Logarithmic Norms and Projections Applied to Linear Differential Systems*

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1. INTRODUCTION

In this paper we study the behavior of solutions to the linear differential equation

\[ u'(t) = A(t) u(t), \quad t \in (-\infty, \infty), \]  

(\text{LDE})

where \( A \) is a continuous \( n \times n \) matrix valued function on \((-\infty, \infty)\). In [6], Wintner uses the Euclidean norm on \( \mathbb{K}^n \) (where \( \mathbb{K} \) is the real or complex field) to give estimates of upper and lower bounds for solutions to (LDE)—see also Cesari [1, p. 48] and references cited therein. Lozinskii [4] uses the logarithmic norm of \( A(t) \) to obtain similar bounds for solutions to (LDE) for any norm on \( \mathbb{K}^n \)—see also Coppel [2, p. 58]. In this paper we apply a class of seminorms on \( \mathbb{K}^n \) to obtain upper and lower bounds for certain families of solutions to (LDE). Applying these bounds to periodic linear equations, we are able to give estimates on the absolute values of the multipliers of (LDE). Our results extend and improve those of Lazer [3] and also improve some of the bounds for linear equations obtained in Martin [5, Propositions 4 and 5].

2. PROJECTION SEMINORMS

Let \( \mathbb{K} \) denote the field of real of complex numbers and let \( | \cdot | \) denote a norm on the vector space \( \mathbb{K}^n \), where \( n \) is a positive integer. Denote by \( \mathcal{L}(\mathbb{K}^n) \) the normed algebra of all linear functions from \( \mathbb{K}^n \) into \( \mathbb{K}^n \) with the norm \| \cdot \| on \( \mathcal{L}(\mathbb{K}^n) \) defined by \( \| A \| = \max\{|Ax|: x \in \mathbb{K}^n, |x| \leq 1\} \). We let \( \theta \) denote the zero of \( \mathbb{K}^n \), \( 0 \) the zero of \( \mathcal{L}(\mathbb{K}^n) \), and \( I \) the identity of \( \mathcal{L}(\mathbb{K}^n) \).

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Throughout this paper it is assumed that $m$ is a positive integer and \{${P}_i: i = 1,..., m$\} is a family of supplementary projections on $\mathcal{L}(\mathcal{X}^n)$, i.e.,

$$P_i \cdot P_i = P_i; \quad P_i \cdot P_j = 0 \quad \text{if} \quad i \neq j; \quad \text{and} \quad \sum_{i=1}^{m} P_i = I.$$

Also, it is assumed that $P_i \neq 0$ for any $i$ (and, hence, $m \leq n$).

**Definition 1.** For each $i$ in \{1,..., $m$\}, define the $[0, \infty)$ valued function $| \cdot |_i$ on $\mathcal{X}^n$ by

$$| x |_i = | P_ix |$$

for each $x$ in $\mathcal{X}^n$, and define the $[0, \infty)$ valued function $\| \cdot \|_i$ on $\mathcal{L}(\mathcal{X}^n)$ by

$$\| A \|_i = \sup \{ | Ax |_i : x \in \mathcal{X}^n, 1 = | x |_i \geq | x |_j \quad \text{for} \quad j \neq i \}$$

for each $A$ in $\mathcal{L}(\mathcal{X}^n)$.

It is trivial to see that $| \cdot |_i$ is a seminorm on $\mathcal{X}^n$ for each $i$ in \{1,..., $m$\}. Also, if $A$ is in $\mathcal{L}(\mathcal{X}^n)$ and $x$ is in $\mathcal{X}^n$ with $1 = | x |_i \geq | x |_j$ for $j \neq i$, then

$$| x | = \left| \sum_{j=1}^{m} P_jx \right| \leq | P_ix | + \sum_{j \neq i} | P_jx | \leq m;$$

so

$$| Ax |_i = | P_iAx | \leq \| P_iA \| \cdot | x | \leq m \cdot \| P_iA \| .$$

Hence $\| A \|_i$ is finite and $\| A \|_i \leq m \cdot \| P_iA \|$. It is also easy to see that $\| \cdot \|_i$ is a seminorm on $\mathcal{L}(\mathcal{X}^n)$. Furthermore, $\| A \|_i$ is the least number $M$ such that the inequality $| Ax |_i \leq M \cdot | x |_i$ is valid for all $x$ in $\mathcal{X}^n$ such that $1 = | x |_i \geq | x |_j$ for $j \neq i$. The inequality $| Ax |_i \leq \| A \|_i \cdot | x |_i$ does not necessarily hold for all $x$ in $\mathcal{X}^n$, and the inequality $\| A \cdot B \|_i \leq \| A \|_i \cdot \| B \|_i$ also does not hold in general. However, if $P_iA = AP_i$, then these inequalities are valid.

**Definition 2.** For each $i$ in \{1,..., $m$\} and $A$ in $\mathcal{L}(\mathcal{X}^n)$, define

$$\mu_i[A] = \lim_{h \to 0^+} (\| I + hA \|_i - 1)/h.$$

Note that $\| I \|_i = 1$, so that $\mu_i[A]$ is the right derivative at $h = 0$ of the convex function $h \to \| I + hA \|_i$. Hence $\mu_i[\cdot]$ is well defined and

$$\mu_i[A] \leq (\| I + hA \|_i - 1)/h$$

for each $h > 0$ (see, e.g., Coppel [2, p. 3]).
PROPOSITION 1. Suppose that \( i \) is in \( \{1, \ldots, m\} \) and \( A \) and \( B \) are in \( \mathcal{L}(\mathcal{H}^n) \). Then

(i) \( \| \mu_i[A] \| \leq \| A \|_i \);

(ii) \( \mu_i[rA] = r \mu_i[A] \) for each \( r > 0 \);

(iii) \( \mu_i[A + B] \leq \mu_i[A] + \mu_i[B] \);

(iv) \( -\mu_i[-A] \leq \mu_i[A] \);

(v) \( \| \mu_i[A] - \mu_i[B] \| \leq \| A - B \|_i \);

(vi) \( \mu_i \left[ \sum_{j=1}^m \alpha_j P_j \right] = \mu_i[\alpha_i P_i] = \Re(\alpha_i) \) for all \( \alpha_1, \ldots, \alpha_m \in \mathcal{H} \);

and

(vii) \( \mu_i \left[ A + \sum_{j=1}^m \alpha_j P_j \right] = \mu_i[A + \alpha_i P_i] = \mu_i[A] + \Re(\alpha_i) \) for all \( \alpha_1, \ldots, \alpha_m \in \mathcal{H} \).

**Indication of Proof.** Part (i) is immediate from the inequality

\[
\| I + hA \|_i - 1 = \| I + hA \|_i - \| I \|_i \leq h \| A \|_i,
\]

and (ii) follows easily since \( rh \to 0^+ \) as \( h \to 0^+ \). Part (iii) follows from (ii) and the estimate

\[
\| I + h(A + B) \|_i - 1 \leq [\| I + h2A \|_i - 1 + \| I + h2B \| - 1]/2.
\]

Part (iv) is immediate from (iii) since \( 0 = \mu_i[A - A] \leq \mu_i[A] + \mu_i[-A] \).

