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On Semi-Krull Domains

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INTRODUCTION

Semi-Krull domains have been introduced by E. Matlis in [M]. An integral domain A is semi-Krull if and only if $A = \bigcap \{A_P\}$, where P ranges over the set of height one primes of A, this intersection has a finite character, and every nonzero ideal of A_P contains a power of PA_P, for every height one prime ideal of A [M, Proposition 4.5]. Krull domains and Cohen–Macaulay noetherian domains are semi-Krull.

We recall that Krull and noetherian domains are Mori domains, that is, domains with the ascending chain condition on divisorial ideals. On the other hand, we prove that in a semi-Krull domain every height one prime is a maximal t-ideal and is divisorial (Theorem 1.7). Hence, like a Mori domain, a semi-Krull domain is the intersection, with a finite character, of its localizations at the maximal divisorial primes. Moreover it has always the ascending chain condition on principal ideals, for short a.c.c.p. (Theorem 1.10). This leads us to investigate the relations between the class of semi-Krull domains and that one of Mori domains. It turns out that neither class is included in the other (Examples 1.9 and 2.4) and that semi-Krull Mori domains are those Mori domains in which every maximal divisorial ideal is of height one (Theorem 2.1). Using this characterization, we are able to produce a new class of examples of semi-Krull domains that are neither Krull nor noetherian. A way of constructing semi-Krull

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domains of this type is given in Example 2.3. Example 2.5 shows that an a.c.c.p. domain with every maximal t-ideal divisorial and of height one is not necessarily semi-Krull.

In Section 3 we deal with the complete integral closure of a semi-Krull domain. We prove that a completely integrally closed semi-Krull domain is Krull (Theorem 3.1). It is well known that a similar result holds for Mori domains and, as in the case of Mori domains, it seems not easy to find out whether the complete integral closure A^* of a semi-Krull domain A is Krull. We solve this problem in the seminormal case. Indeed we prove that if A is a seminormal semi-Krull domain, then A^* is completely integrally closed (Corollary 3.7) and A^* is Krull if and only if A is also a Mori domain (Theorem 3.12). We leave open the question if a seminormal semi-Krull domain is necessarily Mori. We observe that, if there is a counterexample, it cannot be a conducive domain.

We also prove that a polynomial ring over a semi-Krull domain is not necessarily semi-Krull. Indeed if A is integrally closed and $A[X]$ is semi-Krull, then A is Krull (Proposition 3.4). Conversely if A is any domain and $A[\mathbf{X}]$ is semi-Krull for a given set of indeterminates $\mathbf{X}$, then A is semi-Krull (Proposition 3.5).

In Section 4 we study generalized quotient rings of semi-Krull domains. We recall that flat overrings, Nagata transforms, and in general intersections of localizations of a domain A are generalized quotient rings of A. In [M, Propositions 4.8 and 4.9] it has been shown that flat overrings and Nagata transforms of semi-Krull domains are intersections of localizations at height one primes and moreover a flat overring of a semi-Krull domain is semi-Krull. We prove these statements for any generalized quotient ring of a semi-Krull domain (Theorems 4.6 and 4.7). We also investigate when a generalized quotient ring of a semi-Krull domain is a quotient ring. A necessary and sufficient condition for that is given in Proposition 4.12. In Theorem 4.14 we prove that, when a semi-Krull domain A has just finitely many strongly divisorial height one primes, any generalized quotient ring of A is a quotient ring if and only if the class group of A is a torsion group. This generalizes a result well known for Krull domains [F, Proposition 6.7].

Throughout this paper A is always an integral domain and $Q(A)$ its field of quotients. If U is a subset of $Q(A)$, we denote by $(A:U)$ the set $\{x \in Q(A);\ xU \subset A\}$. By an *ideal* we mean a nonzero fractional ideal. We call an *integral ideal* an ideal contained in A and a *proper* ideal an integral ideal properly contained in A.

For any ideal I, I_v is the intersection of all principal ideals containing I, that is, $I_v = (A:(A:I))$. We say that I is *divisorial* if $I = I_v$. The set of divisorial ideals of A is denoted by $D(A)$. A maximal divisorial ideal is an ideal that is maximal among the proper divisorial ideals of A. We denote by $D_m(A)$ the set of the maximal divisorial ideals of A. A divisorial ideal I is called *v-finite* if $I=J_v$ for some finitely generated ideal J of A and I is called a t-ideal if $I=I_t$, where $I_t=\bigcup\{J_v; J\subset I$ and J finitely generated$\}$. We have $I\subset I_t\subset I_v$, so that a divisorial ideal is a t-ideal. A maximal t-ideal of A is an ideal which is maximal among the proper t-ideals of A. The set of all maximal t-ideals of A is denoted by $t_m(A)$. Any integral t-ideal is contained in a maximal t-ideal, so that, if A is not a field, the set $t_m(A)$ is always not empty. Maximal t-ideals, as well as maximal divisorial ideals, are prime. We have $I=\bigcap\{IA_P; P\in t_m(A)\}$ and in particular $A=\bigcap\{A_P; P\in t_m(A)\}$. Two t-ideals I and J are equal if and only if $IA_P=JA_P$ for any $P\in t_m(A)$.

Under the operation $I\circ J=(IJ)_t$ the set of t-ideals of A is a semigroup with unit A. A t-ideal I is said to be *t-invertible*, that is, invertible with respect to this operation, if $(IJ)_t=A$ for some ideal J. The t-inverse of a t-invertible t-ideal I is $(A:I)$. I is t-invertible if and only if I and $(A:I)$ are divisorial and v-finite and IA_P is principal for every $P\in t_m(A)$. A t-invertible prime ideal is a maximal divisorial ideal. A maximal t-ideal P of A is either t-invertible or *strong*, that is, $(A:P)=(P:P)$. We call an ideal *strongly divisorial* if it is strong and divisorial. The group of the t-invertible t-ideals of A is denoted by $T(A)$ and the group of the principal ideals of A is denoted by $P(A)$. We have $P(A)\subset T(A)$ and the quotient group $C(A)=T(A)/P(A)$ is called the *class group* of A. For background on t-ideals we refer to [J, Gr, and G] and general references on the class group are [BvZ, AA, A].

1. Ideal Theoretic Results

LEMMA 1.1. *Let $\mathscr{C}$ be a family of prime ideals of A and $B=\bigcap\{A_P; P\in\mathscr{C}\}$. Then*

(a) *$A=B$ if and only if every proper divisorial ideal I of A is contained in some $P\in\mathscr{C}$.*

(b) *If the intersection $B=\bigcap\{A_P; P\in\mathscr{C}\}$ has a finite character, then $A=B$ if and only if every maximal t-ideal of A is contained in some $P\in\mathscr{C}$.*

Proof. We recall that if $P\in\operatorname{Spec}(A)$ and $x\in Q(A)$, then $x^{-1}A\cap A\subset P$ if and only if $x\notin A_P$; also $x^{-1}A\cap A=A$ if and only if $x\in A$.

(a) Let I be a proper divisorial ideal of A. Then $I\subset x^{-1}A\cap A$ for some $x\in Q(A)\backslash A$. Thus, if $A=B$, then $x\notin A_P$ for some $P\in\mathscr{C}$, that is, $I\subset x^{-1}A\cap A\subset P$ for some $P\in\mathscr{C}$. Conversely, let $x\in\bigcap\{A_P; P\in\mathscr{C}\}$. Since $J=x^{-1}A\cap A$ is a divisorial ideal of A, if $J\neq A$, then by assumption $J\subset P$ for some $P\in\mathscr{C}$. Thus $x\notin A_P$ for some $P\in\mathscr{C}$, contradicting the hypothesis. It follows that $J=A$ and so $x\in A$.

(b) Let $A=B$ and $Q\in t_m(A)$. Suppose that Q is not contained in any $P\in\mathscr{C}$. Let x be a nonzero element of Q and let $P_1, ..., P_n$ be all the primes in $\mathscr{C}$ containing x. By assumption $Q\not\subset\bigcup\{P_i; i=1, ..., n\}$. Let $y\in Q\backslash\bigcup\{P_i; i=1, ..., n\}$ and consider the ideal $I=(x, y)_v=(x, y)_t$. $I\neq A$ because $I\subset Q$ and so $I\subset P_i$ for some $i=1, ..., n$ by (a). A contradiction. Conversely, since $A=\bigcap\{A_P; P\in t_m(A)\}$ [Gr, Proposition 4], then $B\subset A$ and so $B=A$. ▮

We denote by $X^1(A)$ the set of the height one primes of A. A height one prime is a t-ideal by [J, Corollaire 3, p. 31].

Proposition 1.2. *If A is a semi-Krull domain, then $X^1(A)=t_m(A)$.*

Proof. By Lemma 1.1(b). Indeed we recall that $A=\bigcap\{A_P; P\in X^1(A)\}$ and that every prime ideal of height one is a t-ideal. ▮

We note that every proper ideal of a domain A is contained in the union of the maximal t-ideals of A. Thus if $t_m(A)=\{P_i; i=1, ..., n\}$ is a finite set, then every maximal ideal of A is a t-maximal ideal and A is semiquasilocal. Moreover, if I is a t-invertible t-ideal of A, then IA_{P_i} is principal for $i=1, ..., n$ [G, Proposition 1.1]. Hence I is invertible [BvZ, Corollary 2.9] and then it is principal, that is, $C(A)=(0)$.

This, together with Proposition 1.2, allows us to generalize a result well known for Krull domains.

