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Closed Product Formulas for Certain R-polynomials

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R-polynomials get their importance from the fact that they are used to define and compute the Kazhdan–Lusztig polynomials, which have applications in several fields. Here we give a closed product formula for certain R-polynomials valid for every Coxeter group. This result implies a conjecture due to F. Brenti about the symmetric groups.

1. INTRODUCTION

In their main work of 1979 [4] Kazhdan and Lusztig introduced a family of polynomials, indexed by pairs of elements in a Coxeter group, which soon became well known thanks to their applications in different contexts such as in the geometry of Schubert varieties and in representation theory. In order to prove the existence of these polynomials, known as the Kazhdan–Lusztig polynomials, another family of polynomials was defined, the R-polynomials, (see, for example, [3, Section 7.5], or [1]), which are important because their knowledge permits the calculation of the Kazhdan–Lusztig polynomials.

Recently many efforts have been made in trying to calculate the Kazhdan–Lusztig polynomials and, as a consequence, in the attempt to give explicit closed formulas for the R-polynomials. It is not easy even just to conjecture such formulas. One of these conjectures, which appeared in [2], dealt with the R-polynomials of symmetric groups, which are the Coxeter groups that have the largest number of applications. After having proved this conjecture, we realized that a more general result holds for every Coxeter group. Its proof follows the steps of that of the original conjecture except that it needs a further lemma, which is almost straightforward for the symmetric groups and whose proof uses Tits' Word Theorem.

The paper is organized as follows. In Section 2 we recall some definitions and results that are essential for the rest of this paper. In Section 3 we prove the main theorem and the lemma that is necessary for it. In Section 4, as a consequence of the proof of the theorem, we derive an algorithm to compute the exponent that appears in it and then we derive the conjecture already mentioned as a consequence of our result.

2. PRELIMINARIES

In this section we collect some definitions and results that will be used in the rest of this work. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the set of integers; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, \ldots, a\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$).

We follow [1] for general Coxeter groups notation and terminology. In particular, given a Coxeter system (W, S) and $u \in W$, we denote by $l(u)$ the length of u in W, with respect to S, and we let $D_L(u) \stackrel{\text{def}}{=} \{s \in S : l(su) < l(u)\}$, $D_R(u) \stackrel{\text{def}}{=} \{s \in S : l(us) < l(u)\}$, while $T \stackrel{\text{def}}{=} \{usu^{-1} : s \in S, u \in W\}$. We denote by e the identity of W. We will always assume that W is partially ordered by (strong) *Bruhat order*. Recall (see [1, 3]) that $u \leq v$ means that there exist $r \in \mathbf{N}$ and $t_1, \ldots, t_r \in T$ such that $t_r \ldots t_1 u = v$ and $l(t_i \ldots t_1 u) > l(t_{i-1} \ldots t_1 u)$ for $i = 1, \ldots, r$. It is well known that $u \leq v$ if and only if for every reduced expression of v there exists a reduced expression of u which is a subword of that reduced expression of v. Given a set G we let $S(G)$ be the set of all bijections $\pi : G \to G$, and $S_n \stackrel{\text{def}}{=} S([n])$. It is well known

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that (S_n, S), where $S \stackrel{\text{def}}{=} \{(1,2), (2,3), \ldots, (n-1, n)\}$, is a Coxeter system, that $T = \{(i, j) : i < j \leq n\}$ and that every transposition (i, j) admits $s_i s_{i+1} \ldots s_{j-2} s_{j-1} s_{j-2} \ldots s_{i+1} s_i$ as a reduced expression, where $s_k \stackrel{\text{def}}{=} (k, k+1)$. In introducing the R-polynomials, among all the equivalent definitions, we choose the one that is best suited for our purposes. So we define them through the following result:

THEOREM 2.1. *Let (W, S) be a Coxeter system. Then there is a unique family of polynomials $\{R_{u,v}(q)\}_{u,v \in W} \subseteq \mathbf{Z}[q]$ satisfying the following conditions:*

(i) $R_{u,v}(q) = 0$ *if* $u \not\leq v$*;*
(ii) $R_{u,u}(q) = 1$*;*
(iii) *if* $s \in D_L(v)$ *then*

$$R_{u,v}(q) = \begin{cases} R_{su,sv}(q), & \text{if } s \in D_L(u), \\ q R_{su,sv}(q) + (q-1) R_{u,sv}(q), & \text{if } s \notin D_L(u). \end{cases}$$

This theorem–definition will be useful because it will enable us to compute the R-polynomials by induction. Note that a right version of formula (iii) is also true. The following two results give, respectively, the definition of the $\tilde{R}$-polynomials and a tool to compute them analogous to that for the R-polynomials. For their proofs we refer to [1].

