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Some New Results for Bounded and Monotone Properties of Solutions of Second-Order Quasi-Linear Forced Difference Equations

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Abstract—Some new results are obtained for the bounded and monotone properties of solutions of second-order quasi-linear forced difference equations

$$\Delta(p_{n-1}(\Delta y_{n-1})^{\alpha}) = q_n y_n^{\beta} + r_n, \qquad n \geq n_0,$$

where $n_0 \in N = \{0, 1, 2, 3, \dots\}$, $\{p_n\}_{n=n_0-1}^{\infty}$ is a positive sequence, $\{q_n\}_{n=n_0}^{\infty}$ is a nonnegative real sequence with $q_n \not\equiv 0$, $\{r_n\}_{n=n_0}^{\infty}$ is a real sequence, and α and β are quotients of odd positive integers. Some errors in [1] are pointed out and addressed. Examples are given to illustrate the advantages of the new results.

Keywords—Quasi-linear forced difference equation, Monotone, Bounded solution.

1. INTRODUCTION

Asymptotic properties have been extensively investigated for the solutions of second-order quasi-linear difference equations in recent years. For example, see [2–9] and the references cited therein.

In the present paper, we consider the following second-order quasi-linear difference equation

$$\Delta\left(p_{n-1}(\Delta y_{n-1})^{\alpha}\right) = q_n y_n^{\beta} + r_n, \qquad n \geq n_0, \tag{1}$$

where $n_0 \in N = \{0, 1, 2, 3, \dots\}$, Δ denotes the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, $\{p_n\}_{n=n_0-1}^{\infty}$ is a positive sequence, $\{q_n\}_{n=n_0}^{\infty}$ is a nonnegative real sequence with $q_n \not\equiv 0$, $\{r_n\}_{n=n_0}^{\infty}$ is a real sequence, and α and β are quotients of positive odd integers.

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By a solution of equation (1), we mean a real sequence $\{y_n\}$ satisfying equation (1) for $n \geq n_0$. We consider only solutions that are nontrivial for all large n. A solution $\{y_n\}$ of equation (1) is bounded if there exists an $M > 0$ such that $|y_n| \leq M$ for all $n \geq n_0$. A solution $\{y_n\}$ of equation (1) is oscillatory if the terms $\{y_n\}$ are neither eventually positive nor eventually negative and nonoscillatory otherwise. A solution $\{y_n\}$ of equation (1) is said to be weakly oscillatory if $\{y_n\}$ is nonoscillatory whereas $\{\Delta y_n\}$ oscillates. A solution $\{y_n\}$ of equation (1) is said to be of nonlinear limit-cycle type if $\sum_{n=1}^{\infty} y_n^{1+\beta} < \infty$ and of nonlinear limit-point type otherwise, that is, $\sum_{n=1}^{\infty} y_n^{1+\beta} = \infty$.

Equation (1), especially when $r_n = 0$ and $\alpha = 1$, has been extensively studied. For example, see [2–9] and the references cited therein. Recently, Thandapani and Ravi also considered the bounded and monotone properties of solutions of equation (1) in [1]. They defined two sets of nonoscillatory solutions of equation (1) in [1] as follows:

$$A := \{\{y_n\} : \text{there exists an integer } n \in N \text{ such that } y_n \Delta y_n > 0 \text{ for } n \geq N\},$$
$$B := \{\{y_n\} : \text{there exists an integer } n \in N \text{ such that } y_n \Delta y_n < 0 \text{ for } n \geq N\}.$$

Mainly, they obtained the following results (for the sake of readers, we extract them *from the original*).

PROPOSITION 1. *Let $r_n \geq 0$ or < 0 for all large n. Then, every solution of equation (1) is nonoscillatory.*

PROPOSITION 2. *Let $r_n \geq 0$ for all large n and let $\{y_n\}$ be any solution of equation (1). Then, $\{y_n\}$ either belongs to Class A or belongs to Class B.*

REMARK 1. Propositions 1 and 2 include Lemmas 1 and 2 of [10], Theorem 1 of [11], Propositions 1 and 2 of [12], Lemma 3.1 and Corollary 3.1 of [2], as special cases.

THEOREM 1. *Let $r_n \geq 0$ for all large n. Then, every (Class A) solution of equation (1) is bounded if and only if*

$$\sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \frac{q_k + r_k}{p_n} \right]^{1/\alpha} < \infty.$$

REMARK 2. Theorem 1 generalizes Theorem 2 of [11] and Theorem 4 of [10].

THEOREM 2. *Assume that $p_n \equiv 1$ for all large n. If $\sum_{n=1}^{\infty} q_n < \infty$ and $\sum_{n=1}^{\infty} r_n = \infty$, then all solutions of equation (1) are unbounded.*

THEOREM 4. *Let $r_n \geq 0$ for all large n. If $\sum_{n=1}^{\infty}(1/p_n^{1/\alpha}) = \infty$, then any Class B solution $\{y_n\}$ of equation (1) satisfies $\lim_{n\to\infty} p_n(\Delta y_n)^\alpha = 0$.*

REMARK 4. Under the assumption of Theorem 4, it is not difficult to see that $\sum_{n=1}^{\infty}(q_n + r_n) < \infty$ is a necessary condition for a Class B solution of equation (1) to converge to a nonzero limit as $n \to \infty$.

THEOREM 5. *In addition to the hypotheses of Theorem 4, assume that*

$$\sum_{n=1}^{\infty} \left(\sum_{t=n+1}^{\infty} \frac{q_t}{p_n} \right)^{1/\alpha} = \infty.$$

Then, any Class B solution $\{y_n\}$ of equation (1) satisfies $\lim_{n\to\infty} y_n = 0$.

THEOREM 6. *Let $r_n \geq 0$ for all large n. If $\{y_n\}$ is a nonlinear limit-circle solution of equation (1), then $y_n \to 0$ as $n \to \infty$ and $y_n \Delta y_n < 0$ for all large n.*

Here, we would like to point out some errors in the above propositions except for Proposition 1, some in the conclusions and some in the proofs. First, we point out there is a mistake in

Proposition 2, which results in a series of errors contained in [1]. We illustrate the mistakes with the following counterexamples.