Proposition 1.3. *Let A be a semi-Krull domain. If $X^1(A)$ is a finite set, then A is a semiquasilocal domain of dimension one and $C(A)=(0)$.*

We recall that in a semi-Krull domain, by the finiteness character, every nonzero ideal is contained in just finitely many prime ideals of height one.

If P is a prime ideal of A, we denote by $P^{(r)}$ the symbolic rth power of P, that is, $P^{(r)}=P^rA_P\cap A$.

Lemma 1.4. *Let A be a semi-Krull domain and I any proper t-ideal of A. Let $P_1, ..., P_n$ be the prime ideals of height one containing I. Then I has the unique primary decomposition $I=\bigcap\{IA_{P_i}\cap A; i=1, ..., n\}$ and it contains a power of its radical $P_1\cap\cdots\cap P_n$.*

Proof. By Proposition 1.2, $P_1, ..., P_n$ are the only maximal t-ideals containing I, so $I=\bigcap\{IA_{P_i}\cap A; i=1, ..., n\}$ [Gr, Proposition 4]. Each component is primary because IA_{P_i} is a primary ideal of A_{P_i} for $i=1, ..., n$ and the decomposition is unique because P_i is minimal over I for $i=1, ..., n$. Since A is semi-Krull, we have $IA_{P_i}\supset P_i^{r_i}A_{P_i}$ for some $r_i\geqslant 1$, $i=1, ..., n$. Then $I\supset\bigcap\{P_i^{(r_i)}; i=1, ..., n\}\supset\bigcap\{P_i^{r_i}; i=1, ..., n\}$ and finally I contains $(P_1\cap\cdots\cap P_n)^s$ for $s\geqslant\max(r_1, ..., r_n)$. ▮

Remark 1.5. We observe that, keeping the notations of Lemma 1.4 above, if P_i is t-invertible for some $i = 1, ..., n$, then A_{P_i} is a DVR [G, Remark 1.2] and so $IA_{P_i} \cap A = P_i^{r_i} A_{P_i} \cap A = P_i^{(r_i)}$ for some $r_i \geqslant 1$. In particular we see that if every maximal t-ideal P_i containing I is t-invertible, then the unique primary decomposition of I is $I = P_1^{(r_1)} \cap \cdots \cap P_n^{(r_n)}$, thus $I = (P_1^{r_1} \cdots P_n^{r_n})_v$ and so I is t-invertible. To prove the second equality, let $J = P_1^{r_1} \cdots P_n^{r_n}$. Since J_v is t-invertible, we have $J_v A_S = (JA_S)_v$ for every multiplicative part S of A [BvZ, Lemma 1.5]. Thus $J_v A_{P_i} = P_i^{r_i} A_{P_i}$ and $J_v A_Q = A_Q$, when $Q \in t_m(A)$ and $Q \neq P_i$, $i = 1, ..., n$. It follows that $J_v A_P = IA_P$ for every $P \in t_m(A)$ and so $J_v = I$ by [AA, Proposition 1.4]. A similar result has been proved for Mori domains in [BG2, Proposition 1.7].

LEMMA 1.6. *Let A be a semi-Krull domain. Then every proper divisorial ideal of A is contained in a divisorial maximal t-ideal.*

Let I be a proper divisorial ideal of A. By Lemma 1.4 and Proposition 1.2, I is contained in a finite number $P_1, ..., P_n$ of maximal t-ideals of A and I contains $(P_1 \cdots P_n)^s$ for some $s \geqslant 1$. If none of the P_i's is divisorial, then $(A : P_i) = A$ for every $i = 1, ..., n$ [G, Proposition 1.4] and so $(A : (P_1 \cdots P_n)^s) = A$. Thus $A \supset (A : I)$. This is a contradiction because I is divisorial. ∎

THEOREM 1.7. *If A is a semi-Krull domain, then $X^1(A) = t_m(A) = D_m(A)$.*

Proof. By Proposition 1.2, we have $X^1(A) = t_m(A)$. By Lemma 1.6 every proper divisorial integral ideal of A is contained in a divisorial maximal t-ideal. Thus, by Lemma 1.1(a), $A = \bigcap \{A_P ; P \in t_m(A) \cap D(A)\}$ and this intersection has a finite character because $t_m(A) \cap D(A) \subset t_m(A) = X^1(A)$. By Lemma 1.1(b), it follows that every maximal t-ideal of A is divisorial and so $t_m(A) = D_m(A)$. ∎

We recall that in one-dimensional quasilocal domain, the maximal ideal is always a t-ideal but it need not be divisorial.

COROLLARY 1.8. *If A is a semi-Krull domain, then every height one prime ideal has the form* $(a) :_A b$ *for suitable $a, b \in A$.*

Proof. By Theorem 1.7 every height one prime ideal P of A is maximal divisorial. Hence $P = xA \cap A$ for some $x \in Q(A)$. If $x = a/b$, $a, b \in A$, then $P = (a) :_A b$. ∎

In spite of Theorem 1.7, it is not true that in a semi-Krull domain any t-ideal is divisorial, that is, with the notation of [HsZ1], a semi-Krull domain is not necessarily a TV-domain.

EXAMPLE 1.9. A quasilocal one-dimensional semi-Krull domain which is not a TV-domain.

Let K be a field and let $\mathbf{X} = \{X_n; n \geqslant 0\}$ and $\mathbf{Y} = \{Y_n : n \geqslant 1\}$ be two sets of independent indeterminates over K. For any nonzero polynomial f in $K[\mathbf{X}, \mathbf{Y}] = K[\{X_n; n \geqslant 0\}, \{Y_n; n \geqslant 1\}]$ let $v(f)$ be the least degree of a homogeneous component of f. Extend v to a valuation of the field $K(\mathbf{X}, \mathbf{Y})$, which will be also denoted by v. Let V be the valuation ring of v and let P be its maximal ideal. Let $B = K[\mathbf{X}, \{X_m Y_n;\ m \geqslant 1,\ n \geqslant 1,\ m \neq n\}]$. Set $A = B + P^3$, thus A is a ring contained in V. Let M be the ideal of A generated by $\mathbf{X} \cup \{X_m Y_n;\ m \geqslant 1,\ n \geqslant 1,\ m \neq n\} \cup P^3$.

A is a quasilocal domain and M is its maximal ideal. We have $A = K + M$, so M is a maximal ideal of A. Let $t \in M$. We have $1/(1-t) - (1 + t + t^2) = t^3/(1-t) \in P^3$ and $(1 + t + t^2) \in A$, so $1/(1-t) \in A$. Hence A is a quasilocal domain and M is its maximal ideal.

A is one-dimensional and semi-Krull. Let f be a nonzero element of A. Thus $v(f) \geqslant 0$. Set $m = v(f) + 3$. For any element g in M^m, we have $v(g/f) \geqslant 3$, so $g/f \in P^3 \subset A$. It follows that $M^m \subset Af$, so A is one-dimensional and semi-Krull.

A is not a TV-domain. For every $n \geqslant 1$, let $I_n = (X_1, ..., X_n) + P^2$. Set $I = \bigcup \{I_n;\ n \geqslant 1\}$, thus $I = (X_1, X_2, ...) + P^2$. We claim that I is a t-ideal of A which is not divisorial. For the proof we need first some preliminary remarks.

Let D be the ring of homogeneous rational functions in $K(\mathbf{X}, \mathbf{Y})$, thus D is a $\mathbb{Z}$-graded ring. Let f be a nonzero rational function in $K(\mathbf{X}, \mathbf{Y})$ with $v(f) = d$. Then for any integer $n \geqslant d$, we have $f \in D_d + \cdots + D_n + P^{n+1}$. Indeed write $f = g/h$, where g and h are polynomials in $K[\mathbf{X}, \mathbf{Y}]$. Let $h = h_m + h_{m+1} + \cdots$ and $g = g_{m+d} + g_{m+d+1} + \cdots$ the decompositions into homogeneous components of h and g, respectively, such that $h_m \neq 0$ and $g_{m+d} \neq 0$. We have $g_{m+d}/h_m \in D_d$ and, if $f_d = g_{m+d}/h_m$, $f - f_d \in P^{d+1}$. Iterating this argument, we obtain that $f \in D_d + \cdots + D_n + P^{n+1}$. Moreover, for any two integers $n \geqslant d$, the sum $D_d + \cdots + D_n + P^{n+1}$ is a direct sum of additive subgroups of $K(\mathbf{X}, \mathbf{Y})$, so the representation $f = f_d + \cdots + f_n + u$, where $f_i \in D_i$ for $d \leqslant i \leqslant n$ and $u \in P^{n+1}$, is unique.

Now for a given $n \geqslant 0$, let f be a nonzero rational function in $K(\mathbf{X}, \mathbf{Y})$ with $v(f) \leqslant 1$ such that $fX_n \in A$. Write $f = f_d + f_{d+1} + \cdots + f_0 + f_1 + u$, where $f_i \in D_i$ for $d \leqslant i \leqslant 1$ and $u \in P^2$. Hence $fX_n = f_d X_n + f_{d+1} X_n + \cdots + f_1 X_n + uX_n$. Note that B is a graded subring of D and that $A = B_0 \oplus B_1 \oplus B_2 \oplus P^3$. Since $fX_n \in A$, from the uniqueness of the representation for fX_n, we get that $d \geqslant -1$ and $f_i X_n \in B_{i+1}$ for $i = -1, 0, 1$. Clearly these are necessary and sufficient conditions on a rational nonzero function $f \in K(\mathbf{X}, \mathbf{Y})$ with $v(f) \leqslant 1$ in order to belong to $(A : X_n)$. Moreover $P^2 \subset (A : X_n)$. We obtain in particular that, for any integers $m \geqslant 0$ and $n \geqslant 1$, $X_m Y_n \in A$ if and only if $m \geqslant 1$ and $m \neq n$.