By (iii) and (i), we have that

\[
\mu_i[A] \leq \mu_i[B] + \mu_i[A - B] \leq \mu_i[B] + \| A - B \|_i.
\]

Thus \( \mu_i[A] - \mu_i[B] \leq \| A - B \|_i \), and interchanging the roles of \( A \) and \( B \) establishes (v). Since \( \| P_j \|_i \) is 0 if \( j \neq i \) and is 1 if \( j = i \),

\[
\| I + h \sum_{j=1}^m \alpha_j P_j \|_i = \| I + h\alpha_i P_i \|_i = \| 1 + h\alpha_i \|;
\]

(vi) follows easily from the fact that

\[
\lim_{h \to 0^+} (1 + h\alpha_i) = \Re(\alpha_i).
\]

Similarly, it is easy to see that

\[
\mu_i \left[ A + \sum_{j=1}^m \alpha_j P_j \right] = \mu_i[A + \alpha_i P_i].
\]
Also, by (iii) and (vi),
\[ \mu_i[A + \alpha_i P_i] \leq \mu_i[A] + \text{Re}(\alpha_i) \]
and
\[ \mu_i[A] = \mu_i[A + \alpha_i P_i - \alpha_i P_i] \leq \mu_i[A + \alpha_i P_i] + \text{Re}(-\alpha_i), \]
and (vii) is established.

**Proposition 2.** Suppose that $i$ is in \{1, ..., m\} and $x$ is a member of $\mathcal{K}^n$ such that $|x|_i \geq |x|_j$ for $j \neq i$. Then

\[
\lim_{h \to 0^+} (|x + hAx|_i - |x|_i)/h \leq \mu_i[A] |x|_i
\]

for each $A$ in $\mathcal{S}(\mathcal{K}^n)$. Furthermore, for each $M > 0$ and $\epsilon > 0$, there is a number $\delta = \delta(M, \epsilon)$ in $(0, 1)$ such that

\[
\lim_{h \to 0^+} (|x + hAx|_i - |x|_i)/h \leq (\mu_i[A] + \epsilon) |x|_i,
\]

whenever $A$ is in $\mathcal{S}(\mathcal{K}^n)$ with $\|A\| \leq M$ and $x$ is in $\mathcal{K}^n$ with

$|x|_i \geq (1 - \delta) |x|_j$ for $j \neq i$.

**Proof.** Note first that the limits in (1) and (2) both exist since $h \to |x + hAx|_i$ is convex. Also, if $|x|_i \geq |x|_j$ for $j \neq i$, then

\[
|x + hAx|_i \leq \|I + hA\|_i |x|_i,
\]

so

\[
\lim_{h \to 0^+} (|x + hAx|_i - |x|_i)/h \leq \lim_{h \to 0^+} (\|I + hA\|_i |x|_i - |x|_i)/h
\]

\[
= \mu_i[A] |x|_i.
\]

This establishes (1). For the proof of (2), let $\delta$ be a number in $(0, 1)$ such that $\delta(1 - \delta)^{-1} \|P_i\| M(m - 1) \leq \epsilon$, and suppose that $x \in \mathcal{K}^n$ with

$|x|_i \geq (1 - \delta) |x|_j$ for all $j \neq i$.

Let

\[ x' = x - \delta \sum_{j \neq i} P_j x. \]

Then

\[ P_i x' = P_i x \quad \text{and} \quad P_j x' = P_j x - \delta P_j x \quad \text{if} \quad j \neq i. \]
Thus

$$|x'_i|_i = |x_i|_i$$

and

$$|x'_j|_j = |P_jx - \delta P_jx| = (1 - \delta) |x_i|_j \leq |x'_i|_i \quad \text{if} \quad j \neq i.$$ 

By (1),

$$\lim_{h \to 0^+} (|x' + hAx|_i - |x'|_i)/h \leq \mu[A] |x'_i|.$$ 

Also,

$$|x + hAx|_i = |P_i x + hP_i Ax| = |P_i x' + hP_i Ax|$$

$$\leq |P_i x' + hP_i Ax'| + h |P_i A(x - x')|$$

$$\leq |x' + hAx|_i + h \|P_i A\| |x - x'|,$$

and

$$\|P_i A\| |x - x'| \leq \|P_i\| \mathcal{M}(1 - \delta)^{-1} (m - 1) |x|_i.$$ 

Consequently,

$$\lim_{h \to 0^+} (|x + hAx|_i - |x|_i)/h$$

$$\leq \lim_{h \to 0^+} (|x' + hAx'|_i - |x'|_i)/h + \|P_i\| \mathcal{M}(1 - \delta)^{-1} (m - 1) |x|_i$$

$$\leq \mu[A] |x'|_i + \epsilon |x|_i = (\mu[A] + \epsilon) |x|_i.$$ 

This establishes (2), and the proof of the proposition is complete.

For the proof of our main theorems, we will find the following lemma helpful.

**Lemma 1.** Suppose that $\Omega$ is a nonempty subset of $\{1, \ldots, m\}$ and

$$|x|_\Omega = \max\{|x|_i: i \in \Omega\}.$$ 

Then $|\cdot|_\Omega$ is a seminorm on $\mathcal{X}^n$. Furthermore, if $A \in \mathcal{L}(\mathcal{X}^n), x \in \mathcal{X}^n$, and $\mathcal{A} = \{j \in \Omega: |x_j| = |x|_\Omega\}$, then

$$\lim_{h \to 0^+} (|x + hAx|_\Omega - |x|_\Omega)/h = \max\{\lim_{h \to 0^+} (|x + hAx|_j - |x|_j)/h: j \in \mathcal{A}\},$$

and

$$- \lim_{h \to 0^+} (|x - hAx|_\Omega - |x|_\Omega)/h = \min\{-\lim_{h \to 0^+} (|x - hAx|_j - |x|_j)/h: j \in \mathcal{A}\}.$$
**Proof.** It is easy to see that $| \cdot |_Q$ is a seminorm on $\mathcal{X}^n$. Note also that each of the above limits exists and that the second assertion is a direct consequence of the first with $A$ replaced by $-A$. Let

$$I' = \max\{\lim_{h \to 0^+}(| x + hAx |_i - | x |_i)/h; j \in \Delta\}.$$  

If $j \in \Delta$, then $| x |_j = | x |_\Omega$,

$$(| x + hAx |_j - | x |_j)/h \leq (| x + hAx |_\Omega - | x |_\Omega)/h;$$

so

$$\lim_{h \to 0^+}(| x + hAx |_\Omega - | x |_\Omega)/h \geq I'.$$

Since $| x |_j < | x |_\Omega$ for $j \in \Omega - \Delta$, we have that

$$| x + hAx |_\Omega = \max\{| x + hAx |_j; j \in \Delta\}$$

for $h > 0$ sufficiently small. Since $\Delta$ is finite, there must exist a sequence $(h_k)_k$ of positive numbers and some $i \in \Delta$ such that $\lim_{k \to \infty} h_k = 0$ and $| x + h_kAx |_\Omega = | x + h_kAx |_i$ for all $k = 1, 2, \ldots$. Also, $| x |_\Omega = | x |_i$ and we have that

$$\lim_{h \to 0^+}(| x + hAx |_\Omega - | x |_\Omega)/h = \lim_{h \to 0^+}(| x + h_kAx |_i - | x |_i)/h_k$$

$$= \lim_{h \to 0^+}(| x + hAx |_i - | x |_i)/h$$

$$\leq I'.$$

This completes the proof of Lemma 1.

3. Estimates of Solutions

In this section we use the notions introduced in Section 2 to establish our main results concerning upper and lower bounds for solutions to (LDE). Some examples of these results are given in Section 5. The connection between the ideas of Section 2 and the solutions to (LDE) is established by the following lemma.

