Hyperspaces of separable Banach spaces with the Wijsman topology

Wiesław Kubiś a,b,1, Katsuro Sakai c,2,*, Masato Yaguchi c

a Institute of Mathematics, University of Silesia, Katowice, Poland
b Department of Mathematics, Ben Gurion University of the Negev, Beer-Sheva, Israel
c Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571, Japan

Received 12 April 2002; received in revised form 24 February 2004; accepted 23 July 2004

Abstract

Let $X$ be a separable metric space. By $\text{Cld}_W(X)$, we denote the hyperspace of non-empty closed subsets of $X$ with the Wijsman topology. Let $\text{Fin}_W(X)$ and $\text{Bdd}_W(X)$ be the subspaces of $\text{Cld}_W(X)$ consisting of all non-empty finite sets and of all non-empty bounded closed sets, respectively. It is proved that if $X$ is an infinite-dimensional separable Banach space then $\text{Cld}_W(X)$ is homeomorphic to ($\approx$) the separable Hilbert space $\ell_2$ and $\text{Fin}_W(X) \approx \text{Bdd}_W(X) \approx \ell_2 \times \ell_{f,2}$, where

$$\ell_{f,2} = \{ (x_i)_{i \in \mathbb{N}} \in \ell_2 \mid x_i = 0 \text{ except for finitely many } i \in \mathbb{N} \}.$$ 

Moreover, we show that if the complement of any finite union of open balls in $X$ has only finitely many path-components, all of which are closed in $X$, then $\text{Fin}_W(X)$ and $\text{Cld}_W(X)$ are ANR’s. We also give a sufficient condition under which $\text{Fin}_W(X)$ is homotopy dense in $\text{Cld}_W(X)$.

© 2004 Elsevier B.V. All rights reserved.

MSC: 54B20; 54C55; 57N20

* Corresponding author.
E-mail addresses: kubis@math.bgu.ac.il (W. Kubiś), sakaktr@sakura.cc.tsukuba.ac.jp (K. Sakai), masato@math.tsukuba.ac.jp (M. Yaguchi).
1 The author was supported by KBN Grant (No. 5P03A04420).
2 The author is supported by Grant-in-Aid for Scientific Research (No. 14540059).

0166-8641/$ – see front matter © 2004 Elsevier B.V. All rights reserved.
1. Introduction

Let $\text{Cld}(X)$ be the set of all non-empty closed sets in a topological space $X$. By $\text{Cld}_V(X)$, we denote the space $\text{Cld}(X)$ with the Vietoris topology, the most typical hyperspace topology. The Curtis–Schori–West Hyperspace Theorem is a celebrated result in infinite-dimensional topology which states that $\text{Cld}_V(X)$ is homeomorphic to $(\sim)$ the Hilbert cube $Q = [-1, 1]^\omega$ if and only if $X$ is a non-degenerate, connected and locally connected compact metrizable space ([11,24]; cf. [18, Theorem 8.4.5]). For a non-compact metric space $X$, since $\text{Cld}_V(X)$ is non-metrizable, we have to consider hyperspace topologies different from the Vietoris topology. By $\text{Cld}_F(X)$, $\text{Cld}_{AW}(X)$ and $\text{Cld}_W(X)$, we denote the spaces $\text{Cld}(X)$ with the Fell topology, the Attouch–Wets topology and the Wijsman topology, respectively, where the last two topologies are defined in case $X$ is a metric space. When $X$ is compact, these three topologies coincide with the Vietoris one. For a study of hyperspace topologies, we refer to the book [5] (cf. [17]).

In the paper [21], it is shown that $\text{Cld}_F(X) \approx Q \setminus \{0\}$ if and only if $X$ is a locally compact, locally connected separable metrizable space with no compact components. In [3], it is shown that if $X$ is an infinite-dimensional Banach space with weight $w(X)$ then $\text{Cld}_{AW}(X) \approx \ell_2(2^{w(X)})$, where $\ell_2(\tau)$ is the Hilbert space with weight $\tau$. In [14], the first author proved that if $X$ is an infinite-dimensional separable Banach space then $\text{Cld}_W(X)$ is an AR.\(^3\) It should be remarked that $\text{Cld}_F(X) = \text{Cld}_{AW}(X) = \text{Cld}_W(X)$ for a finite-dimensional normed linear space $X$. In fact, this equality holds for a metric space $X$ such that bounded closed subsets of $X$ are compact [5, Theorem 3.1.4 and Exercise 5.1.10(b)]. In this paper, we prove the following:

**Theorem I.** If $X$ is an infinite-dimensional separable Banach space, then $\text{Cld}_W(X)$ is homeomorphic to the separable Hilbert space $\ell_2$.

Let $\text{Fin}(X)$, $\text{Comp}(X)$ and $\text{Bdd}(X)$ be the subsets of $\text{Cld}(X)$ consisting of all non-empty finite sets, of all non-empty compact sets and of all non-empty bounded closed sets, respectively. By $\text{Fin}_V(X)$, $\text{Fin}_W(X)$, etc., we respectively denote the subspaces of $\text{Cld}_V(X)$, $\text{Cld}_W(X)$, etc. In [10], it is proved that $\text{Fin}_V(X) \approx \ell_2^I$ if and only if $X$ is a non-degenerate, strongly countable-dimensional,\(^4\) connected, locally path-connected, $\sigma$-compact metrizable space, where

$$\ell_2^I = \{(x_i)_{i \in \mathbb{N}} \in \ell_2 \mid x_i = 0 \text{ except for finitely many } i \in \mathbb{N}\}.$$  

\(^3\) AR (ANR) = absolute retract (absolute neighborhood retract) for metrizable spaces.

\(^4\) It is said that $X$ is strongly countable-dimensional if $X$ is a countable union of finite-dimensional closed subsets.
In [21], it is also shown that $\text{Fin}_F(X) \approx \ell_f^2$ if and only if $X$ is a strongly countable-dimensional, locally compact, locally connected, separable metrizable space with no compact components. In [20], it is proved that if $X$ is an infinite-dimensional Banach space with weight $w(X)$ then

$$\text{Fin}_{AW}(X) \approx \ell^2(w(X)) \times \ell_f^2 \quad \text{and} \quad \text{Bdd}_{AW}(X) \approx \ell^2(2w(X)) \times \ell_f^2.$$ 

The following is also shown in this paper:

**Theorem II.** If $X$ is an infinite-dimensional separable Banach space, then

$$\text{Fin}_W(X) \approx \text{Bdd}_W(X) \approx \ell^2 \times \ell_f^2.$$ 

It is said that $Y \subset X$ is homotopy dense in $X$ if there is a homotopy $h : X \times I \to X$ such that $h_0 = \text{id}$ and $h_t(Y) \subset Y$ for every $t > 0$, where $I = [0, 1]$. It is well known that every homotopy dense set in an AR (respectively an ANR) is also an AR (respectively an ANR). Proving Theorem II for $\text{Fin}_W(X)$, we also obtain the following more general result:

**Theorem III.** Let $X$ be a separable metric space. If the complement of any finite union of open balls in $X$ has only finitely many path-components and all of them are unbounded closed (respectively each of them is compact or unbounded closed), then $\text{Cld}_W(X)$ is an AR (respectively an ANR) and $\text{Fin}_W(X)$ is homotopy dense in $\text{Cld}_W(X)$. In particular, $\emptyset$ is an AR (respectively an ANR) if $\text{Fin}_W(X) \subset \emptyset \subset \text{Cld}_W(X)$.

In case $X$ is an infinite-dimensional separable Banach space, it is shown in [14] that $\text{Cld}_W(X)$ is an AR. This is a direct consequence of Theorem III above. But it also follows that $\text{Comp}_W(X)$ and $\text{Fin}_W(X)$ are also AR’s.

To prove Theorem III, we give a condition on a metrizable Lawson semilattice $X$ and its subsemilattice $Y$ under which $X$ is an ANR and $Y$ is homotopy dense in $X$ (Theorem 5.1). See Section 5 for the definition of a Lawson semilattice. In Theorem III, if it is not demanded that $\text{Fin}_W(X)$ is homotopy dense in $\text{Cld}_W(X)$, the assumption can be a little weakened, that is,

**Theorem IV.** Let $X$ be a separable metric space. If the complement of any finite union of open balls in $X$ has only finitely many path-components, all of which are closed in $X$, then $\text{Fin}_W(X)$ and $\text{Cld}_W(X)$ are ANRs.

For a closed subspace $Y \subset X$, we have $\text{Cld}(Y) \subset \text{Cld}(X)$ as sets. Due to [3, Proposition 2.1], the Attouch–Wets topology on $\text{Cld}(Y)$ coincides with the subspace topology of the Attouch–Wets topology on $\text{Cld}(X)$ as well as the Vietoris and Fell topologies. However, the Wijsman topology on $\text{Cld}(Y)$ does not necessarily coincide with the subspace topology inherited from $\text{Cld}_W(X)$ (see Example in Section 2). We also discuss the space $\text{Cld}(Y)$ with the subspace topology inherited from $\text{Cld}_W(X)$, which we call the relative Wijsman topology. We prove Theorem IV in this setting.

---

5 This fact follows from [12, Chapter IV, Theorem 6.3].
2. Preliminaries

Let \( X = (X, d) \) be a metric space. The open ball and the closed ball centered at \( x \) with radius \( \varepsilon \) are denoted by \( B(x, \varepsilon) \) and \( \overline{B}(x, \varepsilon) \), respectively. For \( A \subseteq X \) and \( x \in X \), we define \( d(x, A) = \inf_{a \in A} d(x, a) \). By \( C(X) \), we denote the set of all continuous real-valued functions on \( X \). By a 'map', we mean a continuous function (unless otherwise indicated). By identifying each \( A \in \text{Cld}(X) \) with the map \( X \ni x \mapsto d(x, A) \in \mathbb{R} \), we can regard \( \text{Cld}(X) \subseteq C(X) \), whence \( \text{Cld}(X) \) has various topologies inherited from \( C(X) \). The Wijsman topology on \( \text{Cld}(X) \) is the topology of point-wise convergence, which depends on the metric \( d \) for \( X \). For each \( x \in X \) and \( r > 0 \), we define

\[
U^-(x, r) = \{ A \in \text{Cld}(X) \mid d(x, A) < r \};
\]
\[
U^+(x, r) = \{ A \in \text{Cld}(X) \mid d(x, A) > r \}.
\]

These are open sets in \( \text{Cld}(X) \) which form an open subbasis for \( \text{Cld}(X) \). Moreover, to generate the Wijsman topology, it suffices to pick up points \( x \) from a dense subset of \( X \).

For each \( k \in \mathbb{N} \), we denote

\[
\text{Fin}^k(X) = \{ A \in \text{Fin}(X) \mid \text{card } A \leq k \}.
\]

It is easily observed that the correspondence \( X \ni x \leftrightarrow \{ x \} \in \text{Fin}^1_W(X) \) is a homeomorphism. By this correspondence, we can regard \( X = \text{Fin}^1_W(X) \subseteq \text{Cld}(X) \).

It is well known that \( \text{Cld}(X) \) is metrizable if and only if \( X \) is separable, whence \( \text{Cld}(Y) \) is also separable (cf. [5, Theorem 2.1.5]). By using a countable dense subset \( \{ x_i \mid i \in \mathbb{N} \} \) of \( X \), we can define an admissible metric \( d_W \) for \( \text{Cld}(X) \) as follows:

\[
d_W(A, B) = \sup_{i \in \mathbb{N}} \min\{2^{-i}, |d(x_i, A) - d(x_i, B)|\}.
\]

Notice that \( d_W \) is incomplete even if \( d \) is complete [5, Example 2.1.6]. However, the following holds [5, Theorem 2.5.4]:

**Proposition 2.1.** For every separable complete metric space \( X \), the space \( \text{Cld}(X) \) is separable and completely metrizable.

For a closed subspace \( Y \subseteq X \), we have \( \text{Cld}(Y) \subseteq \text{Cld}(X) \). The space \( \text{Cld}(Y) \) admits the Wijsman topology defined by using the metric \( d_Y = d|Y^2 \) inherited from \( X \). On the other hand, as mentioned in Introduction, \( \text{Cld}(Y) \) also has the subspace topology inherited from \( \text{Cld}(X) \), called the relative Wijsman topology. These two topologies do not coincide. In fact, the following example shows that \( \text{Fin}^2_W(Y) \) may not be the subspace of \( \text{Fin}^2_W(X) \).