COUNTEREXAMPLE 1. Consider the difference equation of the form

$$\Delta\left(\Delta y_{n-1}\right) = 4y_n + 8\left[1 + (-1)^{n+1}\right], \qquad n \geq 1. \tag{C$_1$}$$

Clearly, $r_n = 8[1 + (-1)^{n+1}] \geq 0$ for all $n \geq 1$. We find that $y_n = -2 + (-1)^n$, $n \geq 0$, is a solution of equation (C_1). But $\{y_n\} \notin A \cup B$ because $y_n < 0$ for $n \geq 0$ whereas $\Delta y_n = 2(-1)^{n+1}$ is oscillatory.

COUNTEREXAMPLE 2. Consider the difference equation

$$\Delta(n(n-1)\Delta y_{n-1}) = 4n^2 y_n + 8n\left[1 + (-1)^{n+1}\right], \qquad n \geq 2. \tag{C$_2$}$$

Obviously, $r_n = 8n[1+(-1)^{n+1}] \geq 0$ for all $n \geq 2$. One can easily verify that $y_n = [-2+(-1)^n]/n$, $n \geq 1$, is a solution of equation (C_2) whereas $\{y_n\} \notin A \cup B$ because

$$\Delta y_n = \frac{(2n+1)(-1)^{n+1} + 2}{n(n+1)}$$

oscillates.

In fact, the proof of Proposition 2 in [1] holds only for eventually positive solutions of equation (1) and it is invalid for eventually negative solutions of equation (1). From the two counterexamples shown above, it follows that it is necessary to supplement the classes of nonoscillatory solutions of equation (1) with the following Class C

$$C := \{\{y_n\} : \{y_n\} \text{ is eventually weakly oscillatory}\}.$$

Secondly, we present the following counterexamples showing that Theorem 1 is incorrect.

COUNTEREXAMPLE 3. Consider the following difference equation

$$\Delta\left(\left(4n^3 + n + 1\right)\left(\Delta y_{n-1}\right)^{1/5}\right) = (12n + 18)y_n + 6n - 5, \qquad n \geq 1. \tag{C$_3$}$$

Using the notation of equation (1), we have $r_n = 6n - 5 > 0$ for all $n \geq 1$, $p_{n-1} = 4n^3 + n + 1$, $\alpha = 1/5$, $q_n = 12n + 18$, $n \geq 1$. By simple calculation, we see that

$$\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\frac{q_k + r_k}{p_n}\right]^{1/\alpha} = \sum_{n=1}^{\infty}\left(\frac{9n^2 + 22n}{4(n+1)^3 + n + 2}\right)^5 < \infty,$$

which says that the sufficient conditions in Theorem 1 are satisfied. We can easily verify that $y_n = -n$, $n \geq 0$, is a Class A solution of equation (C_3), but y_n is obviously unbounded.

COUNTEREXAMPLE 4. Consider the following difference equation

$$\Delta\left((n^3 + 1)(\Delta y_{n-1})^{1/3}\right) = (3n + 2)y_n + n + 1, \qquad n \geq 1. \tag{C$_4$}$$

Here, $r_n = n + 1 > 0$ for all $n \geq 1$, $p_{n-1} = n^3 + 1$, $\alpha = 1/3$, $q_n = 3n + 2$. By elementary calculation, we have that

$$\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\frac{q_k + r_k}{p_n}\right]^{1/\alpha} = \sum_{n=1}^{\infty}\left(\frac{2n^2 + 5n}{(n+1)^3 + 1}\right)^3 < \infty,$$

which indicates that the sufficient conditions in Theorem 1 are satisfied. We find that $y_n = n$, $n \geq 0$, is a solution of equation (C_4) with $\{y_n\} \in A$. However, y_n is evidently unbounded.

Because of Proposition 2 and Theorem 1 being wrong, Remarks 1 and 2 are also incorrect. In Section 2, we will state some new results for the bounded and monotone properties of solutions for equation (1).

Thirdly, the conclusion of Theorem 2 is true while there are two evident errors in the proof of Theorem 2 in [1]. For the sake of readers, we restate their proof in Appendix A.

One error is that because $\sum_{n=1}^{\infty} r_n = \infty$ cannot ensure that $r_n \geq 0$ for all large n, it is incorrect to use Proposition 1 in the beginning of the proof. We can demonstrate this with the following counterexample.

COUNTEREXAMPLE 5. Consider the sequence $\{r_n\}$ with

$$r_n = \begin{cases} \dfrac{1}{n}, & n \text{ odd}, \\ -\dfrac{1}{n^2}, & n \text{ even}, \end{cases} \qquad n \geq 1. \tag{C_5}$$

Obviously, one cannot find an $N_0 \in N$ such that $r_n \geq 0$ for all $n \geq N_0$ while $\sum_{n=1}^{\infty} r_n = \infty$.

The other error is that Proposition 2 cannot ensure that Δy_n is eventually of one sign. Counterexamples 1 and 2 make evident that Δy_n is not eventually of one sign.

The following example also displays a limitation of Theorem 2.

EXAMPLE 1. Consider the difference equation

$$\Delta(\Delta y_{n-1}) = 2y_n^{1/3} + 4n, \qquad n \geq 1. \tag{E_1}$$

Since $q_n = 2$ implies that $\sum_{n=1}^{\infty} q_n = \infty$, Theorem 2 is not definite in answering whether all solutions of equation (E_1) are unbounded or not.

In the sequel we will give a correct proof of Theorem 2 and derive some new results.

We think that the method of the proof of Theorem 4 in [1] (see Appendix B) is suitable only for the case where $\{y_n\}$ is eventually positive and it is invalid for the case of $\{y_n\}$ being eventually negative. We will state a new result for the proposition in Section 2.

We now manifest the incorrectness of Remark 4 and Theorem 5 using the following counterexample.