**Lemma 2.** Suppose that $i$ is in $\{1, \ldots, m\}$ and $u$ is a solution to (LDE). If $p_i(t) = | u(t) |_i$ for each $t$ in $(-\infty, \infty)$, then $p_i$ is both right and left differentiable, and, for each $t$ in $(-\infty, \infty)$,

$$(p_i)'_+(t) = \lim_{h \to 0^+}(| u(t) + hA(t)u(t) |_i - | u(t) |_i)/h$$

and

$$(p_i)'_-(t) = -\lim_{h \to 0^+}(| u(t) - hA(t)u(t) |_i - | u(t) |_i)/h.$$  

(3)

(4)
Furthermore, if \( t \) is in \((-\infty, \infty)\) and \(| u(t)|_i \geq |u(t)|_j\) for all \( j \neq i \), then

\[
(p_i\rangle^+ (t) \leq \mu_i[A(t)] p_i(t) \tag{5}
\]

and

\[
(p_i\rangle^- (t) \geq -\mu_i[-A(t)] p_i(t). \tag{6}
\]

**Indication of Proof.** Since \( u'(t) = A(t) u(t) \), assertions (3) and (4) are easy to establish (see, e.g., Coppel [2, p. 3]). Also, if \(| u(t)|_i \geq |u(t)|_j\) for all \( j \neq i \), then

\[
|u(t) + hA(t) u(t)|_i \leq \|I + hA(t)\|_i |u(t)|_i
\]

and

\[
|u(t) - hA(t) u(t)|_i \leq \|I - hA(t)\|_i |u(t)|_i,
\]

and assertions (5) and (6) are immediate from (3) and (4).

Let \( U \) denote the unique continuously differentiable function from \((-\infty, \infty)\) into \( \mathcal{L}(\mathcal{X}^n) \) which satisfies \( U(0) = I \) and \( U'(t) = A(t) U(t) \) for all \( t \) in \((-\infty, \infty)\). Then \( U(t) \) is invertible for each \( t \) in \((-\infty, \infty)\) and if \( u \) is a solution to (LDE) and \( \tau \) is in \((-\infty, \infty) \), \( u(t) = U(t) U(\tau)^{-1} u(\tau) \) for all \( t \) in \((-\infty, \infty) \).

**Theorem 1.** Suppose that \( i \) is in \( \{1, \ldots, m\} \) and \( \Omega_+ \) and \( \Omega_- \) are disjoint subsets of \( \{1, \ldots, m\} \setminus \{i\} \) such that \( \Omega_+ \cup \Omega_- = \{1, \ldots, m\} \setminus \{i\} \). Suppose further that

\[
\max\{\mu_j[A(t)]: j \in \Omega_-\} < -\mu_i[-A(t)]
\]

and

\[
\mu_i[A(t)] < \min\{-\mu_i[-A(t)]: j \in \Omega_+\} \tag{7}
\]

for all \( t \) in \((-\infty, \infty)\). Then there is a subspace \( \mathcal{X}_i \) of \( \mathcal{X}^n \) such that the dimension of \( \mathcal{X}_i \) is the rank of \( P_i \) (i.e., the dimension of the range of \( P_i \)) and each of the following are fulfilled.

(i) \(|U(t) x|_i \geq |U(t) x|_j\) for all \((t, x) \in (-\infty, \infty) \times \mathcal{X}_i\) and \(j \in \{1, \ldots, m\}\);

(ii) for each \( x \in \mathcal{X}_i \), the function

\[
t \rightarrow |U(t) x|_i \exp \left( -\int_0^t \mu_i[A(s)] \, ds \right)
\]

is nonincreasing on \((-\infty, \infty)\);

(iii) for each \( x \in \mathcal{X}_i \), the function

\[
t \rightarrow |U(t) x|_i \exp \left( \int_0^t \mu_i[-A(s)] \, ds \right)
\]

is nondecreasing on \((-\infty, \infty)\);
(iv) \(|x_0| \exp \left( -\int_0^t \mu_i[-A(s)] \, ds \right) \)
\[ \leq |U(t) x_0| \leq |x_0| \exp \left( \int_0^t \mu_i[A(s)] \, ds \right) \]
for all \(x \in \mathbb{R}^n\) and \(t \geq 0\); and

(v) \(|x_0| \exp \left( \int_0^t \mu_i[A(s)] \, ds \right) \)
\[ \leq |U(t) x_0| \leq |x_0| \exp \left( -\int_0^t \mu_i[-A(s)] \, ds \right) \]
for all \(x \in \mathbb{R}^n\) and \(t \leq 0\).

Remark. In Theorem 1, we allow the possibility that \(\Omega_+\) and/or \(\Omega_-\) may be empty. In the case that \(\Omega_-\) is empty, we define \(\max\{\mu_j[A(t)]: j \in \Omega_-\}\) to be \(-\infty\), and in the case that \(\Omega_+\) is empty, we define \(\min\{-\mu_j[-A(t)]: j \in \Omega_+\}\) to be \(+\infty\). In particular, if \(m = 1\) and \(P_4 = I\), then \(|x_0| = |x|\) and we have Theorem 3 of Coppel [2, p. 58]. Note also that, by part (v) of Proposition 1, the functions \(t \rightarrow \mu_j[A(t)]\) and \(t \rightarrow -\mu_j[-A(t)]\) are continuous on \((-\infty, \infty)\) for each \(j \in \{1, \ldots, m\}\).

The following two lemmas are convenient for the proof of Theorem 1. The suppositions of Theorem 1 are assumed to be fulfilled in each of these lemmas.

**Lemma 3.** For each \(x \in \mathbb{R}^n\), define
\[
N_-[x] = \max\{|x_j|: j \in \Omega_-\},
\]
\[
N_-[x] = \max\{|x_j|: j \in \Omega_- \cup \{i\}\}, \quad N_+[x] = \max\{|x_j|: j \in \Omega_+\},
\]
and
\[
N_+^{-}[x] = \max\{|x_j|: j \in \Omega_+ \cup \{i\}\}.
\]
(If \(\Omega_-\) is empty, define \(N_-[x] = -\infty\), and if \(\Omega_+\) is empty, define \(N_+[x] = -\infty\)). Let \(u\) be a nontrivial solution to (LDE) and let \(\tau\) be in \((-\infty, \infty)\).

(i) If \(N_+^{-}[u(\tau)] \geq N_-[u(\tau)]\), then \(N_+^{-}[u(t)] \geq N_-[u(t)]\) for all \(t \geq \tau\),
and

(ii) if \(N_+^{-}[u(\tau)] \geq N_-[u(\tau)]\), then \(N_+^{-}[u(t)] \geq N_-[u(t)]\) for all \(t \leq \tau\).

**Proof.** Suppose that \(N_+^{-}[u(\tau)] \geq N_-[u(\tau)]\) and let \(\lambda = \{\lambda \geq \tau: N_+^{-}[u(t)] \geq N_-[u(t)]\} \) for all \(t \in [\tau, \lambda]\).
Part (i) will be established if we show that \( A = [\tau, \infty) \). If \( A \neq [\tau, \infty) \), then \( A \) is of the form \( [\tau, \gamma) \) or \( [\tau, \gamma] \). Since \( N^+, N^- \) and \( u \) are continuous, \( A \) must be of the form \( [\tau, \gamma) \) and \( N^+_+[u(\gamma)] = N^-_[u(\gamma)] \). By (7), there is an \( \varepsilon > 0 \) and \( \delta > 0 \) such that

\[
\max\{\mu_j[A(t)]: j \in \Omega_\varepsilon\} + \varepsilon < \min\{-\mu_j[-A(t)]: j \in \Omega_\varepsilon \cup \{i\}\} - \varepsilon \quad (8)
\]

for all \( t \in [\gamma, \gamma + \delta] \). Let \( p(t) = N^+_+[u(t)] \) and \( q(t) = N^-_[u(t)] \) for \( t \in [\gamma, \gamma + \delta] \). Let

\[
A_p(t) = \{j \in \Omega_\varepsilon \cup \{i\}: p(t) = |u(t)|_j\}
\]

and

\[
A_q(t) = \{j \in \Omega_\varepsilon: q(t) = |u(t)|_j\}.
\]