---

6 Note that \( d(x, A) < r \iff B(x, r) \cap A \neq \emptyset \) and \( d(x, A) > r \iff \overline{B}(x, r) \cap A = \emptyset \). However, \( \overline{B}(x, r) \cap A = \emptyset \nRightarrow d(x, A) > r \). In fact, let \( A = \bigcup_{n \in \mathbb{N}} [1 + \frac{1}{n} - 1, \infty) e_n \subseteq \text{Cld}(\ell_2) \), where \( \{ e_n \mid n \in \mathbb{N} \} \) is the canonical orthonormal basis for \( \ell_2 \). Then \( \overline{B}(0, 1) \cap A = \emptyset \) but \( d(0, A) = 1 \).

7 In [5], the following metric is used but it is uniformly equivalent to ours:

\[
d_W(A, B) = \sum_{i \in \mathbb{N}} 2^{-i} \min\{1, |d(x_i, A) - d(x_i, B)|\}.
\]
Example. Let $X = \ell_2$ be the Hilbert space, $Y = \{ x \in \ell_2 \mid \| x \| \leq 1 \}$ the unit closed ball of $\ell_2$ and $\{ e_n \mid n \in \mathbb{N} \}$ the canonical orthonormal base of $\ell_2$. Fix $\delta > 0$ and define
\[
\alpha_n = \frac{1}{\sqrt{1 + \delta^2}}(e_{n+1} + \delta e_1), \quad n \in \mathbb{N}.
\]
Then we have $A_n = \{ 0, \alpha_n \} \subseteq \text{Fin}^2(Y)$, which does not converge to $\{ 0 \}$ in $\text{Cld}_W(X)$. Indeed, consider $v = te_1 \in X (= \ell_2)$, where
\[
t > \frac{\sqrt{1 + \delta^2}}{2\delta}, \quad \text{i.e.,} \quad \frac{2t\delta}{\sqrt{1 + \delta^2}} > 1.
\]
For each $n \in \mathbb{N}$, we have
\[
\| v - \alpha_n \|^2 = \left( t - \frac{\delta}{\sqrt{1 + \delta^2}} \right)^2 \| e_1 \|^2 = \left( 1 - \frac{2t\delta}{\sqrt{1 + \delta^2}} + 1 \right)^2 = \| v \|^2 - \frac{4t^2}{1 + \delta^2}.
\]
Thus, $d(v, A_n) = \| v - \alpha_n \| \neq \| v \| = d(v, \{ 0 \})$.

However, we can find $\delta > 0$ so that $A_n \to \{ 0 \}$ in $\text{Cld}_W(Y)$. In order to find such $\delta > 0$, consider
\[
x = (x_1, \ldots, x_k, 0, 0, \ldots) \in Y \cap \ell_2^k \subset X = \ell_2.
\]
Then $\| x \|^2 = \sum_{i=1}^k x_i^2$ and
\[
\| x - \alpha_n \|^2 = \left( x_1 - \frac{\delta}{\sqrt{1 + \delta^2}} \right)^2 + \sum_{i=2}^k x_i^2 + \frac{1}{1 + \delta^2} - x_1^2 - \frac{1}{1 + \delta^2}
\]
for $n \geq k$. Thus, $d(x, A_n) \to d(x, \{ 0 \})$ if and only if $\| x - \alpha_n \|^2 \geq \| x \|^2$ for $n \geq k$. Since $|x_1| \leq \| x \| \leq 1$, it follows that
\[
\| x - \alpha_n \|^2 - \| x \|^2 = \left( x_1 - \frac{\delta}{\sqrt{1 + \delta^2}} \right)^2 + \frac{1}{1 + \delta^2} - x_1^2 = 1 - \frac{2x_1\delta}{1 + \delta^2} \geq 1 - \frac{2\delta}{\sqrt{1 + \delta^2}}.
\]
Choose $0 < \delta \leq 1/\sqrt{3}$. Then $\| x - \alpha_n \|^2 \geq \| x \|^2$, hence $d(x, A_n) \to d(x, \{ 0 \})$ for every $x \in Y \cap \ell_2^k$. Since $Y \cap \ell_2^k$ is dense in $Y$, it follows that $A_n \to \{ 0 \}$ in $\text{Cld}_W(Y)$.

**Proposition 2.2.** Let $X$ be an arbitrary metric space. For each closed subset $Y \subseteq X$, $\text{Cld}(Y)$ is closed in $\text{Cld}_W(X)$.

**Proof.** We may assume that $Y \neq X$. For each $A \subseteq \text{Cld}_W(X) \setminus \text{Cld}(Y)$, we have $a \in A \setminus Y$. Let $0 < r < d(a, Y)$. Then
\[
A \subseteq U^{-r}(a, r) \subseteq \text{Cld}_W(X) \setminus \text{Cld}(Y).
\]
Therefore, $\text{Cld}_W(X) \setminus \text{Cld}(Y)$ is open in $\text{Cld}_W(X)$, that is, $\text{Cld}(Y)$ is closed in $\text{Cld}_W(X)$. \qed
To prove Theorems I and II, we use characterizations of $\ell_2$ and $\ell_2 \times \ell_2^f$. The following characterization of $\ell_2$ is due to Toruńczyk [22] (cf. [23]):

Theorem 2.3. In order that $X \approx \ell_2$, it is necessary and sufficient that $X$ is a separable completely metrizable AR which has the discrete approximation property,\footnote{This is often called the strong discrete approximation property (cf. [1]).} that is, each map $f : \bigoplus_{n \in \mathbb{N}} Y^n \to X$ is approximated by maps $g : \bigoplus_{n \in \mathbb{N}} Y^n \to X$ such that $\{g(Y^n) \mid n \in \mathbb{N}\}$ is discrete in $X$.

Here, given a collection $G$ of maps from a space $Y$ to $X$, a map $f : Y \to X$ is approximated by maps in $G$ (or arbitrarily close to maps in $G$) if, for each open cover $\mathcal{U}$ of $X$, $f$ is $\mathcal{U}$-close to some $g \in G$, that is, each $f(y)$ and $g(y)$ are contained in some $U \in \mathcal{U}$. In case $X = (X, d)$ is a metric space, this is equivalent to the condition that for each $\alpha : X \to (0, 1)$, there is $g \in G$ such that $d(f(y), g(y)) < \alpha(f(y))$ for every $y \in Y$.

To state the characterization of $\ell_2 \times \ell_2^f$ due to Bestvina and Mogilski [7], we need some notions. A metrizable space $X$ is $\sigma$-completely metrizable if $X$ is a countable union of completely metrizable closed subsets. A closed set $A \subset X$ is a (strong) $Z$-set in $X$ if there are maps $f : X \to X \setminus A$ arbitrarily close to id (such that $A \cap \text{cl}(f(X)) = \emptyset$). An embedding is called a (strong) $Z$-embedding if its image is a (strong) $Z$-set. A countable union of (strong) $Z$-sets is called a (strong) $Z_\sigma$-set. When $X$ itself is a (strong) $Z_\sigma$-set in $X$, we call $X$ a (strong) $Z_\sigma$-space. For a class $\mathcal{C}$ of spaces, $X$ is said to be universal for $\mathcal{C}$ if each map $f : A \to X$ of $A \in \mathcal{C}$ is approximated by $Z$-embeddings. It is said that $X$ is strongly universal for $\mathcal{C}$ if given a map $f : A \to X$ of $A \in \mathcal{C}$ such that $f|B$ is a $Z$-embedding of a closed set $B \subset A$, there exist $Z$-embeddings $g : A \to X$ such that $g|B = f|B$ and which are arbitrarily close to $f$. The following is Corollary 6.3 in [7].

Theorem 2.4. In order that $X \approx \ell_2 \times \ell_2^f$, it is necessary and sufficient that $X$ is a separable $\sigma$-completely metrizable AR which is a strong $Z_\sigma$-space and it is strongly universal for separable completely metrizable spaces.

In case $X$ is a homotopy dense set in $\ell_2$, $X$ has the discrete approximation property (cf. [4, Theorem I.3.2] or [2]), hence every $Z$-set in $X$ is a strong $Z$-set by [7, Proposition 1.7]. Note that the class $\mathcal{M}_1$ of separable completely metrizable spaces is hereditary with respect to both closed subsets and open subsets (i.e., if $X \in \mathcal{M}_1$ and $A \subset X$ is closed or open then $A \in \mathcal{M}_1$). Then, by [7, Proposition 2.2] (or [4, Theorem 3.2.19]), $X$ is strongly universal for $\mathcal{M}_1$ if and only if every open subset of $X$ is universal for $\mathcal{M}_1$. Thus, we have the following:

Theorem 2.5. For a homotopy dense set $X$ in $\ell_2$, if $X$ is a $\sigma$-completely metrizable $Z_\sigma$-space and every open subset of $X$ is universal for separable completely metrizable spaces, then $X \approx \ell_2 \times \ell_2^f$. 
3. Proof of Theorem I

We have already observed that \( \text{Cld}_W(X) \) is separable and completely metrizable for every separable complete metric space \( X \). It has been proved in [14] that \( \text{Cld}_W(X) \) is an AR for an infinite-dimensional separable Banach space \( X \). To show Theorem I, it remains to verify the discrete approximation property in Theorem 2.3.

Let \( X = (X, \| \cdot \|) \) be a normed linear space and \( d \) the metric induced by the norm \( \| \cdot \| \) (i.e., \( d(x, y) = \| x - y \| \)). By \( B_X \) and \( S_X \), we denote the unit closed ball and the unit sphere of \( X \), respectively. Then for \( x \in X \) and \( \varepsilon > 0 \), \( \overline{B}(x, \varepsilon) = x + \varepsilon B_X \) and \( \partial B(x, \varepsilon) = x + \varepsilon S_X \). For \( \text{Cld}_W(X) \), the metric \( d_W \) is defined by a countable dense set \( \{ x_i \mid i \in \mathbb{N} \} \) in \( X \), where \( x_1 = 0 \).

To prove Theorem I, we need the following lemma, which we formulate in a general setting to apply in the proof of Theorem II.

**Lemma 3.1.** Let \( W \) be an open set in a subspace \( \mathcal{S}_\varepsilon \) of \( \text{Cld}_W(X) \). For each map \( \alpha : W \to (0, 1) \), there exists a map \( \gamma : W \to (1, \infty) \) such that

\[
A \in W, \quad A' \in \mathcal{S}_\varepsilon, \quad A \cap \gamma(A)B_X = A' \cap \gamma(A)B_X \\
\implies A' \in W, \quad d_W(A, A') < \alpha(A). \tag{*}
\]

**Proof.** For each \( A \in W \), we define

\[
i(A) = \min\{i \in \mathbb{N} \mid 2^{-i} < \min\{\alpha(A), \ d_W(A, \mathcal{S}_\varepsilon \setminus W)\}\}
\]

and

\[
r(A) = \max\{|x_i| \mid i \leq i(A)\},
\]

where we mean \( d_W(A, \emptyset) = \infty \). Then \( r : W \to [0, \infty) \) is upper semi-continuous. Indeed, let \( A \in W \) and \( r(A) < t \). By the definition of \( i(A) \) and the continuity of \( \alpha \) and \( d_W(\cdot, \mathcal{S}_\varepsilon \setminus W) \), we can choose \( \delta > 0 \) so that

\[
A' \in W, \quad d_W(A, A') < \delta \implies 2^{-i(A)} < \min\{\alpha(A'), \ d_W(A', \mathcal{S}_\varepsilon \setminus W)\},
\]

whence \( i(A') \leq i(A) \) by the definition of \( i(A') \), so \( r(A') \leq r(A) < t \) by the definition of \( r(A') \).