THEOREM 2.2. *Let (W, S) be a Coxeter system. Then there is a unique family of polynomials $\{\tilde{R}_{u,v}(q)\}_{u,v \in W} \subseteq \mathbf{N}[q]$ such that:*

$$R_{u,v}(q) = q^{\frac{l(v)-l(u)}{2}} \tilde{R}_{u,v}(q^{\frac{1}{2}} - q^{-\frac{1}{2}}).$$

COROLLARY 2.3. *Let (W, S) be a Coxeter system, $u, v \in W$ $u \leq v$. Then $\tilde{R}_{u,v}(q)$ is a monic polynomial of degree $l(v) - l(u)$. Furthermore if $s \in D_L(v)$ then*

$$\tilde{R}_{u,v}(q) = \begin{cases} \tilde{R}_{su,sv}(q), & \text{if } s \in D_L(u), \\ \tilde{R}_{su,sv}(q) + q\tilde{R}_{u,sv}(q), & \text{if } s \notin D_L(u). \end{cases}$$

Obviously, again, there is also a right version of Corollary 2.3.

Now we give a result due to J. Tits that will be useful for Lemma 3.1. Given $s, s' \in S$ let $\alpha_{s,s'} \stackrel{\text{def}}{=} \underbrace{ss'ss'\ldots}_{m(s,s')\text{letters}}$ where $m(s, s')$ is the order (if $< \infty$) of the product ss'. Two expressions are said to be linked by a braid move if it is possible to obtain the first from the second by changing a factor $\alpha_{s,s'}$ with a factor $\alpha_{s',s}$.

THEOREM 2.4 (TITS' WORD THEOREM). *Let $u \in W$. Then every two reduced expressions of u are linked by a finite sequence of braid moves.*

We refer to [1, 3] for more details concerning general Coxeter group theory.

3. THE MAIN RESULT

In this section we give the proof of the main theorem of this paper. We first need the following:

LEMMA 3.1. *Given a Coxeter system (W, S), let $s, t_i \in S$ $s \neq t_i$ $\forall i \in [n]$ and $l(t_1 \ldots t_n) = n$. Furthermore let $t_{i_1} \ldots t_{i_h}$ be a reduced subword of $t_1 \ldots t_n$ such that $st_{i_1} \ldots t_{i_h} \leq t_1 \ldots t_n s$. Then s commutes with every $t_{i_1}, \ldots, t_{i_h}$.*

PROOF. Being $s \neq t_i$ $\forall i \in [n]$, $st_{i_1} \dots t_{i_h}$ and $t_1 \dots t_n s$ are reduced expressions. Then there exists a reduced subword $t_{j_1} \dots t_{j_{h+1}}$ of $t_1 \dots t_n s$ such that

$$t_{j_1} \dots t_{j_{h+1}} = st_{i_1} \dots t_{i_h}.$$

First of all we observe that $t_{j_{h+1}}$ must be s because s must appear in $t_{j_1} \dots t_{j_{h+1}}$ which is a subword of $t_1 \dots t_n s$ and $s \neq t_i$ $\forall i \in [n]$. By Tits' Word Theorem $st_{i_1} \dots t_{i_h}$ and $t_{j_1} \dots t_{j_h} s$ are linked by a (finite) sequence of braid moves. The analysis of this construction will give us the assertion.

Let us start from $st_{i_1} \dots t_{i_h}$. We do all the braid moves until we encounter a braid move that involves s. There must be such a move in the sequence because at the end s will be in the rightmost place. So we reach an expression of the following type:

$$st_{i'_1} \dots t_{i'_h}$$

and the next braid move involves s and (necessarily) $t_{i'_1}$. Being $t_{i'_2} \neq s$, it must be $\alpha_{s,t_{i'_1}} = st_{i'_1}$, namely s commutes with $t_{i'_1}$. So we do that move and we obtain $t_{i'_1} st_{i'_2} \dots t_{i'_h}$.