Let $n>1$ and let f be a nonzero element in $(A:(X_1,...,X_n))$. Thus $v(f)\geqslant -1$. Set $f=f_{-1}+f_0+f_1+u$, where $f_i\in B_i$ for $i=-1,0,1$ and $u\in P^2$. Since $n\geqslant 2$, the elements $f_{-1}X_1$ and $f_{-1}X_2$ are in $B_0=K$, so $f_{-1}\in(1/X_1)K\cap(1/X_2)K=(0)$ and $f_{-1}=0$. Also f_0X_1 and f_0X_2 are in $B_1=\operatorname{span}_K(\{X_0,X_1,...\})$. Thus $f_0\in(1/X_1)B_1\cap(1/X_2)B_1=K$. Finally, for all $1\leqslant i\leqslant n$,

$$f_1X_i\in B_2=\operatorname{span}_K(\{X_iX_j;i\geqslant 0,j\geqslant 0\}\cup\{X_iY_j;i\geqslant 1,j\geqslant 1,i\neq j\}.$$

Thus $f_1\in(1/X_1)B_2\cap(1/X_2)B_2$ and so f_1 is a K-linear form in the indeterminates $\mathbf{X}\cup\mathbf{Y}$. Since $X_iY_j\in A$ if and only if $i\neq j$, we obtain that $f_1\in\operatorname{span}_K(\mathbf{X}\cup\{Y_m;m>n\})$.

We conclude that $(A:(X_1,...,X_n))=K\oplus\operatorname{span}_K(\mathbf{X}\cup\{Y_m;m>n\})\oplus P^2$.

By similar arguments we obtain that $A:(A:(X_1,...,X_n))=(X_1,...,X_n)+P^2=I_n$. Thus $I_n=(X_1,...,X_n)_v$ is a v-finite divisorial ideal. Since $I=\bigcup\{I_n;n\geqslant 1\}$ is an ascending union of v-finite divisorial ideals, we see that I is a t-ideal.

On the other hand,

$$\begin{aligned}(A:I)&=\bigcap\{(A:I_n);n\geqslant 2\}\\&=\bigcap\{K\oplus\operatorname{span}_K(\mathbf{X}\cup\{Y_m;m>n\})\oplus P^2;n\geqslant 2\}\\&=K\oplus\operatorname{span}_K(\mathbf{X})\oplus P^2=A+P^2.\end{aligned}$$

Hence $X_0\in A:(A:I)$, but $X_0\notin I$, so I is not a divisorial ideal. We conclude that A is not a TV-domain.

THEOREM 1.10. *Let A be a semi-Krull domain, then A satisfies the ascending chain condition on principal ideals.*

Proof. Since an intersection with a finite character of domains satisfying a.c.c.p. satisfies a.c.c.p., it is enough to prove the theorem for the case that A is quasilocal and one-dimensional. Let P be the maximal ideal of A. Let $a_1,a_2,...$ be nonzero elements in P such that $Aa_1\subsetneq Aa_2\subsetneq\cdots$. Thus for any $m\geqslant 1$, $(a_m/a_{m+1})\in P$. Let $k\geqslant 1$ such that $P^k\subset Aa_1$. Hence $(a_1/a_{k+1})=(a_1/a_2)(a_2/a_3)\cdots(a_k/a_{k+1})\in P^k\subset Aa_1$. It follows that $1/a_{k+1}\in A$, a contradiction. ∎

By the previous theorem we get in particular that a semi-Krull domain A is Archimedean, that is, $\bigcap\{x^nA;n\geqslant 1\}=(0)$ for every nonzero nonunit $x\in A$. This can be obtained also as a consequence of the fact that a semi-Krull domain is an intersection with a finite character of one-dimensional (and so Archimedean) domain or by [ABD, Proposition 3.7] observing that the Principal Ideal Theorem holds in A.

PROPOSITION 1.11. *If A is a semi-Krull domain, then $\bigcap \{I^n; n \geqslant 1\} = (0)$ for every proper t-ideal I of A.*

Proof. Let $P \in X^1(A)$. A_P is one-dimensional, so it is Archimedean and we have $\bigcap \{x^n A_P; n \geqslant 1\} = (0)$ for every nonzero $x \in P$. Since A is semi-Krull, then the ideal xA_P contains a power of PA_P. It follows that $\bigcap \{P^n A_P; n \geqslant 1\} = (0)$ and so $\bigcap \{P^n;\ n \geqslant 1\} = (0)$. Since any t-ideal I is contained in some height one prime (Proposition 1.2), we also get $\bigcap \{I^n; n \geqslant 1\} = (0)$. ∎

2. A CLASS OF EXAMPLES

We have proved in Section 1 that a semi-Krull domain is a domain with the ascending chain condition on principal ideals such that $X^1(A) = t_m(A) = D_m(A)$. The converse is not true, as Example 2.5 below shows. However, we now prove that, when the ascending chain condition holds in the set of all divisorial ideals of A, that is, A is a Mori domain, and $X^1(A) = D_m(A)$, then A is semi-Krull. We recall that in a Mori domain every t-ideal is divisorial [BG2, Proposition 1.1], thus Example 1.9 shows that conversely a semi-Krull domain is not necessarily Mori.

THEOREM 2.1. *Let A be a Mori domain. Then A is semi-Krull if and only if $X^1(A) = D_m(A)$.*

Proof. By Theorem 1.7, in a semi-Krull domain $X^1(A) = D_m(A)$. Conversely, if A is a Mori domain and $X^1(A) = D_m(A)$, then $A = \bigcap \{A_P; P \in X^1(A)\}$ and this intersection has a finite character [BG1, Proposition 2.2, b]. Moreover, A_P is a Mori domain for each $P \in X^1(A)$ [Q, Sect. 3, Corollaire 1]. Since in a Mori domain every integral divisorial ideal contains a power of its radical [Ra, I, Théorème 5], if I is a nonzero integral ideal of A_P and $x \neq 0$ is an element of I, then $I \supset (x) \supset (PA_P)^n$ for some $n \geqslant 1$. Hence A is a semi-Krull domain. ∎

Remark 2.2. In any domain, a prime ideal P is divisorial if and only if P is an associated prime to a principal ideal [Y, Proposition 10]. Thus in a Mori domain A the condition $X^1(A) = D_m(A)$ means that each associated prime to a principal ideal is of height one (recall that in a Mori domain each height one prime is divisorial). In particular, if A is a noetherian domain, the condition $X^1(A) = D_m(A)$ is equivalent to the condition that each proper principal ideal is unmixed of height one. Thus by Theorem 2.1 the examples of noetherian semi-Krull domains given in [M, p. 397] are the only ones possible.

Using Theorem 2.1, it is not difficult to give examples of non-noetherian semi-Krull domains. Indeed we can consider any non-noetherian Mori domain of dimension one, such as $A = \mathbb{Q} + X\mathbb{R}[[X]]$ (cf. [B1]). This example can be generalized to the non-quasilocal case in the following way.

EXAMPLE 2.3. Let $C = \bigcap \{C_P;\ P \in X^1(C)\}$ be a non-local Dedekind domain. Consider a finite subset $\mathscr{X} = \{P_1, ..., P_n\}$ of $X^1(C)$ and let A_i be a Mori pullback of C_{P_i} for each $i = 1, ..., n$ (cf. [BG1, Sect. 4]). We claim that $A = C \cap \{A_i;\ i = 1, ..., n\}$ is a non-quasilocal semi-Krull Mori domain of dimension one.

Suppose first that $\mathscr{X} = \{P_1\} \cdot A = C \cap A_1$ is a Mori domain as an intersection of two Mori domains. Moreover A is the pull-back of the diagram

$$\begin{array}{ccc} & & (C/P_1) \cap k_1 \\ & & \downarrow \\ C & \longrightarrow & C/P_1 \end{array}$$

where k_1 is the residue field of A_1 and the intersection is made in the residue field of C_{P_1}. Since $\dim C = 1$, then C/P_1 is a field and so also $k_1 \cap (C/P_1)$ is a field. It follows that $\dim A = 1$ and, by Theorem 2.1, A is semi-Krull. By [Fn, Theorem 1.4, (c)] the canonical map $\mathrm{Spec}(C) \to \mathrm{Spec}(A)$ induces a one-to-one correspondence between the primes of C that do not contain P_1 and the primes of A that do not contain P_1. Thus A is not quasilocal. Moreover, by [Pr, Corollary 8], $A_{P_1} = A_1$. Thus, if A_1 is not noetherian, that is, if the extension $k_1 \subset C_{P_1}/P_1 C_{P_1}$ is not finite, then A is not noetherian.

Iterating these arguments, we can easily prove our claim.

A Mori domain is not necessarily semi-Krull and, besides Example 1.9, there are simpler examples of semi-Krull non-Mori domains, as shown below.

EXAMPLES 2.4. (a) Any Mori domain with a maximal divisorial ideal of height greater than one is not semi-Krull by Theorem 2.1. For example, we can consider $A = k + Xk[X, Y]$, where k is a field [BG1, Examples (4.6, b)].

