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Pro-finite MV-spaces

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Abstract

In this work it is shown that every MV-space, i.e. the prime ideal space of an MV-algebra, is pro-finite if and only if it is a completely normal dual Heyting space.

An example is given showing that MV-spaces and completely normal spectral spaces are not pro-finite.

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1. Introduction

An MV-*algebra* is a structure $A=(A,\oplus,\odot,^*,0,1)$ such that $(A,\oplus,0)$ is an abelian monoid, and, moreover, $x\oplus 1=1$, $0^*=1$, $x^{**}=x$, $x\odot y=(x^*\oplus y^*)^*$, and $(x^*\oplus y)^*\oplus y=(y^*\oplus x)^*\oplus x$, for all $x,y\in A$. Following the tradition, for every $x,y\in A$, let us write $x\vee y=(x\odot y^*)\oplus y$ and $x\wedge y=(x\oplus y^*)\odot y$. Then as proved in [4], $(A,\vee,\wedge,0,1)$ is a bounded distributive lattice. We refer to [4,6] for background on MV-algebras.

Let A be an MV-algebra. By an ideal of A we mean the kernel of a homomorphism. By a *prime ideal* J we mean an ideal of A such that the quotient MV-algebra A/J is totally ordered. Following [2], let $\operatorname{Spec} A$ denote the set of all prime ideals of A. As it is well known, $\operatorname{Spec} A$ equipped with set-theoretical inclusion is a root system. For any $x\in A$ let $\operatorname{supp}(x)=\{J\in\operatorname{Spec} A\mid x\notin J\}$. Then $S(A)=\{\operatorname{supp}(x)\mid x\in A\}$ is a basis for a topology on $\operatorname{Spec} A$. The resulting topological space is also denoted by $\operatorname{Spec} A$. Moreover, $\operatorname{Spec} A$ is a completely normal spectral space [2,3], where $S(A)$ is a basis of open sets of $\operatorname{Spec} A$. Further, the basis $S(A)$ of open sets, ordered by inclusion, is a bounded distributive lattice and

$$\operatorname{Spec} A\approx\operatorname{Spec} S(A).$$

A topological space X is said to be an MV-*space* iff there exists an MV-algebra A such that $\operatorname{Spec} A$ and X are homeomorphic.

A topological space X is called *spectral* if X is a compact T_0-space, every nonempty irreducible closed subset of X is the closure of a unique point, and the set $\mathscr{D}(X)$ of compact open subsets of X constitutes a basis for the topology of X and is closed under finite unions and intersections. Note that $\mathscr{D}(X)$ is a bounded distributive lattice. A spectral space is said to be *completely normal* if whenever points x and y are in the closure of a singleton $\{z\}$, then either x is in the closure of $\{y\}$ or y is in the closure of $\{x\}$.

If L is a bounded distributive lattice, then the space $\operatorname{Spec} L$—the set of prime lattice ideals of L endowed with the hull-kernel topology—is a spectral space. It is a result of Stone [22], that for any spectral space X, $\operatorname{Spec}\mathscr{D}(X)$ is homeomorphic to X.

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Later on the motivation for studying spectral spaces comes from algebra. It is part of the folklore of ring theory that the hull–kernel topology on the prime ideals of a commutative ring with identity is spectral, with the associated specialization order being set inclusion. Much deeper results, contained in the classic paper of Hochster [15], show that every spectral space arises in this manner.

Given spectral spaces X and Y, a map $f: X \to Y$ is said to be *strongly continuous* iff $f^{-1}(V)$ is a compact open set for every compact open set $V \subseteq Y$.

Proposition 1 (Balbes and Dwinger [1], Davey and Priestley [8], Speed [21]). *The category of bounded distributive lattices and lattice homomorphisms is dually equivalent to the category of spectral spaces and strongly continuous maps.*

We can define the category of completely normal spectral spaces (CNSS) as a subcategory of spectral spaces. Let X, Y be completely normal spectral spaces. Then a strongly continuous map $f: X \to Y$ is a morphism of CNSS if $\forall x \in X, \forall y_1, y_2 \in \mathrm{cl}\{f(x)\}$ either $y_1 \in \mathrm{cl}\{y_2\}$ or $y_2 \in \mathrm{cl}\{y_1\}$. Let us observe, that if X and Y are completely normal spectral spaces, then any strongly continuous map from X to Y satisfies the property mentioned above. It means that the category of completely normal spectral spaces is a full subcategory of the category of spectral spaces.

The *specialization order* associated with the topology on X, which we shall denote by $\leqslant$, is given by

$$x \leqslant y \Leftrightarrow y \in \mathrm{cl}\{x\}.$$

As it is well known, any bounded distributive lattice can be represented as a lattice of compact open down subsets of a spectral space X with the specialization order. The set of all compact open down subsets of X is a lattice under union and intersection. It constitutes a basis for the spectral topology of X.

A *root system* is a partially ordered set (a poset) P such that for each $x \in P$, the final section $\{y \in P : y \geqslant x\}$ is totally ordered. Let P be a poset and $Q \subseteq P$. We then say that Q is a *down-set* if, whenever $x \in Q$, $y \in P$ and $y \leqslant x$, it follows that $y \in Q$. Dually, $W \subseteq P$ is called an *up-set* if, whenever $x \in W$, $y \in P$ and $x \leqslant y$, $y \in W$. We denote by $U\downarrow$ ($U\uparrow$) the smallest down-set (up-set) containing a given subset U of P; instead of $\{x\}\downarrow$ ($\{x\}\uparrow$) we write $x\downarrow$ ($x\uparrow$).

Note that a spectral space X is completely normal if and only if X, endowed with the specialization order, is a root system (see [8]).

A poset $(X, \leqslant)$ will be called *spectral* if there exists a spectral topology on X such that $\leqslant$ coincides with the specialization order. Note that any finite poset, and, in particular any finite root system, is spectral (see [8]).

It is well known and easy to check that for each bounded distributive lattice L the specialization order of $\mathrm{Spec}\, L$ coincides with the set theoretical inclusion of prime lattice ideals of L (see [8]).

In what follows, we shall consider that all spectral spaces are equipped with the specialization order. Then for each subset U of a spectral space X, $U\downarrow$ ($U\uparrow$) will always mean the initial (final) section of U in the specialization order of X.

