that it is true for intervals of length 5 and 6 in the symmetric group. We also obtain explicit formulas for the R-polynomials and for the Kazhdan–Lusztig polynomials associated with any interval of length 5 in any Coxeter group, showing in particular what they look like in the symmetric group.

Keywords: Coxeter group; Symmetric group; Bruhat order; Kazhdan–Lusztig polynomial; Combinatorial invariance conjecture

1. Introduction

In [9] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials with integer coefficients, indexed by pairs of elements of W, which have become known as the Kazhdan–Lusztig polynomials of W. They are related to the algebraic geometry and topology of Schubert varieties and also play a crucial role in representation theory (see, e.g., [6, Chapter 7], [1, Chapter 5] and the references cited there). In order to prove the existence of these polynomials, Kazhdan and Lusztig used another family of polynomials which arise from the multiplicative structure of the Hecke algebra associated with W. These are known as the R-polynomials of W. The well-known combinatorial invariance conjecture, made by Lusztig and independently by

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Dyer more than twenty years ago, states that the Kazhdan–Lusztig polynomial associated with a pair (x, y), supposedly depends only on the poset structure of the interval $[x, y]$. The conjecture is equivalent to the same statement for the R-polynomial and it is known to be true for intervals up to length 4 and for lattices. Recently, it has been proved to hold for lower intervals, that is, intervals $[e, y]$, where e is the identity of W (see [2]).

In this paper, we solve the conjecture for the first open cases, showing that it is true for intervals of length 5 and 6 in the symmetric group. The main tools are a pictorial way of representing the Bruhat order in the symmetric group, namely the diagram of a pair of permutations, introduced in [8] and developed in [7] and an algorithm for computing the R-polynomial associated with (x, y), based on the diagram of (x, y), introduced in [7]. We also obtain explicit formulas for the R-polynomials and for the Kazhdan–Lusztig polynomials associated with any interval of length 5 in any Coxeter group and in particular in the symmetric group.

The structure of the paper is as follows. Some basic definitions and results are collected in Section 2. In Section 3, we give some preliminary results, showing how from the diagram of (x, y) it is possible to get information about the poset structure of $[x, y]$ and about the R-polynomial associated with (x, y). In Section 4, we state and prove the first main result of the paper, concerning the combinatorial invariance of Kazhdan–Lusztig polynomials. Finally, in Section 5, we obtain the explicit formulas for the R-polynomials and for the Kazhdan–Lusztig polynomials.

2. Preliminaries

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$. For $n \in \mathbf{N}$, let $[n] = \{1, 2, \ldots, n\}$ and for $n, m \in \mathbf{N}$, with $n \leqslant m$, let $[n, m] = \{n, n+1, \ldots, m\}$.

We refer to [10] for general poset theory. Given a poset P, we denote by $\lhd$ the covering relation. The *Hasse diagram* of P is the directed graph having P as vertex set and such that there is an edge from x to y if and only if $x \lhd y$. We denote by $-P$ the poset having the same elements of P but the reverse order and we say that P and $-P$ belong to *dual* isomorphism classes. Given $x, y \in P$, with $x < y$, we set $[x, y] = \{z \in P\colon\ x \leqslant z \leqslant y\}$, and call it an *interval* of P. An *atom* (respectively, *coatom*) of $[x, y]$ is an element $z \in [x, y]$ such that $x \lhd z$ (respectively, $z \lhd y$). We denote by $a(x, y)$ and $c(x, y)$, respectively, the number of atoms and coatoms of $[x, y]$. We also introduce the *capacity* of the interval $[x, y]$:

$$\operatorname{cap}(x, y) = \min\bigl\{a(x, y), c(x, y)\bigr\}.$$

We refer to [6] for basic notions about Coxeter groups. Given a Coxeter group W, with set of generators S, the set of *reflections* is

$$T = \bigl\{wsw^{-1}\colon\ w \in W,\ s \in S\bigr\}.$$

Given $x \in W$, the *length* of x, denoted by $\ell(x)$, is the minimal k such that x can be written as a product of k generators, whereas the *absolute length* of x, denoted by $a\ell(x)$, is the minimal k such that x can be written as a product of k reflections. The *Bruhat graph* of W, denoted by $\mathrm{BG}(W)$ (or simply BG), is the directed graph having W as vertex set and such that there is an edge $x \to y$ if and only if $y = xt$, with $t \in T$, and $\ell(x) < \ell(y)$. We label the edge by the reflection t and write

$$x \xrightarrow{t} y.$$

A *Bruhat path* is a (directed) path in the Bruhat graph of W. The *Bruhat order* of W is the partial order which is the transitive closure of BG: given $x, y \in W$, $x \leqslant y$ in the Bruhat order if and only if there is Bruhat path from x to y

$$x = x_0 \to x_1 \to \cdots \to x_k = y.$$

It is known that W, partially ordered by the Bruhat order, is a graded poset with rank function given by the length. Given $x, y \in W$, with $x < y$, we set $\ell(x, y) = \ell(y) - \ell(x)$, and call it the *length* of the pair (x, y). Note that $\ell(x, y)$ is the distance between x and y in the Hasse diagram of $[x, y]$. In [7] we introduced the *absolute length* of the pair (x, y), denoted by $a\ell(x, y)$, which is the (directed) distance from x to y in BG.

We follow [1, Chapter 5] for the definition of Kazhdan–Lusztig polynomials and R-polynomials.

Theorem 2.1. *There exists a unique family of polynomials* $\{R_{x,y}(q)\}_{x,y\in W} \subseteq \mathbf{Z}[q]$ *satisfying the following conditions*:

(1) $R_{x,y}(q) = 0$, *if* $x \nleqslant y$;
(2) $R_{x,y}(q) = 1$, *if* $x = y$;
(3) *if* $x < y$ *and* $s \in S$ *is such that* $ys \lhd y$ *then*

$$R_{x,y}(q) = \begin{cases} R_{xs,ys}(q), & \text{if } xs \lhd x, \\ q R_{xs,ys}(q) + (q-1) R_{x,ys}(q), & \text{if } xs \rhd x. \end{cases}$$

The existence of such a family is a consequence of the invertibility of certain basis elements of the Hecke algebra $\mathcal{H}$ of W and is proved in [6, Sections 7.4, 7.5]. The polynomials whose existence and uniqueness are guaranteed by Theorem 2.1 are called the *R-polynomials* of W. Theorem 2.1 can be used to compute the polynomials $\{R_{x,y}(q)\}_{x,y\in W}$, by induction on $\ell(y)$.

Theorem 2.2. *There exists a unique family of polynomials* $\{P_{x,y}(q)\}_{x,y\in W} \subseteq \mathbf{Z}[q]$ *satisfying the following conditions*:

(1) $P_{x,y}(q) = 0$, *if* $x \nleqslant y$;
(2) $P_{x,y}(q) = 1$, *if* $x = y$;
(3) *if* $x < y$ *then* $\deg(P_{x,y}(q)) < \ell(x, y)/2$ *and*

$$q^{\ell(x,y)} P_{x,y}\big(q^{-1}\big) - P_{x,y}(q) = \sum_{x<z\leqslant y} R_{x,z}(q)\, P_{z,y}(q).$$

A proof of Theorem 2.2 appears in [6, Sections 7.9–7.11]. The polynomials whose existence and uniqueness are guaranteed by Theorem 2.2 are called the *Kazhdan–Lusztig polynomials* of W. By Theorem 2.2, knowing the R-polynomials is equivalent to knowing the Kazhdan–Lusztig polynomials. In fact, part (3) can be recursively used to compute one family from the other, by induction on $\ell(x, y)$.