Now, let \( \gamma : W \to (1, \infty) \) be a map such that

\[
\gamma(A) > 3 \max\{r(A), \ d(0, A)\} \quad \text{for each} \ A \in \mathcal{S}_\varepsilon.
\]

To see that \( \gamma \) satisfies condition (*), suppose

\[
A \in W, \quad A' \in \mathcal{S}_\varepsilon \quad \text{and} \quad A \cap \gamma(A)B_X = A' \cap \gamma(A)B_X.
\]

Then \( A' \cap \frac{1}{\gamma(A)}B_X = A \cap \frac{1}{\gamma(A)}B_X \neq \emptyset \) because \( d(0, A) < \frac{1}{\gamma(A)} \). For each \( i \leq i(A) \), since \( \|x_i\| \leq r(A) < \frac{1}{\gamma(A)} \), it follows from [3, Fact 2] that

\[
d(x_i, A) = d(x_i, A \cap \gamma(A)B_X) = d(x_i, A' \cap \gamma(A)B_X) = d(x_i, A'),
\]

which implies that \( d_W(A, A') < 2^{-i(A)} \). Thus, we have \( d_W(A, A') < \alpha(A) \). Moreover, \( d_W(A, A') < d_W(A, \mathcal{S}_\varepsilon \setminus W) \), which means \( A' \in W \). \( \square \)
Proof of Theorem 1. It remains to show that $\text{Cld}_W(X)$ has the discrete approximation property. By the argument in [8, p. 203], it suffices to show that there are maps $f: \text{Cld}_W(X) \times \mathbb{N} \rightarrow \text{Cld}_W(X)$ arbitrarily close to id such that $\{f(\text{Cld}_W(X) \times \{n\}) \mid n \in \mathbb{N}\}$ is locally finite.

For each map $\alpha: \text{Cld}_W(X) \rightarrow (0,1)$, let $\gamma: \text{Cld}_W(X) \rightarrow (1,\infty)$ be the map obtained by Lemma 3.1, where $\mathcal{F} = W = \text{Cld}_W(X)$. On the other hand, $S_X$ has a countable-infinite $\frac{1}{2}$-discrete set $\{e_n \mid n \in \mathbb{N}\}$ by Riesz’ Theorem (cf. [16, p. 16]). Taking $v \in S_X$ and a map $\beta: \text{Cld}_W(X) \rightarrow (1,\infty)$ such that $\beta > \gamma$, we define $f: \text{Cld}_W(X) \times \mathbb{N} \rightarrow \text{Cld}_W(X)$ as follows:

$$f(A,n) = (A \cap \beta(A)B_X) \cup \beta(A)S_X \cup \{(\beta(A)+2)v + e_n\}.$$ 

By $(*)$ in Lemma 3.1, $d_W(f(A,n),A) < \alpha(A)$ for every $A \in \text{Cld}_W(X)$ and $n \in \mathbb{N}$. Observe that

$$d(x,f(A,n)) = \min\{\beta(A) - \|x\|, d(x,A)\} \quad \text{if} \quad \|x\| \leq \beta(A),$$

$$\min\{\|x\| - \beta(A), (\beta(A)+2)v + e_n - x\} \quad \text{if} \quad \|x\| \geq \beta(A).$$

Then $(A,n) \mapsto d(x,f(A,n))$ is continuous for each $x \in X$, which implies that $f$ is continuous.

We show that $\{f(\text{Cld}_W(X) \times \{n\}) \mid n \in \mathbb{N}\}$ is locally finite. On the contrary, assume that there exist $A_j \in \text{Cld}_W(X)$, $j \in \mathbb{N}$, and $n_1 < n_2 < \cdots \in \mathbb{N}$ such that $(f(A_j,n_j))_{j \in \mathbb{N}}$ converges to $A \in \text{Cld}_W(X)$. If $\sup_{j \in \mathbb{N}} \beta(A_j) = \infty$, by taking a subsequence, we may assume that $\beta(A_j) \rightarrow \infty$ as $j \rightarrow \infty$. Observe that $d_W(f(A_j,n_j),A_j) \rightarrow 0$. Then it follows that $A_j \rightarrow A$, hence $\beta(A_j) \rightarrow \beta(A)$ by the continuity of $\beta$. This is a contradiction.

When $\sup_{j \in \mathbb{N}} \beta(A_j) < \infty$, it can be assumed that $\lim_{j \rightarrow \infty} \beta(A_j) = r \geq 1$, whence we show that $A \subset rB_X$. Suppose that $\|a\| > r$ for some $a \in A$. Then we can choose $i_0 \leq j_0 \in \mathbb{N}$ so that $d(x_{i_0},a) < s/6$ and

$$j \geq j_0 \implies |\beta(A_j) - r| < s/2, \quad d_W(f(A_j,n_j),A) < \min\{s/6, 2^{-n_j}\},$$

where $s = \min\{\|a\| - r, 1/4\}$. Then, for every $j \geq j_0$,

$$d(a,f(A_j,n_j)) \leq d(a,x_{i_0}) + d(x_{i_0},f(A_j,n_j)) \leq d(a,x_{i_0}) + d(x_{i_0},A) + |d(x_{i_0},f(A_j,n_j)) - d(x_{i_0},A)| \leq 2d(a,x_{i_0}) + d_W(A,f(A_j,n_j)) < 2s/6 + s/6 = s/2.$$

On the other hand, since $\|a\| > r + s/2 \geq \beta(A_j)$,

$$d(a,\beta(A_j)B_X) \geq \|a\| - \beta(A_j) > \|a\| - r - s/2 \geq s - s/2 = s/2.$$ 

Therefore, $d(a,\beta(A_j)B_X) \leq \|a\| - \beta(A_j) > \|a\| - r - s/2 \geq s/2$. Then we have

$$d(a,(r+2)v + e_{n_j}) \leq d(a,\beta(A_j)B_X) + d((\beta(A_j)+2)v + e_{n_j},(r+2)v + e_{n_j}) < s/2 + |\beta(A_j) - r| < s.$$

If $j \neq j' \geq j_0$ then
\[ \frac{1}{2} < d(e_{nj}, e_{nj'}) = d((r + 2)v + e_{nj}, (r + 2)v + e_{nj'}) \]
\[ \leq d(a, (r + 2)v + e_{nj}) + d(a, (r + 2)v + e_{nj'}) < 2s < \frac{1}{2}, \]

which is a contradiction. Therefore, \( A \subset rB_X \).

Now, choose \( i_1 \in \mathbb{N} \) so that \( d(x_{i_1}, (r + 2)v) < \frac{1}{6} \), whence
\[ d(x_{i_1}, A) \geq d(x_{i_1}, rB_X) \]
\[ \geq d((r + 2)v, rB_X) - d(x_{i_1}, (r + 2)v) > 2 - \frac{1}{6} = \frac{11}{6}. \]

Moreover, choose \( j_1 \in \mathbb{N} \) so that if \( j \geq j_1 \) then \( |\beta(A_j) - r| < \frac{1}{6} \), whence
\[ d(x_{i_1}, f(A_j, n_j)) \leq d(x_{i_1}, (\beta(A_j) + 2)v + e_{nj}) \]
\[ \leq d(x_{i_1}, (r + 2)v) + d((r + 2)v, (\beta(A_j) + 2)v) + \|e_{nj}\| \]
\[ < \frac{1}{6} + |\beta(A_j) - r| + 1 < \frac{4}{3}. \]

hence it follows that
\[ d_W(f(A_j, n_j), A) \geq \min\{2^{-i_1}, |d(x_{i_1}, f(A_j, n_j)) - d(x_{i_1}, A)|\} \]
\[ \geq \min\{2^{-i_1}, \frac{11}{6} - \frac{4}{3}\} = 2^{-i_1}. \]

This contradicts the fact that \( f(A_j, n_j) \to A \). Thus, \( \{f(CldW(X) \times \{n\}) \mid n \in \mathbb{N}\} \) is locally finite in \( CldW(X) \). \( \Box \)

4. Proof of Theorem II for \( BddW(X) \)

In this section, Theorem II is proved for the space \( BddW(X) \). Since Theorem I has been proved, we can use Theorem 2.5. We first show the following:

**Proposition 4.1.** For every separable normed linear space \( X \), \( BddW(X) \) is homotopy dense in \( CldW(X) \).

**Proof.** We define \( \theta : Cl(W(X) \times I \to Cl(W(X) by \( \theta_0 = id \) and \( \theta(A, t) = \{A \cap t^{-1}B_X\} \cup \{t^{-1}S_X\} \) for \( t > 0 \).

When \( t \neq 0 \), observe that

\[ d(x, \theta(A, t)) = \begin{cases} \min\{t^{-1} - \|x\|, d(x, A)\} & \text{if } \|x\| \leq t^{-1}, \\ \|x\| - t^{-1} & \text{if } \|x\| > t^{-1}. \end{cases} \]

It is easy to see that \( (A, t) \mapsto d(x, \theta(A, t)) \) is continuous for each \( x \in X \). Thus, \( \theta \) is continuous. \( \Box \)

Notice that \( BddW(X) = \bigcup_{k \in \mathbb{N}} Cl(kB_X) \).
Proposition 4.2. For every separable Banach space $X$, $\text{Bdd}_W(X)$ is $\sigma$-completely metrizable.

Proof. The space $\text{Cld}_W(X)$ is completely metrizable (Proposition 2.1). For each $k \in \mathbb{N}$, $\text{Cld}(kB_X)$ is closed in $\text{Cld}_W(X)$ by Proposition 2.2, hence it is completely metrizable. Then it follows that $\text{Bdd}_W(X)$ is $\sigma$-completely metrizable. □

Proposition 4.3. For every separable normed linear space $X$, $\text{Bdd}_W(X)$ is a $\text{Z}_{\sigma}$-space.

Proof. Let $\theta : \text{Cld}_W(X) \times [0,1] \to \text{Cld}_W(X)$ be the homotopy defined in the proof of Proposition 4.1. Then $\theta(Bdd_W(X) \times [0,1]) \subseteq Bdd_W(X)$. For each map $\alpha : Bdd_W(X) \to (0,1)$, define $\eta : Bdd_W(X) \to (0,1]$ as follows:

$$\eta(A) = \sup\{t > 0 | \text{diam}_{d_W} \theta([A] \times [0,t]) < \alpha(A)\}.$$ 

Then $\eta$ is lower semi-continuous. Indeed, when $\eta(A) > s$, let $s < s' < \eta(A)$ and

$$\varepsilon = \alpha(A) - \text{diam}_{d_W} \theta([A] \times [0,1]) > 0.$$ 

By the continuity of $\theta$ and $\alpha$, we have $\delta > 0$ such that if $A' \in Bdd_W(X)$ and $d_W(A, A') < \delta$ then $|\alpha(A) - \alpha(A')| < \frac{1}{2}\varepsilon$ and $d_W(\theta(A,t), \theta(A',t)) < \frac{1}{2}\varepsilon$ for all $t \in [0,s']$, whence

$$\text{diam}_{d_W} \theta([A'] \times [0,s']) < \text{diam}_{d_W} \theta([A] \times [0,s']) + \frac{2}{3}\varepsilon$$

$$= \alpha(A) - \frac{1}{3}\varepsilon < \alpha(A'),$$

which means that $\eta(A') \geq s' > s$.

For each $k \in \mathbb{N}$, taking a map $\beta : Bdd_W(X) \to (0,1)$ such that

$$\beta(A) < \min\{(k + 1)^{-1}, \eta(A)\},$$

we define a map $f : Bdd_W(X) \to Bdd_W(X)$ as follows:

$$f(A) = \theta(A, \beta(A)) = (A \cap \beta(A)^{-1}B_X) \cup \beta(A)^{-1}S_X.$$ 

For each $A \in Bdd_W(X)$, it follows from $\beta(A)^{-1} > k + 1$ that $f(A) \not\subseteq kB_X$, that is, the image of $f$ misses $\text{Cld}(kB_X)$. Moreover, since $\beta(A) < \eta(A)$, we have

$$d_W(f(A), A) \leq \text{diam}_{d_W} \theta([A] \times [0, \beta(A)]) < \alpha(A).$$

Thus, $\text{Cld}(kB_X)$ is a $\text{Z}_{\sigma}$-set in $Bdd_W(X)$. Then it follows that $Bdd_W(X)$ is a $\text{Z}_{\sigma}$-space. □

It remains to prove the strong universality of $Bdd_W(X)$. To this end, we use the following well-known fact [6, Chapter VI, Theorem 5.1]:

Proposition 4.4. Let $S_X$ be the unit sphere of an infinite-dimensional separable Banach space $X$. Then $S_X \approx \ell_2$.