COUNTEREXAMPLE 6. Consider the following difference equation

$$\Delta((n+1)\Delta y_{n-1}) = 4y_n + 4 + \frac{4n-1}{n(n+1)}, \qquad n \geq 1, \tag{C_6}$$

where $r_n > 0$ for all $n \geq 1$, $p_{n-1} = n+1$, $\alpha = 1$, $q_n = 4$. Clearly, $\sum_{n=1}^{\infty}(1/p_n^{1/\alpha}) = \infty$ and $\sum_{n=1}^{\infty}(\sum_{t=n+1}^{\infty}(q_t/p_n))^{1/\alpha} = \infty$. So, all conditions of Theorems 4 and 5 are satisfied; but we find that $y_n = -1 - 1/(n+1)$, $n \geq 0$, is a Class B solution of equation (C_6). Nevertheless, we have $\lim_{n\to\infty} y_n = -1 \neq 0$ and $\sum_{n=1}^{\infty}(q_n + r_n) = \infty$.

Theorem 6 is also incorrect. The following counterexample can show this.

COUNTEREXAMPLE 7. Consider the difference equation of the form

$$\Delta(n(n-1)\Delta y_{n-1}) = 6n(n+3)y_n + 12n + 36 + (10m + 18)(-1)^{n+1}, \qquad n \geq 2. \tag{C_7}$$

Clearly, $r_n = 12n + 36 + (10n + 18)(-1)^{n+1} > 0$ for all $n \geq 2$ and $\beta = 1$. We find that $y_n = (-2 + (-1)^n)/n$, $n \geq 1$, is a nonlinear limit-circle solution of equation (C_7). Nevertheless, there does not exist an $N_0 \in N$ such that $y_n \Delta y_n < 0$ for $n \geq N_0$ because $y_n < 0$ whereas

$$\Delta y_n = \frac{(2n+1)(-1)^{n+1} + 2}{n(n+1)}$$

is oscillatory for $n \geq 1$.

All these examples above demonstrate that it is worth doing further investigations for equation (1). To the authors' knowledge, however, there are not yet any results to correct the errors. The motivation of this note is to continue the study and to present some new results for the solutions of equation (1), mainly, the bounded and monotone properties of solutions, the existence of Class A solutions, and the discriminating method of nonlinear limit-point solutions. Our results can easily solve the above problems. Some examples are also presented to illustrate the advantages of the new results.

Throughout this paper, we use the convention

$$\sum_{n=i}^{j} y_n = 0, \qquad \text{whenever } j \le i-1.$$

2. MAIN RESULTS

2.1. Bounded and Monotone Properties of Solutions

First, we revise Proposition 2 to derive the following result.

THEOREM 2.1.1. *Let $r_n \ge 0$ (< 0) for all large n and let $\{y_n\}$ be any solution of equation (1). Then, $\{y_n\} \in A \cup B \cup C$.*

PROOF. Since $r_n \ge 0$ (< 0) for all large n, by Proposition 1, y_n is nonoscillatory. Thus, there is an $n_1 \ge n_0$ such that either

$$y_n > 0, \qquad \text{for } n \ge n_1, \tag{2}$$

or

$$y_n < 0, \qquad \text{for } n \ge n_1. \tag{3}$$

If Δy_n oscillates, then $\{y_n\} \in C$. Otherwise, there is an $n_2 \ge n_0$ such that either

$$\Delta y_n > 0, \qquad \text{for } n \ge n_2, \tag{4}$$

or

$$\Delta y_n < 0, \qquad \text{for } n \ge n_2. \tag{5}$$

If (2) and (4) or (3) and (5) hold, then $\{y_n\} \in A$. If (2) and (5) or (3) and (4) are valid, then $\{y_n\} \in B$. So, $\{y_n\} \in A \cup B \cup C$.

Next, we give a correct proof for Theorem 2 of [1].

THEOREM 2.1.2. *(See Theorem 2 in [1].) Assume that $p_n \equiv 1$ for all large n. If $\sum_{n=1}^{\infty} q_n < \infty$ and $\sum_{n=1}^{\infty} r_n = \infty$, then all solutions of equation (1) are unbounded.*

PROOF. Let $\{y_n\}$ be a bounded solution of equation (1). Then, there exists an $M > 0$ such that $|y_n| \le M$ for $n \ge n_0$. Summing equation (1) from $n_0 + 1$ to n gives

$$\begin{aligned}(\Delta y_n)^\alpha &= (\Delta y_{n_0})^\alpha + \sum_{k=n_0+1}^{n} q_k y_k^\beta + \sum_{k=n_0+1}^{n} r_k \\ &> (\Delta y_{n_0})^\alpha - M^\beta \sum_{k=n_0+1}^{n} q_k + \sum_{k=n_0+1}^{n} r_k \to \infty, \qquad \text{as } n \to \infty,\end{aligned}$$

which, in turn, implies that $\lim_{n\to\infty} y_n = \infty$, a contradiction.

Finally, we state two new results.

THEOREM 2.1.3. *Assume that for any given $k \in R$, $l > 0$ and for some $k_0 \in N$,*

$$\sum_{n=k_0}^{\infty} \left\{ \frac{1}{p_n} \left[k + \sum_{s=k_0}^{n} (r_s - lq_s) \right] \right\}^{1/\alpha} = \infty.$$

Then, all solutions of equation (1) are unbounded.

PROOF. Suppose, there exists a sequence $\{y_n\}$ which is a bounded solution of equation (1). Then, there are a positive constant M and an $n_1 \in N$ such that $|y_n| \leq M$ for $n \geq n_1$. By summing both sides of equation (1) up from $n_1 + 1$ to n, one can get that

$$\begin{aligned} p_n (\Delta y_n)^\alpha &= p_{n_1} (\Delta y_{n_1})^\alpha + \sum_{k=n_1+1}^{n} \left(q_k y_k^\beta + r_k \right) \\ &\geq p_{n_1} (\Delta y_{n_1})^\alpha + \sum_{k=n_1+1}^{n} (r_k - M^\beta q_k), \end{aligned}$$

which leads to

$$y_n \geq y_{n_1+1} + \sum_{t=n_1+1}^{n-1} \left\{ \frac{1}{p_n} \left[p_{n_1} (\Delta y_{n_1})^\alpha + \sum_{s=n_1+1}^{t} \left(r_s - M^\beta q_s \right) \right] \right\}^{1/\alpha} \to \infty, \qquad \text{as } n \to \infty.$$

This is contrary to the assumption that $\{y_n\}$ is a bounded solution of equation (1).