It follows that \( \delta \) can be chosen sufficiently small so that \( A_p(t) \subseteq A_p(\gamma) \) and \( A_q(t) \subseteq A_q(\gamma) \) for all \( t \in [\gamma, \gamma + \delta] \). Also, by Lemmas 1 and 2, Proposition 2, and (8), we can also assume that \( \delta \) is sufficiently small so that

\[
p_-(t) = \min\{-\lim_{h \to 0^+}(|u(t) - hA(t)u(t)|_j - |u(t)|_j): j \in A_p(t)\}
\]

\[
\geq \min\{-\mu_j[-A(t)] - \varepsilon \mid u(t)|_j: j \in A_p(t)\}
\]

\[
\geq \max\{\mu_j[A(t)] + \varepsilon: j \in \Omega_\varepsilon\} p(t),
\]

and

\[
q_+(t) = \max\{\lim_{h \to 0^+}(|u(t) + hA(t)u(t)|_j - |u(t)|_j): j \in A_q(t)\}
\]

\[
\leq \max\{\mu_j[A(t)] + \varepsilon \mid u(t)|_j: j \in A_q(t)\}
\]

\[
\leq \max\{\mu_j[A(t)] + \varepsilon: j \in \Omega_\varepsilon\} q(t)
\]

for all \( t \in [\gamma, \gamma + \delta] \). Letting \( \alpha(t) = \max\{\mu_j[A(t)] + \varepsilon: j \in \Omega_\varepsilon\} \) have by solving the above differential inequalities that, for each \( t \in [\gamma, \gamma + \delta] \),

\[
N^+_+[u(t)] = p(t) \geq \rho(t) \exp \left( \int_\gamma^t \alpha(s) \, ds \right)
\]

and

\[
N^-_[u(t)] = q(t) \leq \rho(t) \exp \left( \int_\gamma^t \alpha(s) \, ds \right).
\]

Since \( \rho(t) = q(t) \), we conclude that \( N^+_+[u(t)] \geq N^-_[u(t)] \) for all \( t \in [\gamma, \gamma + \delta] \). Thus, \( [\tau, \gamma + \delta] \supset A \), and it follows that \( A = [\tau, \infty) \). This establishes (i). The proof of (ii) is completely analogous if we define

\[
A = \{\lambda \leq \tau: N^+_+[u(t)] \geq N^-_[u(t)] \text{ for all } t \in [\lambda, \tau]\}
\]

and show that assuming \( A \) is of the form \( [\gamma, \tau] \) leads to a contradiction.
Lemma 4. For each j in \{1,...,m\}, let \(n_j\) denote the rank of \(P_j\) and define

\[
n_+ = n_0 + \sum_{j \in \Omega_+} n_j \quad \text{and} \quad n_- = n_0 + \sum_{j \in \Omega_-} n_j.
\]

Then, if \(N_+, N_-, N_-^i\) and \(N_-\) are as in Lemma 3,

(i) there is an \(n_+\) dimensional subspace \(\mathcal{X}_+\) of \(\mathcal{X}^n\) such that

\[
N_+^i[U(t) x] \geq N_- [U(t) x] \quad \text{for all} \quad x \in \mathcal{X}_+ \quad \text{and} \quad t \in (-\infty, \infty),
\]

and

(ii) there is an \(n_-\) dimensional subspace \(\mathcal{X}_-\) of \(\mathcal{X}^n\) such that

\[
N_-^i[U(t) x] \geq N_+ [U(t) x] \quad \text{for all} \quad x \in \mathcal{X}_- \quad \text{and} \quad t \in (-\infty, \infty).
\]

Proof. If \(x = (\xi_j)^n\) and \(y = (\eta_j)^n\) are in \(\mathcal{X}^n\), define \(\langle x, y \rangle = \sum_{j=1}^n \xi_j \overline{\eta_j}\). For each \(\tau < 0\), let \(\mathcal{X}_+(\tau)\) be the range of \(U(\tau)^{-1} [P_1 + \sum_{j \in \Omega_+} P_j]\). Then \(\mathcal{X}_+(\tau)\) has dimension \(n_+\); so let \(\{x_j(\tau) : j = 1,..., n_+\}\) be an orthonormal basis for \(\mathcal{X}_+(\tau)\) (i.e., \(\langle x_j(\tau), x_k(\tau) \rangle = 0\) if \(j \neq k\) and is 1 if \(j = k\)). Since closed and bounded subsets of \(\mathcal{X}^n\) are compact, there is a sequence \(\{\tau_k\}_k\) in \((-\infty, 0]\) such that \(\lim_{k \to \infty} \tau_k = -\infty\) and \(\lim_{k \to \infty} x_j(\tau_k) = x_j\) for \(j = 1,..., n_+.\) It is easy to see that \(\{x_j : j = 1,..., n_+\}\) is also orthonormal and, hence, linearly independent. Let \(\mathcal{X}_+\) be the subspace generated by \(\{x_j : j = 1,..., n_+\},\) If \(x \in \mathcal{X}_+\), then there is a subset \(\{\alpha_j : j = 1,..., n_+\}\) of \(\mathcal{X}\) such that \(x = \sum_{j=1}^{n_+} \alpha_j x_j.\) Then \(x = \lim_{k \to \infty} x_k\), where \(x_k = \sum_{j=1}^{n_+} \alpha_j x_j(\tau_k)\) is in \(\mathcal{X}_+(\tau_k).\) By the definition of \(\mathcal{X}_+(\tau_k),\) we have \(U(\tau_k) x_k\) is in the range of \(P_1 + \sum_{j \in \Omega_+} P_j;\) and, hence, \(P_j U(\tau_k) x_k = \theta\) for all \(j \in \Omega_.\) Hence \(N_+^i[U(\tau_k) x_k] \geq 0 \geq N_- [U(\tau_k) x_k],\) and part (i) of Lemma 3 implies that \(N_+^i[U(t) x_k] \geq N_- [U(t) x_k]\) for all \(t \geq \tau_k.\) Since \(\lim_{k \to \infty} \tau_k = -\infty\) and \(\lim_{k \to \infty} U(t) x_k = U(t) x\) for all \(t \in (-\infty, \infty),\) it follows that \(N_+^i[U(t) x] \geq N_- [U(t) x]\) for all \(t \in (-\infty, \infty).\) This completes the proof of part (i). The proof of part (ii) is directly analogous, if for each \(\tau > 0\) we choose \(\mathcal{X}_-(\tau)\) to be the range of \(U(\tau)^{-1} [P_1 + \sum_{j \in \Omega_-} P_j]\) and define \(\mathcal{X}_-\) in a analogous manner as \(\mathcal{X}_+\) except that, in this case, \(\lim_{k \to \infty} \tau_k = +\infty.\)

Proof of Theorem 1. If \(\mathcal{X}_+\) and \(\mathcal{X}_-\) are as in Lemma 4 and \(\mathcal{X}_i = \mathcal{X}_+ \cap \mathcal{X}_-\), then \(\mathcal{X}_i\) is a linear subspace of \(\mathcal{X}^n\) which has dimension no smaller than \(n_+ + n_- - n = n_i\) it can be shown that \(\mathcal{X}_i\) has dimension exactly \(n_i\). Let \(\mathcal{X}_i\) be an \(n_i\) dimensional subspace of \(\mathcal{X}_i\). If \(N_+^i\) and \(N_-^i\) are as in Lemma 3 and \(x \in \mathcal{X}_i\), then \(x \in \mathcal{X}_+\) and \(x \in \mathcal{X}_-\), and it follows from Lemma 4 that \(N_+^i[U(t) x] = N_-^i[U(t) x]\) for all \(t \in (-\infty, \infty).\) The claim is that
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\[ N_+ t[U(t) x] = N_- t[U(t) x] = |U(t) x|_i \text{ for all } t \in (-\infty, \infty). \]

Assume, for contradiction, that this is not the case. Let

\[ p(t) = N_+ t[U(t) x], \quad q(t) = N_- t[U(t) x], \]

\[ A_p(t) = \{ j \in \Omega_+ \cup \{i\} : p(t) = |U(t) x|_j \}, \]

and

\[ A_q(t) = \{ j \in \Omega_- \cup \{i\} : q(t) = |U(t) x|_j \}. \]