Note that open subsets of a spectral space are down-sets and closed subsets are up-sets (see [8]).

By a *direct* (*inverse*) *system* in a category $\mathscr{A}$ we mean a family $\{A_i, \varphi_{ij}\}_{i \in I}$ ($\{B_i, \psi_{ij}\}_{i \in I}$) of objects, indexed by a directed poset I, together with a family of morphisms $\varphi_{ij} : A_i \to A_j$ ($\psi_{ij} : B_j \to B_i$), for each $i \leqslant j$, satisfying the following conditions:

(i) $\varphi_{ij} \circ \varphi_{hi} = \varphi_{hj}$ ($\psi_{hj} = \psi_{hi} \circ \psi_{ij}$) for all $h \leqslant i \leqslant j$;
(ii) $\varphi_{ii} = 1_{A_i}$ ($\psi_{ii} = 1_{B_i}$) for every $i \in I$.

For brevity we say that $\{A_i, \varphi_{ij}\}_I$ ($\{B_i, \psi_{ij}\}_I$) is a direct (inverse) system in $\mathscr{A}$. We shall omit to specify in which category we take a direct (inverse) system when this is evident from the context. A *direct* (*inverse*) *limit* of the direct (inverse) system is an object A (B) together with a family $\varphi_i : A_i \to A$ ($\psi_i : B \to B_i$) (which often is denoted by $\{A, \varphi_i\}$ ($\{B, \psi_i\}$)) of morphisms satisfying the conditions: $\varphi_j \circ \varphi_{ij} = \varphi_i$ ($\psi_{ij} \circ \psi_j = \psi_i$) when $i \leqslant j$ and having the following universal property: for any object C (D) of $\mathscr{A}$ together with a family of morphisms $\rho_i : A_i \to C$ ($\lambda_i : D \to B_i$), if $\rho_j \circ \varphi_{ij} = \rho_i$ ($\psi_{ij} \circ \lambda_j = \lambda_i$) for $i \leqslant j$, then there exists a morphism $\rho : A \to C$ ($\lambda : D \to B$) such that $\rho \circ \varphi_i = \rho_i$ ($\psi_i \circ \lambda = \lambda_i$) for any $i \in I$. The *direct* (*inverse*) *limit* of the above system is denoted by $\varinjlim\{A_i, \varphi_{ij}\}_I$ ($\varprojlim\{B_i, \psi_{ij}\}_I$), and its elements by $[a_i]$, with $a_i \in A_i$ and $i \in I$. If φ_{ij} (ψ_{ij}) is understood, we may simply write $\varinjlim\{A_i,\}_I$ ($\varprojlim\{B_i,\}_I$).

Call an object of some concrete category *ind-finite* (*pro-finite*) (comp. Johnstone [17]) if it is isomorphic to the direct (inverse) limit of finite objects of the category.

It is well known that the category of bounded distributive lattices is ind-finite and, respectively (by duality), the category of spectral spaces is pro-finite (see [21]).

In this paper we characterize pro-finite MV-spaces, i.e. those completely normal spectral spaces for which there exist corresponding MV-algebras with homeomorphic prime ideal spaces. For this aim we define the category of MV-spaces, whose objects are the MV-spaces and morphisms are the continuous strongly isotone maps, that is the maps whose inverse images preserve up-sets. At the end of the paper two examples are given showing that the categories of MV-spaces and completely normal spectral spaces are not pro-finite.

2. Heyting and dual Heyting algebras

A *Heyting algebra* is a bounded distributive lattice H with an additional binary operation implication $\rightharpoonup: H \times H \to H$ such that for any $a, b \in H$

$$x \leqslant a \rightharpoonup b \Leftrightarrow a \wedge x \leqslant b$$

(here $x \leqslant y$ iff $x \wedge y = x$ iff $x \vee y = y$). If, in addition, H satisfies the (linearity) equation $(x \rightharpoonup y) \vee (y \rightharpoonup x) = 1$, then H is said to be a *linear Heyting algebra* (or, in other terminology, *Gödel algebra*), for short, an $\mathscr{L}$*-algebra*. $\mathscr{L}$-algebras were considered by Horn [16]. They can be equivalently defined as the subvariety of Heyting algebras generated by all totally ordered Heyting algebras. Let $\mathscr{H}$ denote the category of Heyting algebras having as morphisms lattice homomorphisms which preserve implication.

A *dual Heyting algebra* is a bounded distributive lattice H such that, for each $x, y \in H$, there exists the *pseudo-difference* of x and y, i.e. there is a smallest element $z \in H$ satisfying the inequality $x \vee z \geqslant y$. Then we denote this element z by $x \rightharpoondown y$ and say that z is the *pseudo-difference* of x and y. Let $\mathscr{H}^{\mathrm{op}}$ denote the category of dual Heyting algebras having as morphisms the lattice homomorphisms which preserve pseudo-difference. Any spectral space is an object dual to a dual Heyting algebra if for all compact open U and V $(U \backslash V) \downarrow$ is compact open [7]. This is a consequence of the restriction of the well-known Priestley duality [20] to dual Heyting algebras. We denote by $\mathscr{P}\mathscr{H}^{\mathrm{op}}$ this kind of spectral spaces.

Let $(X, \leqslant)$ and $(Y, \leqslant')$ be posets. We say that a map $f: X \to Y$ is *strongly isotone* [13] if for any elements $x \in X$, $y \in Y$

$$f(x) \leqslant' y \Rightarrow \exists x' \in X (x \leqslant x' \,\&\, f(x') = y).$$

It is easy to see that any strongly isotone map is isotone (i.e. order-preserving).

Lemma 2 (Esakia [14]). *Let $(X, \leqslant)$, $(Y, \leqslant')$ be arbitrary posets and $f: X \to Y$ be an isotone map. Then the following are equivalent*:

(1) *f is strongly isotone*;
(2) $\forall y \in Y \ (f^{-1}(y)) \downarrow = f^{-1}(y \downarrow)$;
(3) $\forall x \in X \ (f(x)) \uparrow = f(x \uparrow)$;
(4) *if U is an up-set of X (of Y, resp.), then $f(U)$ ($f^{-1}(U)$, resp.) is an up-set of Y (of X, resp.)*.