Proposition 4.5. If $X$ is an infinite-dimensional separable Banach space, then every open set $W \subset Bdd_W(X)$ is universal for separable completely metrizable spaces.
Proof. Let $Y$ be a separable completely metrizable space and $f : Y \to W$ a map. For each map $\alpha : W \to (0, 1)$, let $\gamma : W \to (1, \infty)$ be the map obtained by Lemma 3.1, where $\mathfrak{S} = \text{Bdd}_W(X)$. We have a map $\beta : W \to (0, 1)$ such that $\beta(A)^{-1} > \gamma(A)$ for each $A \in W$. Since $S_X \approx \ell_2$ is universal for separable completely metrizable spaces (Proposition 4.4), we have a closed embedding $h : Y \to S_X$. By taking $v \in S_X$ and using the homotopy $\theta : \text{Cld}_W(X) \times I \to \text{Cld}_W(X)$ defined in the proof of Proposition 4.1, we define a map $g : Y \to W$ as follows:

$$g(y) = \theta(f(y), \beta(f(y))) \cup \{\beta(f(y))^{-1} + 2\}v + h(y).$$

Then it follows from $(\ast)$ in Lemma 3.1 that $d_W(f(y), g(y)) < \alpha(f(y))$ for every $y \in Y$. As is easily observed, $g$ is injective.

To see that $g$ is a closed embedding, let $y_i \in Y$, $i \in \mathbb{N}$, such that $g(y_i) \to A \in W$. We may assume that $\lim_{i \to \infty} \beta(f(y_i)) = b \in I$. If $b = 0$ then $\beta(f(y_i))^{-1} \to \infty$, which implies $d_W(f(y_i), g(y_i)) \to 0$, so $f(y_i) \to A$. Hence, $\beta(f(y_i)) \to \beta(A) \neq 0$, which is a contradiction. Therefore, $b \neq 0$. Furthermore, $A \cap ((b^{-1} + 2)v + \frac{1}{2}B_X) \neq \emptyset$. In fact, if $A \cap (U^+(b^{-1} + 2)v, \frac{1}{2})$ then $g(y_i) \in U^+(b^{-1} + 2)v, \frac{1}{2})$ for sufficiently large $i \in \mathbb{N}$. On the other hand, for sufficiently large $i \in \mathbb{N}$, $|\beta(f(y_i))^{-1} - b^{-1}| < \frac{1}{2}$, whence

$$\|\beta(f(y_i))^{-1} + 2\|v + h(y_i) - (b^{-1} + 2)v\|$$

$$\leq |\beta(f(y_i))^{-1} - b^{-1}| + 1 < \frac{3}{2}.$$

This is a contradiction.

Now, let $c \in A \cap ((b^{-1} + 2)v + \frac{1}{2}B_X)$. Then $\|c\| > b^{-1} + \frac{1}{2}$. For each $0 < \epsilon < \frac{1}{4}$, we can choose $i_0 \in \mathbb{N}$ so that if $i \geq i_0$ then $g(y_i) \in U^-(c, \epsilon)$ and $|\beta(f(y_i))^{-1} - b^{-1}| < \epsilon < \frac{1}{4}$. For each $i \geq i_0$, since

$$d(c, \theta(f(y_i), \beta(f(y_i)))) \geq \|c\| - \beta(f(y_i))^{-1}$$

$$\geq \frac{1}{2} + b^{-1} - \beta(f(y_i))^{-1} > \frac{1}{4} > \epsilon,$$

it follows that

$$d(c, \beta(f(y_i))^{-1} + 2)v + h(y_i)) = d(c, g(y_i)) < \epsilon.$$

Thus, $(\beta(f(y_i))^{-1} + 2)v + h(y_i) \to c$. Since $\beta(f(y_i))^{-1} \to b^{-1}$, it follows that $h(y_i) \to c - (b^{-1} + 2)v$, hence $(y_i)_{i \in \mathbb{N}}$ is convergent in $Y$ because $h$ is a closed embedding. Therefore, $g$ is a closed embedding.

By Proposition 4.3, $g(Y)$ is a $Z_\sigma$-set in $W$. Since $g(Y) \approx Y$ is completely metrizable, $g(Y)$ is a $Z$-set in $W$ by [9, Lemma 2.4]. Thus, $g : Y \to W$ is a $Z$-embedding. This completes the proof. \qed

Proof of Theorem II for Bdd$_W(X)$. By Theorem I and Proposition 4.1, Bdd$_W(X)$ can be regarded as a homotopy dense set in $\ell_2$. It has been shown that Bdd$_W(X)$ is a $\sigma$-completely metrizable $Z_\sigma$-space (Propositions 4.2 and 4.3), and every open set in Bdd$_W(X)$ is universal for separable completely metrizable spaces (Proposition 4.5). Thus, we have $\text{Bdd}_W(X) \approx \ell_2 \times \ell_2$ by Theorem 2.5. \qed
5. Homotopy dense subsemilattices of Lawson semilattices

A topological semilattice is a topological space $S$ equipped with a continuous operator $\lor : S \times S \to S$ which is idempotent, commutative and associative (i.e., $x \lor x = x$, $x \lor y = y \lor x$, $(x \lor y) \lor z = x \lor (y \lor z)$). A topological semilattice $S$ is called a Lawson semilattice if $S$ admits an open basis consisting of subsemilattices [15].

According to Proposition 3.2 in [3], a metrizable Lawson semilattice $X$ is an ANR (respectively an AR) if and only if it is $LC^0$, i.e., locally path-connected (respectively connected and $LC^0$). Here, we introduce a relative version of local path-connectedness. A subset $Y \subset X$ is relatively $LC^0$ in $X$ if for every $x \in X$, each neighborhood $U$ of $x$ in $X$ contains a smaller neighborhood $V$ of $x$ such that every two points of $V \cap Y$ can be joined by a path in $U \cap Y$. If $Y \subset X$ is relatively $LC^0$ in $X$ then $Y$ is $LC^0$, but the converse is not true. For example, $Y = \{ n^{-1} \mid n \in \mathbb{N} \} \subset \mathbb{R}$ is $LC^0$ but $Y$ is not relatively $LC^0$ in $\mathbb{R}$. In this section, we shall prove the following theorem.

**Theorem 5.1.** Let $X$ be a metrizable Lawson semilattice with $Y \subset X$ a dense subsemilattice. If $Y$ is relatively $LC^0$ in $X$ (and path-connected), then $X$ is an ANR (an AR) and $Y$ is homotopy dense in $X$, hence $Y$ is also an ANR (an AR).

To prove Theorem 5.1, we use the following result from [19]:

**Theorem 5.2.** Let $X$ be a metric space with $Y \subset X$ a dense set. Assume that there exist a sequence $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ of open covers of $X$ and a map $f : |TN(\mathcal{U})| \to Y$ such that

1. $\lim_{n \to \infty} \text{mesh} U_n = 0$,
2. $f(U) \in U \cap Y$ for $U \in \bigcup_{n \in \mathbb{N}} U_n = TN(\mathcal{U})^{(0)}$ and
3. $\lim_{n \to \infty} \text{mesh}(f(\sigma) \mid \sigma \in N(U_n \cup U_{n+1})) = 0$,

where $TN(\mathcal{U}) = \bigcup_{n \in \mathbb{N}} N(U_n \cup U_{n+1})$ is the simplicial complex defined as the union of the nerves of the covers $U_n \cup U_{n+1}$ (we regard $U_i \cap U_j = \emptyset$ as the sets of vertices if $i \neq j$). Then $X$ is an ANR and $Y$ is homotopy dense in $X$.

**Lemma 5.3.** Every Lawson semilattice $X$ is $k$-aspherical for any $k > 0$, that is, each map $f : S^k \to X$ extends to $\tilde{f} : B^{k+1} \to X$. Hence, if $X$ is path-connected then it is $k$-connected for every $k \geq 0$.

**Proof.** Identify $S^k = \text{Fin}_V^n(S^k) \subset \text{Fin}_V(S^k)$ and $X = \text{Fin}_V(X) \subset \text{Fin}_V(X)$. Then $f$ extends to a map $f_\ast : \text{Fin}_V(S^k) \to \text{Fin}_V(X)$. Since $\text{Fin}_V(X)$ is a free Lawson semilattice over $X$, we have a retraction $r : \text{Fin}_V(X) \to X$ as an extension of the identity map of $X$ (see [3, Section 3]). By [10, Lemma 3.3], we have $\varphi : B^{k+1} \to \text{Fin}_V^n(S^k)$ with $\varphi|S^k = \text{id}$. Then $\tilde{f} = rf_\ast \varphi : B^{k+1} \to X$ is an extension of $f$. \qed

**Proof of Theorem 5.1.** Let $d$ be an admissible metric for $X$. For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be an open cover of $X$ such that $\text{mesh} \mathcal{V}_n < 2^{-n}$ and each $V \in \mathcal{V}_n$ is a subsemilattice of $X$. Since $Y$ is relatively $LC^0$ in $X$, each $\mathcal{V}_n$ has an open refinement $U_n$ such that each $U \in U_n$ is
contained in some \( V \in \mathcal{V}_n \) so that every pair of points in \( U \cap Y \) can be connected by a path in \( V \cap Y \). Applying Lemma 5.3, we can inductively construct maps \( f_n : \mathcal{N}(U_n \cup U_{n+1}) \rightarrow Y \), \( n \in \mathbb{N} \), so that

1. \( f_n|\mathcal{N}(U_{n+1}) = f_{n+1}|\mathcal{N}(U_{n+1}) \)
2. \( f_n(U) \in U \cap Y \) for \( U \in \mathcal{U}_n \cup \mathcal{U}_{n+1} \)
3. For every simplex \( \sigma \in \mathcal{N}(U_n \cup U_{n+1}) \), the image \( f_n(\sigma) \) is contained in a member of \( \mathcal{V}_n \).

Then we define a map \( f : |TN(U)| \rightarrow Y \) by \( f|\mathcal{N}(U_{n+1}) = f_n \), which satisfies the assumptions of Theorem 5.2, hence \( X \) is an ANR and \( Y \) is homotopy dense in \( X \).

Moreover, if \( X \) or \( Y \) is path-connected, then it is \( k \)-connected for every \( k \geq 0 \) by Lemma 5.3. Thus \( X \) is an AR. \( \square \)

Let \( X = (X, d) \) be a metric space. To apply Theorem 5.1 to the space \( \text{Cld}_W(X) \), we show the following:

**Proposition 5.4.** For an arbitrary metric space \( X \), the space \( \text{Cld}_W(X) \) is a Lawson semilattice with respect to the union operator \( \cup \).