The R-polynomials satisfy the following relation (see, e.g., [1, Exercise 5.11]).

Proposition 2.3. *Let* $x, y \in W$, *with* $x < y$. *Then*

$$\sum_{x\leqslant z\leqslant y} (-1)^{\ell(x,z)} R_{x,z}(q) R_{z,y}(q) = 0.$$

Note that Proposition 2.3 allows to compute the R-polynomial associated with an interval $[x, y]$ of even length ℓ, once the R-polynomials associated with the subintervals of $[x, y]$ of length at most $\ell - 1$ are known.

In order to give a combinatorial interpretation of the R-polynomials, another family of polynomials, known as the *$\tilde{R}$-polynomials*, has been introduced. The following is [1, Proposition 5.3.1].

Proposition 2.4. *Let $x, y \in W$, with $x < y$. Then there is a unique polynomial $\tilde{R}_{x,y}(q) \in \mathbf{Z}_{\geqslant 0}[q]$ such that*

$$R_{x,y}(q) = q^{\ell(x,y)/2}\tilde{R}_{x,y}\big(q^{1/2} - q^{-1/2}\big).$$

The advantage of the $\tilde{R}$-polynomials over the R-polynomials is that they have nonnegative integer coefficients. Indeed, there is a nice combinatorial interpretation for them. We refer to [1, Section 5.2] for basic notions about reflection orderings. Given $x, y \in W$, with $x < y$, we denote by $\mathrm{BP}(x, y)$ the set of all Bruhat paths from x to y. The *length* of $\Delta = (x_0, x_1, \ldots, x_k) \in \mathrm{BP}(x, y)$, denoted by $|\Delta|$, is the number k of its edges. Now let $\prec$ be a fixed reflection ordering on the set T of reflections. A path $\Delta = (x_0, x_1, \ldots, x_k) \in \mathrm{BP}(x, y)$, with

$$x_0 \xrightarrow{t_1} x_1 \xrightarrow{t_2} \cdots \xrightarrow{t_k} x_k,$$

is said to be *increasing* with respect to $\prec$ if $t_1 \prec t_2 \prec \cdots \prec t_k$. We denote by $\mathrm{BP}^{\prec}(x, y)$ the set of all paths in $\mathrm{BP}(x, y)$ increasing with respect to $\prec$. The following is a refinement of a result of Dyer (see [4] and [7, Corollary 2.6]).

Theorem 2.5. *Let W be a Coxeter group and let $x, y \in W$, with $x < y$. Set $\ell = \ell(x, y)$ and $a\ell = a\ell(x, y)$. Then*

$$\tilde{R}_{x,y}(q) = q^{\ell} + c_{\ell-2}\, q^{\ell-2} + \cdots + c_{a\ell+2}q^{a\ell+2} + c_{a\ell}q^{a\ell},$$

where

$$c_k = \big|\big\{\Delta \in \mathrm{BP}^{\prec}(x, y)\colon\ |\Delta| = k\big\}\big| \geqslant 1,$$

for every $k \in [a\ell, \ell - 2]$, $k \equiv \ell$ (mod 2).

In [3] Dyer proved that the relation $x \to y$ is determined by the poset structure of $[x, y]$. Combining that with the definition of absolute length of a pair, we have the following.

Theorem 2.6. *The absolute length of a pair is a combinatorial invariant. That is, $a\ell(x, y)$ depends only on the poset structure of the interval* $[x, y]$.

We denote by S_n the *symmetric group* over n elements, that is the set of all bijections of $[n]$ onto itself, and call its elements *permutations*. The symmetric group S_n is known to be a Coxeter group, with generators given by the simple transpositions $(i, i + 1)$, for $i \in [n - 1]$. To denote a permutation $x \in S_n$ we use the *one-line notation*: we write $x = x_1x_2 \ldots x_n$, to mean that $x(i) = x_i$ for all $i \in [n]$. The *diagram* of a permutation $x \in S_n$ is the subset of $\mathbf{N}^2$ defined by

$$\mathrm{Diag}(x) = \big\{\big(i, x(i)\big)\colon\ i \in [n]\big\}.$$

For $x \in S_n$ and $(h, k) \in [n]^2$, we set

$$x[h, k] = \big|\big\{i \in [n]\colon\ i \leqslant h,\ x(i) \geqslant k\big\}\big| \tag{1}$$

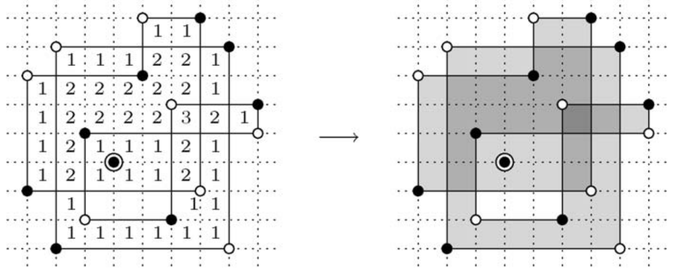


Fig. 1. Diagram of a pair of permutations.

and given $x, y \in S_n$ and $(h, k) \in [n]^2$, we set

$$(x, y)[h, k] = y(h, k) - x(h, k). \tag{2}$$

Then, we have the following well-known characterization of the Bruhat order in the symmetric group (see, e.g., [1, Theorem 2.1.5]).

Theorem 2.7. *Let $x, y \in S_n$. Then*

$$x \leqslant y \quad \Leftrightarrow \quad (x, y)[h, k] \geqslant 0, \quad \textit{for every } (h, k) \in [n]^2.$$

Following [7], we extend the notation in (1) and (2) to every $(h, k) \in \mathbf{R}^2$ and call the mapping $(h, k) \mapsto (x, y)[h, k]$, which associates with every $(h, k) \in \mathbf{R}^2$ the integer $(x, y)[h, k]$, the *multiplicity mapping* of the pair (x, y). Theorem 2.7 can be reformulated as follows:

$$x \leqslant y \quad \Leftrightarrow \quad (x, y)[h, k] \geqslant 0, \quad \text{for every } (h, k) \in \mathbf{R}^2.$$

Then, the *diagram* of the pair (x, y) is the collection of:

(1) the diagram of x;
(2) the diagram of y;
(3) the multiplicity mapping $(h, k) \mapsto (x, y)[h, k]$.

We pictorially represent the diagram of a pair (x, y) with the following convention: the diagram of x (respectively, y) is denoted by black dots (respectively, white dots) and, if $x < y$, then the mapping $(h, k) \mapsto (x, y)[h, k]$ is represented by coloring the preimages of different positive integers with different levels of gray, with the rule that with a lower integer corresponds a lighter gray. In Fig. 1, the diagram of the pair (x, y), where $x = 315472986$ and $y = 782496315$, is illustrated.