Remark 1. Lemma 2 has been also proved in [11,5]

Taking into account item (3) of Lemma 2, we can define strongly isotone maps between spectral spaces. Namely, having spectral spaces X and Y, a map $f: X \to Y$ is said to be *strongly isotone* if

$$\mathrm{cl}\{f(x)\} = f(\mathrm{cl}\{x\})$$

for every $x \in X$. As we see, a strongly isotone map is defined in spectral spaces with respect to the specialization order.

Let X, Y be posets and $f: X \to Y$ be an onto strongly isotone map. Then f^{-1} generates a partition of X. Let E denote the corresponding equivalence relation. Let $E(W) = \bigcup_{x \in W} E(x)$ for any up-set W of X, call $E(W)$ the *saturation* of W (by E). Then, according to condition (4), with the above notations, we have:

Theorem 3. *The saturation $E(W)$ of any up-set $W \subseteq X$ is an up-set.*

Proof. Let us observe that $E(W) = f^{-1}(f(W))$. Then the assertion immediately follows from item (4) of Lemma 2. □

Theorem 4. *Let X be a poset and E be an equivalence relation on X such that the E-saturation of any up-set of X is up-set. Then the E-saturated down-sets of X are closed under pseudo-difference $Y_1 \rightharpoondown Y_2 = (Y_1 \backslash Y_2) \downarrow$, where Y_1 and Y_2 are any E-saturated down-sets of X.*

Proof. Let $f : X \to X/E$ be the natural map such that $f(x) = x/E$ for every $x \in X$. Then the partition corresponding to the equivalence relation E is Ker f, and $(f^{-1})(x/E) = E(x)$. Since both Y_1 and Y_2 are E-saturated, then according to Lemma 2(2) $(Y_1 \setminus Y_2) \downarrow$ is also E-saturated. □

The specialization of the Priestley duality [20] to the case of the category of $\mathscr{H}^{\mathrm{op}}$ and the category $\mathscr{P}\mathscr{H}^{\mathrm{op}}$ with strongly isotone maps is essentially a part of the folklore of Duality theory. So we have

Theorem 5. *The category $\mathscr{H}^{\mathrm{op}}$ is dually equivalent to the category $\mathscr{P}\mathscr{H}^{\mathrm{op}}$.*

We let $\mathscr{L}^*$ denote the subvariety of dual Heyting algebras satisfying the equation $(x \rightharpoonup y) \wedge (y \rightharpoonup x) = 0$. Algebras in $\mathscr{L}^*$ shall be naturally called $\mathscr{L}^*$-algebras. Let H, H' be $\mathscr{L}^*$-algebras. Then any D_{01}-homomorphism h from H to H' such that $h(x \rightharpoonup y) \leqslant h(x) \rightharpoonup h(y)$ for all $x, y \in H$, shall be called an *$\mathscr{L}^*$-homomorphism*. From Theorem 4 we have that the E-saturated down-sets of X form an $\mathscr{L}^*$-algebra. In other words the map $f : X \to X/E$ is strongly isotone. By abuse of notation, we also let $\mathscr{L}^*$ denote the category whose objects are the $\mathscr{L}^*$-algebras and whose morphisms are the $\mathscr{L}^*$-homomorphisms. A PH^{op}-space X is said to be $\mathscr{L}^*$-space if its clopen down-sets satisfy the linearity condition:

$$(U \setminus V) \downarrow \cap (V \setminus U) \downarrow = \emptyset.$$

Observe that the order duals of dual Heyting algebras are the orders of Heyting algebras. Horn [16] showed that $\mathscr{L}$-algebras can be characterized among Heyting algebras in terms of the order on prime filters (co-ideals). Specifically, a Heyting algebra is an $\mathscr{L}$-algebra iff its set of prime lattice filters is a root system (ordered by inclusion). We can say the same for $\mathscr{L}^*$-algebras (according to Horn's assertion): a dual Heyting algebra is an $\mathscr{L}^*$-algebra iff its set of prime lattice ideals is a root system. We can restrict the Theorem 5 to the categories of $\mathscr{L}^*$-algebras and $\mathscr{L}^*$-spaces.

Theorem 6. *The category $\mathscr{L}^*$ is dually equivalent to the category of $\mathscr{L}^*$-spaces.*

Remark 2. Bounded distributive lattices L such that Spec L are root systems are known as dual completely normal lattices (see [18,5]).

Let $\mathscr{F}\mathscr{L}^*$ denote the full subcategory of $\mathscr{L}^*$ whose objects are the finite $\mathscr{L}^*$-algebras, and $\mathscr{F}\mathscr{R}$ denote the category having finite root systems as objects and the strongly isotone maps as morphisms.

Theorem 7 (Di Nola and Grigolia [11]). *The categories $\mathscr{F}\mathscr{R}$ and $\mathscr{F}\mathscr{L}^*$ are dually equivalent.*

As in the case of $\mathscr{L}$-algebras it is easy to prove that the variety of all $\mathscr{L}^*$-algebras is locally finite, i.e. any finitely generated $\mathscr{L}^*$-algebra is finite. From this we conclude:

Theorem 8 (Di Nola and Grigolia [11]). *Any $\mathscr{L}^*$-algebra is isomorphic to a direct limit of a family of finite $\mathscr{L}^*$-algebras.*

Dually it holds.

Theorem 9 (Di Nola and Grigolia [11]). *Any $\mathscr{L}^*$-space is homeomorphic to an inverse limit of a family of finite $\mathscr{L}^*$-spaces with strongly isotone mapping as bonding maps. In other words any $\mathscr{L}^*$-space is pro-finite.*

3. MV-spaces

On each MV-algebra A, a binary relation $\equiv$ is defined by the following stipulation: $x \equiv y$ iff $\mathrm{supp}(x) = \mathrm{supp}(y)$. As proved in [2], $\equiv$ is a congruence with respect to $\oplus$ and $\wedge$. The resulting set $\beta(A)(=A/\equiv)$ of equivalence classes is a bounded distributive lattice, called the *Belluce lattice* of A. For each $x \in A$ let us denote by $\beta(x)$ the equivalence class of x. Let $f : A \to B$ be an MV-homomorphism. Then $\beta(f)$ is a lattice homomorphism from $\beta(A)$ to $\beta(B)$ defined as follows: $\beta(f)(\beta(x)) = \beta(f(x))$. We stress that β defines a covariant functor from the category of MV-algebras to the category of bounded distributive lattices (see [2]). Let

$$\mathrm{Spec}\, \beta(A)$$

be the space of all prime ideals of the Belluce lattice $\beta(A)$, equipped with the hull-kernel topology. In [2] (Theorem 20) it is proved that Spec A and Spec $\beta(A)$ are homeomorphic.