(b) Let $A = k + XD + X^2K[[X]]$, where $k \subset K$ are two distinct fields and D is a domain, but not a field, such that $k \subset D \subset K$. A is a quasilocal one-dimensional domain with maximal ideal $M = XD + X^2K[[X]]$. Moreover A is not a Mori domain [BDF*n*, Example 17]. To prove that A is a semi-Krull domain, it is enough to show that, for any nonzero integral ideal I of A, $X^n K[[X]] \subset I$ for some $n \geqslant 1$. Let $h \in I$ be an element of order s. It is easy to show that there exists an element $g \in K[[X]]$ of order 2 such that $hg = X^{s+2}$. Hence $X^{s+2}K[[X]] = hgK[[X]] \subset hX^2K[[X]] \subset I$, where the last inclusion holds because $X^2K[[X]] \subset A$.

EXAMPLE 2.5. An example of a one-dimensional quasilocal domain (A, P) satisfying a.c.c.p. such that P is divisorial, but A is not semi-Krull.

Let F be a field and let T be the ring of (generalized) polynomials $F[\mathbb{R}^+] = F[\{X^r;\ r \in \mathbb{R},\ r > 0\}]$. For any nonzero polynomial f in T, let $v(f)$ be the minimal s such that X^s occurs in f. We may uniquely extend $v\colon T \to \mathbb{R}$ to a valuation of $Q(T)$, which will be also denoted by v. Let $Q = \{f \in Q(T);\ v(f) > 1\}$. Let $(\gamma_n)_{n \geqslant 1}$ be an infinite sequence of real numbers in the interval $(0, 1)$ such that $\lim_{n \to \infty} (\gamma_n) = 0$ and $1, \gamma_1, \gamma_2, \ldots$ are linearly independent over $\mathbb{Q}$. Let Γ be the additive subsemigroup of $\mathbb{R}$ generated by $\{\gamma_n;\ n \geqslant 1\}$ and let $R = F[\Gamma] = F[X^{\gamma_n};\ n \geqslant 1]$. Set $A = R + Q$. A is a ring. Consider the ideal P of A generated by $\{X^{\gamma_n};\ n \geqslant 1\} \cup Q$. Thus $A = F + P$ and P is a maximal ideal of A. Moreover (A, P) is quasilocal. Indeed, if $t \in P$, for $n \gg 0$, we have $(1/(1-t)) - (1 + t + \cdots + t^n) = t^{n+1}/(1-t)$ and $v(t^{n+1}/(1-t)) = (n+1)\,v(t) > 1$. Hence $t^{n+1}/(1-t) \in A$. It follows that $1/(1-t) \in A$ and so (A, P) is quasilocal.

To show that (A, P) has the desired properties, we will use the following facts.

(a) For any nonzero $g \in A$, $X^{v(g)} \in A$.

This is clear if $v(g) > 1$. Assume that $v(g) \leqslant 1$. We can write $g = f + q$, with $f \in R$ and $q \in Q$. We have $v(f) \leqslant 1 < v(q)$, thus $v(g) = v(f)$. By definition, $v(f)$ is the minimal s such that X^s occurs in f, hence $v(f) \in \Gamma$ and $X^{v(g)} = X^{v(f)} \in A$.

(b) For any real number r, $X^r \in A$ if and only if $r \in \Gamma$ or $r > 1$.

If $r \in \Gamma$ or $r > 1$, clearly $X^r \in A$. Conversely assume that $X^r \in A$ and $r \leqslant 1$. Write $X^r = f + q$, with $f \in R$ and $q \in Q$. Let $f = f_0 + f_1$ such that $s \leqslant 1$ for any X^s occurring in f_0 and $s > 1$ for any X^s occurring in f_1. Thus $f_1 \in Q$ and $X^r = f + q'$, where $q' = f_1 + q \in Q$. If $q' \neq 0$, then $1 \geqslant v(X^r - f_0) = v(q') > 1$, a contradiction. Thus $q' = 0$ and $X^r = f_0 \in R$. It follows that $r \in \Gamma$.

(c) If $v(f) = 0$, then f is invertible in A.

If $v(f) = 0$, then $f \notin P$ and so f is invertible.

A is one-dimensional. Let f and g be any nonzero elements in P. Thus $v(g) > 0$ and for $n \gg 0$ we have $v(g^n) = nv(g) > v(f) + 1$. Hence, for $n \gg 0$, $v(g^n/f) > 1$ and so $g^n/f \in Q \subset A$. It follows that $P \in \operatorname{rad}(Af)$ for any $f \in P$ and thus P is of height one.

P is divisorial. By hypothesis the elements $1, \gamma_1, \gamma_2, \ldots$ are linearly independent over $\mathbb{Q}$. Then $1 \notin \Gamma$ and so $X \notin A$ by (b). On the other hand $X = X^3/X^2$ is in the quotient field of A. We have $XP \subset P$. Thus $P \subset (1/X)A$. Since P is maximal, we have $P = (1/X)A \cap A$ and so P is divisorial.

A *satisfies a.c.c.p.* Let $Af_1 \subsetneq Af_2 \subsetneq \cdots$ be an infinite strictly ascending chain of nonzero principal ideals in A. Thus, for all $n \geqslant 1$, $f_n/f_{n+1} \in A$ and is not invertible. Hence $v(f_n/f_{n+1}) > 0$ by (c) and so $v(f_n) > v(f_{n+1})$. Set $g_1 = f_1/f_3$ and $g_2 = f_2/f_3$. We have that g_1, g_2 are in A and $Ag_1 \subsetneq Ag_2$. Hence $v(g_1) > v(g_2)$ and $g_1/g_2 = f_1/f_2 \in A$. Thus, by (a), $X^{v(g_1)}$, $X^{v(g_2)}$, $X^{v(g_1) - v(g_2)}$ are in A.

Since the sequence $\{v(f_n)\}_{n \geqslant 1}$ is bounded and decreasing, we have $\lim_{n \to \infty} (v(f_n) - v(f_{n-1})) = 0$. Thus, for $n \gg 0$, $v(f_n) - v(f_{n-1}) < 1$ and also $v(f_n) - v(f_{n-2}) < 1$. We may assume that this holds for all $n \geqslant 1$. Hence $v(g_1) < 1$ and, since $X^{v(g_1)} \in A$, by (b) we have $v(g_1) \in \Gamma$. Similarly we have that $v(g_2)$ and $v(g_1) - v(g_2) = v(g_1/g_2)$ are in Γ. This is a contradiction because any element in Γ can be uniquely represented as a linear combination of the elements $\{\gamma_n; n \geqslant 1\}$ over $\mathbb{Q}$ and the coefficients are nonnegative.

A is not semi-Krull. Assume the contrary. Then for some positive integer m, we have $P^m \subset AX^{\gamma_1}$. Since $\lim_{n \to \infty} (\gamma_n) = 0$, we can choose n such that $m\gamma_n < \gamma_1$. Thus $v((X^{\gamma_n})^m / X^{\gamma_1}) = m\gamma_n - \gamma_1 < 0$ and so $(X^{\gamma_n})^m \notin AX^{\gamma_1}$, a contradiction.

3. On the Complete Integral Closure of a Semi-Krull Domain

We denote by A^* the complete integral closure of A, that is, $A^* = \bigcup \{(I:I);\ I \text{ a fractional ideal of } A\}$. A is completely integrally closed if $A = A^*$. The next proposition shows that a completely integrally closed semi-Krull domain is Krull.

We recall that A is a pseudo v-multiplication domain, in short a PVMD, if the set of divisorial v-finite ideals of A is a group.

THEOREM 3.1. *Let A be a semi-Krull domain. Then the following are equivalent*:

(i) *A is completely integrally closed*;

(ii) *A is a PVMD*;

(iii) *A is Krull.*

Proof. (iii) $\Rightarrow$ (ii) is well known. (ii) $\Rightarrow$ (i). If A is a semi-Krull PVMD, then A_P is a one-dimensional valuation domain for every $P \in X^1(A)$, by [Gr, Theorem 5] and Proposition 1.2. Thus A_P is completely integrally closed for every $P \in X^1(A)$ and A is completely integrally closed as an intersection of completely integrally closed domains. (i) $\Rightarrow$ (iii). Since, by Theorem 1.7, $t_m(A) = D_m(A)$, we can apply [G, Corollary 2.8]. ▮

A local version of Theorem 3.1 is the following:

PROPOSITION 3.2. *Let A be a semi-Krull domain. Then the following are equivalent*:

(i) *A is completely integrally closed*;

(ii) *A_P is completely integrally closed for every $P \in X^1(A)$*;

(iii) *A_P is a DVR for every $P \in X^1(A)$*;

(iv) *P is t-invertible for every $P \in X^1(A)$*.

Proof. (i) $\Rightarrow$ (iii) by (i) $\Rightarrow$ (iii) in Theorem 3.1. (ii) $\Rightarrow$ (i) because an intersection of completely integrally closed domains is completely integrally closed. (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) follows from [G, Corollary 2.10] because $t_m(A) = D_m(A)$ by Theorem 1.7. ▮

Remark 3.3. We recall that a maximal t-ideal P of any domain A can be either strong or t-invertible [G, Theorem 1.3]. Thus, by Theorem 1.7, we have that the height one primes of a semi-Krull domain are either t-invertible or strongly divisorial. Hence a semi-Krull domain A is Krull if and only if no (height one) prime ideal of A is strongly divisorial. A similar result holds for Mori domains (cf. [BG1, p. 104]).

Using Theorem 3.1, we can show that, if A is a semi-Krull domain, then the polynomial ring $A[X]$ is not necessarily semi-Krull.