**Proof.** For each \( x \in X \),

\[
(A, B) \mapsto d(x, A \cup B) = \min\{d(x, A), d(x, B)\}
\]

is continuous, which means that the union operator \( \cup \) is continuous (cf. [5, Exercise 2.1.3(a)]). Observe that \( U^-(x, r) \) and \( U^+(x, r) \) are subsemilattices of \( \text{Cld}_W(X) \) for each \( x \in X \) and \( r > 0 \). Since any intersection of subsemilattices is again a subsemilattice, \( \text{Cld}_W(X) \) has an open basis consisting of subsemilattices. Thus, \( \text{Cld}_W(X) \) is a Lawson semilattice. \( \square \)

For an arbitrary subset \( Y \subset X \), we have \( \text{Fin}_W(Y) \subset \text{Comp}(Y) \subset \text{Cld}_W(X) \). It should be noticed that \( \text{Cld}(Y) \not\subset \text{Cld}(X) \) unless \( Y \) is closed in \( X \). As we mentioned in Preliminaries, \( \text{Fin}_W(Y) \) (or \( \text{Cld}_W(Y) \)) is not a subspace of \( \text{Cld}_W(X) \) even if \( Y \) is closed in \( X \), where \( Y \) inherits the metric from \( X \). However, we have the following:

**Proposition 5.5.** Let \( Y \) be a dense subspace of an arbitrary metric space \( X \). Then \( \text{Cld}_W(Y) \) can be naturally embedded in \( \text{Cld}_W(X) \) by the closure operator \( \text{cl}_X : \text{Cld}_W(Y) \rightarrow \text{Cld}_W(X) \). Namely, \( \text{Cld}_W(Y) \) can be identified with the subspace

\[
\{ A \in \text{Cld}(X) | \text{cl}_X(A \cap Y) = A \} \subset \text{Cld}_W(X).
\]

**Proof.** The continuity of \( \text{cl}_X : \text{Cld}_W(Y) \rightarrow \text{Cld}_W(X) \) can be seen as follows: For each \( \varepsilon > 0 \) and \( x \in X \), choose \( y \in Y \) so that \( d(x, y) < \frac{1}{4}\varepsilon \). For each \( A, A' \in \text{Cld}_W(Y) \),

\[
|d(y, A) - d(y, A')| < \frac{1}{4}\varepsilon \quad \text{implies}
\]

\[
|d(x, \text{cl}_X A) - d(x, \text{cl}_X A')| \leq 2d(x, y) + |d(y, A) - d(y, A')| < \varepsilon.
\]
On the other hand, the restriction operator \( C(X) \ni f \mapsto f|Y \in C(Y) \) is continuous with respect to the topology of point-wise convergence. Restricting this operator to the image of \( \text{cl}_X \), we have the inverse of the operator \( \text{cl}_X \). Thus, we have the result.

**Remark.** In the separable case, we can use a countable dense subset of \( Y \) to define the metric \( d_W \) for \( \text{Cld}_W(X) \), whence the above embedding is an isometry.

**Proposition 5.6.** For a dense subset \( Y \) of an arbitrary metric space \( X \), \( \text{Fin}_W(Y) \) is a dense subsemilattice of \( \text{Cld}_W(X) \).

**Proof.** Let \( A \in \text{Cld}_W(X) \). For each neighborhood \( W \) of \( A \) in \( \text{Cld}_W(X) \), we have

\[
A \in \bigcap_{i=1}^{m} U^-(p_i, r_i) \cap \bigcap_{j=m+1}^{n} U^+(p_j, r_j) \subset W.
\]

For each \( 1 \leq i \leq m \), \( B(p_i, r_i) \setminus \bigcup_{j=m+1}^{n} \overline{B}(p_j, r_j) \neq \emptyset \) because it contains a point of \( A \). Since \( Y \) is dense in \( X \), we can find \( y_i \in Y \cap B(p_i, r_i) \setminus \bigcup_{j=m+1}^{n} \overline{B}(p_j, r_j) \).

Thus, we have \( \{y_1, \ldots, y_m\} \in \text{Fin}_W(Y) \cap W \). Therefore, \( \text{Fin}_W(Y) \) is dense in \( \text{Cld}_W(X) \).

As an application of Theorem 5.1, we have the following:

**Corollary 5.7.** Let \( X \) be a separable metric space with \( Y \subset X \) a dense subset. If \( \text{Fin}_W(Y) \) is relatively \( \text{LC}^0 \) in \( \text{Cld}_W(X) \) (and \( \text{Fin}_W(Y) \) is path-connected), then \( \text{Cld}_W(X) \) is an ANR (an AR) and \( \text{Fin}_W(Y) \) is homotopy dense in \( \text{Cld}_W(X) \), hence \( \mathcal{H} \) is an ANR (an AR) if \( \text{Fin}_W(Y) \subset \mathcal{H} \subset \text{Cld}_W(Y) \).

### 6. Proof of Theorem III

In this section, we prove Theorem III by using Corollary 5.7. Given a metric space \( X \), let \( d \) be the metric for \( X \).

To prove the next lemma, we shall use a well-known combinatorial fact, called König’s Lemma: Every finitely-branching infinite tree contains an infinite branch (cf. [13, (4.12)]). A tree is a partially ordered set \((T, \prec)\) such that for every \( x \in T \), the set \( \{y \in T \mid y \prec x\} \) is well-ordered (in our case: finite and linearly ordered). It is said that \( T \) is finitely-branching if each element of \( T \) has only finitely many immediate successors. A branch of \( T \) is a maximal linearly ordered subset of \( T \). In the proof of the following theorem, we consider a tree of sets with the reversed inclusion as the partial order.

**Lemma 6.1.** Let \( X \) be a path-connected unbounded metric space. Suppose that the complement of each ball in \( X \) has only finitely many path-components. Then, for each \( x \in X \), there exists an infinite path \( f : [1, \infty) \to X \) such that \( f(1) = x \) and \( \lim_{t \to \infty} d(x, f(t)) = \infty \).
Let \( T = \{ H^n_i \mid n \in \mathbb{N}, \ i = 1, \ldots, k(n) \} \). Then \( (T, \supseteq) \) is a finitely-branching infinite tree. By König’s Lemma, \( T \) contains an infinite branch \( H^1 \supset H^2 \supset \cdots \).

For each \( n \in \mathbb{N} \), pick \( v_n \in H^n_i \) and a path \( f_n : [1, \infty) \rightarrow H^n_{\supseteq} \) with \( f_n(0) = v_{n-1} \) and \( f_n(1) = v_0 \), where \( H^n_{\supseteq} = X \) and \( v_0 = x \). By joining these paths, we can define an infinite path \( f : [1, \infty) \rightarrow X \), that is, \( f(t) = f_n(t - n) \) for \( n \leq t \leq n + 1 \). Then \( f(1) = x \) and \( \lim_{t \to \infty} d(x, f(t)) = \infty \). Indeed, for each \( m \in \mathbb{N} \),

\[
\begin{align*}
f([m + 1, \infty)) & \subseteq \bigcup_{n=m}^{\infty} f_n([1, \infty)) \subseteq H^n_{\supseteq} \setminus X(x, m),
\end{align*}
\]

that is, \( d(x, f(t)) \geq m \) for \( t \geq m + 1 \).  

Applying Lemma 6.1, we can prove the following:

**Proposition 6.2.** Let \( X \) be a metric space which has only finitely many path-components \( X_1, \ldots, X_m \), each of which is unbounded and satisfies the condition of Lemma 6.1, that is, the complement of each ball in \( X_i \) has finitely many path-components. Then \( \text{Fin}_W(X) \) is path-connected.

**Proof.** Take any \( a \in X_1 \). It suffices to show that each \( B \in \text{Fin}_W(X) \) can be connected to \( \{a\} \) by a path in \( \text{Fin}_W(X) \). By Lemma 6.1, we have an infinite path \( f_0 : [1, \infty) \rightarrow X_1 \) such that \( f_0(1) = a \) and \( \lim_{t \to \infty} d(a, f_0(t)) = \infty \). Observe that \( \lim_{t \to \infty} d(x, f_0(t)) = \infty \) for every \( x \in X \). Then \( B \) can be connected to \( B \cup \{a\} \) by the path \( \varphi : [0, \infty) \rightarrow \text{Fin}_W(X) \) defined as follows:

\[
\varphi(t) = \begin{cases} 
B & \text{if } t = 0, \\
B \cup \{ f_0(t-1) \} & \text{if } t > 0.
\end{cases}
\]

For each \( x \in X \), \( d(x, \varphi(t)) = d(x, B) \) for sufficiently small \( t > 0 \), which implies that \( \varphi \) is continuous at \( t = 0 \).

Each \( z \in B \) is contained in some \( X_{k(z)} \). By Lemma 6.1, we have an infinite path \( f_z : [1, \infty) \rightarrow X_{k(z)} \) such that \( f_z(1) = z \) and \( \lim_{t \to \infty} d(z, f_z(t)) = \infty \) (whence \( \lim_{t \to \infty} d(x, f_z(t)) = \infty \) for any \( x \in X \)). Then \( \{a\} \) is connected to \( B \cup \{a\} \) by the path \( \psi : [0, \infty) \rightarrow \text{Fin}_W(X) \) defined similarly to \( \varphi \), that is,

\[
\psi(t) = \begin{cases} 
\{a\} & \text{if } t = 0, \\
\{a\} \cup \{ f_z(t-1) \} & \text{if } t > 0.
\end{cases}
\]

By joining the paths \( \varphi \) and \( \psi \), we can obtain a path in \( \text{Fin}_W(X) \) from \( B \) to \( \{a\} \).  

If \( X \) is path-connected then so is \( \text{Fin}_W(X) \). In fact, we can prove the following in a little more general setting for applications:

**Proposition 6.3.** Let \( Y \) be a path-connected subset of an arbitrary metric space \( X \). For each pair \( A, B \in \text{Fin}(Y) \), there exists a path \( \varphi : [0, \infty) \rightarrow \text{Fin}_W(X) \) such that \( \varphi(0) = A \), \( \varphi(1) = B \),
\(A \subset \varphi(t)\) for \(t \leq \frac{1}{2}\), \(B \subset \varphi(t)\) for \(t \geq \frac{1}{2}\), and \(\varphi(t) \subset Y\) for each \(t \in \mathbf{I}\), i.e., \(\varphi(t) \in \text{Fin}(Y)\). In particular, if \(X\) is path-connected then so is \(\text{Fin}_W(X)\).

**Proof.** For each \(a \in A\) and \(b \in B\), we have a path \(f_{a,b} : \mathbf{I} \to Y\) from \(a\) to \(b\). Now, we define a path \(\varphi' : \mathbf{I} \to \text{Fin}_W(X)\) by

\[
\varphi'(t) = A \cup \bigcup \{ f_{a,b}(t) \mid a \in A, b \in B \}.
\]

Then \(\varphi'\) is a path in \(\text{Fin}_W(X)\) from \(A\) to \(B\) such that \(A \subset \varphi'(t) \subset Y\) for each \(t \in \mathbf{I}\).

By the same argument, we can find a path \(\varphi'' : \mathbf{I} \to \text{Fin}_W(X)\) from \(B\) to \(A\) such that \(B \subset \varphi''(t) \subset Y\) for each \(t \in \mathbf{I}\). By joining the paths \(\varphi'\) and \(\varphi''\), we obtain the desired path \(\varphi\) connecting \(A\) and \(B\). \(\square\)

It is said that \(A, B \subset X\) are **strongly disjoint** if

\[\text{dist}(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\} > 0.\]

This notion is used in the proof of the following proposition:

**Proposition 6.4.** Let \(X\) be a metric space. Suppose that the complement of any finite union of open balls in \(X\) has only finitely many path-components, each of which is compact or unbounded closed. Then \(\text{Fin}_W(X)\) is relatively \(LC^0\) in \(\text{Cld}_W(X)\).

**Proof.** Let \(A \in \text{Cld}_W(X)\) and \(U\) a neighborhood of \(A\) in \(\text{Cld}_W(X)\). Then

\[A \in \bigcap_{i=1}^{m} U^{-}(p_i, r_i) \cap \bigcap_{j=1}^{n} U^{+}(q_j, s_j) \subset U,\]

for some \(p_i \in A, q_j \in X \setminus A\) and \(r_i, s_j > 0\). Let

\[0 < \varepsilon < \min\{d(q_j, A) - s_j \mid j = 1, \ldots, n\}\]

and let \(X_1, \ldots, X_l\) be all path-components of \(X \setminus \bigcup_{j=1}^{m} B(q_j, s_j + \varepsilon)\), where \(X_k \cap A \neq \emptyset\) for \(k \leq l_0\) and \(X_k \cap A = \emptyset\) for \(k > l_0\). Moreover, we can assume that \(X_k\) is unbounded for \(l_0 < k \leq l_1\) and \(X_k\) is compact for \(k > l_1\). Note that each \(X_k\) is not only closed but also open in \(X \setminus \bigcup_{j=1}^{m} B(q_j, s_j + \varepsilon)\) and \(A\) is strongly disjoint from \(\bigcup_{j=1}^{m} B(q_j, s_j + \varepsilon)\). For each \(k = 1, \ldots, l_0\), we can choose \(p_{m+k} \in A \cap X_k\) and \(r_{m+k} > 0\) so that \(B(p_{m+k}, r_{m+k}) \subset X_k\).