THEOREM 2.1.4. *Assume that r_n is eventually of definite sign. Let $\{y_n\}$ be any solution of equation (1) with the same sign as r_n eventually. Then, $\{y_n\}$ either belongs to Class A or belongs to Class B.*

PROOF. We prove only the case where r_n is eventually negative. The proof for the other case is similar and will be omitted here. By Proposition 1, we know that $\{y_n\}$ is nonoscillatory. So, there exists an $n_1 \in N$ such that $r_n < 0$, $y_n < 0$ for $n \geq n_1$. Consider the sequence $\{W_n\}$ defined by

$$W_n = y_n p_n (\Delta y_n)^\alpha, \qquad n \geq n_1. \tag{6}$$

Then,

$$\begin{aligned} \Delta W_n &= y_{n+1} \Delta \left(p_n (\Delta y_n)^\alpha \right) + p_n (\Delta y_n)^{\alpha+1} \\ &= y_{n+1} q_{n+1} y_{n+1}^\beta + r_{n+1} y_{n+1} + p_n (\Delta y_n)^{\alpha+1} > 0. \end{aligned} \tag{7}$$

If there exists an integer $N_1 \geq n_1$ such that $\Delta y_{N_1} < 0$, then from (6) and (7) we see that $\Delta y_n < 0$ for $n \geq N_1$, which means that $\{y_n\} \in A$.

If there does not exist an integer $N_1 \geq n_1$ such that $\Delta y_{N_1} < 0$, then it must be $\Delta y_n > 0$ for $n \geq n_1$, which implies that $\{y_n\} \in B$. The proof is complete.

2.2. Existence and Boundedness of Class A Solutions

The existence of Class B solutions of equation (1) has been investigated in [1]. However, there are no results for the existence of Class A solutions of equation (1). Now, we give a result.

THEOREM 2.2.1. *Suppose that r_n is eventually of definite sign. If for some $n_1 \geq n_0$*

$$\sum_{n=n_1}^{\infty} \left(\sum_{s=1}^{n-1} \frac{q_s + r_s \operatorname{sign}(r_s)}{p_{n-1}} \right)^{1/\alpha} < \infty,$$

where sign$(\cdot)$ *is the sign function, then there exits at least one solution of equation (1) belonging to Class A.*

PROOF. We first prove the case when r_n is eventually negative. Choose $N_0 \in N$ large enough so that

$$\sum_{k=N_0}^{n} \left(\sum_{s=1}^{k-1} \frac{q_s + r_s \operatorname{sign}(r_s)}{p_{k-1}} \right)^{1/\alpha} \le \frac{1}{2^{\beta/\alpha}}, \qquad n \ge N_0. \tag{8}$$

Let A_N be the Banach space of all bounded sequences $Y = \{y_n\}$, $n \ge N_0$, with supremum norm $\|Y\| = \sup_{n \ge N_0} |y_n|$. Set

$$S = \{Y \in A_N : -2 \le y_n \le -1,\ n \ge N_0\},$$

and define an operator $T : S \to A_N$ as follows

$$(Ty)_n = -1 + \sum_{k=N_0}^{n} \left(\sum_{s=1}^{k-1} \frac{q_s y_s^\beta + r_s}{p_{k-1}} \right)^{1/\alpha}, \qquad n \ge N_0.$$

Clearly, $(Ty)_n \le -1$ for any $\{y_n\} \in S$. Again, according to (8), we have for $n \ge N_0$

$$\begin{aligned}
(Ty)_n &\ge -1 + \sum_{k=N_0}^{n} \left(\sum_{s=1}^{k-1} \frac{q_s(-2)^\beta + r_s}{p_{k-1}} \right)^{1/\alpha} \\
&= -1 + (-2)^{\beta/\alpha} \sum_{k=N_0}^{n} \left(\sum_{s=1}^{k-1} \frac{q_s + r_s \operatorname{sign}(r_s)/2^\beta}{p_{k-1}} \right)^{1/\alpha} \\
&\ge -1 + (-2)^{\beta/\alpha} \sum_{k=N_0}^{n} \left(\sum_{s=1}^{k-1} \frac{q_s + r_s \operatorname{sign}(r_s)}{p_{k-1}} \right)^{1/\alpha} \\
&\ge -1 + (-2)^{\beta/\alpha} \frac{1}{2^{\beta/\alpha}} = -2.
\end{aligned}$$

This means that $TS \subset S$. It follows from the definition of T that T is an increasing mapping. Hence, by the Knater-Tarski fixed-point theorem [13,14], there exists a $Y \in S$ such that $TY = Y$. That is,

$$y_n = -1 + \sum_{k=N_0}^{n} \left(\sum_{s=1}^{k-1} \frac{q_s y_s^\beta + r_s}{p_{k-1}} \right)^{1/\alpha}, \qquad n \ge N_0.$$

From this, it is clear that $\{y_n\}$ is eventually a solution of equation (1). Since $-2 \le y_n \le -1$ and

$$\Delta y_n = \left(\sum_{s=1}^{n} \frac{q_s y_s^\beta + r_s}{p_n} \right)^{1/\alpha} < 0, \qquad n \ge N_0,$$

one can see that $\{y_n\} \in A$.

For the proof when r_n is eventually positive, it suffices to replace S and T, respectively, by

$$S' = \{Y \in A_N : 1 \le y_n \le 2,\ n \ge N_0\}$$

and

$$(T'y)_n = 1 + \sum_{k=N_0}^{n} \left(\sum_{s=1}^{k-1} \frac{q_s y_s^\beta + r_s}{p_{k-1}} \right)^{1/\alpha}, \qquad n \ge N_0.$$

THEOREM 2.2.2. *Assume that r_n is eventually of definite sign and that $\beta \le \alpha$. Then, every solution $\{y_n\}$ of Class A of equation (1) with the same sign as r_n eventually is bounded if and only if for any given positive real numbers k and l and for some $N_0 \in N$,*

$$\sum_{n=N_0}^{\infty} \left\{ \frac{1}{p_n} \left[k + \sum_{s=N_0}^{n} (q_s + l \operatorname{sign}(r_s) r_s) \right] \right\}^{1/\alpha} < \infty, \tag{9}$$

where $\operatorname{sign}(\cdot)$ *is the sign function.*

PROOF. We only prove the case where r_n is eventually negative. The proof for the other case is similar. Let $\{y_n\}$ be an arbitrary Class A solution of equation (1) with the same sign as r_n eventually. Then, there exists an $n_1 \in N$ such that

$$r_n < 0, \quad y_n < 0, \quad \Delta y_n < 0, \qquad \text{for } n \ge n_1. \tag{10}$$