Let A, B be MV-algebras and $f:A\to B$ be a homomorphism. Then we can define the map $\tilde{f}$ as follows: for every $P\in\operatorname{Spec}B$:

$$\tilde{f}(P)=\{a\in A\mid f(a)\in P\}.$$

It is easy to check that $\tilde{f}$ is a map from the poset $\operatorname{Spec}B$ to the poset $\operatorname{Spec}A$. Similarly, we can define the map $\tilde{\beta}(f)$ from $\operatorname{Spec}\beta(B)$ to $\operatorname{Spec}\beta(A)$, setting $\tilde{\beta}(f)(P')=\{\beta(a)\in\beta(A)\mid \beta(f)(\beta(a))\in P'\}$ for every $P'\in\operatorname{Spec}\beta(B)$.

Theorem 10. *Let $\{A_i,\varphi_{ij}\}_I$ be a direct system of MV-algebras and $\{A,\varphi_i\}_I$ be its direct limit. Then $\{\beta(A),\ \beta(\varphi_i)\}_I$ is the direct limit of $\{\beta(A_i),\ \beta(\varphi_{ij})\}_I$.*

Proof. Let the map $\tau:\beta(A)\to\varinjlim\{\beta(A_i),\beta(\varphi_{ij})\}_I$ send $\beta[x_i]$ to $[\beta(x_i)]$ for every $x_i\in A_i$, $i\in I$. We shall show that τ is the identity map. Indeed, let $y_j\in A_j$, $i\leqslant j$, be such that $y_j=\varphi_{ij}(x_j)$. Then $\beta(y_j)=\beta(\varphi_{ij}(x_i))=\beta(\varphi_{ij})(\beta(x_i))$, i.e., $\beta(y_j)\in[\beta(x_i)]$. Hence, we have $\beta([x_i])\subseteq[\beta(x_i)]$.

Vice versa, let $k\in[\beta(x_i)]$. Then $k=\beta(\varphi_{ij})(\beta(x_i))=\beta(\varphi_{ij}(x_i))$, i.e., $k\in\beta([x_i])$. Hence $[\beta(x_i)]\subseteq\beta([x_i])$. Now, we prove the commutativity of the following diagram for every $i\leqslant j\in I$:

$$\begin{tikzcd} \beta(A_i) \arrow[rr, "\beta(\varphi_{ij})"] \arrow[dr, "\beta(\varphi_i)"'] & & \beta(A_j) \arrow[dl, "\beta(\varphi_j)"] \\ & \beta(A) & \end{tikzcd}$$

Indeed, let $\beta(x)\in\beta(A)$. Then

$$\beta(\varphi_j)(\beta(\varphi_{ij})(\beta(x)))=\beta(\varphi_j)(\beta(\varphi_{ij}(x)))=\beta(\varphi_j\varphi_{ij}(x))=\beta(\varphi_i(x))=\beta(\varphi_i)(\beta(x)).$$

This concludes the proof. $\square$

Lemma 11. *Let A,B be totally ordered MV-algebras and k be an embedding from A to B. Then $\tilde{k}$ preserves up-sets from the poset* $\operatorname{Spec}B$ *to the poset* $\operatorname{Spec}A$.

Proof. Let U be an up-set of $\operatorname{Spec}B$. Set $\tilde{k}(U)=\{\tilde{k}(P):P\in U\}$. We claim that $\tilde{k}(U)$ is an up-set of $\operatorname{Spec}A$. Assume $Q\in\tilde{k}(U)$. Then $Q=\{a\in A|\tilde{k}(a)\in P\}$ for some $P\in U$. Take $J\in\operatorname{Spec}A$, such that $Q\subset J$. We have to show that $J\in\tilde{k}(U)$. Indeed, we have that $P\subseteq\langle k(J)\rangle$ (where $\langle k(J)\rangle$ is the ideal generated by $k(J)$). Hence $\langle k(J)\rangle\in U$. So $\tilde{k}(\langle k(J)\rangle)\in\operatorname{Spec}A$. Trivially we have that $J\subseteq\tilde{k}(\langle k(J)\rangle)$. We want to show that $J=\tilde{k}(\langle k(J)\rangle)$. Indeed, assuming that there is $y\in\tilde{k}(\langle k(J)\rangle)\backslash J$, we have that $k(y)\in\langle k(J)\rangle$. Hence, there is $j\in J$ such that $k(y)\leqslant k(j)$. Moreover, for every $x\in J$ we have $x\leqslant y$, and then $k(x)\leqslant k(y)$. In particular, $k(j)\leqslant k(y)$. Hence, $k(j)=k(y)$, absurd, because k is an embedding. $\square$

Lemma 12. *Let A, B be totally ordered MV-algebras and h be a homomorphism from A onto B. Then $\tilde{h}$ preserves up-sets from the poset* $\operatorname{Spec}B$ *to the poset* $\operatorname{Spec}A$.

Proof. Let U be an up-set of $\operatorname{Spec}B$. Set $\tilde{h}(U)=\{\tilde{h}(P):P\in U\}$. Assume $Q\in\tilde{h}(U)$. Then $Q=\tilde{h}(P)$ for some $P\in U$. Let $J\in\operatorname{Spec}A$ such that $Q\subseteq J$. Since B is totally ordered and h is a surjective homomorphism, then $h(J)$ is a prime ideal of B. In addition, since h is a surjective homomorphism, it gives a congruence E, corresponding to the ideal $h^{-1}(0)$, which is contained in Q. The equivalence E partitions A on the equivalence classes $h^{-1}(x)$ for every $x\in B$. It is easy to check that the equivalence classes are convex and $a_1/E=a_2/E$ iff $(a_1\odot a_2^*)\oplus(a_2\odot a_1^*)\in h^{-1}(0)$ [4]. Hence there exists an isomorphism $\phi:A/E\to B$, therefore A/E and B have isomorphic prime ideal structures. Therefore the ideal J/E on A/E is isomorphic to the ideal $h(J)$ on B. So, since $J\supseteq Q\supseteq h^{-1}(0)$, $\phi^{-1}(h(J))=J/E$ and $\bigcup_{\theta\in J/E}\theta=J=h^{-1}(h(J))$. Therefore $\tilde{h}(h(J))=J$. Hence $h(J)$ contains P and $h(J)\in U$. Therefore $\tilde{h}(h(J))=J\in\tilde{h}(U)$. $\square$