Since \(\bigcup_{k=l_1+1}^{l} X_k\) is compact and \(X \setminus \bigcup_{k=l_1+1}^{l} X_k\) is open in \(X\),

\[X_k \subset \bigcup_{i=1}^{v} B(y_i, \delta_i) \subset \bigcup_{i=1}^{v} B(y_i, 2\delta_i) \subset X \setminus \bigcup_{k=1}^{l_1} X_k,\]

for some \(y_i \in \bigcup_{k=l_1+1}^{l} X_k\) and \(\delta_i > 0\). Thus, we have the following neighborhood of \(A\):\(^9\)

\[V = \bigcap_{i=1}^{m+l_0} U^{-}(p_i, r_i) \cap \bigcap_{j=1}^{n} U^{+}(q_j, s_j + \varepsilon) \cap \bigcap_{i=1}^{v} U^{+}(y_i, \delta_i) \subset U.\]

\(^9\) See Footnote 6.
Moreover, since $p_i \in A$ for every $i = 1, \ldots, m + l_0$, we have
\[ A_0 = \{ p_i \mid i = 1, \ldots, m + l_0 \} \in V \cap Fin_W (X). \]

We show that every $B \in V \cap Fin_W (X)$ can be connected to $A_0$ by a path in $U \cap Fin_W (X)$. Then it would follow that $Fin_W (X)$ is relatively $LC^0$ in $Cld_W (X)$. Notice $B \subseteq \bigcup_{k=1}^{l_0} X_k$.

Let $B^* = B \cap \bigcup_{k=l_0+1}^{l_1} X_k$. For $k = 1, \ldots, l_0$, we apply Proposition 6.3 to obtain a path $\varphi_k : I \to Fin_W (X)$ such that $\varphi(0) = B \cap X_k$, $\varphi(1) = A_0 \cap X_k$, $B \cap X_k \subset \varphi(t)$ for $0 \leq t \leq \frac{1}{2}$, $A_0 \cap X_k \subset \varphi(t)$ for $\frac{1}{2} \leq t \leq 1$, and $\varphi(t) \subset X_k$ for each $t \in I$. Then $B$ can be connected to $A_0 \cup B^*$ by the path $\psi : I \to Fin_W (X)$ defined by $\psi(t) = \bigcup_{k=1}^{l_0} \varphi_k(t) \cup B^*$. Since each $\varphi(t)$ contains $A_0$ or $B$ and
\[
\varphi(t) \subseteq \bigcup_{k=1}^{l_0} X_k \cup B^* \subseteq \bigcup_{k=1}^{l_1} X_k \setminus \bigcup_{j=1}^{n} B(q_j, s_j + \varepsilon),
\]

it follows that $\psi (I) \subseteq U \cap Fin_W (X)$.

For each $z \in B^*$, let $X_k(z)$ be the path-component containing $z$, where $l_0 < k(z) \leq l_1$. By Lemma 6.1, we have an infinite path $f_z : [1, \infty) \to X_k(z)$ such that $f_z(1) = z$ and $\lim_{t \to \infty} d(z, f_z(t)) = \infty$ (whence $\lim_{t \to \infty} d(x, f_z(t)) = \infty$ for any $x \in X$). Then $A_0$ can be connected to $A_0 \cup B^*$ by the path $\psi : I \to V$ defined as follows:
\[
\psi(t) = \begin{cases} 
A_0 & \text{if } t = 0, \\
A_0 \cup \bigcup_{z \in B^*} f_z(t^{-1}) & \text{if } t > 0.
\end{cases}
\]

For any $x \in X$, $d(x, \psi(t)) = d(x, A_0)$ for sufficiently small $t > 0$, which means that $\psi$ is continuous at 0. Since $A_0 \subset \psi(t) \subset \bigcup_{k=1}^{l_0} X_k$ for each $t \in I$, it follows that $\psi (I) \subseteq U \cap Fin_W (X)$. By joining paths $\varphi$ and $\psi$, we have a path in $U \cap Fin_W (X)$ from $B$ to $A_0$.

**Proof of Theorem III.** Under the assumption of Proposition 6.4, $Cld_W (X)$ is an ANR and $Fin_W (X)$ is homotopy dense in $Cld_W (X)$ by Corollary 5.7. When the complement of any finite union of open balls in $X$ has only finitely many path-components and all of them are unbounded closed, $X$ satisfies the assumption of Proposition 6.2 (as the number of open balls is zero), hence $Fin_W (X)$ is path-connected, so it is an AR by Corollary 5.7. Thus, we have Theorem III. \hfill \Box

7. **Proof of Theorem II for $Fin_W (X)$**

To prove Theorem II for the space $Fin_W (X)$, we apply Theorem 2.5 like in the case of $Bdd_W (X)$ but we also use Theorem III proved in the previous section.

Now, notice $Fin_W (X) = \bigcup_{k \in \mathbb{N}} Fin^k (X)$.

**Proposition 7.1.** For an arbitrary metric space $X$, each $Fin^k (X)$ is closed in $Cld_W (X)$. Therefore, if $X$ is separable and complete then each $Fin^k (X)$ is completely metrizable, so $Fin_W (X)$ is $\sigma$-completely metrizable.
Proof. For each \( A \in \text{Cld}_W(X) \setminus \text{Fin}^k(X) \), choose distinct \( k + 1 \) many points \( a_1, \ldots, a_{k+1} \in A \) and let \( r = \frac{1}{2} \min\{d(a_i, a_j) \mid i \neq j\} > 0 \). Then
\[
A \in \bigcap_{i=1}^{k+1} U^- (a_i, r) \subset \text{Cld}_W(X) \setminus \text{Fin}^k(X).
\]
Therefore, \( \text{Cld}_W(X) \setminus \text{Fin}^k(X) \) is open in \( \text{Cld}_W(X) \), that is, \( \text{Fin}^k(X) \) is closed in \( \text{Cld}_W(X) \). \( \square \)

Proposition 7.2. For every separable normed linear space \( X \), \( \text{Fin}_W(X) \) is a \( Z_{\sigma} \)-space.

Proof. It suffices to show that each \( \text{Fin}^k(X) \) is a \( Z \)-set in \( \text{Fin}_W(X) \). By using distinct \( k + 1 \) many points \( v_1, \ldots, v_{k+1} \in X \setminus \{0\}, \) we define a homotopy \( \zeta: \text{Fin}_W(X) \times I \to \text{Fin}_W(X) \) as follows:
\[
\zeta(A, t) = \begin{cases} A & \text{if } t = 0, \\ A \cup t^{-1} \{v_1, \ldots, v_{k+1}\} & \text{if } t > 0. \end{cases}
\]
For each map \( \alpha: \text{Fin}_W(X) \to (0, 1) \), define \( \eta: \text{Fin}_W(X) \to (0, 1] \) by
\[
\eta(A) = \sup \{ t > 0 \mid \text{diam}_{d_W} \zeta(A) \times [0, t] < \alpha(A) \}.
\]
Then \( \eta \) is lower semi-continuous. Indeed, if \( \eta(A) > s \) then we have \( s < s' < \eta(A) \) and \( \varepsilon = \alpha(A) - \text{diam}_{d_W} \zeta(A) \times [0, s'] > 0 \).

By the continuity of \( \zeta \) and \( \alpha \), we have \( \delta > 0 \) such that if \( A' \in \text{Fin}_W(X) \) and \( d_W(A, A') < \delta \) then \( |\alpha(A) - \alpha(A')| < \frac{1}{3} \varepsilon \) and \( d_W(\zeta(A, t), \zeta(A', t)) < \frac{1}{3} \varepsilon \) for all \( t \in [0, s'] \), whence
\[
\text{diam}_{d_W} \zeta(A) \times [0, s'] \leq \text{diam}_{d_W} \zeta(A) \times [0, s'] + \frac{2}{3} \varepsilon
\]
\[
= \alpha(A) - \frac{1}{3} \varepsilon < \alpha(A'),
\]
which means that \( \eta(A') \geq s' > s \).

Now, we define a map \( f: \text{Fin}_W(X) \to \text{Fin}_W(X) \setminus \text{Fin}^k(X) \) as follows:
\[
f(A) = \zeta(A, \beta(A)) = A \cup \beta(A)^{-1} \{v_1, \ldots, v_{k+1}\}.
\]
where \( \beta: \text{Fin}_W(X) \to (0, 1) \) is a map such that \( \beta(A) < \eta(A) \). Observe that \( d_W(f(A), A) < \alpha(A) \) for each \( A \in \text{Fin}_W(X) \). This completes the proof. \( \square \)

Remark. In case \( X \) is an infinite-dimensional separable Banach space, we have \( \text{Cld}_W(X) \approx \ell_2 \) by Theorem I. On the other hand, \( \text{Fin}_W(X) \) is homotopy dense in \( \text{Cld}_W(X) \) by Theorem III. Due to the remark before Theorem 2.5, it follows that every \( Z \)-set in \( \text{Fin}_W(X) \) is a strong \( Z \)-set. Therefore, each \( \text{Fin}^k(X) \) is a strong \( Z \)-set in \( \text{Fin}_W(X) \). However, without infinite-dimensionality of \( X \), this can be obtained in the above proof by showing that
\[
\text{Fin}^k(X) \cap \text{cl} f(\text{Fin}_W(X)) = \emptyset.
\]

10 In case \( \dim X > 1 \), we can take \( v_1, \ldots, v_{k+1} \in S_X \).
To this end, assume that there exist $A_n \in \Fin_W(X)$, $n \in \mathbb{N}$, such that $A = \lim_{n \to \infty} f(A_n) \in \Fin^k(X)$. Then we shall find a contradiction in both cases $\inf_{n \in \mathbb{N}} \beta(A_n) = 0$ and $\inf_{n \in \mathbb{N}} \beta(A_n) > 0$.

When $\inf_{n \in \mathbb{N}} \beta(A_n) = 0$, we may assume that $\lim_{n \to \infty} \beta(A_n) = 0$. For each $\varepsilon > 0$, choose $i_0 \in \mathbb{N}$ so that $2^{-i_0} < \varepsilon$ and let

$$c = \min\{\|v_1\|, \ldots, \|v_{k+1}\|\} > 0.$$

Since $d_W(f(A_n), A) \to 0$ and $\beta(A_n)^{-1}c \to \infty$ as $n \to \infty$, we can find $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies d_W(f(A_n), A) < 2^{-i_0-1},$$

$$\beta(A_n)^{-1}c > \max\{\|x_i\| + d(x_i, A) + 2^{-i_0-1} | i = 1, \ldots, in\},$$

where $\{x_i | i \in \mathbb{N}\} \subset X$ is the dense set defining $d_W$. Then, for each $n \geq n_0$ and $i \leq i_0$,

$$\beta(A_n)^{-1}c - \|x_i\| > d(x_i, A) + 2^{-i_0-1} > d(x_i, f(A_n)).$$

hence $d(x_i, f(A_n)) = d(x_i, A_n)$, which implies $d_W(f(A_n), A_n) < 2^{-i_0-1}$. Since $d_W(f(A_n), A) < 2^{-i_0-1}$, it follows that $d_W(A_n, A) < 2^{-i_0} < \varepsilon$. Thus, $(A_n)_{n \in \mathbb{N}}$ converges to $A$, hence $\lim_{n \to \infty} \beta(A_n) = \beta(A) > 0$, which is a contradiction.

When $\inf_{n \in \mathbb{N}} \beta(A_n) > 0$, we may assume that $\lim_{n \to \infty} \beta(A_n) = b > 0$. Now, we shall show that $z_j = b^{-1}v_j \in A$ for each $j = 1, \ldots, k+1$, which would contradict $A \in \Fin^k(X)$.