PROPOSITION 3.4. *Let A be an integrally closed semi-Krull domain. Then the following are equivalent*:

(i) *$A[X]$ is a semi-Krull domain*;

(ii) *$A[X]$ is a Krull domain*;

(iii) *A is a Krull domain.*

Proof. (i) $\Rightarrow$ (iii). By Proposition 1.2, if $A[X]$ is semi-Krull then $X^1(A[X]) = t_m(A[X])$. Thus in this case every prime ideal of height one of $A[X]$ of type $fK[X] \cap A[X]$, with $K = Q(A)$ and f an irreducible element of $K[X]$, is a maximal t-ideal. Since A is integrally closed, then A is a PVMD by [HsZ2, Proposition 3.2]. Therefore A is Krull by Theorem 3.1. (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are well known. ▮

Thus for example, if A is an integrally closed Mori domain of dimension one which is not Krull [B1], then A is semi-Krull but $A[X]$ is not semi-Krull.

On the other hand, we prove in Proposition 3.5 below that if A is any domain and $A[\mathbf{X}]$ is semi-Krull for a given set of indeterminates $\mathbf{X}$, then A is semi-Krull. A similar property holds for Mori domains since $A = A[\mathbf{X}] \cap Q(A)$ and a finite intersection of Mori domains is Mori [Ra, I, Théorème 2]. This simple proof fails for semi-Krull domains since an intersection of two semi-Krull domains is not necessarily semi-Krull, as we will show in Section 4.

PROPOSITION 3.5. *If A is any domain and $A[\mathbf{X}]$ is semi-Krull for a given set of indeterminates $\mathbf{X}$, then A is semi-Krull.*

Proof. First note that, for any nonzero element $a \in A$, $\operatorname{rad}(aA) = \operatorname{rad}(aA[\mathbf{X}]) \cap A$ and so aA contains a power of its radical. It follows that every proper ideal $I \subset A_P$ contains a power of PA_P, for every $P \in X^1(A)$. Setting $K = Q(A)$, we have $A = A[\mathbf{X}] \cap K = \bigcap \{A[\mathbf{X}]_P \cap K;\ P \in X^1(A[\mathbf{X}])\} = \bigcap \{A[\mathbf{X}]_P \cap K;\ P \in X^1(A[\mathbf{X}])$ and $P \cap A \neq (0)\}$. Now, $A[\mathbf{X}]_P \cap K = A_{P \cap A}$ for all primes P in $A[\mathbf{X}]$. Indeed we have $A_{P \cap A} \subset A[\mathbf{X}]_P \cap K$. Conversely, let $c \in A[\mathbf{X}]_P \cap K$. Thus $cs(\mathbf{X}) \in A[\mathbf{X}]$ for some polynomial $s(\mathbf{X}) \in A[\mathbf{X}] \backslash P$. Let $s(\mathbf{X}) = s_1 M_1 + \cdots + s_k M_k$ where M_i are distinct monomials in the indeterminates $\mathbf{X}$ and $s_i \in A$ for $1 \leqslant i \leqslant k$. Since $cs(\mathbf{X}) \in A[\mathbf{X}]$, we have that $cs_i \in A$ for all i. On the other hand there is an i_0 such that $s_{i_0} \notin P$. Thus $cs_{i_0} \in A$ and $s_{i_0} \in A \backslash P$, hence $c \in A_{P \cap A}$. We conclude that $A = \bigcap \{A_{P \cap A};\ P \in X^1(A[\mathbf{X}])$ and $P \cap A \neq (0)\}$ with a finite character. Clearly, if $P \in X^1(A[\mathbf{X}])$ and $P \cap A \neq (0)$, then $P \cap A \in X^1(A)$. Taking into account that every height one prime ideal is a t-ideal, by Lemma 1.1(b), we get that $X^1(A) = \{P \cap A;\ P \in X^1(A[\mathbf{X}])$ and $P \cap A \neq (0)\}$. Thus A is semi-Krull. ∎

We don't know if the complete integral closure of a semi-Krull domain is completely integrally closed. However, this is true if A is seminormal. We recall that a domain A is said to be seminormal when, for any $x \in Q(A)$, if x^2 and x^3 are in A, then x is in A. This is equivalent to say that, for any $x \in Q(A)$, if $x^n \in A$ for $n \gg 0$, then $x \in A$ (cf., e.g., [Ru] or [S]). An integrally closed domain is clearly seminormal.

THEOREM 3.6. *Let A be an intersection with a finite character of quasilocal seminormal one-dimensional domains. Then A^* is completely integrally closed.*

Proof. The complete integral closure of a quasilocal seminormal one-dimensional domain is completely integrally closed by [R3, Theorem 1.8]. Now it is enough to observe that an intersection of completely integrally closed domains is completely integrally closed and if $A = \bigcap \{A_\lambda;\ \lambda \in \Lambda\}$ is an intersection with a finite character, then $A^* = \bigcap \{(A_\lambda)^*;\ \lambda \in \Lambda\}$ [HOP, Lemma 2.2]. ∎

COROLLARY 3.7. *If A is a seminormal semi-Krull domain, then A^* is completely integrally closed.*

Proof. Since any localization of a seminormal domain is seminormal [S, Corollary 4.6], we can use Theorem 3.6 representing A as the intersection of its localizations at the height one primes. ∎

We don't know if the complete integral closure of a semi-Krull domain is also semi-Krull. However, we show in Theorem 3.12 below that if A is a seminormal semi-Krull domain, then A^* is semi-Krull if and only if A is also a Mori domain.

We recall that the pseudoradical of a domain A is the intersection of all nonzero prime ideals of A.

LEMMA 3.8. *Let A be a seminormal domain with a nonzero pseudoradical J. Then $JA^* = J$.*

Proof. Let x be a nonzero element of JA^*. There are nonzero elements a and b in J such that $x = a/b$. Represent x as a sum $x = x_1 p_1 + \cdots + x_r p_r$ with x_i in A^* and p_i in J, for $i = 1, \ldots, r$. Since $J \subset \mathrm{rad}(Aa)$, for a sufficiently large m, $p_i^m \in Aa$ for all i, $1 \leqslant i \leqslant r$. So $x^m \in A^*a$ for $m \gg 0$. Fix such an integer $m \geqslant 2$. Let $\alpha = a^{m-1}$ and $\beta = b^m$. Thus $\alpha/\beta \in A^*$. So there is a nonzero element p in J such that $p(\alpha/\beta)^n \in A$ for all $n \geqslant 1$. Since $a \in J = \mathrm{rad}(Ap)$, $a^k/p \in A$ for some k. Hence $p(\alpha/\beta)^n\, a^k/p = a^{(m-1)n+k}/b^{mn} \in A$ for all $n \geqslant 1$. Since $(m-1)\,n + k \leqslant mn$ for $n \geqslant k$, then $a^{mn}/b^{mn} = (a^m/b^m)^n \in A$ and since A is seminormal, we get $a^m/b^m \in A$. But this holds for all $m \gg 0$ and A is seminormal, so $x = a/b \in A$. Moreover, in the previous notation, $(m-1)\,n + k < mn$ for $n > k$, so $x^{mn}/a = (a^{(m-1)n+k}/b^{mn})(a^{(n-k)}/a) \in A$. Thus $x \in \mathrm{rad}(Aa) = J$. We conclude that $JA^* = J$. ∎

In the conditions of the previous lemma, we have $A^* = (J:J)$. Hence, if A is noetherian, semilocal, and seminormal, $A^* = A'$ is the integral closure of A and the conductor $(A:A')$ is not zero. It follows that A' is a finite A-module.

We observe that if a semi-Krull domain has a nonzero pseudoradical, then by the finiteness character it has just finitely many height one primes. Thus it is semiquasilocal of height one (Proposition 1.3).

LEMMA 3.9. *Let A and B be domains with a common nonzero integral ideal I. If A is seminormal and I is radical in A, then I is radical in B.*

Proof. Let x be an element in the radical of I in B. Thus x^n is in I for $n \gg 0$. Since A and B have the same quotient field and A is seminormal, we conclude that x is in A. Thus x is in I and so I is radical in B. ∎

PROPOSITION 3.10. *Let A be a quasilocal seminormal one-dimensional domain. Then the following are equivalent*:

(i) A^* *is Krull*;

(ii) A^* *is Mori*;

(iii) A *is Mori*.

Proof. (i)$\Leftrightarrow$(ii) because A^* is completely integrally closed by [R3, Theorem 1.8]. (iii)$\Rightarrow$(i). By Lemma 3.8 the conductor $(A:A^*)$ contains the maximal ideal of A. Thus it is not zero. Since A is Mori, then A^* is Krull by [B2, Corollary 18]. (ii)$\Rightarrow$(iii). Let P be the maximal ideal of A and p a nonzero element in P. $P = \text{rad } Ap$ is contained in the radical of A^*p. On the other hand A^*p is contained in $PA^* = P$ (Corollary 3.8) and P is a radical ideal in A^* by Lemma 3.9. Hence P is the radical of A^*p. But the radical of A^*p is a finite intersection of prime ideals, so it is a Mori ideal [R1, Theorem 6.2]. Hence P is a Mori ideal and since it is maximal in A, then A is a Mori domain by [R2, Theorem 4.14]. ∎

LEMMA 3.11. *Let K be a field and $A_\lambda \subset B_\lambda \subset K$ domains with quotient field K, for $\lambda \in \Lambda$. If the intersection $A = \bigcap \{A_\lambda;\ \lambda \in \Lambda\}$ has a finite character, then also the intersection $B = \bigcap \{B_\lambda;\ \lambda \in \Lambda\}$ has a finite character.*

Proof. Let $x \in B$, $x = a/b$ with a, b nonzero elements of A. We have that a is not invertible in just finitely many A_λ's. Since $aB \subset xB$, if x is not invertible in B_λ, then a is not invertible in A_λ. Thus x is invertible only in finitely many B_λ's. ∎

THEOREM 3.12. *If A is a seminormal semi-Krull domain, then the following are equivalent*:

(i) A^* *is semi-Krull*;

(ii) A^* *is Krull*;

(iii) A^* *is Mori*;

(iv) A *is Mori*.