NECESSITY. Suppose that $\{y_n\}$ is bounded. Then, there exists an $M > 0$ such that

$$-M \le y_n \le y_{n_1}, \qquad \text{for } n \ge n_1. \tag{11}$$

Taking the summation on both sides of equation (1) from $n_1 + 1$ to n, we have that

$$\begin{aligned} p_n (\Delta y_n)^\alpha &= p_{n_1} (\Delta y_{n_1})^\alpha + \sum_{k=n_1+1}^{n} \left(q_k y_k^\beta + r_k \right) \\ &\le p_{n_1} (\Delta y_{n_1})^\alpha + \sum_{k=n_1+1}^{n} \left(y_{n_1}^\beta q_k + r_k \right) \\ &= y_{n_1}^\beta \left[\frac{p_{n_1} (\Delta y_{n_1})^\alpha}{y_{n_1}^\beta} + \sum_{k=n_1+1}^{n} \left(q_k + \frac{\operatorname{sign}(r_k) r_k}{(-y_{n_1})^\beta} \right) \right], \end{aligned}$$

which produces

$$y_n \le y_{n_1+1} + y_{n_1}^{\beta/\alpha} \sum_{t=n_1+1}^{n-1} \left\{ \frac{1}{p_n} \left[\frac{p_{n_1}(\Delta y_{n_1})^\alpha}{y_{n_1}^\beta} + \sum_{k=n_1+1}^{t} \left(q_k + \frac{\operatorname{sign}(r_k) r_k}{(-y_{n_1})^\beta} \right) \right] \right\}^{1/\alpha}.$$

And so,

$$\frac{y_n}{y_{n_1}^{\beta/\alpha}} \ge \frac{y_{n_1+1}}{y_{n_1}^{\beta/\alpha}} + \sum_{t=n_1+1}^{n-1} \left\{ \frac{1}{p_n} \left[\frac{p_{n_1}(\Delta y_{n_1})^\alpha}{y_{n_1}^\beta} + \sum_{k=n_1+1}^{t} \left(q_k + \frac{\operatorname{sign}(r_k) r_k}{(-y_{n_1})^\beta} \right) \right] \right\}^{1/\alpha}.$$

From this and (11), one can see that (9) is true.

SUFFICIENCY. From equation (1), we have

$$q_n = \frac{\Delta(p_{n-1}(\Delta y_{n-1})^\alpha)}{y_n^\beta} - \frac{r_n}{y_n^\beta} \ge \Delta \left(\frac{p_{n-1}(y_{n-1})^\alpha}{y_{n-1}^\beta} \right) - \frac{r_n}{y_n^\beta}, \qquad n \ge n_2. \tag{12}$$

Summing both sides of (12) from $n_1 + 1$ to n and noticing $y_n \le y_{n_1} < 0$ for $n \ge n_1$, we derive

$$\begin{aligned} \frac{\Delta y_n}{y_n^{\beta/\alpha}} &\le \left\{ \frac{1}{p_n} \left[\frac{p_{n_1}(\Delta y_{n_1})^\alpha}{y_{n_1}^\beta} + \sum_{k=n_1+1}^{n} \left(q_k + \frac{r_k}{y_k^\beta} \right) \right] \right\}^{1/\alpha} \\ &\le \left\{ \frac{1}{p_n} \left[\frac{p_{n_1}(\Delta y_{n_1})^\alpha}{y_{n_1}^\beta} + \sum_{k=n_1+1}^{n} \left(q_k + \frac{\operatorname{sign}(r_k) r_k}{(-y_{n_1})^\beta} \right) \right] \right\}^{1/\alpha}. \end{aligned}$$

This, together with (9), implies that there exists an $M > 0$ such that

$$\sum_{k=n_1+1}^{n} \frac{\Delta y_k}{y_k^{\beta/\alpha}} \le M, \qquad \text{for all } n \ge n_1 + 1.$$

Also, we have

$$\sum_{k=n_1+1}^{n} \frac{\Delta y_k}{y_k^{\beta/\alpha}} = \sum_{k=n_1+1}^{n} \int_{y_{k+1}}^{y_k} \frac{1}{(-y_k)^{\beta/\alpha}}\, ds \ge \sum_{k=n_1+1}^{n} \int_{y_{k+1}}^{y_k} \frac{1}{(-s)^{\beta/\alpha}}\, ds$$

$$= -\int_{y_{n+1}}^{y_{n_1+1}} \frac{1}{s^{\beta/\alpha}}\, ds = \begin{cases} \ln|y_{n+1}| - \ln|y_{n_1+1}|, & \beta = \alpha, \\ \dfrac{y_{n+1}^{1-\beta/\alpha}}{1-\beta/\alpha} - \dfrac{y_{n_1+1}^{1-\beta/\alpha}}{1-\beta/\alpha}, & \beta < \alpha. \end{cases}$$

Therefore, for $n \ge n_1 + 1$, we have $\ln|y_{n+1}| \le \ln|y_{n_1+1}| + M$ if $\beta = \alpha$, and $|y_{n+1}|^{1-\beta/\alpha} \le y_{n_1+1}^{1-\beta/\alpha} + M(1-\beta/\alpha)$, when $\beta < \alpha$. In either case, $\{y_n\}$ is a bounded solution of equation (1).

We are now in the position to deal with the convergence of solutions in Class B of equation (1) to correct Theorems 4 and 5 in [1]. We have the results as follows.