Suppose $z_j \notin A$ and choose $i_j \in \mathbb{N}$ so that $\|x_{i_j} - z_j\| < \frac{1}{4}d(z_j, A)$, whence

$$d(x_{i_j}, A) \geq d(z_j, A) - \|x_{i_j} - z_j\| > \frac{3}{4}d(z_j, A).$$

For sufficiently large $n \in \mathbb{N}$,

$$|b^{-1} - \beta(A_n)^{-1}| \cdot \|v_j\| < \frac{1}{4}d(z_j, A),$$

and

$$d_W(A, f(A_n)) < \min\left\{2^{-i_j}, \frac{1}{4}d(z_j, A)\right\},$$

whence $|d(x_{i_j}, A) - d(x_{i_j}, f(A_n))| < \frac{1}{4}d(z_j, A)$. Since $\beta_n(A)^{-1}v_j \in f(A_n)$, we have

$$d(z_j, f(A_n)) \leq \|z_j - \beta_n(A)^{-1}v_j\| = |b^{-1} - \beta(A_n)^{-1}| \cdot \|v_j\| < \frac{1}{4}d(z_j, A),$$

whence it follows that

$$d(z_j, A) \leq |d(z_j, A) - d(z_j, f(A_n))| + d(z_j, f(A_n))$$

$$\leq 2\|z_j - x_{i_j}\| + |d(x_{i_j}, A) - d(x_{i_j}, f(A_n))| + d(z_j, f(A_n))$$

$$< \frac{1}{2}d(z_j, A) + \frac{1}{4}d(z_j, A) + \frac{1}{4}d(z_j, A) = d(z_j, A).$$

This is a contradiction.

**Proposition 7.3.** If $X$ is an infinite-dimensional separable Banach space, then every open set $W \subset \Fin_W(X)$ is universal for separable completely metrizable spaces.
Proof. Let $Y$ be a separable completely metrizable space and $f : Y \to W$ a map. For each map $\alpha : W \to (0, 1)$, let $\gamma : W \to (1, \infty)$ be the map obtained by Lemma 3.1, where $\beta = \text{Fin}_W(X)$. We take a map $\beta_0 : W \to (0, 1)$ such that $\beta_0(A)^{-1} > \gamma(A)$ for each $A \in W$. For $n \in \mathbb{N}$, we define a map $\beta_n : W \to I$ as follows:

$$\beta_n(A) = \min\{n + 1, 1, \beta_0(A), \beta_0(A) d\omega(A, \text{Fin}^n(nB_X))\}.$$  

Note that $n + 1 \leq \beta_n(A)^{-1}$ if $\beta_n(A) \neq 0$ and $\beta_n(A)^{-1} \leq \beta_{n+1}(A)^{-1}$ if $\beta_{n+1}(A) \neq 0$. For each $A \in W$, define $k(A) \in \mathbb{N}$ by

$$A \in \text{Fin}^{k(A)}(k(A)B_X) \setminus \text{Fin}^{k(A)-1}((k(A) - 1)B_X).$$

Then $\beta_n(A) \neq 0$ for $n < k(A)$ and $\beta_n(A) = 0$ for $n \geq k(A)$.

Since $S_X \approx \ell_2$ by Proposition 4.4, there exists a closed embedding $h : Y \to S_X$. Take $v_n \in S_X, n \in \mathbb{N} \cup \{0\}$, so that $\|v_n - v_m\| > \frac{1}{2}$ for $n \neq m$ (Riesz’ Theorem [16, p. 16]). For each $n \neq m$ and $2 \leq i \leq s$,

$$\|s v_m - t v_n\| \geq \max\{\|s v_m - \|t v_n\|, \|t v_m - v_n\|, \|s - t\| v_m\} \geq \max\{s - t, 1 - (s - t)\} \geq \frac{1}{2}.$$  

We define a map $g : Y \to \text{Fin}_W(X)$ as follows:

$$g(z) = f(z) \cup \left\{ (\beta_n(f(z))^{-1} + 2)v_n, \right. \left. (\beta_n(f(z))^{-1} + 1)v_n + \frac{1}{8}h(z) \bigg| n < k(f(z)) \right\},$$

where $\beta_0(f(z)) = 0$ for $n \geq k(f(z))$, which guarantees the continuity of $g$. By (i) in Lemma 3.1, $g$ is $\alpha$-close to $f$ and $g(z) \in W$. Observe that

$$(\beta_0(f(z))^{-1} + 2)v_0(f(z))^{-1}$$

is the farthest point of $g(z)$ from $0 \in X$ and, for each $n < k(f(z))$,

$$g(z) \cap \left( (\beta_n(f(z))^{-1} + 1)v_n + \frac{1}{8}S_X \right) = \left\{ (\beta_n(f(z))^{-1} + 1)v_n + \frac{1}{8}h(z) \right\}.$$  

To see that $g$ is injective, let $z \neq z' \in Y$. In case $k(f(z)) \neq k(f(z'))$, assume $k(f(z)) > k(f(z'))$. Then it follows that

$$(\beta_0(f(z))^{-1} + 2)v_0(f(z'))^{-1} \neq (\beta_0(f(z'))^{-1} + 2)B_X$$

and

$$g(z') \subset (\beta_0(f(z'))^{-1} + 2)B_X,$$

so $g(z) \neq g(z')$. When $k(f(z)) = k(f(z')) = k$, if

$$(\beta_{k-1}(f(z))^{-1} + 2)v_{k-1} \neq (\beta_{k-1}(f(z'))^{-1} + 2)v_{k-1}$$

then $g(z) \neq g(z')$ because they are the farthest points of $g(z)$ and $g(z')$ from $0 \in X$. Otherwise, we have

$$(\beta_{k-1}(f(z))^{-1} + 1)v_{k-1} = (\beta_{k-1}(f(z'))^{-1} + 1)v_{k-1},$$
which we denote by $p$. Since $h$ is an embedding, we have
\[ g(z) \cap \left( p + \frac{1}{8} S_X \right) = \left\{ p + \frac{1}{8} h(z) \right\} \not= \left\{ p + \frac{1}{8} h(z') \right\} = g(z') \cap \left( p + \frac{1}{8} S_X \right). \]

hence $g(z) \not= g(z')$.

To see that $g$ is a closed map, let $y_i \in Y$, $i \in \mathbb{N}$, and assume that $g(y_i) \to G \in W$. For each $n \in \mathbb{N} \cup \{0\}$, let
\[ b_n = \liminf_{i \to \infty} \beta_n(f(y_i)) \in [0, (n + 1)^{-1}]. \]
Then $1 \geq b_0 \geq b_1 \geq \cdots \geq 0$. Moreover, $b_0 > 0$. Otherwise, by taking a subsequence, we can assume that $\beta_0(f(y_i)) \to 0$. Then it follows that $f(y_i) \to G$, hence $\beta_0(f(y_i)) \to \beta_0(G) \not= 0$, which is a contradiction.

Assume $b_m \not= 0$ and $b_{m+1} = 0$. By taking a subsequence, we can assume that $\beta_m(f(y_i)) \to b_m$ and $\beta_{m+1}(f(y_i)) \to 0$. Since
\[ d((b_m^{-1} + 2)v_m, G) \leq \lim_{i \to \infty} d((b_m^{-1} + 2)v_m, g(y_i)) \]
\[ \leq \lim_{i \to \infty} \| b_m^{-1} - \beta_m(f(y_i))^{-1} \| = 0, \]
we have $(b_m^{-1} + 2)v_m \in G \subset k(G)B_X$, hence $m + 3 \leq b_m^{-1} + 2 \leq k(G)$, that is, $m \leq k(G) - 3$. On the other hand, since $\liminf_{i \in \mathbb{N}} \beta_0(f(y_i)) = b_0 > 0$ and $b_{m+1} = 0$, we have
\[ d_W(f(y_i), \text{Fin}^{m+1}(mB_X)) \to 0. \]
Moreover, observe that
\[ \liminf_{i \to \infty} d_W(f(y_i), \text{Fin}^m(mB_X)) \geq b_m/b_0 > 0. \]
Thus, we can find
\[ F_i \in \text{Fin}^{m+1}(mB_X) \setminus \text{Fin}^m(mB_X), \quad i \in \mathbb{N}, \]
so that $d_W(F_i, f(y_i)) \to 0$. For each $i \in \mathbb{N}$, let
\[ G_i = F_i \cup \left\{ \beta_n(f(y_i))^{-1} + 2)v_n, \quad n < k(f(y_i)) \right\}. \]
Then $G_i \to G$ as $i \to \infty$. Observe that
\[ \| (\beta_n(f(y_i))^{-1} + 2)v_m \| = \beta_n(f(y_i))^{-1} + 2 > \beta_n(f(y_i))^{-1} + \frac{9}{8} \]
\[ \geq \| (\beta_m(f(y_i))^{-1} + 1)v_m + \frac{1}{8} h(y_i) \| \]
\[ \geq \beta_m(f(y_i))^{-1} + \frac{7}{8} > m + \frac{3}{2}. \]
Since $F_i \subset (m + 1)B_X$ and $\beta_m(f(y))^{-1} \to b_m^{-1}$, it follows that $(h(y))_{i \in \mathbb{N}}$ is convergent, which implies that $(y)_{i \in \mathbb{N}}$ is convergent because $h$ is a closed embedding.

By Proposition 7.2, $g(Y)$ is a $Z_\sigma$-set in $W$. Since $g(Y) \approx Y$ is completely metrizable, $g(Y)$ is a $Z$-set in $W$ by [9, Lemma 2.4]. Thus, $g : Y \to W$ is a $Z$-embedding. This completes the proof. □

**Proof of Theorem II for $\text{Fin}_W(X)$**. By Theorems I and III, $\text{Fin}_W(X)$ can be regarded as a homotopy dense set in $\ell_2$. It has been shown that $\text{Fin}_W(X)$ is $\sigma$-completely metrizable $Z_\sigma$-space (Propositions 7.1 and 7.2), and every open set in $\text{Fin}_W(X)$ is universal for separable completely metrizable spaces (Proposition 7.3). Thus, we have $\text{Fin}_W(X) \approx \ell_2 \times \ell_2^I$ by Theorem 2.5. □

### 8. The relative Wijsman topology

Let $X = (X, d)$ be a metric space. For $\mathcal{F} \subset \text{Cld}_W(X)$ and $Y \subset X$, we denote $\mathcal{F}|Y = \{A \in \mathcal{F} | A \subset Y\}$. Without any condition, we have $\text{Fin}(X)|Y = \text{Fin}(Y)$ and $\text{Comp}(X)|Y = \text{Comp}(Y)$ as sets. Moreover, note that $\text{Cld}(Y) = \text{Cld}(X)|Y$ if and only if $Y$ is closed in $X$. By example in Section 2, even if $Y$ is closed in $X$, $\text{Fin}_W(X)|Y \neq \text{Fin}_W(Y)$ (hence $\text{Comp}(X)|Y \neq \text{Comp}(Y)$ or $\text{Cld}_W(X)|Y \neq \text{Cld}_W(Y)$) as spaces in general.

In this section, we give some sufficient conditions in order that $\text{Cld}_W(X)$ is a metric space. For each $\mathcal{F} \subset \text{Cld}_W(X)$, there exists a path $\varphi : \mathbb{I} \to \text{Cld}_W(X)$ such that $\varphi(0) = A$, $\varphi(1) = B$, $A \subset \varphi(t)$ for $t \leq \frac{1}{2}$ and $B \subset \varphi(t)$ for $t \geq \frac{1}{2}$.

**Lemma 8.1.** Let $X$ be an arbitrary metric space with $Y \subset X$ a separable path-connected closed set. For each $A, B \in \text{Cld}_W(X)|Y$, there exists a path $\varphi : \mathbb{I} \to \text{Cld}_W(X)|Y$ such that $\varphi(0) = A$, $\varphi(1) = B$, $A \subset \varphi(t)$ for $t \leq \frac{1}{2}$ and $B \subset \varphi(t)$ for $t \geq \frac{1}{2}$.