Proof. (ii)$\Leftrightarrow$(iii) because A^* is completely integrally closed by Corollary 3.7. (i)$\Leftrightarrow$(iii) by Corollary 3.7 and Theorem 3.1. (iii)$\Rightarrow$(ii). We have $A = \bigcap \{A_P;\ P \in X^1(A)\}$ with a finite character. Thus $A^* = \bigcap \{(A_P)^*;\ P \in X^1(A)\}$ with a finite character by [HOP, Lemma 2.2] and Lemma 3.11. Since A_P is seminormal [S, Corollary 4.6] and Mori [Q, Sect. 1, Corollaire 1], then $(A_P)^*$ is Krull by Proposition 3.10 and finally A^* is Krull as an intersection with a finite character of Krull domains. (i)$\Rightarrow$(iii). Let $P \in X^1(A)$ and $S = A \backslash P$. Since $A_P \subset A^*_S$, then $(A_P)^* \subset (A^*_S)^*$. But A^*_S is Krull, hence $(A^*_S)^* = A^*_S$. On the other hand, $A^* \subset (A_P)^*$ and so $A_S^* \subset (A_P)^*$. It follows that $(A_P)^* = A_S^*$ is Krull. Since A_P is seminormal, by Proposition 3.10, A_P is Mori and finally A is Mori as an intersection with a finite character of Mori domains [Ra, I, Théorème 2]. ▮

The complete integral closure of a Mori domain is not always Mori nor completely integrally closed [R3]. However, from Theorem 3.12 we get that, if A is a seminormal semi-Krull Mori domain, then A^* is Krull.

We do not know if the complete integral closure of a seminormal Mori domain is a Krull domain and we do not know either if a seminormal (or even an integrally closed) semi-Krull domain is always Mori.

Remark 3.13. Notice that the Example 2.4(b) of a semi-Krull non-Mori domain is not seminormal. Indeed the seminormalization of A is $k + XK[[X]]$ [BDFn, Corollary 3].

More generally a semi-Krull domain is a.c.c.p. (Theorem 1.10) and any conducive a.c.c.p. seminormal domain is a quasilocal Mori domain. We recall that A is a conducive domain if and only if it fits a cartesian square of the type

$$\begin{array}{ccc} A & \longrightarrow & D \\ \downarrow & & \downarrow \\ V & \xrightarrow[f]{} & B \end{array} \qquad (*)$$

where (V, M) is a nontrivial valuation overring of A, $B = V/Q$, with Q an M-primary ideal of V, $f\colon V \to B$ is the canonical projection, and D is a subring of B. By [BDFn, Corollary 13], the seminormalization of an a.c.c.p. domain is an a.c.c.p. (conducive) pseudovaluation domain. Hence, if A is an a.c.c.p. domain, A is such that, in the cartesian square (*), V is a discrete valuation domain, $Q = M$, $B = V/M$, and D is a field. Thus A is a quasilocal Mori domain by [B1, Corollary 3.5].

We also observe that the domain A constructed in Example 1.9 is not seminormal. Indeed, for example, $(Y_1^2)^2$ and $(Y_1^2)^3$ are in A, but Y_1^2 is not in A. In fact the seminormalization of A is $K + P$.

4. Generalized Quotient Rings of Semi-Krull Domains

An intersection of two semi-Krull domains with the same quotient field need not be necessarily semi-Krull. To see this, consider the Example 2.4(a), $A = k + Xk[X, Y]$. A is not a semi-Krull domain. However, A is the intersection of two semi-Krull domains. In fact $A = A_1 \cap A_2$, where $A_1 = k[X, Y]$ is a Krull domain and $A_2 = k + Xk(Y)[X]_{(X)}$ is a one-dimensional Mori domain [B1], thus a semi-Krull domain by Theorem 2.1.

Note that, since A_1 is Krull, A_1 is an intersection of DVRs with a finite character and then A is an intersection of one-dimensional quasilocal semi-Krull domains with a finite character. The next proposition gives a sufficient condition for an intersection of this type to be semi-Krull.

PROPOSITION 4.1. *Let $\{(A_\lambda, P_\lambda);\ \lambda \in \Lambda\}$ be a family of one-dimensional quasilocal domains and let $A = \bigcap \{A_\lambda;\ \lambda \in \Lambda\}$ have a finite character. Set $P'_\lambda = P_\lambda \cap A$ and suppose that $P'_\lambda \not\subset P'_\mu$ whenever $\lambda \neq \mu$. Then $X^1(A) = \{P'_\lambda;\ \lambda \in \Lambda\}$ and A is a semi-Krull domain if and only if A_λ is a semi-Krull domain for any $\lambda \in \Lambda$.*

Proof. If $P'_\lambda \not\subset P'_\mu$ for every pair λ, μ with $\lambda \neq \mu$, then $A_\lambda = A_{P'\lambda}$ for each $\lambda \in \Lambda$ [Pr, Corollary 8] and so $A = \bigcap \{A_{P'\lambda};\ \lambda \in \Lambda\}$ with a finite character. Moreover, $X^1(A) = \{P'_\lambda;\ \lambda \in \Lambda\}$. Indeed $A_\lambda = A_{P'\lambda}$ is one-dimensional and so P'_λ is an height one prime of A. On the other hand, by Lemma 1.1(b), any height one prime of A, being a t-ideal [J, Corollaire 3, p. 31], is contained in P'_λ for some $\lambda \in \Lambda$. Thus, A is semi-Krull if and only if every integral ideal of $A_{P'\lambda}$ contains a power of $P'_\lambda A_{P'\lambda}$, that is $A_{P'\lambda} = A_\lambda$ is semi-Krull for any $\lambda \in \Lambda$. ▮

With the notation of Proposition 4.1, we have that the hypothesis that $P'_\lambda \not\subset P'_\mu$, when $P'_\lambda \neq P'_\mu$ for $\lambda \neq \mu$, is always satisfied if A is semi-Krull and Λ is a finite set.

PROPOSITION 4.2. *Let $\{(A_i, P_i);\ i = 1, ..., n\}$ be a finite family of one-dimensional quasilocal domains and let $A = \bigcap \{A_i;\ i = 1, ..., n\}$. Set $P'_i = P_i \cap A$ and suppose that $P'_i \neq (0)$ for $i = 1, ..., n$ and $P'_i \neq P'_j$ whenever $i \neq j$. Then A is a semi-Krull domain if and only if A_i is a semi-Krull domain for $i = 1, ..., n$ and $P'_i \not\subset P'_j$ for $i \neq j$. In this case A is semiquasilocal of dimension one with maximal ideals $P'_1, ..., P'_n$.*

Proof. Since $A_{P'_i} \subset A_i$ for $i = 1, ..., n$, then $A = \bigcap \{A_{P'_i};\ i = 1, ..., n\}$ and moreover $X^1(A) \subset \{P'_i;\ i = 1, ..., n\}$ [K, Theorem 110]. Hence, if A is semi-Krull, A is semiquasilocal of dimension one (Proposition 1.3) and so $P'_i \not\subset P'_j$ for $i \neq j$. Thus $A_{P'_i} = A_i$ for $i = 1, ..., n$ by [Pr, Corollary 8] and then A_i is semi-Krull. The converse follows from Proposition 4.1. ▮

PROPOSITION 4.3. *Let A be a domain, $\mathscr{X} \subset X^1(A)$, and $B = \bigcap \{A_P;\ P \in \mathscr{X}\}$ with a finite character. Then B is a semi-Krull domain if and only if A_P is a semi-Krull domain for every $P \in \mathscr{X}$. In this case $X^1(B) = \{PA_P \cap B,\ P \in \mathscr{X}\}$.*

Proof. If $PA_P \cap B \subset QA_Q \cap B$ for some $P, Q \in \mathscr{X}$, then $P = PA_P \cap A \subset QA_Q \cap A = Q$. Thus $P = Q$ and we conclude by Proposition 4.1. ▮

COROLLARY 4.4. *Let A be a semi-Krull domain, $\mathscr{X} \subset X^1(A)$, and $B = \bigcap \{A_P;\ P \in \mathscr{X}\}$. Then B is a semi-Krull domain and $X^1(B) = \{PA_P \cap B, P \in \mathscr{X}\}$.*

Proof. A_P is clearly a semi-Krull domain for every $P \in \mathscr{X}$ and the intersection $\bigcap \{A_P;\ P \in \mathscr{X}\}$ has a finite character. Thus we conclude by Proposition 4.3. ∎

Remarks 4.5. Let A be a semi-Krull domain.

(a) By Corollary 4.4, we get that if $\mathscr{X}, \mathscr{X}' \subset X^1(A)$, then $B = \bigcap \{A_P;\ P \in \mathscr{X}\}$ and $C = \bigcap \{A_P;\ P \in \mathscr{X}'\}$ are equal if and only if $\mathscr{X} = \mathscr{X}'$. In particular, the decomposition $A = \bigcap \{A_P;\ P \in X^1(A)\}$ is irredundant, that is, $\{A_P;\ P \in X^1(A),\ P \neq Q\} \not\subset A_Q$ for every $Q \in X^1(A)$.