Let $x, y \in S_n$, with $x < y$. The *support* of the pair (x, y) is

$$\Omega(x, y) = \left\{(h, k) \in \mathbf{R}^2\colon\ (x, y)[h, k] > 0\right\},$$

and the *support index set* of (x, y) is

$$I_\Omega(x, y) = \left\{i \in [n]\colon\ \bigl(i, x(i)\bigr) \in \overline{\Omega(x, y)}\right\},$$

where $\overline{\Omega(x, y)}$ denotes the (topological) closure of the set $\Omega(x, y)$. A pair (x, y) is said to have *full support* if $I_\Omega(x, y) = [n]$. For instance, the pair whose diagram is shown in Fig. 1 has full support. A *fixed point* of (x, y) is a fixed point of $x^{-1}y$, that is an index $i \in [n]$ such that $x(i) = y(i)$. We set

$$\mathrm{Fix}(x,y)=\{i\in[n]\colon\ x(i)=y(i)\},\qquad \mathrm{fix}(x,y)=\left|\mathrm{Fix}(x,y)\right|,$$
$$\mathrm{Fix}_\Omega(x,y)=\mathrm{Fix}(x,y)\cap I_\Omega(x,y),\qquad \mathrm{fix}_\Omega(x,y)=\left|\mathrm{Fix}_\Omega(x,y)\right|.$$

As already pointed out in [7], the diagram of a pair of permutations (x,y) gives information about the atoms and the coatoms of the interval $[x,y]$. We recall that, given $x\in S_n$, a *free rise* of x is a pair $(i,j)\in[n]^2$, with $i<j$ and $x(i)<x(j)$, such that there is no $k\in[n]$, with $i<k<j$ and $x(i)<x(k)<x(j)$. Symmetrically, we can define a *free inversion* of y, as a pair $(i,j)\in[n]^2$, with $i<j$ and $y(i)>y(j)$, such that there is no $k\in[n]$, with $i<k<j$ and $y(i)>y(k)>y(j)$.

Given $a,b,c,d\in[n]$, with $a<b$ and $c<d$, we set

$$\mathrm{Rect}(a,b,c,d)=\{(h,k)\in\mathbf{R}^2\colon\ a\leqslant h<b,\ c<k\leqslant d\}.$$

Now, let $x,y\in S_n$, with $x<y$. A free rise (i,j) of x is said to be *good* with respect to y if $\mathrm{Rect}(i,j,x(i),x(j))\subseteq\Omega(x,y)$. Similarly, a free inversion (i,j) of y is said to be *good* with respect to x if $\mathrm{Rect}(i,j,y(j),y(i))\subseteq\Omega(x,y)$. Then, we have the following characterizations of atoms and coatoms [7, Propositions 3.4, 3.5].

Proposition 2.8. *Let $x,y\in S_n$, with $x<y$. Then z is an atom of $[x,y]$ if and only if $z=x(i,j)$, where (i,j) is a free rise of x which is good with respect to y. Symmetrically, w is a coatom of $[x,y]$ if and only if $w=y(i,j)$, where (i,j) is a free inversion of y which is good with respect to x.*

By Proposition 2.8, if z is an atom of $[x,y]$, then $z=x(i,j)$, with (i,j) a free rise of x which is good with respect to y. Note that in this case the diagram of (z,y) is obtained from that of (x,y) by "removing" the rectangle $\mathrm{Rect}(i,j,x(i),x(j))$ and a similar consideration holds for coatoms. This means that the structure of the interval $[x,y]$, as an abstract poset, reflects a process of "unmounting" the diagram of (x,y), by throwing away rectangles step-by-step.

In [7] we described an algorithm, the *stair method*, which can be applied to produce all the increasing paths between two permutations in the Bruhat graph of the symmetric group, and thus to compute the $\tilde{R}$-polynomial associated with the two permutations. In particular, we derived explicit formulas for the $\tilde{R}$-polynomials, for some general classes of pairs (x,y). Given $x,y\in S_n$, with $x<y$, we say that

(1) (x,y) has the 01-*multiplicity property* if
$$(x,y)[h,k]\in\{0,1\}\quad\text{for every }(h,k)\in\mathbf{R}^2;$$
(2) (x,y) is *simple* if it has the 01-multiplicity property and $\mathrm{Fix}(x,y)=\emptyset$;
(3) (x,y) is a *permutaomino* if it is simple and $\Omega(x,y)$ is connected.

Then, we have the following [7, Theorem 6.2].

Theorem 2.9. *Let $x,y\in S_n$, with $x<y$.*

(1) *If (x,y) is simple then*
$$\tilde{R}_{x,y}(q)=q^{\ell(x,y)}.$$
(2) *If (x,y) has the* 01*-multiplicity property, then*
$$\tilde{R}_{x,y}(q)=(q^2+1)^{\mathrm{fix}_\Omega(x,y)}q^{\ell(x,y)-2\,\mathrm{fix}_\Omega(x,y)}.$$

3. Preliminary results

In this section, we show how from the diagram of a pair of permutations (x, y) it is possible to get further information about the poset structure of the interval $[x, y]$ and about the $\tilde{R}$-polynomial associated with (x, y).

3.1. Symmetries

In every Coxeter group, it is known that the mapping $x \mapsto x^{-1}$ is an automorphism of the Bruhat order. Also, finite Coxeter groups always have a maximum, usually denoted w_0, and the mappings $x \mapsto xw_0$ and $x \mapsto w_0x$ are anti-automorphisms. It follows that $x \mapsto w_0xw_0$ is an automorphism.

In the symmetric group, these facts can be described in a nice pictorial way. In fact, the maximum of S_n is $w_0 = n\, n-1\, \ldots\, 2\, 1$ and, given $x \in S_n$, the diagrams of x^{-1}, xw_0, w_0x can be respectively obtained from the diagram of x by a reflection with respect to the lines $\{(h,k) \in \mathbf{R}^2\colon h = k\}$, $\{(h,k) \in \mathbf{R}^2\colon h = (n+1)/2\}$ and $\{(h,k) \in \mathbf{R}^2\colon k = (n+1)/2\}$. The diagram of w_0xw_0 can be obtained from that of x by a reflection with respect to the point $((n+1)/2, (n+1)/2)$.

So, given $x, y \in S_n$, with $x < y$, the four intervals $[x, y]$, $[x^{-1}, y^{-1}]$, $[w_0xw_0, w_0yw_0]$ and $[w_0x^{-1}w_0, w_0y^{-1}w_0]$ belong to the same isomorphism class and the four intervals $[yw_0, xw_0]$, $[w_0y, w_0x]$, $[y^{-1}w_0, x^{-1}w_0]$ and $[w_0y^{-1}, w_0x^{-1}]$ belong the dual class. This means that, in order to study all possible isomorphism types of intervals that can occur in the symmetric group, it is enough to consider diagrams of pairs of permutations up to these symmetries. In Fig. 2 an example is shown.

In any Coxeter group W, given $x, y \in W$, with $x < y$, it is known that

$$\tilde{R}_{x,y}(q) = \tilde{R}_{x^{-1},y^{-1}}(q),$$

and, if W is finite,

$$\tilde{R}_{x,y}(q) = \tilde{R}_{yw_0,xw_0}(q) = \tilde{R}_{w_0y,w_0x}(q) = \tilde{R}_{w_0xw_0,w_0yw_0}(q).$$

In the symmetric group, this means that the eight pairs obtained from (x, y) by the symmetries represented in Fig. 2 all have the same $\tilde{R}$-polynomial associated. Thus, in order to list all possible $\tilde{R}$-polynomials that can occur in the symmetric group, it is enough to consider diagrams up to those symmetries.