At the mth step we reach an expression of the following type:

$$t_{i_1} \dots t_{i_{m-1}} st_{i_m} \dots t_{i_h}$$

with the knowledge that s commutes with every $t_{i_1}, \dots, t_{i_{m-1}}$. As before, we do all the following braid moves of the sequence till we encounter a move that involves s. Again there must be such a move in the sequence because at the end s will be in the rightmost place. So we reach an expression of the following type:

$$t_{i'_1} \dots t_{i'_{m-1}} st_{i'_m} \dots t_{i'_h}.$$

If the following braid move involves s and $t_{i'_{m-1}}$ we do it and return to the $(m-1)$th step. If it involves s and $t_{i'_m}$, since $s \neq t_{i'_{m+1}}$, it must be $\alpha_{s,t_{i'_m}} = st_{i'_m}$, namely s commutes with $t_{i'_m}$. We do the move obtaining

$$t_{i'_1} \dots t_{i'_m} st_{i'_{m+1}} \dots t_{i'_h}$$

and we pass at the $(m+1)$th step, with the knowledge that s commutes not only with every $t_{i'_1}, \dots, t_{i'_{m-1}}$ but also with $t_{i'_m}$.

At the end of the finite sequence of braid moves we obtain $t_{j_1} \dots t_{j_h} s$ and we have demonstrated that s commutes with every $t_{j_1}, \dots, t_{j_h}$, that is with every $t_{i_1}, \dots, t_{i_h}$. □

Now we are able to prove the main result of this paper.

THEOREM 3.2. *Given a Coxeter system (W, S), let $\forall i \in [n]$ $s_i \in S$ $s_i \neq s_j$ if $i \neq j$ and $u, v \in W$ such that $u \leq v \leq s_1 \dots s_{n-1} s_n s_{n-1} \dots s_1$ where this last expression is reduced. Then there exists $a \in \mathbf{N}$ such that*

$$R_{u,v}(q) = (q-1)^a (q^2 - q + 1)^{\frac{l(v)-l(u)-a}{2}}.$$

PROOF. We proceed by induction on n.

The result is clear for $n \leq 1$.

Now we fix a reduced expression of v that is a subword of $s_1 \dots s_{n-1} s_n s_{n-1} \dots s_1$ and a reduced expression of u that is a subword of this reduced expression of v. For simplicity we will refer to these two fixed expressions as $\overline{v}$ and $\overline{u}$, respectively.

Let us focus our attention on the number and the position of the occurrences of s_1s in $\overline{v}$ and $\overline{u}$. We have to consider the following cases, in which $\hat{s_1}$ means that s_1 has been omitted and in which we do not bother about s_i, $i \neq 1$.

(a) $\begin{cases} \overline{v} = \hat{s_1} \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots \hat{s_1} \\ \overline{u} = \hat{s_1} \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots \hat{s_1}. \end{cases}$

Here we conclude right away by induction since $u \leq v \leq s_2 \ldots s_{n-1} s_n s_{n-1} \ldots s_2$

(b$_1$) $\begin{cases} \overline{v} = s_1 \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots \hat{s_1} \\ \overline{u} = s_1 \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots \hat{s_1}. \end{cases}$

Then by Theorem 2.1 we get $R_{u,v}(q) = R_{s_1u,s_1v}(q)$ and we conclude by induction since $s_1u \leq s_1v \leq s_2 \ldots s_{n-1}s_ns_{n-1} \ldots s_2$.

(b$_2$) $\begin{cases} \overline{v} = s_1 \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots \hat{s_1} \\ \overline{u} = \hat{s_1} \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots \hat{s_1}. \end{cases}$

Then by Theorem 2.1 we get $R_{u,v}(q) = qR_{s_1u,s_1v} + (q-1)R_{u,s_1v}(q)$ and we conclude by induction since $s_1u \not\leq s_1v$ and $u \leq s_1v \leq s_2 \ldots s_{n-1}s_ns_{n-1} \ldots s_2$.

(c$_1$) $\begin{cases} \overline{v} = \hat{s_1} \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots s_1 \\ \overline{u} = \hat{s_1} \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots s_1. \end{cases}$

Like (b$_1$) using the right version of Theorem 2.1.

(c$_2$) $\begin{cases} \overline{v} = \hat{s_1} \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots s_1 \\ \overline{u} = \hat{s_1} \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots \hat{s_1}. \end{cases}$

Like (b$_2$) using the right version of Theorem 2.1.

(d$_1$) $\begin{cases} \overline{v} = s_1 \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots s_1 \\ \overline{u} = s_1 \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots s_1. \end{cases}$

$R_{u,v}(q) = R_{s_1u,s_1v}(q) = R_{s_1us_1,s_1vs_1}(q)$ and we conclude by induction since $s_1us_1 \leq s_1vs_1 \leq s_2 \ldots s_{n-1}s_ns_{n-1} \ldots s_2$.