By assumption, there is a \( \tau \in (-\infty, \infty) \) such that \( i \notin A_p(\tau) \cap A_q(\tau) \). Since \( p(\tau) = q(\tau) \), it is immediate that \( i \notin A_p(\tau) \cup A_q(\tau) \) and, by continuity, \( i \notin A_p(t) \cup A_q(t) \) for all \( t \in [\tau, \tau + \delta] \) for some \( \delta > 0 \). Note also that if \( j \in A_p(t) \cup A_q(t) \) for any \( t \in [\tau, \tau + \delta] \), then \( |U(t) x|_i \geq |U(t) x|_k \) for all \( k \in \{1, \ldots, m\} \). Since \( A_p(t) \subseteq \Omega_+ \) for each \( t \in [\tau, \tau + \delta] \), we have from (7) and Lemmas 1 and 2 that, for each \( t \in (\tau, \tau + \delta] \),

\[
p_i^-(t) = \min\{- \lim_{h \to 0^+} \| u(t) - hA(t) u(t)|_j - u(t)|_j| / h : j \in A_p(t) \}
\]

\[
\geq \min\{- \mu_j[-A(t)] / u(t)|_j : j \in A_p(t) \}
\]

\[
> \mu_j[A(t)] p(t).
\]

Also, since \( A_q(t) \subseteq \Omega_- \) for each \( t \in [\tau, \tau + \delta] \), we have from (7) and Lemmas 1 and 2 that, for each \( t \in [\tau, \tau + \delta] \),

\[
q_i^+(t) = \max\{ \lim_{h \to 0^+} \| u(t) + hA(t) u(t)|_j - u(t)|_j| / h : j \in A_q(t) \}
\]

\[
\leq \max\{ \mu_j[A(t)] / u(t)|_j : j \in A_q(t) \}
\]

\[
< - \mu_j[-A(t)] q(t)
\]

\[
< \mu_j[A(t)] q(t).
\]

Thus, for each \( t \in (\tau, \tau + \delta] \),

\[ p(t) > p(\tau) \exp \left( \int_{\tau}^{t} \mu_s[A(s)] \, ds \right) \quad \text{and} \quad q(t) \leq q(\tau) \exp \left( \int_{\tau}^{t} \mu_s[A(s)] \, ds \right). \]

This is an obvious contradiction to the fact that \( p(t) = q(t) \) for all \( t \in (-\infty, \infty) \). Consequently, \( |U(t) x|_i = N_+ t[U(t) x] = N_- t[U(t) x] \) for all \( t \in (-\infty, \infty) \) and \( x \in \mathcal{Z}_i \). Part (i) of Theorem 1 now follows easily. Furthermore, if \( x \in \mathcal{Z}_i \) and \( p_i(t) = |U(t) x|_i \) for all \( t \in (-\infty, \infty) \), then \( |U(t) x|_i \geq |U(t) x|_j \) for all \( j \in \{1, \ldots, m\} \), and it follows from (5) and (6) in Lemma 2 that

\[ \frac{d^+}{dt} \left( p_i(t) \exp \left( - \int_{0}^{t} \mu_s[A(s)] \, ds \right) \right) \leq 0 \]
and
\[ \frac{d}{dt} \left( P_i(t) \exp \left( \int_0^t \mu_i[-A(s)] \, ds \right) \right) \geq 0 \]
for all \( t \in (-\infty, \infty) \). Parts (ii) and (iii) follow directly from these differential inequalities and parts (iv) and (v) are immediate from (ii) and (iii). This completes the proof of Theorem 1.

**Corollary 1.** Assume that the suppositions of Theorem 1 are fulfilled. Then, for each \( x \in \mathcal{X}_i \),

\( \leq |U(t)x| \leq m \left\| P_i \right\| |x| \exp \left( \int_0^t \mu_i[A(s)] \, ds \right) \)
for all \( t \geq 0 \),

and

\( \leq |U(t)x| \leq m \left\| P_i \right\| |x| \exp \left( - \int_0^t \mu_i[-A(s)] \, ds \right) \)
for all \( t \leq 0 \).

**Proof.** By part (i) of Theorem 1, \( |U(t)x|_i \geq |U(t)x|_j \) for all \( j \in \{1, \ldots, m\} \). Hence,
\[
|U(t)x| = \left\| \sum_{j=1}^m P_j U(t)x \right\| \leq \sum_{j=1}^m |U(t)x|_j \leq m \left\| U(t)x \right\|_i ,
\]
and
\[
|U(t)x|_i = \left\| P_i U(t)x \right\| \leq \left\| P_i \right\| \left\| U(t)x \right\| .
\]
for all \( t \in (-\infty, \infty) \). The Corollary is now an immediate consequence of parts (iv) and (v) of Theorem 1.

**Theorem 2.** Suppose that \( m \geq 2 \) and for each \( i \) in \( \{2, \ldots, m\} \) and \( t \) in \( (-\infty, \infty) \), \( \mu_{i-1}[A(t)] < -\mu_i[-A(t)] \). Then there is a family \( \{Q_i : i = 1, \ldots, m\} \) of supplementary projections on \( \mathcal{X}^n \) such that each of the following holds:

(i) the rank of \( Q_i \) equals the rank of \( P_i \) for each \( i \) in \( \{1, \ldots, m\} \);

(ii) \( |U(t)Q_i x|_i \geq |U(t)Q_j x|_j \) for all \( t \) in \( (-\infty, \infty) \), \( x \) in \( \mathcal{X}^n \), and \( i \) and \( j \) in \( \{1, \ldots, m\} \);
(iii) the function
\[ t \rightarrow |U(t)Q_i x_i| \exp \left( - \int_0^t \mu_i[A(s)] \, ds \right) \]
is nonincreasing on \((-\infty, \infty)\) for each \(x\) in \(\mathcal{X}^n\) and \(i\) in \(\{1, \ldots, m\}\);
(iv) the function
\[ t \rightarrow |U(t)Q_i x_i| \exp \left( \int_0^t \mu_i[-A(s)] \, ds \right) \]
is nondecreasing on \((-\infty, \infty)\) for each \(x\) in \(\mathcal{X}^n\) and \(i\) in \(\{1, \ldots, m\}\);
(v) \[ |Q_i x_i| \exp \left( - \int_0^t \mu_i[-A(s)] \, ds \right) \]
for each \(x\) in \(\mathcal{X}^n\), \(i\) in \(\{1, \ldots, m\}\), and \(t \geq 0\); and
(vi) \[ |Q_i x_i| \exp \left( \int_0^t \mu_i[A(s)] \, ds \right) \]
for each \(x\) in \(\mathcal{X}^n\), \(i\) in \(\{1, \ldots, m\}\), and \(t \leq 0\).