3.2. Simplifications

Note that all the information needed for determining the poset structure of the interval $[x, y]$ and for computing the $\tilde{R}$-polynomial associated is contained in the support of (x, y). Given $x \in S_n$ and $I \subseteq [n]$, with $|I| = m$, we denote by $x|_I$ the permutation of S_m whose diagram is obtained from that of x, by considering only the dots corresponding to the indices in I, removing the others, and renumbering the remaining indices and values.

Proposition 3.1. *Let $x, y \in S_n$, with $x < y$. Set $x_\Omega = x|_{I_\Omega(x,y)}$ and $y_\Omega = y|_{I_\Omega(x,y)}$. Then*

(1) $[x, y] \cong [x_\Omega, y_\Omega]$;
(2) $\tilde{R}_{x_\Omega,y_\Omega}(q) = \tilde{R}_{x,y}(q)$.

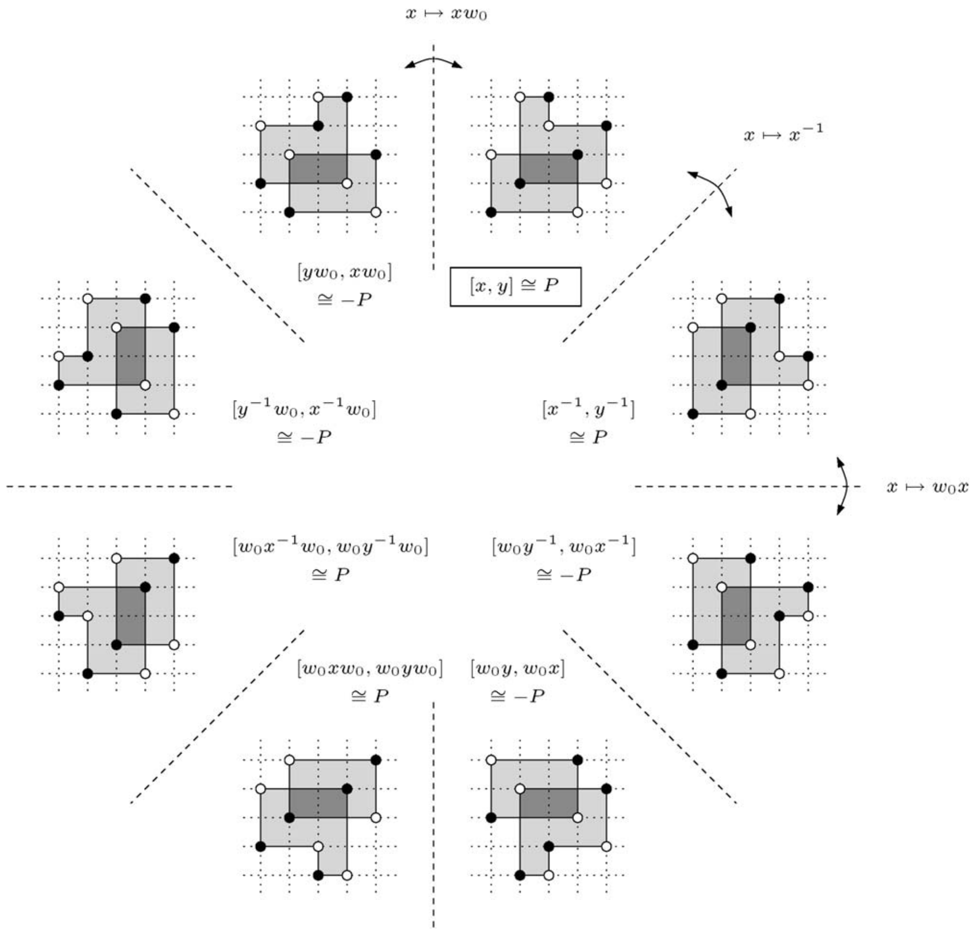
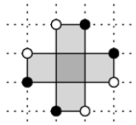


Fig. 2. Symmetries among diagrams and isomorphism classes.

Proof. (1) As already observed, the structure of the interval $[x, y]$ reflects a process of unmounting the diagram of (x, y). Of course, this process only concerns (the closure of) the support. (2) This is [7, Proposition 5.1]. □

By Proposition 3.1, we are allowed, without loss of generality, to consider only diagrams of pairs with full support. An example is described in Fig. 3.

Another kind of simplification is provided by the possibility of decomposing the interval, that is writing it as the direct product of smaller intervals. This happens when the support is not connected and, for example, for the following diagram:



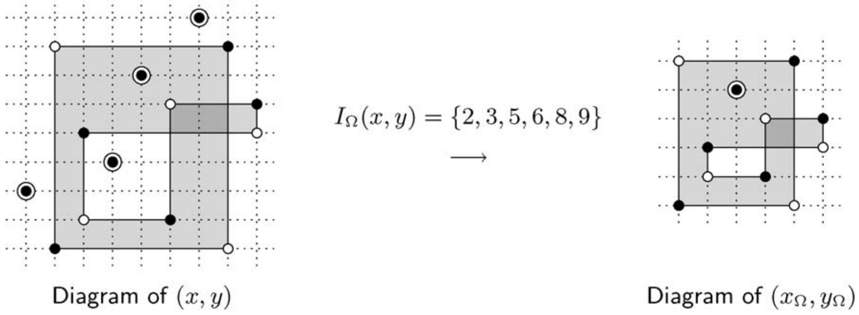


Fig. 3. All the information is contained in the support.

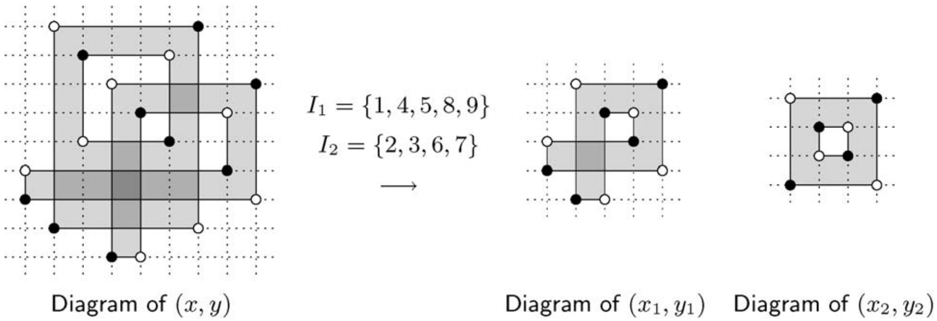


Fig. 4. Trivially decomposable pair.

In this case, there are two free rises of x good with respect to y and they do not interfere with each other. This situation can be generalized. Let $x \in S_n$ and $I \subseteq [n]$. For $(h, k) \in \mathbf{R}^2$, we set

$$x[h, k]|_I = \left|\left\{i \in I\colon\ i \leqslant h,\ x(i) \geqslant k\right\}\right|.$$

Let $x, y \in S_n$ and $I \subseteq [n]$ be such that $x(I) = y(I)$. For $(h, k) \in \mathbf{R}^2$, we set

$$(x, y)[h, k]|_I = y(h, k)|_I - x(h, k)|_I.$$

Then, we set

$$\Omega(x, y)|_I = \left\{(h, k) \in \mathbf{R}^2\colon\ (x, y)[h, k]|_I > 0\right\}.$$

Definition 3.2. Let $x, y \in S_n$, with $x < y$. Let $I_1, I_2 \subseteq [n]$, with $I_1, I_2 \neq \emptyset$, $I_\Omega(x, y) = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$, be such that $x(I_1) = y(I_1)$ and $x(I_2) = y(I_2)$. Set $x_r = x|_{I_r}$, $y_r = y|_{I_r}$ and $\Omega_r = \Omega(x, y)|_{I_r}$, for $r = 1, 2$. Note that, necessarily, $x_1 < y_1$ and $x_2 < y_2$. We say that (x, y) is *trivially decomposable* into the two pairs (x_1, y_1) and (x_2, y_2) if Ω_1 and Ω_2 are either disjoint or if they intersect in a region whose closure does not contain any of the dots of the diagrams of x and y.