THEOREM 2.2.3. *Assume that r_n is eventually of one sign and that $\sum_{n=1}^{\infty} 1/p_n^{1/\alpha} = \infty$. Then, every solution $\{y_n\}$ of equation (1) in Class B with the same sign as r_n eventually satisfies $\lim_{n\to\infty} p_n (\Delta y_n)^\alpha = 0$.*

PROOF. We treat only the case where r_n is eventually negative. Let $\{y_n\}$ be an arbitrary solution of equation (1) in Class B with the same sign as r_n eventually. Then, there exists an $n_1 \in N$ such that $r_n < 0$, $y_n < 0$, $\Delta y_n > 0$ for $n \ge n_1$. From equation (1), we know $p_n(\Delta y_n)^\alpha > 0$ is decreasing when $n > n_1$. So, $\lim_{n\to\infty} p_n(\Delta y_n)^\alpha \triangleq L \ge 0$. If $L > 0$, then there is an $n_2 \in N$ such that $p_n(\Delta y_n)^\alpha > L/2$ for $n \ge n_2$. Summing this inequality gives

$$y_{n+1} \ge y_{n_1} + \left(\frac{L}{2}\right)^{1/\alpha} \sum_{n=n_1}^{n} \frac{1}{p_n^{1/\alpha}} \to \infty, \qquad \text{as } n \to \infty.$$

This is a contradiction.

THEOREM 2.2.4. *Suppose that r_n is eventually of definite sign and that for any given real numbers k and l and for some $N_0 \in N$,*

$$\sum_{n=N_0}^{\infty} \left\{ \frac{1}{p_n} \left[k + \sum_{s=N_0}^{n} (q_s + l \operatorname{sign}(r_s) r_s) \right] \right\}^{1/\alpha} = \infty, \tag{13}$$

where $\operatorname{sign}(\cdot)$ is a sign function. Then, every Class B solution $\{y_n\}$ of equation (1) satisfies $\lim_{n\to\infty} y_n = 0$.

PROOF. Similarly, we only prove the case where r_n is eventually negative. Let $\{y_n\}$ be an arbitrary solution of equation (1) in Class B. Then, the limit $\lim_{n\to\infty} y_n \triangleq L$ exists and is finite. It suffices to verify $L = 0$. We now consider the case $y_n < 0$, $\Delta y_n > 0$ for $n \ge n_3$ for some $n_3 \in N$. Obviously, $L \le 0$ and $y_n \le L$ for $n \ge n_3$. If $L < 0$, then we obtain by summing up both sides of equation (1) from $n_3 + 1$ to n that

$$\begin{aligned} p_n (\Delta y_n)^\alpha &= p_{n_3} (\Delta y_{n_3})^\alpha + \sum_{k=n_3+1}^{n} \left(q_k y_k^\beta + r_k\right) \\ &\le p_{n_3} (\Delta y_{n_3})^\alpha + \sum_{k=n_3+1}^{n} \left(L^\beta q_k + r_k\right) \\ &= L^\beta \left[\frac{p_{n_3} (\Delta y_{n_3})^\alpha}{L^\beta} + \sum_{k=n_3+1}^{n} \left(q_k + \frac{\operatorname{sign}(r_k)\, r_k}{(-L)^\beta} \right) \right]. \end{aligned}$$

Combining this with (13) leads to

$$y_n \le y_{n_3+1} + L^{\beta/\alpha} \sum_{t=n_3+1}^{n-1} \left\{ \frac{1}{p_t} \left[\frac{p_{n_3}(\Delta y_{n_3})^\alpha}{L^\beta} + \sum_{k=n_3+1}^{t} \left(q_k + \frac{\operatorname{sign}(r_k) r_k}{(-L)^\beta} \right) \right] \right\}^{1/\alpha} \to -\infty, \quad \text{as } n \to \infty.$$

This is a contradiction. The proof for the case $y_n > 0$, $\Delta y_n < 0$ for $n \ge n_3 \in N$ is similar.

2.3. Discrimination Method for Nonlinear Limit-Point Solutions

THEOREM 2.3.1. *Suppose that for a sufficiently small positive number a and any real number b,*

$$\sum_{n=N_0}^{\infty} \left\{ \frac{1}{p_n} \left[b + \sum_{s=N_0}^{n} (r_s - a q_s) \right] \right\}^{1/\alpha} = \infty, \qquad \textit{for some } N_0 \in N. \tag{14}$$

Then, any solution of equation (1) is a nonlinear limit-point type one.

PROOF. Assume that there exists a solution $\{y_n\}$ of equation (1) which is a nonlinear limit-circle type one. Then, according to the definition of nonlinear limit-circle type solution, we have $\lim_{n\to\infty} y_n = 0$. So, for an arbitrarily sufficiently small positive number $a \in (0,1)$, there is an $n_1 \in N$ such that $|y_n| \le a$ for $n \ge n_1$. By summing up both sides of equation (1) from $n_1 + 1$ to n, one can get that

$$p_n(\Delta y_n)^\alpha = p_{n_1}(\Delta y_{n_1})^\alpha + \sum_{k=n_1+1}^{n} \left(q_k y_k^\beta + r_k \right) \ge p_{n_1}(\Delta y_{n_1})^\alpha + \sum_{k=n_1+1}^{n} \left(r_k - a^\beta q_k \right),$$

which, together with (14), gives

$$y_n \ge y_{n_1+1} + \sum_{t=n_1+1}^{n-1} \left\{ \frac{1}{p_t} \left[p_{n_1}(\Delta y_{n_1})^\alpha + \sum_{s=n_1+1}^{t} \left(r_s - a^\beta q_s \right) \right] \right\}^{1/\alpha} \to \infty, \qquad \text{as } n \to \infty.$$

This is contrary to $\lim_{n\to\infty} y_n = 0$.

3. EXAMPLES

Now, we give some examples, including Example 1 mentioned previously, to illustrate the advantages of our results.

EXAMPLE 1. Consider the difference equation

$$\Delta(\Delta y_{n-1}) = 2 y_n^{1/3} + 4n, \qquad n \ge 1. \tag{E$_1$}$$

Using the notation in equation (1), we have $\alpha = 1$, $\beta = 1/3$, $p_n = 1$, $q_n = 2$, and $r_n = 4n$.