**Proof.** For the proof of this theorem, we use the proof techniques of Lemma 4 to simultaneously construct subspaces \(Z_i, i = 1, \ldots, m\), such that \(\mathcal{X}^n\) is the direct sum of \(\{Z_i: i = 1, \ldots, m\}\), and the conclusions of Theorem 1 are valid for each \(i\) in \(\{1, \ldots, m\}\). For each \(i\) in \(\{1, \ldots, m\}\) and \(\tau\) in \((-\infty, \infty)\), let \(X^i(\tau)\) be the range of \(U(\tau)^{-1} [\sum_{j=1}^i P_j] \) and \(X^i(\tau)\) be the range of \(U(\tau)^{-1} [\sum_{j=1}^i P_j] \). Note that \(X^i(\tau) \supseteq X^{i+1}(\tau) \) and \(X^i(\tau) \subseteq X^{i+1}(\tau) \). For each \(\tau\) in \((-\infty, \infty)\), let \(\{x_j(\tau): j = 1, \ldots, n\}\) be an orthonormal set in \(\mathcal{X}^n\) such that \(\{x_j(\tau): j = 1, \ldots, n\}\) is a basis for \(X^i(\tau)\) — where \(\sum_{k=1}^n n_k\) is defined to be 0. To see that such an orthonormal set exists, one may begin by constructing an orthonormal basis \(\{x_{n-n+1}(\tau), \ldots, x_n(\tau)\}\) for \(X^i(\tau)\). Since \(X^{m-1}(\tau) \supseteq X^m(\tau)\), one may extend this orthonormal basis to one for \(X^{m-1}(\tau)\). By continuing in this fashion one obtains the desired orthonormal set in \(\mathcal{X}^n\). Using a compactness argument, we can assume the existence of a sequence \((t_k) \in (-\infty, 0)\) such that \(\lim_{k \to \infty} t_k = -\infty\) and \(x_j = \lim_{k \to \infty} x_j(\tau_k)\) exists for each \(j \in \{1, \ldots, m\}\). Let \(X^i(\tau)\) denote the subspace of \(\mathcal{X}^n\) generated by \(\{x_j: j = (1 + \sum_{k=1}^{i-1} n_k), n\}\). Then \(X^i(\tau)\) has dimension \(\sum_{j=1}^n n_j\) and \(X^i(\tau) \supseteq X^{i-1}(\tau)\). Similarly, we can construct subspaces \(X^i(\tau)\) by appropriately choosing convergence orthonormal bases of \(X^i(\tau)\) where \(\lim_{k \to \infty} \tau_k = +\infty\). Now define \(M^i(x) = \max\{x_i: j = i, \ldots, m\}\) and \(M^i = \max\{x_i: j = 1, \ldots, i\}\). Note that, with the notations used in Lemmas 3 and 4,
\[ M^i = N^i, \quad M^{i+1} = N^i, \quad M^i - N^i, \quad \text{and} \quad M^{i-1} = N^i. \]
where $\Omega_+ = \{i + 1, \ldots, m\}$ (or $\Omega_+ = \emptyset$ if $i = m$) and $\Omega_- = \{1, \ldots, i - 1\}$ (or $\Omega_- = \emptyset$ if $i = 1$). Thus, by Lemma 4,

$$M_+^{i+1}[U(t)x] \geq M_-^i[U(t)x] \quad \text{for all } (t, x) \in (-\infty, \infty) \times X_+^{i+1}$$

and

$$M_-^i[U(t)x] \geq M_+^{i+1}[U(t)x] \quad \text{for all } (t, x) \in (-\infty, \infty) \times X_-^i.$$

As in the proof of Theorem 1, let $Z_i$ be an $n_i$ dimensional subspace of $X_-^i \cap X_+^i$. Then the conclusions of Theorem 1 are valid for each $i$ in $\{1, \ldots, m\}$, and if $Q_i$ is any projection from $\mathcal{H}^n$ onto $Z_i$, assertions (i)-(vi) of this theorem are valid. To complete the proof we need to show that the $Q_i$'s can be chosen so that the family $\{Q_i : i = 1, \ldots, m\}$ is supplementary. Since the sum of the dimensions of the $Z_i$ is $n$, this is equivalent to showing that $Z_i \cap \bigcup_{j=i+1}^m Z_j = \{0\}$ for each $i \in \{1, \ldots, m - 1\}$. Note first that $Z_i \subseteq X_-^i$ and $Z_j \subseteq X_+^j \subseteq X_+^{i+1}$ for each $j = i + 1, \ldots, m$. Since $X_+^{i+1}$ is a subspace of $\mathcal{H}^n$, we have that $X_-^i \subseteq X_+^{i+1}$. Thus, the proof is complete if it is shown that $X_-^i \cap X_+^{i+1} = \{0\}$. Let $x \in X_-^i \cap X_+^{i+1}$ and define

$$p(t) = M_+^{i+1}[U(t)x] \quad \text{and} \quad q(t) = M_-^i[U(t)x] \quad \text{for } t \in (-\infty, \infty).$$

By our construction of $X_-^i$ and $X_+^{i+1}$, we have from Lemma 4 that $p(t) \geq q(t)$ and $q(t) \geq p(t)$ for all $t \in (-\infty, \infty)$, and hence $p = q$. Let $t \in (-\infty, \infty)$ and let

$$\Delta_p(t) = \{j : j = i + 1, \ldots, m \text{ and } |U(t)x|_j = M_i^{i+1}[U(t)x]\}.$$

By Lemmas 1 and 2 it follows that

$$p_-'(t) = \min\{- \lim_{h \to 0^+} \left| U(t)x - hA(t)U(t)x \right|_j - \left| U(t)x|_j\right|/h : j \in \Delta_p(t)\}$$

$$\geq \min\{-\mu_j[-A(t)] |U(t)x|_j : j \in \Delta_p(t)\}$$

$$\geq -\mu_{i+1}[-A(t)] p(t),$$

since

$$-\mu_{i+1}[-A(t)] \leq \mu_{i+1}[A(t)] < -\mu_j[-A(t)] \quad \text{for } j = i + 2, \ldots, m.$$

Similarly,

$$q_+'(t) \leq \mu_i[A(t)] q(t).$$

Thus, for each $t \geq 0$,

$$q(0) \exp\left(\int_0^t \mu_i[A(s)]ds\right) \geq q(t) = p(t) \geq p(0) \exp\left(-\int_0^t \mu_{i+1}[-A(t)]ds\right) ,$$
and it is immediate that $q(0) = p(0) = 0$, since $-\mu_{i+1}[-A(s)] > \mu_i[A(s)]$ for all $s$. Consequently $x = 0$, and the proof of Theorem 2 is complete.

As an immediate consequence of Corollary 1 and Theorem 2 we have

**Corollary 2.** Assume that the suppositions of Theorem 2 are fulfilled and $i$ is in $\{1, \ldots, m\}$. Then

(i) $m^{-1} \|P_i\|^{-1} |Q_i x| \exp\left(-\int_0^t \mu_i[-A(s)] \, ds\right)$

$$\leq |U(t)Q_i x| \leq m \|P_i\| \|Q_i x| \exp\left(\int_0^t \mu_i[A(s)] \, ds\right)$$

for all $x$ in $\mathcal{H}$ and $t \geq 0$, and

(ii) $m^{-1} \|P_i\|^{-1} |Q_i x| \exp\left(\int_0^t \mu_i[A(s)] \, ds\right)$

$$\leq |U(t)Q_i x| \leq m \|P_i\| \|Q_i x| \exp\left(-\int_0^t \mu_i[-A(s)] \, ds\right)$$

for all $x$ in $\mathcal{H}$ and $t \leq 0$.

4. **Periodic Systems**

In this section we suppose that $\omega$ is a positive number and $A$ is $\omega$-periodic on $(-\infty, \infty)$ (i.e., $A(t + \omega) = A(t)$ for all $t \in (-\infty, \infty)$). The number $\omega$ is not required to be the least period of $A$. The crucial property of $\omega$-periodic systems which is used is that the function $t \rightarrow U(t + \omega) x$ is a solution to (LDE) for each $x$ in $\mathcal{H}$.

**Theorem 3.** In addition to the suppositions of Theorem 2, suppose that $A$ is $\omega$-periodic. If the family $\{\mathcal{Q}_i; i = 1, \ldots, m\}$ is as in Theorem 2 and $\mathcal{Z}_i$ is the range of $Q_i$ for each $i$ in $\{1, \ldots, m\}$, then $U(\omega) \mathcal{F}_i = \mathcal{Z}_i$ and $\mathcal{Q}_i U(\omega) = U(\omega) \mathcal{Q}_i$ for each $i$ in $\{1, \ldots, m\}$.