An example is shown in Fig. 4. In general, we have the following.

Proposition 3.3. *Let $x, y \in S_n$, with $x < y$, be such that the pair (x, y) is trivially decomposable into (x_1, y_1) and (x_2, y_2). Then*

(1) $[x, y] \cong [x_1, y_1] \times [x_2, y_2]$;
(2) $\tilde{R}_{x,y}(q) = \tilde{R}_{x_1,y_1}(q) \cdot \tilde{R}_{x_2,y_2}(q)$.

Proof. (1) The process of unmounting the diagram of (x, y), that the interval $[x, y]$ reflects, can be done either acting on the diagram of (x_1, y_1), or on that of (x_2, y_2). The fact that the closure of the intersection between the supports does not contain any dot of x or y, guarantees that these two processes are completely independent.

(2) The proof is exactly the same as in [7, Proposition 5.2], provided that in this case Γ and Λ act on subsets of the support, whose intersection does not contain any dot of x or y. This again implies that they do not interfere with each other. □

3.3. Enlarging an interval

Starting from the diagram of a pair (x, y), with $x < y$, it is possible to obtain intervals containing $[x, y]$, by "adding" rectangles. More precisely, the intervals of length $\ell(x, y) + 1$ containing $[x, y]$ are exactly those of the form $[x, y(i, j)]$, with (i, j) a free rise of y, and $[x(i, j), y]$, with (i, j) a free inversion of x. We only consider the first case, the second one being completely symmetric.

Definition 3.4. Let $x, y \in S_n$, with $x < y$. Let (i, j) be a free rise of y and set $z = y(i, j)$. We say that $[x, z]$ is *obtained* from $[x, y]$ by

(1) adding two new dots, if $\{i, j\} \subseteq [n] \backslash I_\Omega(x, y)$;
(2) adding one new dot, if $|\{i, j\} \cap I_\Omega(x, y)| = 1$;
(3) using two old dots, if $\{i, j\} \subseteq I_\Omega(x, y)$.

Examples are shown in Fig. 5. Note that if (x, z) is obtained from (x, y) by adding two new dots, then the pair (x, z) is trivially decomposable into the two pairs (x, y) and $(12, 21)$. Thus, by Proposition 3.3, we get the following.

Corollary 3.5. *Let $x, y \in S_n$, with $x < y$, and let $[x, z]$ be obtained from $[x, y]$ by adding two new dots. Then*

(1) $[x, z] \cong [x, y] \times \{0, 1\}$;
(2) $\tilde{R}_{x,z}(q) = q\tilde{R}_{x,y}(q)$.

In case of adding one new dot, we have a result about the capacity.

Proposition 3.6. *Let $x, y \in S_n$, with $x < y$, and let $[x, z]$ be obtained from $[x, y]$ by adding one new dot. Then*

$$\operatorname{cap}(x, z) \geqslant \operatorname{cap}(x, y) + 1.$$

Proof. Using the characterization of the atoms and the coatoms of an interval, given in Proposition 2.8 it can be checked, by a simple case-by-case verification, that $a(x, z) \geqslant a(x, y) + 1$ and $c(x, z) \geqslant c(x, y) + 1$. □

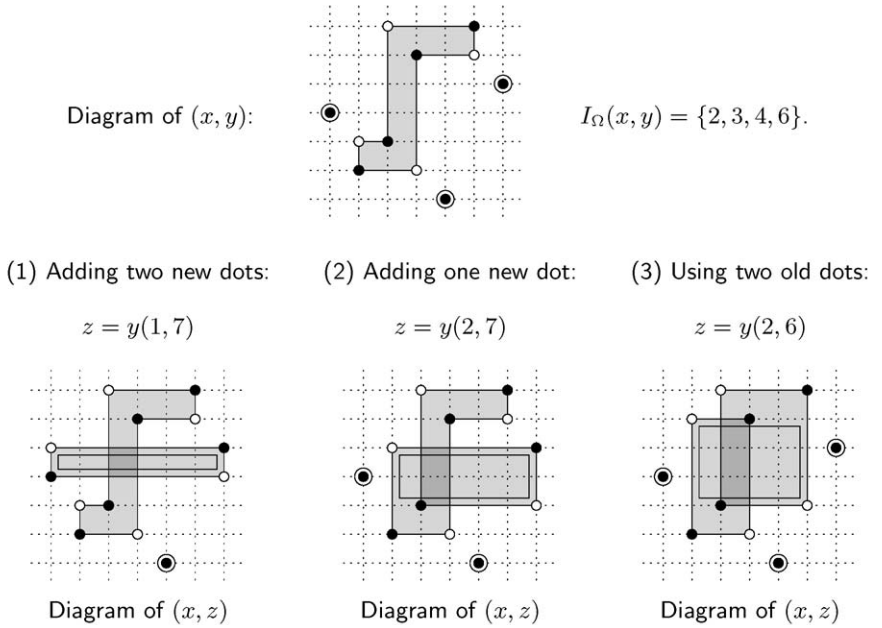


Fig. 5. Enlarging an interval.

4. Combinatorial invariance

In this section, we state and prove the first main results of this paper, concerning the combinatorial invariance of Kazhdan–Lusztig polynomials for intervals of length 5 and 6 in the symmetric group.

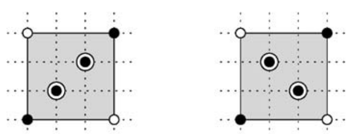
Theorem 4.1. *Let $x, y \in S_n$, with $x < y$, and $u, v \in S_m$, with $u < v$, for some n and m, be such that $\ell(x, y) = \ell(u, v) = 5$. Then*

$$[x, y] \cong [u, v] \quad \Rightarrow \quad P_{x,y}(q) = P_{u,v}(q).$$

Proof. By Theorem 2.2, Proposition 2.4 and by the fact that the combinatorial invariance holds for intervals up to length 4, we only need to show that $\tilde{R}_{x,y}(q) = \tilde{R}_{u,v}(q)$. By Proposition 2.6, we have $a\ell(x, y) = a\ell(u, v) \in \{1, 3, 5\}$.

If $a\ell(x, y) = a\ell(u, v) = 5$, then, by Theorem 2.5, we have $\tilde{R}_{x,y}(q) = \tilde{R}_{u,v}(q) = q^5$.

If $a\ell(x, y) = a\ell(u, v) = 1$, then (x, y) and (u, v) are both Bruhat edges, that is, edges in the Bruhat graph. Essentially, the only possible diagrams for them are



By Theorem 2.9, we have

$$\tilde{R}_{x,y}(q) = \tilde{R}_{u,v}(q) = q^5 + 2q^3 + q.$$

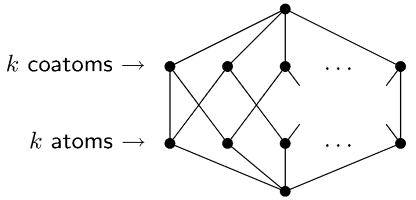
Note that in this case, by Proposition 2.8, $\{a(x,y), c(x,y)\} = \{3,4\}$.