(d$_2$) $\begin{cases} \overline{v} = s_1 \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots s_1 \\ \overline{u} = s_1 \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots \hat{s_1}. \end{cases}$

$R_{u,v}(q) = R_{s_1u,s_1v}(q) = qR_{s_1us_1,s_1vs_1}(q) + (q-1)R_{s_1u,s_1vs_1}$ and we conclude by induction since $s_1us_1 \not\leq s_1vs_1$, $s_1u \leq s_1vs_1 \leq s_2 \ldots s_{n-1}s_ns_{n-1} \ldots s_2$.

(d$_3$) $\begin{cases} \overline{v} = s_1 \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots s_1 \\ \overline{u} = \hat{s_1} \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots s_1. \end{cases}$

Like (d$_2$).

(d$_4$) $\begin{cases} \overline{v} = s_1 \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots s_1 \\ \overline{u} = \hat{s_1} \ldots \hat{} \ldots \hat{} \ldots s_n \ldots \hat{} \ldots \hat{} \ldots \hat{s_1}. \end{cases}$

We have to distinguish two subcases:

(1) $s_1u \not\leq s_1v$.

Then we get

$$R_{u,v}(q) = qR_{s_1u,s_1v}(q) + (q-1)R_{u,s_1v} = (q-1)[qR_{us_1,s_1vs_1}(q) + (q-1)R_{u,s_1vs_1}]$$

and we conclude by induction since $us_1 \not\leq s_1vs_1$,$u \leq s_1vs_1 \leq s_2 \ldots s_{n-1}s_ns_{n-1} \ldots s_2$.

(2) $s_1u \leq s_1v$.

Then we get

$$\begin{aligned} R_{u,v}(q) &= qR_{s_1u,s_1v}(q) + (q-1)R_{u,s_1v} \\ &= qR_{s_1us_1,s_1vs_1}(q) + (q-1)[qR_{us_1,s_1vs_1}(q) + (q-1)R_{u,s_1vs_1}(q)] \\ &= (q^2 - q + 1)R_{u,s_1vs_1}(q) \end{aligned}$$

being, by Lemma 3.1, $u = s_1us_1$ and $us_1 \not\leq s_1vs_1$. So we conclude by induction since $u \leq s_1vs_1 \leq s_2 \ldots s_{n-1}s_ns_{n-1} \ldots s_2$. □

4. Consequences

In this section we derive some consequences from our main result. First we give an algorithm to calculate the exponent a that appears in the formula for $R_{u,v}(q)$ and finally we present the conjecture appearing in [2] as a corollary of Theorem 3.2.

Since the algorithm reflects the induction in Theorem 3.2, it depends on the particular reduced expressions $\overline{v}$ and $\overline{u}$ chosen for v and u. In order to simplify the notation, for every reduced expression $\overline{x}$ and for every $s \in S$, we define $\overline{x}\,(s)$ as the number of factors s appearing in $\overline{x}$.

THEOREM 4.1. *Fix a reduced expression $\overline{v}$ of v that is a subword of $s_1 \dots s_{n-1} s_n s_{n-1} \dots s_1$ and a reduced expression $\overline{u}$ of u that is a subword of $\overline{v}$. We have the following formula for the exponent a of Theorem* 3.2*:*

$$a = \sum_{i=1}^{n} a_i$$

where

$$a_i = \begin{cases} 0, & \text{if } \overline{v}(s_i) = 2, \overline{u}(s_i) = 0 \text{ and } s_i \text{ commutes} \\ & \text{with every } s_j\ j > i \text{ such that } \overline{u}(s_j) \neq 0 \\ \overline{v}(s_i) - \overline{u}(s_i), & \text{otherwise.} \end{cases}$$

PROOF. We calculate a guided by the proof of Theorem 3.2.

In every case, except in the subcase (2) of case (d_4), the R-polynomial indexed by u and v is equal to the R-polynomial indexed by the elements we obtain from $\overline{u}$ and $\overline{v}$ by deleting all the factors s_1, multiplied by $(q-1)^{\overline{v}(s_1)-\overline{v}(s_1)}$. Subcase (2) of case ($d_4$) happens when $\overline{v}\,(s_1) = 2$, $\overline{u}\,(s_1) = 0$ and $s_1 u \leq s_1 v$, namely, by Lemma 3.1, when $\overline{v}\,(s_1) = 2$, $\overline{u}\,(s_1) = 0$ and s_1 commutes with every s_j $j > 1$ such that $\overline{u}\,(s_j) \neq 0$. There the R-polynomial indexed by u and v is equal to the R-polynomial indexed by the elements we obtain from $\overline{u}$ and $\overline{v}$ by deleting the factors s_1 multiplied by $(q^2 - q + 1)$. So we have no contribution to the exponent of $(q-1)$.