Lemma 13. *Let A,B be MV-algebras and f be a homomorphism from A to B. Then $\tilde{f}$ preserves the up-sets from the poset* $\operatorname{Spec}B$ *to the poset* $\operatorname{Spec}A$, *i.e. $\tilde{f}$ is strongly isotone.*

Proof. We can represent $f:A\to B$ as a composition εf_1 of two morphisms: $f_1:A\to f(A)$ and an embedding $\varepsilon f(A)\to B$, where $\varepsilon(x)=x$ for every $x\in f(A)$. Dually, we have $\tilde{f}:\operatorname{Spec}B\to\operatorname{Spec}A$ which can be represented as a composition $\tilde{f_1}\tilde{\varepsilon}$ of strongly continuous maps $\tilde{\varepsilon}:\operatorname{Spec}B\to\operatorname{Spec}f(A)$ and $\tilde{f_1}:\operatorname{Spec}f(A)\to\operatorname{Spec}A$.

Notice that every MV-algebra is isomorphic to a subdirect product of totally ordered MV-algebras. And every totally ordered MV-algebra is isomorphic to a quotient MV-algebra by some prime ideal. Remind also that an MV-space forms a root system. So, if we have any MV-homomorphism $g : A_2 \to A_1$, then the domain (co-domain, respectively) $\mathrm{Spec}\, A_1$ ($\mathrm{Spec}\, A_2$, respectively) of $\tilde{g}$ can be represented as $\bigcup_{\xi \in \mathrm{MinSpec}\, A_1} \xi\uparrow$ ($\bigcup_{\zeta \in \mathrm{MinSpec}\, A_2} \zeta\uparrow$, respectively), where $\mathrm{MinSpec}\, A_1$ ($\mathrm{MinSpec}\, A_2$, respectively) is the set of all minimal points of $\mathrm{Spec}\, A_1$ ($\mathrm{Spec}\, A_2$, respectively); and the map $\tilde{g}$ can be represented as $\biguplus_{\xi \in \mathrm{MinSpec}\, A_1} \tilde{g}_\xi$, where $\tilde{g}_\xi$ is the restriction of $\tilde{g}$ on the $\xi\uparrow$ and $(\biguplus_{\xi \in \mathrm{MinSpec}\, A_1} \tilde{g}_\xi)(x) = \tilde{g}(x)$ for $x \in \xi\uparrow$. Let us note also that if $\tilde{g}_\xi$ preserves up-sets from the poset $\xi\uparrow$ to the poset $\zeta\uparrow$, where $\tilde{g}_\xi(\xi\uparrow) \subseteq \zeta\uparrow$ for some $\zeta \in \mathrm{MinSpec}\, A_2$, for every $\xi \in \mathrm{MinSpec}\, A_1$, then $\tilde{g}$ preserves up-sets from the poset $\mathrm{Spec}\, A_1$ to the poset $\mathrm{Spec}\, A_2$.

Now we shall consider the map $\tilde{f}_1 : \mathrm{Spec}\, f(A) \to \mathrm{Spec}\, A$, which is an embedding. So, for every prime ideal $\xi \in \mathrm{MinSpec}\, f(A)$ $\tilde{f}_{1\xi}(\xi\uparrow) \subseteq \zeta\uparrow$ for some prime ideal $\zeta \in \mathrm{MinSpec}\, A$, is also an embedding. And what is more, $f(A)/\xi$ is a totally ordered MV-algebra with $\mathrm{Spec}\,(f(A)/\xi) \cong \xi\uparrow$, and $\mathrm{Spec}\,(A/\tilde{f}_1(\xi)$ is isomorphic (as ordered set) to $(\tilde{f}_1(\xi))\uparrow \subseteq \zeta\uparrow$. Then we can define a homomorphism $f_{1\xi} : A/\tilde{f}_1(\xi) \to f(A)/\xi$ setting $f_{1\xi}(x/\tilde{f}_1(\xi)) = f_1(x)/\xi$. Therefore for onto MV-homomorphism $f_{1\xi} : A/\tilde{f}_1(\xi) \to f(A)/\xi$ we have an embedding $\tilde{f}_{1\xi} : \xi\uparrow \to \zeta\uparrow$ which, according to Lemma 11, preserves up-sets.

Similar argument we apply to show that $\tilde{\varepsilon}$ preserves up-sets. So, the map $\tilde{f} = \tilde{\varepsilon}\tilde{f}_1 :\ \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ preserves up-sets, i.e. $\tilde{f}$ is strongly isotone. □

Let $\mathscr{MS}$ be the category of MV-spaces whose objects are the MV-spaces and the morphisms are the continuous strongly isotone maps.

We cannot assert that for any spectral root system $(R, \leqslant)$ there exists an MV-algebra A such that $\mathrm{Spec}\, A$ is homeomorphic to R. Indeed, Delzel and Madden [9] give an example of a spectral root system which does not correspond to any MV-space (see also [7]).

Recall that Mundici's functor Γ is a categorical equivalence (a full, faithful, dense functor) between abelian lattice-ordered groups (for short ℓ-groups) with strong unit and MV-algebras [19]. Namely, if we have an abelian ℓ-group with strong unit u, then, by definition,

$$\Gamma(G, u) = [0, u] = \{x \in G \mid 0 \leqslant x \leqslant u\}$$

is the MV-algebra A, obtained by equipping the unit interval $[0, u]$ of G with the following operations: $x \oplus y = (x + y) \wedge u$, $x \odot y = (x + y - u) \vee 0$, $x^* = u - x$, and u is the element 1 of A. As proved in [19], up to isomorphism, for any MV-algebra A there is a unique abelian ℓ-group G with strong unit u such that A is isomorphic to $\Gamma(G, u)$.