Finally suppose $a\ell(x,y) = a\ell(u,v) = 3$. In this case $\tilde{R}_{x,y}(q) = q^5 + bq^3$, for some $b \in \mathbf{N}$. We will actually prove that

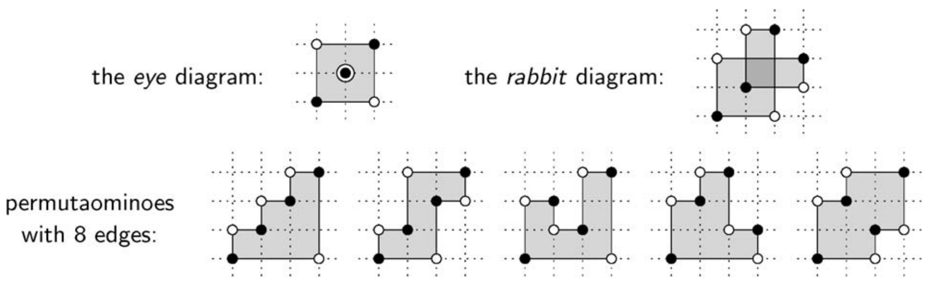
$$\tilde{R}_{x,y}(q) = \begin{cases} q^5 + 2q^3, & \text{if } a(x,y) = c(x,y) = 3, \\ q^5 + q^3, & \text{if } \operatorname{cap}(x,y) \geqslant 4 \end{cases}$$

and this formula covers all cases. Thus, $[x,y] \cong [u,v]$ will imply $\tilde{R}_{x,y}(q) = \tilde{R}_{u,v}(q)$.

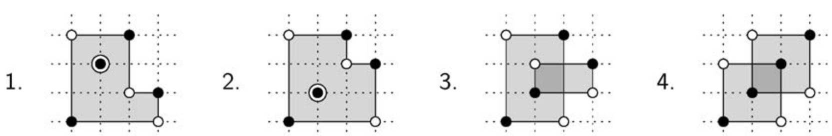
We recall that a *k-crown*, with $k \geqslant 2$, is a poset isomorphic to:



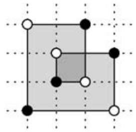
For general Coxeter groups, it is known that only k-crowns can occur as intervals of length 3. Moreover (see [1, Exercise 5.36d]) the intervals $[x,y]$ of any length such that $\tilde{R}_{x,y} \neq q^{\ell(x,y)}$ are exactly those containing a 2-crown as subinterval. In the symmetric group, only 2-, 3- and 4-crowns occur as intervals of length 3, and the diagrams of pairs (x,y), with $\ell(x,y) = 3$, which have full support and are not trivially decomposable are, up to symmetries, exactly the following:



The only diagram with full support corresponding to a 2-crown is the eye diagram (and the only one corresponding to a 4-crown is the last permutaomino). Therefore, we can obtain all intervals $[x,y]$ in S_n, with $\tilde{R}_{x,y}(q) \neq q^{\ell(x,y)}$, by successively enlarging, as described in Subsection 3.3, the eye diagram. Note that we cannot use two old dots to enlarge it, since 123 does not have free inversions and 321 does not have free rises. In case we add two dots, we obtain diagrams which are trivially decomposable. Adding one new dot, we obtain, up to symmetries, four diagrams, which are exactly those corresponding to pairs (x,y) of length 4, with $\tilde{R}_{x,y}(q) \neq q^4$, which have full support and are not trivially decomposable:



We call them the *essential diagrams* of length 4. For all the pairs (x,y) corresponding to them, we have $\operatorname{cap}(x,y) = 3$ and $\tilde{R}_{x,y}(q) = q^4 + q^2$. Finally, we can obtain all the diagrams of pairs (x,y) of length 5, with $\tilde{R}_{x,y}(q) \neq q^5$, which have full support and are not trivially decomposable, by enlarging the essential diagrams of length 4, either using two old dots, or adding one new dot. If we do it using two old dots, we obtain, up to symmetries, two diagrams, one is that corresponding to the Bruhat edges, already considered, the other one is the following:

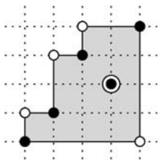
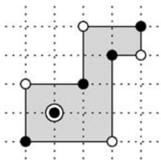
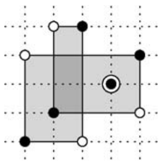
the *heart* diagram: 

In this case, by Proposition 2.8, we get $a(x, y) = c(x, y) = 3$. Applying the stair method, it turns out that

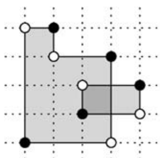
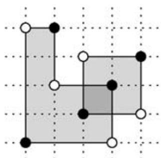
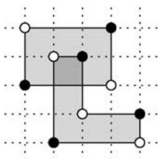
$$\tilde{R}_{x,y}(q) = q^5 + 2q^3.$$

Otherwise, if we enlarge the essential diagrams of length 4, by adding one new dot, then only the following two general situations can occur:

(1) the diagram of (x, y) is obtained from one of the diagrams of length 3, except the eye diagram, by adding one fixed point with multiplicity 1. Here are some examples:

(2) the diagram of (x, y) is the "overlapping" of a permutaomino with 4 edges and one with 6 edges. A few examples:

In all these cases, by Proposition 3.6, we have $\mathrm{cap}(x, y) \geqslant 4$. Applying the stair method, we get

$$\tilde{R}_{x,y}(q) = q^5 + q^3.$$

In case the pair (x, y) is trivially decomposable, then, by Proposition 3.3, the computation of the capacity and of the $\tilde{R}$-polynomial can easily be reduced to smaller diagrams, obtaining in every case $\mathrm{cap}(x, y) \geqslant 4$ and $\tilde{R}_{x,y}(q) = q^5 + q^3$. $\square$

We mention that in [6] all the isomorphism types of intervals of length 5 in the symmetric group have been listed. In Table 1 we collect the expressions of the $\tilde{R}$-polynomial and of the R-polynomial, for each of these types. We recall that for a graded poset P of rank k, the *f-vector* of P is $(f_1, f_2, \dots, f_{k-1})$, where f_i is the number of elements of rank i, for $i \in [k-1]$. In this case, the f-vector uniquely determines the poset structure of the interval, except when it is $(5, 10, 10, 5)$. In the table, we associate with type 9 the *boolean algebra* of rank 5, denoted by $\mathcal{B}_5$, that is, the family of all subsets of $[5]$ partially ordered by inclusion.

The information contained in Table 1 can be summarized in the following result.