(b) As noted in Remark 3.3, we have $X^1(A) = \mathscr{X} \cup \mathscr{Y}$ where $\mathscr{X} = \{P \in X^1(A);\ P \text{ is } t\text{-invertible}\}$ and $\mathscr{Y} = X^1(A) \backslash \mathscr{X} = \{Q \in X^1(A);\ Q \text{ is strongly divisorial}\}$. If $P \in \mathscr{X}$, then A_P is a DVR and so $B = \bigcap \{A_P;\ P \in \mathscr{X}\}$ is a Krull domain. On the other hand $C = \bigcap \{A_Q;\ Q \in \mathscr{Y}\}$ is a semi-Krull domain with no t-invertible height one prime. Indeed if $Q' \in X^1(C)$ and $Q' = QA_Q \cap C$, with $Q \in X^1(A)$, then $C_{Q'} = A_Q$ is not a DVR. Also $A = B \cap C$. Thus for semi-Krull domains we have a "canonical decomposition" similar to the one given in [BG1, Theorem 3.3] for Mori domains.

In what follows Σ denotes a multiplicative system of ideals of A and A_Σ the generalized quotient ring of A with respect to Σ, that is, $A_\Sigma = \bigcup \{(A:J);\ J \in \Sigma\}$. If I is an ideal of A and $\Sigma = \{I^n,\ n \geqslant 1\}$, then $A_\Sigma = \bigcup \{(A:I^n);\ n \geqslant 1\}$ is the Nagata transform of A with respect to I. If A is any domain, then every flat overring of A is an intersection of localizations of A [Rc, Corollary to Theorem 2] and any intersection of localizations of A is a generalized quotient ring of A [HOP, Proposition 4.3].

As for Krull domains, if A is a semi-Krull domain and $\mathscr{X} \subset X^1(A)$, we say that $B = \bigcap \{A_P;\ P \in \mathscr{X}\}$ is a *subintersection* of A. A subintersection of a semi-Krull domain A is a generalized quotient ring of A. Now we prove the converse, that is, a generalized quotient ring of a semi-Krull domain A is a subintersection of A. Thus by Corollary 4.4 we obtain that generalized quotient rings of semi-Krull domains are semi-Krull.

THEOREM 4.6. *Let A be a semi-Krull domain and A_Σ a generalized quotient ring of A. Let $\mathscr{X}$ be the set of height one prime ideals P of A such that $P \not\supset J$ for any $J \in \Sigma$. Then $A_\Sigma = \bigcap \{A_P;\ P \in \mathscr{X}\}$ (if $\mathscr{X} = \varnothing$, we define $\bigcap \{A_P;\ P \in \mathscr{X}\} = K$).*

Proof. Set $B = \bigcap \{A_P;\ P \in \mathscr{X}\}$ and take $x \in B$. If $x \in A$, then $x \in A_\Sigma$. Otherwise, consider the ideal $I = x^{-1}A \cap A$. I is contained in just finitely many ideals $Q_1, ..., Q_n \in X^1(A)$ and contains a power of its radical (Lemma 1.4). Since $x \in A_P$, then $I \not\subset P$ for any $P \in \mathscr{X}$. Hence Q_i contains some ideal J_i in Σ, for $i = 1, ..., n$. If $J = J_1 \cdots J_n$, then $J \in \Sigma$ and I contains a power J^s of J. Therefore $J^s \subset x^{-1}A$, that is, $x \in (A : J^s)$ and so $x \in A_\Sigma$. Conversely, let $x \in A_\Sigma$ and $J \in \Sigma$ be such that $xJ \subset A$. Then $J \not\subset P$ for any $P \in \mathscr{X}$. Let $y_P \in J \backslash P$ for any $P \in \mathscr{X}$. Then $xy_P \in A$ and $x \in A_P$ for any $P \in \mathscr{X}$. ∎

THEOREM 4.7. *If A is a semi-Krull domain, then any generalized quotient ring A_Σ of A is a semi-Krull domain. Moreover, if $\mathscr{X}$ is the set of height one prime ideals P of A such that $P \not\supset J$ for any $J \in \Sigma$, then $X^1(A_\Sigma) = \{PA_P \cap A_\Sigma ; P \in \mathscr{X}\}$.*

Proof. By Theorem 4.6, $A_\Sigma = \bigcap \{A_P ;\ P \in \mathscr{X}\}$. Thus we can use Corollary 4.4. ∎

COROLLARY 4.8. *If A is a semi-Krull domain and I is any ideal of A, then the Nagata transform of A with respect to I is a semi-Krull domain.*

COROLLARY 4.9. [M, *Proposition* 4.8]. *If A is a semi-Krull domain, then any flat overring of A is a semi-Krull domain.*

A generalized quotient ring of a semi-Krull domain A is not always flat. Indeed, by Theorem 4.6 and [M, Proposition 4.8], $A_\Sigma = \bigcap \{A_P ; P \in \mathscr{X}\}$ is flat if and only if $QA_\Sigma = A_\Sigma$ for every prime ideal $Q \in X^1(A) \backslash \mathscr{X}$. However, if A is one-dimensional, then every generalized quotient ring of A is flat over A by [M, Proposition 5.7]. An example of a noetherian Krull domain A with a generalized quotient ring which is not flat is given in [F, p. 32].

With the usual notation, we say that a domain A is a QQR-domain if every overring of A is an intersection of quotient rings of A, that A is a GQR-domain if every overring of A is a generalized quotient ring of A, and finally that A is a QR-domain if every overring of A is a quotient ring of A.

PROPOSITION 4.10. *Let A be a semi-Krull domain and consider the following conditions*:

(i) *A is a GQR-domain*;

(ii) *A is a QQR-domain*;

(iii) *Each overring of A is an intersection of localizations of A*;

(iv) *Each overring of A is a subintersection of A*;

(v) *Each overring of A is flat over A*;

(vi) *A is a Prufer domain*;

(vii) *A is a Dedekind domain*;

(viii) *Each overring of A is a Dedekind domain*;

(ix) *Each overring of A satisfies a.c.c.p.*

Then (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) and (v)$\Leftrightarrow$(vi)$\Leftrightarrow$(vii)$\Leftrightarrow$(viii)$\Rightarrow$(ix). *Moreover, if A is integrally closed, then all the conditions are equivalent.*

Proof. (iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i) are trivial and (i)$\Rightarrow$(iv) by Theorem 4.6. (v)$\Leftrightarrow$(vi) by [Rc, Theorem 4]. (vi)$\Rightarrow$(vii). A is a Krull domain by (ii)$\Rightarrow$(iii) in Theorem 3.1 and it is well known that a Prufer Krull domain is Dedekind. (vii)$\Leftrightarrow$(viii) is well known (cf. [D, p. 198]). (vii)$\Rightarrow$(vi) and (viii)$\Rightarrow$(ix) are trivial. When A is integrally closed, (ii)$\Leftrightarrow$(vi) follows from [D, p. 197 and Corollary 1] and (ix)$\Rightarrow$(vi) by [BD, Lemma 3.2]. ∎

We observe that if (P) is any property such that Dedekind$\Rightarrow$ $(P)\Rightarrow$a.c.c.p., like the property of being semi-Krull, Krull, or Mori, then, in the integrally closed case, any condition in Proposition 4.10 is equivalent to the following: each overring of A satisfies (P).

PROPOSITION 4.11. *If A is a semi-Krull domain, then A is a QR-domain if and only if A is a Dedekind domain with torsion class group.*

Proof. A QR-domain is always Prufer [D, pp. 197–198]. Hence, if A is semi-Krull, then A is Dedekind by (vi)$\Rightarrow$(vii) in Proposition 4.10. But a Dedekind domain is a QR-domain if and only if it has torsion class group [D, p. 200]. ∎

We recall that a QR-domain which is not noetherian need not have torsion class group [H]. On the other hand, if A is a Krull domain, then every subintersection of A is a ring of quotients if and only if the divisorial class group of A is a torsion group [F, Proposition 6.7]. We generalize this statement to semi-Krull domains with a finite number of strongly divisorial height one primes (Theorem 4.14 below).

PROPOSITION 4.12. *If A is a semi-Krull domain, then* $A_\Sigma=\bigcap\{A_P;\ P\in\mathscr{X}\}$, $\mathscr{X}\subset X^1(A)$, *is a quotient ring of A if and only if* $Q\not\subset\bigcup\{P;\ P\in\mathscr{X}\}$ *for every prime ideal* $Q\in X^1(A)\backslash\mathscr{X}$. *In this case* $A_\Sigma=A_S$, *where* $S=A\backslash\bigcup\{P;\ P\in\mathscr{X}\}$.

Proof. By Theorem 4.6, $A_S=\bigcap\{A_Q;\ Q\in X^1(A)$ and $Q\cap S=\varnothing\}$. Thus $A_S\subset A_\Sigma$ and, by Remark 4.5, $A_S=A_\Sigma$ if and only if $Q\cap S\neq\varnothing$, that is, $Q\not\subset\bigcup\{P;\ P\in\mathscr{X}\}$, for any $Q\in X^1(A)\backslash\mathscr{X}$. To finish, it is easy to prove that if $A_\Sigma=A_T$ for some multiplicative part $T\subset A$, then $T\subset S$ and so $A_\Sigma=A_T=A_S$. ∎

The following lemma generalizes [MuS, Lemma 4.3].