Obviously, Theorem 2 in [1] cannot determine whether or not all solutions of equation (E_1) are unbounded because $\sum_{n=1}^{\infty} q_n = \infty$. Whereas one can easily verify that for any given $k \in R$, $l > 0$ and for some $k_0 \in N$,

$$\sum_{n=k_0}^{\infty} \left\{ \frac{1}{p_n} \left[k + \sum_{s=k_0}^{n} (r_s - l q_s) \right] \right\}^{1/\alpha} = \infty.$$

So, by Theorem 2.1.3, all solutions of equation (E_1) are unbounded. In fact, $y_n = n^3$, $n \ge 0$, is one such solution.

EXAMPLE 2. Consider the difference equation

$$\Delta\left(\left(2n-\frac{1}{n}\right)(\Delta y_{n-1})^3\right)=\frac{1}{n^2(n+1)}y_n+2,\qquad n\geq 1,\quad n\in N. \tag{E$_2$}$$

Corresponding to equation (1), we get $p_{n-1}=2n-1/n$, $\alpha=3$, $\beta=1$, $q_n=1/n^2(n+1)$, $r_n=2$, $n\geq 1$.

Clearly, Theorem 2 in [1] cannot be used to determine whether all solutions of equation (E$_2$) are unbounded or not because $p_n\not\equiv 1$. However, it is easy to see that for any given $k\in R$, $l>0$, and $k_0\in N$,

$$\sum_{n=k_0}^{\infty}\left\{\frac{1}{p_n}\left[k+\sum_{s=k_0}^{n}(r_s-lq_s)\right]\right\}^{1/\alpha}=\infty.$$

So, all conditions of Theorem 2.1.3 hold, which in turn means that all solutions $\{y_n\}$ of equation (E$_2$) are unbounded. Indeed, $y_n=n$, $n\geq 0$, is such a solution.

EXAMPLE 3. Consider the difference equation

$$\Delta\left((4n^3-3n+1)(\Delta y_{n-1})^{1/3}\right)=(12n+11)y_n+n+1,\qquad n\geq 2, \tag{E$_3$}$$

where $p_{n-1}=4n^3-3n+1$, $\alpha=1/3$, $\beta=1$, $q_n=12n+11$, $r_n=n+1$, $n\geq 2$, the same notations as in (1).

It is clear that

$$\sum_{n=2}^{\infty}\left(\sum_{s=1}^{n-1}\frac{q_s+r_s}{p_{n-1}}\right)^{1/\alpha}=\sum_{n=2}^{\infty}\left(\frac{(13n+24)(n-1)}{2(4n^3-3n+1)}\right)^3<\infty.$$

Thus, all assumptions in Theorem 2.2.1 hold. It follows that there exists at least one solution of equation (E$_3$) which belongs to Class A. Actually, $y_n=n$, $n\geq 1$, is such a solution of equation (E$_3$).

EXAMPLE 4. Consider the difference equation

$$\Delta(3n(n-1)^2\Delta y_{n-1})=y_n+1+\frac{1}{n},\qquad n\geq 2. \tag{E$_4$}$$

According to equation (1), $\alpha=\beta=1$, $p_{n-1}=3n(n-1)^2$, $q_n\equiv 1$, $r_n=1+1/n$, $n\geq 2$. One can easily prove that

$$\sum_{n=2}^{\infty}\left(\sum_{s=1}^{n-1}\frac{q_s+r_s}{p_{n-1}}\right)^{1/\alpha}<\sum_{n=2}^{\infty}\frac{1}{n(n-1)}<\infty,$$

and that for any given positive real numbers k and l

$$\sum_{n=N_0}^{\infty}\left\{\frac{1}{p_n}\left[k+\sum_{s=N_0}^{n}(q_s+l\operatorname{sign}(r_s)r_s)\right]\right\}^{1/\alpha}<\infty,\qquad \text{for some } N_0\in N.$$

Therefore, it follows from Theorems 2.2.1 and 2.2.2, respectively, that there exists at least one solution of equation (E$_4$) in Class A and every eventually positive solution of equation (E$_4$) in Class A is bounded. Indeed, $y_n=2-1/n$, $n\geq 1$, is such a solution.

EXAMPLE 5. Consider the difference equation

$$\Delta\left(4n^4(n+1)^3(\Delta y_{n-1})^3\right)=y_n-1-\frac{1}{n+1},\qquad n\geq 1. \tag{E$_5$}$$

Evidently, $\alpha = 3 > \beta = 1$, $p_{n-1} = 4n^4(n+1)^3$, $q_n \equiv 1$, $r_n = -1 - 1/(n+1) < 0$, $n \geq 2$.

One can easily prove that (9) holds for any $k, l > 0$ and for some $N_0 \in N$. Thus, it follows from Theorem 2.2.2 that all eventually negative solutions of equation (E_5) in Class A are bounded. Certainly, $y_n = -3 + 1/(n+1)$, $n \geq 0$, is one of such solutions of equation (E_5).

EXAMPLE 6. Consider the difference equation

$$\Delta((n-1)\Delta y_{n-1}) = \frac{1}{2(n+1)} y_n + \frac{1}{n(n+1)}, \qquad n \geq 2. \tag{E_6}$$

Clearly, we see that $\alpha = \beta = 1$, $p_{n-1} = n-1$, $q_n = 1/(2(n+1))$, $r_n = 1/(n(n+1)) > 0$, $n \geq 2$.

One can easily show that for any $k, l > 0$ and for some $N_0 \in N$

$$\sum_{n=N_0}^{\infty} \left\{ \frac{1}{p_n} \left[k + \sum_{s=N_0}^{n} (q_s + l \operatorname{sign}(r_s) r_s) \right] \right\}^{1/\alpha} > \sum_{n=N_0}^{\infty} \frac{k}{n} = \infty.$$

Thus, it follows from Theorem 2.2.4 that all eventually positive solutions of equation (E_6) in Class B satisfy $\lim_{n\to\infty} y_n = 0$. Indeed, $y_n = 2/n$, $n \geq 1$ is such a solution.

Because $\sum_{n=1}^{\infty} 1/p_n^{1/\alpha} = \sum_{n=1}^{\infty} 1/n = \infty$ and for the solution $\lim_{n\to\infty} p_n(\Delta y_n)^\alpha = \lim_{n\to\infty} -2/(n+1) = 0$, Example 6 also illustrates Theorem 2.2.3.