Table 1
R-polynomials for intervals of length 5 in S_n

Type	f-vector	$\mathrm{cap}(x, y)$	$\tilde{R}_{x,y}(q)$	$R_{x,y}(q)$
1.	(3, 5, 6, 4)	3	$q^5 + 2q^3 + q$	$(q-1)(q^2-q+1)^2$
2.	(3, 5, 5, 3)	3	$q^5 + 2q^3$	$(q-1)^3(q^2+1)$
3.	(4, 7, 7, 4)	4	$q^5 + q^3$	$(q-1)^3(q^2-q+1)$
4.	(4, 8, 9, 5)	4		
5.	(4, 9, 10, 5)	4		
6.	(4, 10, 12, 6)	4		
7.	(5, 10, 10, 5)	5		
8.	(5, 10, 11, 6)	5		
9.	(5, 10, 10, 5)	5	q^5	$(q-1)^5$
10.	(6, 13, 13, 6)	6		
11.	(6, 14, 15, 7)	6		
12.	(6, 14, 16, 8)	6		
13.	(6, 15, 18, 9)	6		
14.	(7, 17, 17, 7)	6		
15.	(7, 17, 18, 8)	7		

Corollary 4.2. *Let $x, y \in S_n$, for some n, with $x < y$ and $\ell(x, y) = 5$. Set $a = a(x, y)$, $c = c(c, y)$ and $\mathrm{cap} = \mathrm{cap}(x, y)$. Then*

$$\tilde{R}_{x,y}(q) = \begin{cases} q^5 + 2q^3 + q, & \textit{if } \{a, c\} = \{3, 4\}, \\ q^5 + 2q^3, & \textit{if } a = c = 3, \\ q^5 + q^3, & \textit{if } \mathrm{cap} \in \{4, 5\} \textit{ but } [x, y] \ncong \mathcal{B}_5, \\ q^5, & \textit{if } \mathrm{cap} \in \{6, 7\} \textit{ or } [x, y] \cong \mathcal{B}_5 \end{cases}$$

and this formula actually covers all possible cases.

As a consequence of Theorem 4.1, we also have the combinatorial invariance of Kazhdan–Lusztig polynomials for intervals of length 6 in the symmetric group.

Corollary 4.3. *Let $x, y \in S_n$, with $x < y$, and $u, v \in S_m$, with $u < v$, for some n and m, be such that $\ell(x, y) = \ell(u, v) = 6$. Then*

$$[x, y] \cong [u, v] \quad \Rightarrow \quad P_{x,y}(q) = P_{u,v}(q).$$

Proof. By Proposition 2.3, the combinatorial invariance of the R-polynomials for intervals up to length 5 implies that for intervals of length 6. Then, by Theorem 2.2, we have the combinatorial invariance of the Kazhdan–Lusztig polynomials. □

5. Explicit formulas

In this section, we give explicit formulas for the $\tilde{R}$-polynomials (from which those for the R-polynomials can easily be deduced) and for the Kazhdan–Lusztig polynomials associated with any interval of length 5 in any Coxeter group. In particular, we show what they look like in the symmetric group.

Let W be any Coxeter group and let $\prec$ be a fixed reflection ordering on its set of reflections. For $x, y \in W$, with $x < y$, we denote by $be_k(x, y)$ the number of Bruhat edges of length k contained in $[x, y]$:

$$be_k(x, y) = \left|\left\{(z, w)\colon\ x \leqslant z \to w \leqslant y,\ \ell(z, w) = k\right\}\right|,$$

and by $be_k^{\prec}(x, y)$ the number of those contained in a Bruhat path from x to y, which is increasing with respect to $\prec$. For intervals of length 5 the following holds.

Proposition 5.1. *Let $x, y \in W$, with $x < y$ and $\ell(x, y) = 5$. Then*

$$\tilde{R}_{x,y}(q) = q^5 + be_3^{\prec}(x, y)q^3 + be_5(x, y)q.$$

Proof. It is a consequence of Theorem 2.5. In fact, by the EL-shellability of W, $be_3^{\prec}(x, y)$ is the number of increasing paths from x to y of length 3, and $be_5(x, y)$ is 1 or 0 depending on whether (x, y) is a Bruhat edge or not. $\square$

The expressions of the R-polynomials and of the Kazhdan–Lusztig polynomials are known for intervals up to length 4 (see, e.g., [1, exercises of Chapter 5]). In the following we give unified formulas for them.

Proposition 5.2. *Let $x, y \in W$, with $x < y$ and $\ell(x, y) \leqslant 4$. Then*

$$R_{x,y}(q) = \begin{cases} (q-1)^{\ell(x,y)}, & \text{if } \ell(x, y) = 1, 2, \\ (q-1)[q^2 + (be_3(x, y) - 2)q + 1], & \text{if } \ell(x, y) = 3, \\ (q-1)^2[q^2 + (\frac{be_3(x,y)}{2} - 2)q + 1], & \text{if } \ell(x, y) = 4 \end{cases}$$

and

$$P_{x,y}(q) = \begin{cases} 1, & \text{if } \ell(x, y) = 1, 2, \\ 1 + [c(x, y) + be_3(x, y) - 3]q, & \text{if } \ell(x, y) = 3, \\ 1 + [c(x, y) + \frac{be_3(x,y)}{2} - 4]q, & \text{if } \ell(x, y) = 4. \end{cases}$$

Let $x, y \in W$, with $x < y$, and set $\ell = \ell(x, y)$. For $i \in [\ell - 1]$, set $F_i(x, y) = \{z \in [x, y]\colon \ell(x, z) = i\}$ and $f_i(x, y) = |F_i(x, y)|$. So, in particular, $f_1(x, y) = a(x, y)$ and $f_{\ell-1}(x, y) = c(x, y)$. For $i, j \in [\ell - 1]$, with $i < j$, set

$$f_{ij}(x, y) = \left|\left\{(z, w) \in F_i(x, y) \times F_j(x, y)\colon\ z < w\right\}\right|,$$

and, if $j - i = 3$, set

$$be_3^{ij}(x, y) = \left|\left\{(z, w) \in F_i(x, y) \times F_j(x, y)\colon\ z \to w\right\}\right|.$$

The next result gives a formula for $P_{x,y}(q)$, when $\ell(x, y) = 5$. In its statement and proof the dependence of the parameters from (x, y) is for simplicity omitted.

Theorem 5.3. *Let $x, y \in W$, with $x < y$ and $\ell(x, y) = 5$. Then*

$$P_{x,y}(q) = 1 + \left(c + be_3^{\prec} - 5\right)q + \left(10 - 3a - 3c + f_{14} + be_3 - \frac{be_3^{14}}{2} - 3be_3^{\prec} + be_5\right)q^2.$$

Proof. By Theorem 2.2, the polynomial $P_{x,y}(q)$ is uniquely determined by the condition $\deg(P_{x,y}(q)) \leqslant 2$ and by the relation

$$q^5 P_{x,y}\big(q^{-1}\big) - P_{x,y}(q) = \sum_{x<z\leqslant y} R_{x,z}(q) P_{z,y}(q). \tag{3}$$

If we set $A_i = \sum_{z\in F_i(x,y)} R_{x,z}(q) P_{z,y}(q)$, for $i \in [4]$, Eq. (3) can be written:

$$q^5 P_{x,y}\big(q^{-1}\big) - P_{x,y}(q) = A_1 + A_2 + A_3 + A_4 + R_{x,y}(q). \tag{4}$$