Let us define a functor $\mathscr{M}$ from the category of finite root systems $\mathscr{FR}$ to the category $\mathbf{MV}(C)$ whose objects are algebras from the variety generated by Chang algebra $C \cong \Gamma(Z \otimes Z, (1, 0))$, where Z is the cyclic ℓ-group and $Z \otimes Z$ is the lexicographic product of Z by Z. Let us note that the MV-algebra $\Gamma(Z_0 \otimes Z_n \otimes \cdots \otimes Z_1,\ (1, 0, \ldots, 0))$ is an object of $\mathbf{MV}(C)$ where $Z_0 \otimes Z_n \otimes \cdots \otimes Z_1$ is the lexicographic product of $Z_0, Z_n, \ldots, Z_1$ and $Z \cong Z_0 \cong Z_n \cong \cdots \cong Z_1$, $(1, 0, \ldots, 0))$ is strong unit of $Z_0 \otimes Z_n \otimes \cdots \otimes Z_1$. Let R be $(k+1)$-element totally ordered poset. Then let $\mathscr{M}(R)$ be $\Gamma(Z_0 \otimes Z_k \otimes \cdots \otimes Z_1, (1, 0, \ldots, 0))$, which is generated by k generators $(0, 0, \ldots, 1), \ldots, (0, 1, 0, \ldots, 0)$. Let us note that $\mathrm{Spec}\, \Gamma(Z_0 \otimes Z_k \otimes \cdots \otimes Z_1, (1, 0, \ldots, 0))$ is isomorphic to R, where the elements of $\mathrm{Spec}\, \Gamma(Z_0 \otimes Z_k \otimes \cdots \otimes Z_1, (1, 0, \ldots, 0))$ are prime ideals ordered by inclusion and generated by the generators $(0, 0, \ldots, 0), (0, 0, \ldots, 1), \ldots, (0, 1, 0, \ldots, 0)$ [10] (Theorem 2).

Theorem 14. *Let R be a finite root. Then there exists a cardinal sum $\sum R_i$ of totally ordered roots and a strongly isotone onto map $f : \sum R_i \to R$.*

Proof. Let $\min R$ denote the set of all minimal elements of R. Let $R_i = m_i\uparrow$, where $m_i \in \min R$. Denote an element x belonging to $m_i\uparrow$ by x_i. Then, it is clear that $\bigcup_{m_i \in \min R} m_i\uparrow = R$. Hence the map $f : \sum_{m_i \in \min R} R_i \to R$ defined by: $f(x_i) = x$ for every $m_i \in \min R$, is a strongly isotone. □

An MV-algebra A is called *perfect* if it has a unique maximal ideal and it coincides with the union of the maximal ideal and the maximal co-ideal (=filter).

Theorem 15. *Let R be a finite root system. Then there exists an MV-algebra A such that* $\mathrm{Spec}\, A$ *is isomorphic to R.*

Proof. According to Theorem 14, any finite root R is an image of a strongly isotone map f of a cardinal sum $\sum_{m_i \in \min R} R_i$ of totally ordered roots R_i. Let A_i be a perfect totally ordered MV-algebra to R_i which is isomorphic to $\Gamma(Z_0 \otimes Z_k \otimes \cdots \otimes Z_1, (1, 0, \ldots, 0))$ for some $k \in \omega$. Then the product $\prod_{m_i \in \min R} A_i$ corresponds to the cardinal sum $\sum_{m_i \in \min R} R_i$. □

Let $s_{i1}, \ldots, s_{in_i}$ be the generators of A_i, where $s_{i1} \leqslant \cdots \leqslant s_{in_i}$. Recall that $s_{ij} = (0, 0, \ldots, 1, \ldots, 0), 1 \leqslant j \leqslant n_i$. Then the elements $(0, \ldots, 0, s_{ij}, 0, \ldots, 0) \in \prod_{m_i \in \min R} A_i$ generate the algebra $\prod_{m_i \in \min R} A_i$. Let S be the set of elements

$(0,\ldots,s_{i_1j_1},0,\ldots,0,s_{i_2j_2},0,\ldots,0,s_{i_kj_k},0,\ldots,0)\in\prod_{m_i\in\min R}A_i$ such that

$$f(x_{i_1})=f(x_{i_2})=\cdots=f(x_{i_k})=x\in R,$$

where $f^{-1}(x)=\{x_{i_1},\ldots,x_{i_k}\}$. Observe that, since $s_{i1},\ldots,s_{in_i}$ are the generators of A_i, the subalgebra A of $\prod_{m_i\in\min R}A_i$, generated by the set S, is a subdirect product of the MV-algebras A_i. Observe, also, that $(s_{1n_1},s_{2n_2},\ldots,s_{pn_p})\in S$ generate a maximal prime ideal of A, where $\{m_1,\ldots,m_p\}=\min R$, s_{in_i} generates a maximal ideal of A_i; $(0,\ldots,0,s_{i1},0,\ldots,0)\in S$ generate a minimal prime ideal of A. It easy to check that every element $s\in S$ generates a prime ideal P_s of A. Moreover the poset $(\{P_s\}_{s\in S},\subseteq)$ is isomorphic to R.

Now let R be a finite root system, i.e. R is a cardinal sum of finitely many roots : $R=\sum_{i=1}^m R_i$. Let A_i be an MV-algebra corresponding to the root R_i. Then the MV-algebra $\prod_{i=m}^m A_i$ corresponds to the root system R, since finite cardinal sum of finite spectral spaces (with the specialization orders), say X_1 and X_2, is homeomorphic to the spectral space of product $A_1\times A_2$ of algebras A_1 and A_2 corresponding to X_1 and X_2, respectively.

Theorem 16. *If R_1,R_2 are finite root systems and $f:R_1\to R_2$ is a strongly isotone map, then there exist MV-algebras A_1,A_2 and an MV-homomorphism $h:A_1\to A_2$ such that* $\operatorname{Spec}A_i\cong R_i$, $i=12$.