LEMMA 4.13. *Let A be any domain and $Q_1, ..., Q_n \in t_m(A)$. Then for every t-invertible t-ideal I of A there exists $x \in Q(A)$ such that $xI \subset A$ and $xI \not\subset Q_i$ for $i = 1, ..., n$.*

Proof. Since I is t-invertible, then IA_P is principal for every $P \in t_m(A)$. Let $S = A \setminus \bigcup \{Q_i\}$. The domain $B = A_S = \bigcap \{A_{Q_i}; i = 1, ..., n\}$ is semi-quasilocal with maximal ideals $Q_i' = Q_i A_{Q_i} \cap B$, $i = 1, ..., n$. The ideal IB is a t-invertible t-ideal of B [BvZ, Lemma 2.6] and $IB_{Q_i'} = IA_{Q_i}$ is principal for every $i = 1, ..., n$. Thus IB is principal [BvZ, Corollary 2.9]. Let $IB = aB$, $a \in Q(A)$. Since I is v-finite in A, there is $s \in S$ such that $sI \subset aA$. Then $a^{-1}sI \subset A$ and moreover $a^{-1}sI \not\subset Q_i$ because $a^{-1}sIA_{Q_i} = sA_{Q_i} = A_{Q_i}$, $i = 1, ..., n$. Thus $x = a^{-1}s$. ∎

THEOREM 4.14. *Let A be a semi-Krull domain with a finite number of strongly divisorial height one primes. Then the following are equivalent:*

(i) *Every subintersection of A is a ring of quotients;*

(ii) $Q \not\subset \bigcup\{P;\ P \in X^1(A)$ *and* $P \neq Q\}$ *for every* $Q \in X^1(A)$;

(iii) Q *is the radical of a principal integral ideal for every* $Q \in X^1(A)$;

(iv) $C(A)$ *is a torsion group.*

Proof. (i) $\Leftrightarrow$ (ii) by Proposition 4.12. (ii) $\Leftrightarrow$ (iii) because any ideal minimal over a principal ideal is a t-ideal [J, Corollaire 3, p. 31], thus it is a height one prime by Proposition 1.2. (iii) $\Rightarrow$ (iv). It is enough to prove that for any integral t-invertible t-ideal I of A, there exist $k \geqslant 1$ such that $(I^k)_v$ is principal [BvZ, Proposition 3.1]. By Lemma 4.13 we can assume that I is not contained in any strongly divisorial height one prime of A. Then, by Remark 1.5, $I = P_1^{(r_1)} \cap \cdots \cap P_n^{(r_n)} = (P_1^{r_1} \cdots P_n^{r_n})_v$ for some t-invertible height one primes $P_1, ..., P_n$ and $r_i \geqslant 1$. Suppose that $P_i = \operatorname{rad}(x_i A)$ for $i = 1, ..., n$. Then $x_i A = P_i^{e_i} A_{P_i} \cap A = P_i^{(e_i)}$ for some $e_i \geqslant 1$. It follows that $(I^k)_v$ is principal for some $k \geqslant 1$. (iv) $\Rightarrow$ (iii). Let $Q \in X^1(A)$. There exists $x \in Q$ such that x is not contained in any strongly divisorial height one prime of A different from Q. If Q is the only t-maximal ideal containing x, then $\operatorname{rad}(xA) = Q$, and there is nothing to prove. Otherwise, $xA = (xA_Q \cap A) \cap P_1^{(r_1)} \cap \cdots \cap P_n^{(r_n)}$, where P_i is a t-invertible height one prime and $r_i \geqslant 1$ for $i = 1, ..., n$. Set $I = xA_Q \cap A$ and $J = P_1^{(r_1)} \cap \cdots \cap P_n^{(r_n)} = (P_1^{r_1} \cdots P_n^{r_n})_v$. We claim that $xA = (IJ)_v$. Indeed $\operatorname{rad}(xA) = \operatorname{rad}(I \cap J) = \operatorname{rad}(IJ)$ and, since $IJ \subset (IJ)_v \subset xA$, $\operatorname{rad}(xA) = \operatorname{rad}((IJ)_v)$. We have $IJA_Q = IA_Q = xA_Q$ and $IJA_{P_i} = JA_{P_i} = xA_{P_i}$ for $i = 1, ..., n$. It follows that $xA_P = (IJ)_v A_P$ for every $P \in X^1(A)$ and thus $xA = (IJ)_v$ by [AA, Proposition 1.4]. Since J is t-invertible, by hypothesis $(J^k)_v = yA$ for some $y \in A$

and $k \geqslant 1$. Thus $x^k A = (I^k)_v y$ and so $(I^k)_v$ is principal. Since $\operatorname{rad}(I^k) = \operatorname{rad}(I) = Q$ and $(I^k)_v \subset Q$, then $\operatorname{rad}(I^k)_v = Q$ and we are done. ∎

The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) proved in Theorem 4.14 above hold for any semi-Krull domain. In the case of one-dimensional noetherian domains, (iii) $\Leftrightarrow$ (iv) has been proved in [PSh, Remark 2]. Note that if A is one-dimensional, then every t-invertible t-ideal of A is invertible and so $C(A)$ is the Picard group of A [BvZ, Corollary 2.10].

References

[A] D. F. Anderson, A general theory of class groups, *Comm. Algebra* **16** (1988), 805–847.

[AA] D. D. Anderson and D. F. Anderson, Some remarks on star operations and the class group, *J. Pure Appl. Algebra* **16** (1988), 805–847.

[ABD] D. F. Anderson, V. Barucci, and D. E. Dobbs, Coherent Mori domains and the principal ideal theorem, *Comm. Algebra* **15**, No. 6 (1989), 1119–1156.

[B1] V. Barucci, On a class of Mori domains, *Comm. Algebra* **11** (1983), 1989–2001.

[B2] V. Barucci, Strongly divisorial ideals and complete integral closure of an integral domain, *J. Algebra* **99** (1986), 132–142.

[BD] V. Barucci and D. E. Dobbs, On chain conditions in integral domains, *Canad. Math. Bull.* **27** (1984), 351–359.

[BDFn] V. Barucci, D. E. Dobbs, and M. Fontana, Conducive integral domains as pullbacks, *Manuscripta Math.* **54** (1986), 261–277.

[BG1] V. Barucci and S. Gabelli, How far is a Mori domain from being a Krull domain? *J. Pure Appl. Algebra* **45** (1987), 101–112.

[BG2] V. Barucci and S. Gabelli, On the class group of a Mori domain, *J. Algebra* **108** (1987), 161–173.

[BvZ] A. Bouvier and M. Zafrullah, On some Class Group of an integral domain, *Bull. Soc. Math. Grèce* **29** (1988), 45–59.

[D] E. D. Davis, Overrings of commutative rings, II, *Trans. Amer. Math. Soc.* **110** (1964), 196–212.

[Fn] M. Fontana, Topologically defined classes of commutative rings, *Ann. Mat. Pura Appl.* **123** (1980), 331–335.

[F] R. Fossum, "The Divisor Class Group of a Krull Domain," Springer-Verlag, New York/Berlin, 1973.

[G] S. Gabelli, Completely integrally closed domains and t-ideals, *Boll. Un. Mat. Ital. B(7)* **3** (1989), 327–343.

[Gr] M. Griffin, Some results on v-multiplication rings, *Canad. J. Math.* **19** (1967), 710–722.

[H] W. Henzer, Quotient overrings of integral domains, *Mathematika* **17** (1970), 139–148.

[HOP] W. Heinzer, J. Ohm, and R. L. Pendleton, On integral domains of the form $\bigcap D_P$, P minimal, *J. Reine Angew. Math.* **241** (1970), 147–149.

[HsZ1] E. Houston and M. Zafrullah, Integral domains in which each t-ideal is divisorial, *Michigan Math. J.* **35** (1988), 291–300.

[HsZ2] E. Houston and M. Zafrullah, On t-invertibility, II, *Comm. Algebra* **17**, No. 8 (1989), 1955–1969.

[J] P. Jaffard, "Les Systèmes d'Idéaux," Dunod, Paris, 1960.

[K] I. KAPLANSKY, "Commutative Rings," Allyn & Bacon, Boston, 1970.

[M] E. MATLIS, Some properties of commutative ring extensions, *Illinois J. Math.* **31**, No. 3 (1987), 374–418.

[MuS] M. P. MURTHY AND R. G. SWAN, Vector bundles over affine surfaces, *Invent. Math.* **36** (1976), 125–165.

[PSh] J. V. PAKALA AND T. S. SHORES, On compactly packed rings, *Pacific J. Math.* **97** (1981), 197–201.

[Pr] B. PREKOWITZ, Intersections of quasilocal domains, *Trans. Amer. Math. Soc.* **181** (1973), 329–339.

[Q] J. QUERRÉ, Sur les anneaux reflexifs, *Canad. J. Math.* **6** (1975), 1222–1228.

[Rc] F. RICHMAN, Generalized quotient rings, *Proc. Amer. Math. Soc.* **16** (1965), 794–799.

[Ra] N. RAILLARD (DESSAGNES), Sur les anneaux de Mori, *C. R. Acad. Sci. Paris* **280** (1975), 1571–1573.

[R1] M. ROITMAN, On Mori domains and commutative rings with $CC^{\perp}$, I, *J. Pure Appl. Algebra* **56** (1989), 247–268.

[R2] M. ROITMAN, On Mori domains and commutative rings with $CC^{\perp}$, II, *J. Pure Appl. Algebra* **61** (1989), 53–77.

[R3] M. ROITMAN, On the complete integral closure of a Mori domain, *J. Pure Appl. Algebra* **66** (1990), 55–79.

[Ru] D. E. RUSH, Seminormality, *J. Algebra* **67** (1980), 377–384.

[S] R. G. SWAN, On semi-normality, *J. Algebra* **67** (1980), 210–229.

[Y] K. YOSHIDA, On birational-integral extension of rings and prime ideals of depth one, *Japan J. Math.* **8** (1982), 49–70.