By iterating this procedure, the result follows. □

Note that in the algorithm given by Theorem 4.1, for the computation of the a_is it does not matter under which case we view a problem if it can be viewed under more than one case.

EXAMPLE. Let us calculate the R-polynomial indexed by $u = s_1 s_2 s_5 s_1$ and $v = s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1$ in S_7. We immediately find that $a_1 = 0$, $a_2 = 1$, $a_3 = 0$ (giving a factor $q^2 - q + 1$), $a_4 = 2$, $a_5 = 0$, $a_6 = 1$, $a = \sum_{i=1}^{6} a_i = 4$ and therefore $R_{u,v}(q) = (q-1)^4 (q^2 - q + 1)$.

Now we give the proof of Conjecture 7.7 in [2] written in the equivalent way in terms of the R-polynomials.

COROLLARY 4.2. *Let $u, v \in S_n$ be such that $u \leq v \leq (i, j)$ for some $i, j \in [n]$, $i \neq j$. Then there exists $a \in \mathbf{N}$ such that*

$$R_{u,v}(q) = (q-1)^a (q^2 - q + 1)^{\frac{l(v)-l(u)-a}{2}}.$$

PROOF. It is straightforward from Theorem 3.2. In fact the transposition (i, j) admits $s_i s_{i+1} \dots s_{j-2} s_{j-1} s_{j-2} \dots s_{i+1} s_i$ as a reduced expression, where $s_k = (k, k+1)$. □

5. Remarks

We would like to mention the following equivalence that deals with R-polynomials of the type studied in this paper. This result is valid for every Coxeter group W and for every element $w \in W$.

THEOREM 5.1. *Given a Coxeter system* (W, S)*, let* $w \in W$*. Then the following are equivalent:*

(i) $a(u, sv) = a(su, sv) + 1$ *for all* $u, v \leq w$ *and* $s \in S$ *such that* $u < su \leq sv < v$*;*
(ii) $R_{u,v}(q) = (q-1)^{a(u,v)}(q^2 - q + 1)^{\frac{l(v)-l(u)-a(u,v)}{2}}$ *for all* $u \leq v \leq w$*;*

where, for $x, y \in W$ $x \leq y$*,* $(q-1)^{a(x,y)}$ *is the largest power of* $(q-1)$ *that divides* $R_{x,y}(q)$*.*

PROOF. Let us prove that (i) implies (ii) by induction on $l(v)$. Let $s \in D_L(v)$. If $s \in D_L(u)$ or if $s \notin D_L(u)$ but $su \not\leq sv$ then we conclude by induction. Otherwise $R_{u,v}(q) = q R_{su,sv}(q) + (q-1) R_{u,sv}(q)$ that, by inductive assumption, is equal to $q[(q-1)^{a(su,sv)}(q^2 - q + 1)^{\frac{l(sv)-l(su)-a(su,sv)}{2}}] + (q-1)[(q-1)^{a(u,sv)}(q^2 - q + 1)^{\frac{l(sv)-l(u)-a(u,sv)}{2}}]$. By hypothesis, this polynomial is equal to $(q-1)^{a(su,sv)}(q^2 - q + 1)^{\frac{l(sv)-l(su)-a(su,sv)}{2}}[q + (q-1)^2]$.

Conversely fix (if there are) $s \in S$ such that $u < su \leq sv < v$. Then $R_{u,v}(q) = q R_{su,sv}(q) + (q-1) R_{u,sv}(q) = q[(q-1)^{a(su,sv)}(q^2 - q + 1)^{\frac{l(sv)-l(su)-a(su,sv)}{2}}] + (q-1)[(q-1)^{a(u,sv)}(q^2 - q + 1)^{\frac{l(sv)-l(u)-a(u,sv)}{2}}]$. But $R_{u,v}(q) = (q-1)^{a(u,v)}(q^2 - q + 1)^{\frac{l(v)-l(u)-a(u,v)}{2}}$ and an easy argument of divisibility shows that this is possible only if $a(u, sv) = a(su, sv) + 1$. □

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