Proof. Let R,R' be totally ordered roots and $f:R\to R'$—an onto strongly isotone map. Then R is partitioned into convex classes, i.e. for any $y_1,y_2\in f^{-1}(x)$ if $y_1\leqslant z\leqslant y_2$, then $z\in f^{-1}(x)$. Suppose that R (R', respectively) consists of $k+1$ elements ($k'+1$ elements, respectively, with $k\geqslant k'$). Recall that $\mathscr{M}(R)=\Gamma(Z_0\otimes Z_k\otimes\cdots\otimes Z_1,(1,0,\ldots,0))$ ($\mathscr{M}(R')=\Gamma(Z_0\otimes Z_{k'}\otimes\cdots\otimes Z_1,(1,0,\ldots,0))$, respectively) and identify the elements of R (R', respectively) with generators $g_0=(0,0,\ldots,0),g_1=(0,0,\ldots,1),\ldots,g_k=(0,1,0,\ldots,0)$ ($g_0'=(0,0,\ldots,0),g_1'=(0,0,\ldots,1),\ldots,g_{k'}'=(0,1,0,\ldots,0)$, respectively) of the algebra $\mathscr{M}(R)$ ($\mathscr{M}(R')$, respectively).

Then the map $\varphi:\mathscr{M}(R')\to\mathscr{M}(R)$, which sends the generator g_j' of $\mathscr{M}(R')$ to the generator $g_{j'}$ being the least in the class $f^{-1}(g_j')$ (which is totally ordered and convex), is an injective homomorphism. For example, let $R=(x_0<x_1<x_2),R'=(x_0'<x_1')$ and the map $f:R\to R'$, defined as follows: $f(x_2)=f(x_1)=x_1'$ and $f(x_0)=x_0'$, is strongly isotone. Then $\mathscr{M}(R)=\Gamma(Z_0\otimes Z_2\otimes Z_1,(1,0,0)),\mathscr{M}(R')=\Gamma(Z_0\otimes Z_1,(1,0))$. In this case, the map $\varphi:\Gamma(Z_0\otimes Z_1,(1,0))\to\Gamma(Z_0\otimes Z_2\otimes Z_1,(1,0,0))$, sending $(0,0)\to(0,0,0)$ and $(0,1)\to(0,1,0)$, is an embedding corresponding to f.

Now let R_1,R_2 be finite root systems and $f:R_1\to R_2$ be an onto strongly isotone map. Then the map f induces an equivalence relation $E=\operatorname{Ker}f$ such that $R_1/E\cong R_2$. According to Theorem 14, there exists a strongly isotone onto map $f':\sum_{x\in\min R_1}x\uparrow\to R_1$. Therefore there exists an equivalence relation $E'=\operatorname{Ker}f'$ such that $\sum_{x\in\min R_1}x\uparrow/E'\cong R_1$. Define an equivalence relation E_d on $\sum_{x\in\min R_1}x\uparrow$ as follows:

$$y_1E_xy_2\Leftrightarrow y_1Ey_2$$

for any $y_1,y_2\in x\uparrow$; $E_d=\bigvee_{x\in\min R_1}E_x$. Then we assert that

$$\sum_{x\in\min R_1}x\uparrow/E_d/E'\cong\sum_{x\in\min R_1}x\uparrow/E'/E_d\cong R_2.$$

But, as we know, $\mathscr{M}(\sum_{x\in\min R_1}x\uparrow/E_d)$ is the direct product $\prod_{x\in\min R_1}\mathscr{M}(x\uparrow/E_x)$ and R_2 is isomorphic to the subalgebra of $\prod_{x\in\min R_1}\mathscr{M}(x\uparrow/E_x)$ which is also a subdirect product of $\mathscr{M}(x\uparrow/E_x)$, $x\in\min R_1$.

Now, suppose that R,R' are finite root systems and $f:R\to R'$ is a strongly isotone map, which is an embedding. Then $f(R)$ is an upper cone of R' and $h:\mathscr{M}(R')\to\mathscr{M}(R)$ is a homomorphism corresponding to the MV-ideal $P_1\cap\cdots\cap P_k$, where $P_1,\ldots,P_k$ are all prime ideals corresponding to all minimal elements of $f(R)\subseteq R'$. $\square$

Now we formulate the main.

Theorem 17. *An MV-space is pro-finite in the category $\mathscr{MS}$ if and only if it is an $\mathscr{L}^*$-space.*

Proof. Let us suppose that the MV-space X (which is a spectral root system) is pro-finite in $\mathscr{MS}$. Then X is homeomorphic to an inverse limit of a family of finite MV-spaces (which are spectral root systems) with strongly isotone maps as bonding maps. Therefore X is an $\mathscr{L}^*$-space, since it is an inverse limit of finite $\mathscr{L}^*$-spaces (since they are finite spectral root systems) and strongly isotone maps are $\mathscr{L}^*$-morphisms. Conversely, let X be an $\mathscr{L}^*$-space. Then it is pro-finite in the category **CNSS**. It means that X is homeomorphic to an inverse limit of a family of finite $\mathscr{L}^*$-spaces with strongly isotone maps as bonding maps. Then, according to the Theorem 16, we can construct the direct limit A of the family of MV-algebras, with corresponding finite spectral spaces (which are spectral root systems), such that $\operatorname{Spec}A\cong X$. $\square$

Example 1. Let us consider the MV-algebra $\mathbf{2}^\omega\times\mathbf{C}$, where $\mathbf{2}$ is the two-element Boolean algebra and C is the MV-algebra described by Chang (see [4]) as follows: C is the set of formal symbols $\{0,c,c+c,c+c+c,\ldots,\ldots,1-c-c-c,1-c-$

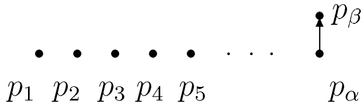

Fig. 1.

$c, 1-c, 1\}$. For abbreviation, we define $0\cdot c=0$ and $n\cdot c=c+c\cdots+c$ n-times and $1-0\cdot c=1$ and $1-n\cdot c=1-c-c-\cdots-c$ n-times.

(ia) If $x=n\cdot c$ and $y=m\cdot c$ then $x\oplus y=(m+n)\cdot c$;
(ib) if $x=n\cdot c$, $y=1-(m\cdot c)$ and $m\leqslant n$, then $x\oplus y=1$;
(ic) if $x=n\cdot c$, $y=1-(m\cdot c)$ and $n<m$, then $x\oplus y=1-(m-n)\cdot c$;
(id) if $x=1-(m\cdot c)$, $y=n\cdot c$ and $m\leqslant n$, then $x\oplus y=1$;
(ie) if $x=1-(m\cdot c)$, $y=n\cdot c$ and $n<m$, then $x\oplus y=1-(m-n)\cdot c$;
(if) if $x=1-(n\cdot c)$, $y=1-m\cdot c$, then $x\oplus y=1$;
(iia) if $x=n\cdot c$, then $x^*=1-n\cdot c$;
(iib) if $x=1-n\cdot c$, then $x^*=n\cdot c$.