**Proof.** Note that the two conclusions of Theorem 3 are equivalent. Furthermore, since $U(\omega)$ is invertible, if $U(\omega) \mathcal{F}_i \subseteq \mathcal{Z}_i$, then $U(\omega) \mathcal{F}_i = \mathcal{Z}_i$. Thus we need to show that $U(\omega) \mathcal{F}_i \subseteq \mathcal{Z}_i$ for each $i \in \{1, \ldots, m\}$. Let $i$ be in $\{1, \ldots, m\}$ and assume, for contradiction, that $x \in \mathcal{Z}_i$ and $U(\omega) x \notin \mathcal{Z}_i$. Since $\mathcal{H}$ is the direct sum of $\{\mathcal{Z}_i; j = 1, \ldots, m\}$, we have that

$$U(\omega) x = \sum_{j=1}^m z_j,$$
where \( z_j \in \mathcal{P}_j \) for each \( j \in \{1, \ldots, m\} \). Also, \( U(\omega)x \notin \mathcal{P}_i \) implies that \( z_k \neq \theta \) for some \( k \neq i \). Since \( \psi(t) - U(t + \omega)x \) is also a solution to (LDE) and \( \psi(0) = \sum_{j=1}^{m} z_j \), we have that

\[
\psi(t) = \sum_{j=1}^{m} U(t) z_j
\]

for all \( t \in (\infty, \infty) \). We consider two cases. Suppose first that \( z_k \neq \theta \) for some \( k > i \). Suppose also that \( k \) is the largest number in \( \{1, \ldots, m\} \) such that \( z_k \neq \theta \). Since \( A \) is \( \omega \)-periodic, there is an \( \epsilon > 0 \) such that

\[
\mu_j[A(t)] + \epsilon \leq -\mu_k[-A(t)] \quad \text{for all } t \in (\infty, \infty) \quad \text{and } j < k.
\]

Since \( Qx = x \), it follows from part (i) of Corollary 2 that

\[
\lim_{t \to +\infty} |U(t + \omega)x| \exp \left( \int_0^t \mu_k[-A(s)] \, ds \right) 
\leq \lim_{t \to +\infty} m \|P_\epsilon\| \|x\| \exp \left( \int_0^{t+\omega} \mu_k[A(s)] \, ds \right) \exp \left( \int_0^t \mu_k[-A(s)] \, ds \right) 
= 0.
\]

Similarly, if \( j < k \), then \( z_j = Q_j z_j \) and

\[
\lim_{t \to +\infty} |U(t)z_j| \exp \left( \int_0^t \mu_k[-A(s)] \, ds \right) = 0. \tag{10}
\]

By applying part (i) of Corollary 2 again, we also have that

\[
\liminf_{t \to +\infty} |U(t)z_k| \exp \left( \int_0^t \mu_k[-A(s)] \, ds \right) \geq m^{-1} \|P_k\|^{-1} |z_k| > 0. \tag{11}
\]

Since

\[
U(t + \omega)x = \sum_{j=1}^{m} U(t)z_j
\]

with \( z_j = \theta \) for \( j > k \) and \( z_k \neq \theta \), we have from (10) and (11) that

\[
\liminf_{t \to +\infty} |U(t + \omega)x| \exp \left( \int_0^t \mu_k[-A(s)] \, ds \right) \geq m^{-1} \|P_k\|^{-1} |z_k| > 0.
\]

This is obviously a contradiction to (9). Now suppose that \( z_k \neq \theta \) for some \( k < i \), and suppose also that \( k \) is the smallest number in \( \{1, \ldots, m\} \) such that \( z_k \neq \theta \). The argument in this case is completely analogous to the above case.
if we let \( t \to -\infty \) and use the estimates in part (ii) of Corollary 2. In particular, as above, it can be shown that

\[
\lim_{t \to -\infty} \left| U(t + \omega) x \right| \left/ \left| U(t) z_k \right| \right. = 0,
\]

but

\[
\liminf_{t \to -\infty} \left| \sum_{j=1}^{m} U(t) z_j \right| \left/ \left| U(t) z_k \right| \right. > 0,
\]

which is impossible. We can now conclude that \( U(\omega) x \in \mathcal{Z}_i \) for each \( x \in \mathcal{Z}_i \) and the assertions of Theorem 3 follow.

**Theorem 4.** Assume that the suppositions of Theorem 3 are fulfilled and let \( n_i \) denote the rank of \( P_i \) for each \( i \in \{1, \ldots, m\} \). Let \( \{\gamma_1, \ldots, \gamma_n\} \) denote the eigenvalues of \( U(\omega) \) (i.e., the multipliers of (LDE)), where each distinct eigenvalue is counted as many times as its multiplicity. Then there is a family \( \{\Gamma_i: i = 1, \ldots, m\} \) of mutually disjoint subsets of \( \{1, \ldots, n\} \) such that

(i) \( \Gamma_i \) contains exactly \( n_i \) elements and \( \bigcup_{i=1}^{m} \Gamma_i = \{1, \ldots, n\} \);

(ii) if \( i \) is in \( \{1, \ldots, m\} \) and \( j \) is in \( \Gamma_i \), then each eigenvalue \( x_j \) corresponding to \( \gamma_j \) is in \( \mathcal{Z}_i \); and

(iii) if \( i \) is in \( \{1, \ldots, m\} \) and \( j \) is in \( \Gamma_i \), then

\[
\exp \left( - \int_{0}^{\omega} \mu_i[-A(s)] \, ds \right) \leq \left| \gamma_j \right| \leq \exp \left( \int_{0}^{\omega} \mu_i[A(s)] \, ds \right).
\]

In particular, if \( \lambda_j \) is a characteristic exponent of (LDE) corresponding to the multiplier \( \gamma_j \), where \( j \) is in \( \Gamma_i \), then \( \text{Re}(\lambda_j) = -\omega^{-1} \ln(\left| \gamma_j \right|) \), and we have the estimate

\[
-\omega^{-1} \int_{0}^{\omega} \mu_i[-A(s)] \, ds \leq \text{Re}(\lambda_j) \leq \omega^{-1} \int_{0}^{\omega} \mu_i[A(s)] \, ds.
\]

**Proof.** By Theorem 3, \( U(\omega) \mathcal{Z}_i = \mathcal{Z}_i \) for each \( i \in \{1, \ldots, m\} \), and since the dimension of \( \mathcal{Z}_i \) is \( n_i \), it is immediate that there is a mutually disjoint family \( \{\Gamma_i: i = 1, \ldots, m\} \) such that the number of elements in \( \Gamma_i \) is \( n_i \) and assertions (i) and (ii) hold. Now let \( i \in \{1, \ldots, m\} \) and \( j \in \Gamma_i \). Let \( x_j \) be a nonzero member of \( \mathcal{Z}_i \) such that \( U(\omega) x_j = \gamma_j x_j \). Since \( Q \frac{d}{dt} x_j = x_j \) and \( U(\omega) x_j = \gamma_j x_j \), we have from part (v) of Theorem 2 (with \( t = \omega \)) that

\[
\left| x_j \right| \exp \left( - \int_{0}^{\omega} \mu_i[-A(s)] \, ds \right) \leq \left| \gamma_j \right| \left| x_j \right| \leq \left| x_j \right| \exp \left( \int_{0}^{\omega} \mu_i[A(s)] \, ds \right).
\]

(12)
Since $|x_j|_i \geq |x_j|_k$ for all $k \in \{1, \ldots, m\}$, we have that

$$|x_j| = \left| \sum_{k=1}^{m} P_k x_j \right| \leq \sum_{k=1}^{m} |x_j|_k \leq m |x_j|_i.$$  

Since $|x_j| > 0$, $|x_j|_i > 0$ and assertion (iii) is immediate from (12). All of the assertions of Theorem 4 are now seen to be valid.