EXAMPLE 7. Consider the difference equation

$$\Delta((n+1)(n+2)\Delta y_{n-1}) = y_n + k + \frac{1}{n+2}, \qquad k > 1, \quad n \geq 1. \tag{E_7}$$

Here, $\alpha = \beta = 1$, $p_{n-1} = (n+1)(n+2)$, $q_n \equiv 1$, $r_n = k + 1/(n+2) > 0$, $n \geq 1$.

One can easily prove that for any $a \in (0,1)$ and any real number b,

$$\sum_{n=1}^{\infty} \left\{ \frac{1}{p_n} \left[b + \sum_{s=1}^{n} (r_s - a q_s) \right] \right\}^{1/\alpha} > \sum_{n=1}^{\infty} \frac{b + (k-a)n}{(n+2)(n+3)} = \infty.$$

Therefore, it follows from Theorem 2.3.1 that all solutions of equation (E_7) are of nonlinear limit-point type. In fact, $y_n = -k - 1/(n+2)$, $n \geq 0$, is such a solution of equation (E_7).

REMARK 3.1. The properties shown above for the solutions of (E_1)–(E_6) cannot be derived from the known results.

APPENDIX A

PROOF OF THEOREM 2 IN [1]. Let $\{y_n\}$ be a solution of equation (1). By Proposition 1, we may assume that first $y_n > 0$ and $\Delta y_n > 0$ for all $n \geq N \in \mathbf{N}$. If possible, let $\{y_n\}$ be bounded. So, there exist constants M_1 and M_2 such that $0 < M_1 \leq y_n \leq M_2$ for $n \geq N$. Now, summing equation (1) from $N+1$ to n, we obtain

$$(\Delta y_n)^\alpha = (\Delta y_N)^\alpha + \sum_{s=N+1}^{n} q_s y_s^\beta + \sum_{s=N+1}^{n} r_s > \sum_{s=N+1}^{n} r_s + M_1^\beta \sum_{s=N+1}^{n} q_s.$$

Consequently, $\lim_{n\to\infty} \Delta y_n = \infty$. This, in turn implies that $\lim_{n\to\infty} y_n = \infty$, a contradiction.

Suppose $y_n > 0$ and $\Delta y_n < 0$ for all $n \geq N \in \mathbf{N}$. Summing equation (1), we have

$$(-\Delta y_n)^\alpha = (-\Delta y_N)^\alpha - \sum_{s=N+1}^{n} q_s y_s^\beta - \sum_{s=N+1}^{n} r_s \leq (-\Delta y_N)^\alpha - M_1^\beta \sum_{s=N+1}^{n} q_s - \sum_{s=N+1}^{n} r_s < 0,$$

for large n, a contradiction. Hence, $\{y_n\}$ cannot be bounded. This completes the proof of the theorem.

APPENDIX B

PROOF OF THEOREM 4 IN [1]. Suppose that $\{y_n\}$ is a Class B solution of equation (1), say, $y_n > 0$ for all $n \geq N \in \mathbf{N}$. Then, $p_n(\Delta y_n)^\alpha < 0$ and increasing. If $p_n(\Delta y_n)^\alpha \not\to 0$, there exists $K > 0$ such that $p_n(\Delta y_n)^\alpha \leq -K$ for $n \geq N$. Summing, we have

$$y_{n+1} \leq y_{n_1} - \frac{K^{1/\alpha} \sum_{n=n_1}^{n} 1}{p_n^{1/\alpha}} \to -\infty,$$

as $n \to \infty$, which is a contradiction.

REFERENCES

1. E. Thandapani and K. Ravi, Bounded and monotone properties of solutions of second order quasilinear forced difference equations, *Computers Math. Applic.* **38** (2), 113–121, (1999).
2. J.W. Hooker and W.T. Paltula, A second order nonlinear difference equation: Oscillation and asymptotic behavior, *J. Math. Anal. Appl.* **91**, 9–29, (1983).
3. S.S. Cheng, H.J. Li and W.T. Patula, Bounded and zero convergent solutions of second order difference equations, *J. Math. Anal. Appl.* **141**, 463–483, (1989).
4. H.J. Li and C.C. Yeh, Existence of positive nondecreasing solutions of nonlinear difference equations, *Nonlinear Analysis* **22**, 1271–1284, (1994).
5. J.R. Graef and P.W. Spikes, Boundedness and asymptotic behavior of solutions of a forced difference equations, *J. Math. and Math. Sci.* **17**, 397–400, (1994).
6. E. Thandapani and R. Arul, Oscillation theory for a class of second order quasilinear difference equations, *Tamkang J. Math.* **28** (3), 229–238, (1997).
7. E. Thandapani, M.M. Manuel, J.R. Graef and P.W. Spikes, Monotone properties of certain classes of solutions of second-order difference equations, *Advances in Difference Equations II,* Special Issue of *Computers Math. Applic.,* (Edited by R.P. Agarwal) **36** (10–12), 291–297, (1998).
8. G. Zhang and S.S. Cheng, Elementary nonexistence criteria for a recurrence relation, *Chinese J. Math.* **24**, 229–235, (1996).
9. P.J.Y. Wong and R.P. Agarwal, Oscillation theorems and existence of positive monotone solutions for second order nonlinear difference equations, *Mathl. Comput. Modelling* **21** (3), 63–84, (1995).
10. S.S. Cheng, H.J. Li and W.T. Patula, Bounded and zero convergent solutions of second order difference equations, *J. Math. Anal. Appl.* **141**, 463–483, (1989).
11. H.J. Li and S.S. Cheng, Asymptotically monotone solutions of a nonlinear difference equation, *Tamkang J. Math.* **24**, 269–282, (1993).
12. E. Thandapani, J.R. Graef and P.W. Spikes, On the oscillation of solutions of second order quasilinear difference equations, *Nonlinear World* **3**, 545–565, (1996).
13. J.R. Graef and P.W. Spikes, Some asymptotic properties of solutions of $(a(t)x')' - q(t)f(x) = r(t)$, In *Differential Equations: Qualitative Theory,* (Szeged, 1984), *Colloq, Math. Soc. János Bolyai, Volume 47* North-Holland, Amsterdam, 347–359, (1987).
14. J. Dugundji and A. Granas, *Fixed Point Theory,* Warzawa, (1982).