Let A be a subalgebra of $\mathbf{2}^\omega\times\mathbf{C}$ generated by the following elements:

$$a=((x_1,x_2,\ldots),y),$$

where the set $\{i: x_i=1\}$ is finite and $y=y\wedge y^*$. Observe that any element $p_i=((x_1,x_2,\ldots),1)$, where $x_i=0$ and $x_j=1$ for $i\neq j$, generates a principal prime ideal $(p_i]$ of the algebra A. We denote the principal ideal by its generating element p_i. Then, there exist also two non-principal prime ideals: p_α, which is generated by elements $((x_1,x_2,\ldots),0)$, where the set $\{i: x_i=1\}$ is finite, and p_β, which is generated by $((x_1,x_2,\ldots),y)$, where the set $\{i: x_i=1\}$ is finite and $y=y\wedge y^*$. It is clear that $p_\alpha\subseteq p_\beta$. It is easy to check that there are no other prime ideals. Hence $X=\{p_i: i\in\omega\backslash\{0\}\}\cup\{p_\alpha,p_\beta\}=\operatorname{Spec}A$ is a completely normal spectral space (see Fig. 1). The open sets of the space are

(1) all subsets of $X_\omega=\{p_i: i\in\omega\backslash\{0\}\}$;
(2) all sets of the form $Y\cup\{p_\alpha\}$, where Y is a cofinite subset of X_ω; and
(3) all sets of the form $Z\cup\{p_\alpha,p_\beta\}$, where Z is a cofinite subset of X_ω.

For each $x\in X_\omega\cup\{p_\beta\}$, the singleton $\{x\}$ is closed, and the closure of $\{p_\alpha\}$ is $\{p_\alpha,p_\beta\}$. The compact open sets are the finite subsets of X_ω and the sets of the form (2) and (3). We claim that:

- $\operatorname{Spec}A$ *is not an* $\mathscr{L}^*$*-space.*

Indeed, $(X\backslash(X_\omega\cup\{p_\alpha\}))\downarrow=\{p_\alpha,p_\beta\}$, which is not a compact open set.

In the category of spectral spaces (and in the category of completely normal spectral spaces CNSS as well) $\operatorname{Spec}A$ is pro-finite. In fact, the compact open down-sets form a bounded distributive lattice, which is isomorphic to $\beta(A)$. We identify the isomorphic elements. Now we shall construct a direct family, whose direct limit will be isomorphic to $\beta(A)$. In other words we shall show that $\beta(A)$ is ind-finite. Let $a_k=\{p_i: i\geqslant k\}\cup\{p_\alpha\}$ and $b_k=a_k\cup\{p_\beta\}$. It is obvious that a_k,b_k and $\{p_{i_1},\ldots,p_{i_n}\}$ belong to $\beta(A)$.

Let A_k be the sublattice of $\beta(A)$ generated by $\{p_1\},\ldots,\{p_k\},a_k,b_k$. Observe that A_k is a sublattice of A_{k+1} and, hence, the family $\{A_k\}$ is directed. It is obvious that $\bigcup_{k=1}^\infty A_k=\beta(A)$. Therefore in the dual category we obtain that $\operatorname{Spec}A$ is pro-finite, i.e. $\operatorname{Spec}A$ is an inverse limit of finite spectral spaces. However, to the sublattice A_k of the lattice $\beta(A)$ corresponds the partition E_k on X, the only non-trivial class of which is a_k, and the strongly continuous map (which is natural map) is not strongly isotone, since the saturation of the up-set $\{p_k\}$ under the partition E_k is not an up-set. Moreover, according to Theorem 17, $\operatorname{Spec}A$ is not pro-finite in the category of MV-spaces. So we have

Claim. *The category of* MV*-spaces is not pro-finite.*

Example 2. Now we shall give an example showing that completely normal spectral spaces are not necessarily pro-finite in the category CNSS of completely normal spectral spaces. Indeed, let us consider the spectral space $\operatorname{Spec}F_{\mathbf{MV}}(1)$ of the free cyclic MV-algebra [12]. The space MaxSpec $F_{\mathbf{MV}}(1)$ (i.e., the subspace of maximal ideals of the free cyclic

MV-algebra $F_{\mathbf{MV}}(1)$) is connected [19] (Lemma 8.1). Then $X = \mathrm{cl}(\mathrm{MaxSpec}\, F_{\mathbf{MV}}(1))$ is a closed (up-)set of $\mathrm{Spec}\, F_{\mathbf{MV}}(1)$ and, hence, the spectral space X is completely normal, connected and contains more than one maximal point (i.e. maximal ideal). Let us suppose that X is pro-finite in the category CNSS of completely normal spectral spaces. Therefore X is (homeomorphic to) an inverse limit $\varprojlim\{X_i, \psi_{ij}\}_I$, where X_i is a finite completely normal spectral space and ψ_{ij} are bonding maps ($i, j \in I$). Since X_i is a cardinal sum of finite roots for every $i \in I$, without loss of generality we can suppose that the inverse limit has the property that the projection maps from the inverse limit are surjections onto the coordinate spaces, and hence the bonding maps are also surjections. Thus the inverse limit of the connected space X, has each of the coordinate spaces necessarily connected (i.e., it is a finite root), and since they are finite T_0 completely normal, each coordinate space would contain exactly one closed point (i.e., a point whose singleton set is a closed set). Now, continuous surjections between spaces having exactly one closed point must map the closed point of the domain to the closed point in the range. Thus the point in the product of the coordinate spaces, whose coordinates are the unique closed points from the corresponding coordinate spaces, is a thread (i.e. the element of the inverse limit, in other terms an element $(x_i)_{i\in I}$ of the product is called a *thread* if $\psi_{ij}(x_j) = x_i$ for $i \leqslant j$). Moreover this thread is the unique closed point in the inverse limit. But this contradicts the fact that the space X contains more than one maximal points. Therefore the completely normal spectral space $X = \mathrm{cl}(\mathrm{MaxSpec}F_{\mathbf{MV}}(1))$ is not pro-finite. So we have

Claim. *The category CNSS is not pro-finite.*

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