**Proof.** It suffices to construct a path $\xi : \mathbb{I} \to \text{Cld}_W(X)|Y$ from $A$ to $Y$ such that each $\xi(t)$ contains $A$. Let $\{y_n | n \in \mathbb{N}\}$ be dense in $Y$ with $y_1 \in A$. For each $n \in \mathbb{N}$, we have a path $\xi_n : \mathbb{I} \to Y$ with $\xi_n(0) = y_n$ and $\xi_n(1) = y_n+1$. The desired path $\xi : \mathbb{I} \to \text{Cld}_W(X)$ can be defined as follows:

$$
\xi(t) = \begin{cases} 
Y & \text{if } t = 0, \\
A \cup \bigcup_{i=1}^{n-1} \xi_i(\mathbb{I}) \cup \xi_n\left([0, t^{-1} - n]\right) & \text{if } (n+1)^{-1} < t \leq n^{-1}.
\end{cases}
$$

Clearly, $\xi$ is continuous at each $t > 0$. To see the continuity of $\xi$ at $t = 0$, for each $x \in X$ and $\varepsilon > 0$, we can choose $n \in \mathbb{N}$ so that $d(x, y_n) < d(x, Y) + \varepsilon$, whence $0 < t \leq n^{-1}$ implies

$$
d(x, Y) \leq d\left(x, \xi(t)\right) \leq d(x, y_n) < d(x, Y) + \varepsilon.
$$

This means that $\xi$ is continuous at $t = 0$. □

In the following result, the case $Y = X$ is Theorem IV for $\text{Cld}_W(X)$.
Theorem 8.2. Let $X$ be a separable metric space and $Y \subset X$ a closed set such that for any finitely many open balls $B_1, \ldots, B_n$ in $X$, $Y \setminus \bigcup_{i=1}^{n} B_i$ has only finitely many path-components, all of which are closed in $X$. Then the space $\text{Cld}_W(X)|Y$ is an ANR.

Proof. By in [3, Proposition 3.2] (i.e., the case $X = Y$ in Theorem 5.1) it suffices to show that $\text{Cld}_W(X)|Y$ is locally path-connected. This can be proved similarly to Proposition 6.4 but one should note that for any finitely many open balls $B_1, \ldots, B_n$ in $X$, no components of $Y \setminus \bigcup_{i=1}^{n} B_i$ are here assumed to be compact nor unbounded.

Let $A \in \text{Cld}_W(X)$ and $\mathcal{U}$ a neighborhood of $A$ in $\text{Cld}_W(X)$. Then

$$A \in \bigcap_{i=1}^{m} U^{-}(p_i, r_i) \cap \bigcap_{j=1}^{n} U^{+}(q_j, s_j) \subset \mathcal{U},$$

for some $p_i \in A$, $q_j \in X \setminus A$ and $r_i, s_j > 0$. Choose

$$0 < \epsilon < \min \{ d(q_j, A) - s_j \mid j = 1, \ldots, n \}$$

and let $Y_1, \ldots, Y_r$ be all path-components of $Y \setminus \bigcup_{j=1}^{n} B(q_j, s_j + \epsilon)$, where $Y_k \cap A \neq \emptyset$ for $k \leq l_0$ and $Y_k \cap A = \emptyset$ for $k > l_0$. Note that each $Y_k$ is open in $Y \setminus \bigcup_{j=1}^{n} B(q_j, s_j + \epsilon)$ and $A$ is strongly disjoint from $\bigcup_{j=1}^{n} B(q_j, s_j + \epsilon)$. For each $k = 1, \ldots, l_0$, we can choose $p_{m+k} \in Y_k \cap A$ and $r_{m+k} > 0$ such that $B(p_{m+k}, r_{m+k}) \cap Y \subset Y_k$. Since $A \subset \bigcup_{k=1}^{n} Y_k$, it follows that $\bigcup_{k=l_0+1}^{n} Y_k$ is covered by open balls which are strongly disjoint from $A$. It can be assumed that $l_0 < k \leq l_1$ if and only if $Y_k$ cannot be covered by finitely many open balls which are strongly disjoint from $A$. We can find $y_1, \ldots, y_v \in X$ and $\delta_1, \ldots, \delta_v > 0$ such that

$$\bigcup_{k=l_1+1}^{l_0} Y_k \subset \bigcup_{i=1}^{v} B(y_i, \delta_i) \quad \text{and} \quad A \in \bigcap_{i=1}^{v} U^{+}(y_i, \delta_i).$$

Thus, we have the following neighborhood of $A$:

$$\mathcal{V} = \bigcap_{i=1}^{m+l_0} U^{-}(p_i, r_i) \cap \bigcap_{j=1}^{n} U^{+}(q_j, s_j + \epsilon) \cap \bigcap_{i=1}^{v} U^{+}(y_i, \delta_i) \subset \mathcal{U}.$$ 

For each $B \in \mathcal{V}|Y$, we construct a path in $\mathcal{U}|Y$ from $B$ to $A$. Then it would follow that $\text{Cld}_W(X)|Y$ is locally path-connected. Note that $B \subset \bigcup_{k=1}^{n} Y_k$. Let

$$A = \{ k' \mid l_0 < k' \leq l_1, \ B \cap Y_{k'} \neq \emptyset \}.$$ 

For each $k' \in A$, take $z_{k'} \in B \cap Y_{k'}$ and define $B^* = \{ z_{k'} \mid k' \in A \}$. By Lemma 8.1, for each $k = 1, \ldots, l_0$, we have a path $\phi_k : I \to \text{Cld}_W(X)|Y_k$ such that $\phi_k(0) = B \cap Y_k$, $\phi_k(1) = A \cap Y_k$, $B \cap Y_k \subset \phi_k(t)$ for $t \leq \frac{1}{2}$ and $A \cap Y_k \subset \phi_k(t)$ for $t \geq \frac{1}{2}$. By Lemma 8.1 again, for each $k' \in A$, we have a path $\phi_{k'} : I \to \text{Cld}_W(X)|Y_{k'}$ such that $\phi_{k'}(0) = B \cap Y_{k'}$, $\phi_{k'}(1) =$

\footnotesize
\begin{enumerate}
\item See Footnote 6.
\end{enumerate}
\[
\{ z_k \} \text{ and } z_k' \in \psi_k(t) \text{ for each } t \in I. \text{ Then } B \text{ can be connected to } A \cup B^* \text{ by the path } \\
\varphi : I \to \text{Cld}_W(X)/Y \text{ defined by } \\
\varphi(t) = \bigcup_{k=1}^{l_0} \psi_k(t) \cup \bigcup_{k' \in A} \psi_k'(t) \quad \text{each } t \in I.
\]

Since each \( \varphi(t) \) contains \( A \) or \( B \) and
\[
\varphi(t) \subseteq \bigcup_{k=1}^{l_1} Y_k \subseteq Y \setminus \bigcup_{j=1}^{n} B(q_j, s_j + \varepsilon),
\]
it follows that \( \varphi(I) \subseteq U \setminus Y \).

For each \( k' \in A \), we shall construct an infinite path \( \psi_{k'} : [1, \infty) \to Y_{k'} \) with \( \psi_{k'}(1) = z_{k'} \) which has the following property:

\((\dagger)\) for each \( n \in \mathbb{N} \), there is some \( t_0 > 0 \) such that
\[
i \geq t_0 \implies d(x_i, \psi_{k'}(t)) > d(x_i, A) - \frac{1}{n} \quad \text{for } i \leq n,
\]
where \( \{ x_i \mid i \in \mathbb{N} \} \) is a countable dense subset of \( X \) defining the metric \( d_W \). Then we can define a path \( \psi : I \to V \) from \( A \) to \( A \cup B^* \) as follows:
\[
\psi(s) = \begin{cases} 
A & \text{if } s = 0, \\
A \cup \bigcup_{k' \in A} \psi_{k'}(s^{-1}) & \text{if } s > 0.
\end{cases}
\]
For each \( n \in \mathbb{N} \), we use the property \((\dagger)\) to find \( t_0 > 0 \) so that
\[
i \geq t_0 \implies d(x_i, \psi_{k'}(t)) > d(x_i, A) - \frac{1}{n} \quad \text{for } i \leq n \text{ and } k' \in A.
\]
Then, \( s \leq t_0^{-1} \) implies \( d_W(\psi(t), A) < 1/n \) because for \( i \leq n \),
\[
d(x_i, A) - \frac{1}{n} < \min\{d(x_i, A), d(x_i, \psi_{k'}(s^{-1})) \mid k' \in A\} \\
= d(x_i, \psi(s)) \leq d(x_i, A),
\]
hence \( |d(x_i, \psi(s)) - d(x_i, A)| < 1/n \).

Now, let \( k' \in A \) be fixed. We construct an infinite path \( \psi_{k'} : [1, \infty) \to Y_{k'} \) with \( \psi_{k'}(1) = z_{k'} \) and property \((\dagger)\). Enumerate as \( B_1, B_2, \ldots \) all open balls of the form \( B(x_i, \alpha) \), where \( 0 < \alpha < d(x_i, A) \) and \( \alpha \in Q \). By the assumption, for each \( n \in \mathbb{N} \), \( Y_{k'} \setminus \bigcup_{i=1}^{n} B_i \) has finitely many path-components \( H_{n}^{1}, \ldots, H_{n}^{a(n)} \). Let \( T = \{ H_{n}^{i} \mid n \in \mathbb{N}, i = 1, \ldots, a(n) \} \).

Since every \( B_i \) is strongly disjoint from \( A \) and \( l_0 < k' < l_1 \), \( Y_{k'} \) cannot be covered by finitely many \( B_i \)'s, hence \( T \) is infinite. Thus, \( (T, \supseteq) \) is a finitely-branching infinite tree. By König’s Lemma, \( T \) contains an infinite branch \( H_{i(1)}^{1} \supseteq H_{i(2)}^{2} \supseteq \cdots \).

For each \( n \in \mathbb{N} \), pick \( v_n \in H_{i(n)}^{n} \) and a path \( f_n : I \to H_{i(n-1)}^{n-1} \) with \( f_n(0) = v_{n-1} \) and \( f_n(1) = v_n \), where \( H_{i(n)}^{n} = Y_{i(n-1)} \) and \( v_0 = z_{k'} \). By joining the paths \( f_1, f_2, \ldots \), we can define a path \( \psi_{k'} : [1, \infty) \to Y_{k'} \), that is, \( \psi_{k'}(t) = f_n(t - n) \) for \( n \leq t \leq n + 1 \). Then \( \psi_{k'}(1) = z_{k'} \).

To verify condition \((\dagger)\), let \( n \in \mathbb{N} \). For each \( i = 1, \ldots, n \), choose \( \alpha_i \in Q \) so that \( d(x_i, A) -
$1/n < \alpha_i < d(x_i, A)$. Then we have $m \in \mathbb{N}$ such that all balls $B(x_1, \alpha_1), \ldots, B(x_n, \alpha_n)$ appear in $B_1, \ldots, B_m$. For $t > m$, choose $m' \geq m$ so that $m' \leq t < m'+1$, whence

$$\psi_k'(t) = f_{m'}(t - m') \in H^m_{i(m')} \subset H^m_{i(m)} \subset Y_k \setminus \big( B(x_1, \alpha_1) \cup \cdots \cup B(x_n, \alpha_n) \big).$$

Therefore, $d(x_i, \psi_k'(t)) \geq \alpha_i > d(x_i, A) - 1/n$ for every $i \leq n$. Thus, $\psi$ has property (‡).

In the above, we use Proposition 6.3 instead of Lemma 8.1 to obtain the following result, which in the case $Y = X$ is Theorem IV for $	ext{Fin}_W(X)$.

**Theorem 8.3.** Let $X$ be a separable metric space and $Y \subset X$ a path-connected subset such that for any finitely many open balls $B_1, \ldots, B_n$ in $X$, $Y \setminus \bigcup_{i=1}^n B_i$ has only finitely many path-components which are closed in $Y \setminus \bigcup_{i=1}^n B_i$. Then the space $	ext{Fin}_W(X) | Y$ is an ANR.

In the same setting as above, Theorem III can be generalized as follows:

**Theorem 8.4.** Let $X$ be a separable metric space and $Y$ a path-connected subset of $X$ such that for any finitely many open balls $B_1, \ldots, B_n$ in $X$, $Y \setminus \bigcup_{i=1}^n B_i$ has finitely many path-components which are compact or unbounded closed in $Y \setminus \bigcup_{i=1}^n B_i$ (respectively all are unbounded closed in $Y \setminus \bigcup_{i=1}^n B_i$). Then $\text{Fin}_W(X) | Y$ is homotopy dense in $\text{Cld}_W(X) | Y$ and $\text{Cld}_W(X) | Y$ is an ANR (respectively an AR).

The proof of this theorem is left to the readers.

**Acknowledgement**

The authors would like to express their sincere thanks to the referee for diligent reading of the manuscript and pointing out several errors contained in the earlier versions. They appreciate his helpful comments and suggestions.

**References**