By Proposition 5.2, we get

$$\begin{aligned}
A_1 &= \sum_{z\in F_1} (q-1)\left[1 + \left(c(z,y) + \frac{be_3(z,y)}{2} - 4\right) q\right] \\
&= (q-1)\left[f_1 + \left(f_{14} + \frac{be_3^{14}}{2} + be_3^{25} - 4f_1\right) q\right], \\
A_2 &= \sum_{z\in F_2} (q-1)^2\big[1 + \big(c(z,y) + be_3(z,y) - 3\big) q\big] \\
&= (q-1)^2\big[f_2 + \big(f_{24} + be_3^{25} - 3f_2\big) q\big], \\
A_3 &= \sum_{z\in F_3} (q-1)\big[q^2 + \big(be_3(x,z) - 2\big) q + 1\big] \\
&= (q-1)\big[f_3 q^2 + \big(be_3^{03} - 2f_3\big) q + f_3\big], \\
A_4 &= \sum_{z\in F_4} (q-1)^2\left[q^2 + \left(\frac{be_3(x,z)}{2} - 2\right) q + 1\right] \\
&= (q-1)^2\left[f_4 q^2 + \left(be_3^{03} + \frac{be_3^{14}}{2} - 2f_4\right) q + f_4\right].
\end{aligned}$$

By Propositions 2.4 and 5.1, we have

$$R_{x,y}(q) = (q-1)\big[q^4 + \big(be_3^{\prec} - 4\big) q^3 + \big(6 - 2be_3^{\prec} + be_5\big) q^2 + \big(be_3^{\prec} - 4\big) q + 1\big].$$

So Eq. (4) becomes:

$$q^5 P_{x,y}\big(q^{-1}\big) - P_{x,y}(q) = (q-1)\big(q^4 + \alpha q^3 + \beta q^2 + \gamma q + \delta\big), \tag{5}$$

where

$$\begin{aligned}
\alpha &= f_4 + be_3^{\prec} - 4, \\
\beta &= f_{24} - 3f_2 + f_3 - 3f_4 + be_3^{03} + \frac{be_3^{14}}{2} + be_3^{25} + 6 - 2be_3^{\prec} + be_5, \\
\gamma &= f_{14} - f_{24} - 4f_1 + 4f_2 - 2f_3 + 3f_4 + be_3^{\prec} - 4, \\
\delta &= f_1 - f_2 + f_3 - f_4 + 1 = 1.
\end{aligned}$$

By a symmetry condition, we have $\alpha = \gamma$, which implies

$$f_{24} = f_{14} + 2f_3 - 2f_4,$$

and substituting this value in β, we get

$$\beta = f_{14} - 3f_1 - 2f_4 + be_3 - \frac{be_3^{14}}{2} + 6 - 2be_3^{\prec} + be_5.$$

Finally, from (5), we obtain $P_{x,y}(q) = 1 + (\alpha - 1)q + (\beta - \alpha)q^2$, as desired. □

It is worth noting that, if $[x, y]$ and $[u, v]$ are intervals of length 4 or 5, belonging to dual isomorphism classes, then the associated Kazhdan–Lusztig polynomials may only differ in the linear term. More precisely, we have the following.

Proposition 5.4. *Let $x, y \in S_n$, with $x < y$, and $u, v \in S_m$, with $u < v$, for some n and m, be such that $\ell(x, y) = \ell(u, v) \in \{4, 5\}$ and $[u, v] \cong -[x, y]$. Then*

$$P_{u,v}(q) = P_{x,y}(q) + \big[a(x, y) - c(x, y)\big]q.$$

Proof. It is known that $\tilde{R}_{x,y}(q) = \tilde{R}_{u,v}(q)$. In particular this implies $be_3^{\prec}(x, y) = be_3^{\prec}(u, v)$. Thus, for the length 4 case the result follows from Proposition 5.2. For the length 5 case, it follows from Theorem 5.3, observing that the parameters $a + c$, f_{14}, be_3, be_3^{14} and be_5 are the same for (x, y) and (u, v). □

In the symmetric group interesting formulas come out. For $a, b \in \mathbf{N}$, we denote by a mod b the remainder of the integer division between a and b, that is

$$a \ (\mathrm{mod}\ b) = a - b\left\lfloor \frac{a}{b} \right\rfloor.$$

Then, we have the following.

Theorem 5.5. *Let $x, y \in S_n$, with $x < y$ and $\ell(x, y) = 5$. Then*

$$\tilde{R}_{x,y}(q) = q^5 + \left\lfloor \frac{be_3}{3} \right\rfloor q^3 + be_5 q,$$

$$P_{x,y}(q) = 1 + \left(c + \left\lfloor \frac{be_3}{3} \right\rfloor - 5\right)q + \left(10 - 3a - 3c + f_{14} + be_3 \ (\mathrm{mod}\ 3) - \frac{be_3^{14}}{2} + be_5\right)q^2.$$

Proof. Counting the Bruhat edges of length 3 that appear in every type (see column 4 of Table 2) and comparing with the coefficient of q in the corresponding $\tilde{R}$-polynomial (see column 4 of Table 1), it turns out that in every case

$$be_3^{\prec} = \left\lfloor \frac{be_3}{3} \right\rfloor. \tag{6}$$

So the result follows from Proposition 5.1 and Theorem 5.3. □

Note that Eq. (6), valid for intervals of length 5 in S_n, does not hold in general for Coxeter groups. For example, for the signed permutations $x = [1, 3, 2]$ and $y = [-1, 2, -3] \in B_3$, we have $\ell(x, y) = 5$, $be_3(x, y) = 9$, but $be_3^{\prec}(x, y) = 2$.

Table 2
Kazhdan–Lusztig polynomials for intervals of length 5 in S_n

Type	f-vector	f_{14}	be_3	be_3^{14}	be_5	$P_{x,y}(q)$
1.	(3, 5, 6, 4)	12	6	4	1	$1+q$
−1.	(4, 6, 5, 3)					1
2.	(3, 5, 5, 3)	9	6	2	0	1
3.	(4, 7, 7, 4)	14	4	2	0	1
4.	(4, 8, 9, 5)	17	4	2	0	$1+q$
−4.	(5, 9, 8, 4)					1
5.	(4, 9, 10, 5)	17	3	0	0	$1+q$
−5.	(5, 10, 9, 4)					1
6.	(4, 10, 12, 6)	20	4	0	0	$1+2q+q^2$
−6.	(6, 12, 10, 4)					$1+q^2$
7.	(5, 10, 10, 5)	19	5	2	0	$1+q$
−7.	(5, 10, 10, 5)					$1+q$
8.	(5, 10, 11, 6)	23	5	4	0	$1+2q$
−8.	(6, 11, 10, 5)					$1+q$
9.	(5, 10, 10, 5)	20	0	0	0	1
10.	(6, 13, 13, 6)	26	0	0	0	$1+q$
11.	(6, 14, 15, 7)	29	0	0	0	$1+2q$
−11.	(7, 15, 14, 6)					$1+q$
12.	(6, 14, 16, 8)	32	0	0	0	$1+3q$
−12.	(8, 16, 14, 6)					$1+q$
13.	(6, 15, 18, 9)	36	0	0	0	$1+4q+q^2$
−13.	(9, 18, 15, 6)					$1+q+q^2$
14.	(7, 17, 17, 7)	32	0	0	0	$1+2q$
15.	(7, 17, 18, 8)	36	0	0	0	$1+3q+q^2$
−15.	(8, 18, 17, 7)					$1+2q+q^2$

We finally collect in Table 2 all the information needed, according to Theorem 5.3, for computing the Kazhdan–Lusztig polynomials, for every isomorphism class. We refer to the types as in Table 1, and we denote by $-k$ the type dual to type k.

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