5. Examples

In this section a few simple examples are given which illustrate when these techniques may apply, and also we connect these results with those of Lazer [3].

Let the member $A$ of $\mathcal{L}(\mathcal{H}^n)$ be associated with the $n \times n$ matrix $(a_{jk})_{1 \leq j, k \leq n}$, where $a_{jk} \in \mathcal{H}$, and let $\{\Gamma_i; i = 1, \ldots, m\}$ be a family of mutually disjoint, nonempty subsets of $\{1, \ldots, n\}$ such that $\bigcup_{i=1}^{m} \Gamma_i = \{1, \ldots, n\}$. For each $i$ in $\{1, \ldots, m\}$, let $P_i$ be associated with the diagonal $n \times n$ matrix $\text{diag}(p_i^j)_{1 \leq j \leq n}$, where $p_i^j = 1$ if $j \in \Gamma_i$ and $p_i^j = 0$ otherwise. Obviously the family $\{P_i; i = 1, \ldots, m\}$ is supplementary on $\mathcal{H}^n$.

**Proposition 3.** Suppose that $A$, $\{\Gamma_i; i = 1, \ldots, m\}$, and $\{P_i; i = 1, \ldots, m\}$ are as above and for each $x = (\xi_j)^n \in \mathcal{H}^n$, let $|x| = \max\{|\xi_j|; j = 1, \ldots, n\}$. Then, for each $i$ in $\{1, \ldots, m\}$,

(i) $|x|_i = \max\{|\xi_j|; j \in \Gamma_i\}$;

(ii) $\|A\|_i = \max \left\{ \sum_{k=1}^{n} |a_{jk}|; j \in \Gamma_i \right\}$; and

(iii) $\mu_i[A] = \max \left\{ \text{Re}(a_{jj}) + \sum_{k=1}^{n} |a_{jk}|; j \in \Gamma_i \right\}$.

**Proof.** Part (i) is trivial. Let $M_i$ denote the maximum in (ii) and note that $P_i A$ is the matrix $B = (b_{jk})$, where $b_{jk} = a_{jk}$ if $(j, k) \in \Gamma_i \times \{1, \ldots, n\}$ and $b_{jk} = 0$ otherwise. Now, if $x = (\xi_j)^n \in \mathcal{H}^n$ and $1 = |x|_i \geq |x|_j$ for $j = 1, \ldots, m$, then $|\xi_k| \leq 1$ for each $k$ in $\{1, \ldots, n\}$ and

$$|Ax|_i = |P_iAx| = \max \left\{ \sum_{k=1}^{n} b_{jk} \xi_k; j = 1, \ldots, n \right\}$$

$$= \max \left\{ \sum_{k=1}^{n} a_{jk} \xi_k; j \in \Gamma_i \right\}$$

$$\leq \max \left\{ \sum_{k=1}^{n} |a_{jk}|; j \in \Gamma_i \right\} = M_i.$$
Now, for each \( j \in \Gamma_i \) and \( k \in \{1, \ldots, n\} \), let \( \xi_{jk} = a_{jk}/a_{jk} \) if \( a_{jk} \neq 0 \) and \( \xi_{jk} = 1 \) if \( a_{jk} = 0 \). If \( x_j = (\xi_{jk})_{k=1}^n \), then \( 1 = |x_j| \geq |x_j|_k \) for each \( k \in \{1, \ldots, m\} \); so

\[
\|A\|_t \geq |Ax_j|_t = \left| P_i Ax_j \right| \geq \left| \sum_{k=1}^n a_{jk} \xi_{jk} \right| = \sum_{k=1}^n |a_{jk}|
\]

for each \( j \in \Gamma_i \). Thus, \( \|A\|_t \geq M_t \) and (ii) established. Using (ii) (with \( A \) replaced by \( I + hA \)), we have that

\[
\|I + hA\|_t - 1 = \max \left\{ 1 + ha_{jj} - 1 + h \sum_{k=1}^n |a_{jk}| : j \in \Gamma_i \right\}
\]

for each \( h > 0 \). Dividing each side of this equation by \( h \) and letting \( h \to 0^+ \) establishes (iii).

Note that, with the suppositions of Proposition 3, if \( x \in \mathbb{K}^n \) and \( |x|_t \geq |x|_j \) for each \( j \in \{1, \ldots, m\} \), then \( |x| = |x|_t \). Thus, in this case, Theorem 2 is also true with \( | \cdot |_t \) replaced by \( | \cdot | \). Furthermore, Proposition 3 gives the connection between the results presented here and those of Lazer [3], as can be seen by the following example.

**Example 1.** Suppose that the function \( A \) in (LDE) is associated with the \( n \times n \) matrix valued function \( t \mapsto (a_{jk}(t)) \) on \((-\infty, \infty)\). Let

\[
r_j(t) = \sum_{k=1}^n |a_{jk}(t)|
\]

for each \( t \in (-\infty, \infty) \) and \( j \in \{1, \ldots, n\} \). Also let \( c_j(t) = \text{Re}(a_{jj}(t)) - r_j(t) \) and \( d_j(t) = \text{Re}(a_{jj}(t)) + r_j(t) \) for all \( t \in (-\infty, \infty) \) and \( j \in \{1, \ldots, n\} \). For each \( j \in \{1, \ldots, n\} \), define

\[
S_j = \bigcup_{t \in (-\infty, \infty)} \{t\} \times [c_j(t), d_j(t)],
\]

where

\[
[c_j(t), d_j(t)] = \{ \gamma \in (-\infty, \infty) : c_j(t) \leq \gamma \leq d_j(t) \}.
\]

Let \( S = \bigcup_{j=1}^n S_j \) and, for each \( j \) in \( \{1, \ldots, n\} \), let \( C_j \) denote the maximal connected subset of \( S \) which contains \( S_j \). Let \( j_1 \) be an integer in \( \{1, \ldots, n\} \) such that \( c_{j_1}(0) = \min\{c_j(0) : j = 1, \ldots, n\} \), and let \( \Gamma_1 = \{ j : S_j \subset C_{j_1} \} \). Let \( j_2 \) be an integer in \( \{1, \ldots, n\} - \Gamma_1 \) such that \( c_{j_2}(0) = \min\{c_j(0) : j = 1, \ldots, n, j \notin \Gamma_1 \} \), and let \( \Gamma_2 = \{ j : S_j \subset C_{j_2} \} \). Continuing in this fashion, we see that there is a family \( \{ \Gamma_i : i = 1, \ldots, m \} \) of disjoint, nonempty subsets of \( \{1, \ldots, n\} \) such that
\[ \bigcup_{i=1}^{m} \Gamma_i = \{1, \ldots, n\} \] and, if \( m \geq 2 \) and \( j \in \Gamma_{i-1} \) and \( k \in \Gamma_i \), then \( d_j(t) < c_k(t) \) for all \( t \in (-\infty, \infty) \). Let \( P_i = \text{diag}(p_{ij}); j, i \leq n \), where \( p_{ij} = 1 \) if \( j \in \Gamma_i \) and \( p_{ij} = 0 \) otherwise. By Proposition 3, if \( |x| = \max\{|\xi_j|; j = 1, \ldots, n\} \) for each \( x = (\xi_j)_{j \in \mathbb{N}} \in \mathcal{X}^n \), then

\[-\mu_i[-A(t)] = \min\{c_j(t); j \in \Gamma_i\} \quad \text{and} \quad \mu_i[A(t)] = \max\{d_j(t); j \in \Gamma_i\}.

If \( m \geq 2 \), the suppositions of Theorem 2 are fulfilled, and, if \( A \) is \( \omega \)-periodic, those of Theorems 3 and 4 are fulfilled. (Professor A. C. Lazer has informed the author by letter that he has obtained a result similar to this example, but with the assumption that \( A \) is bounded.)

One should note that if, in Theorem 1, it is assumed that \( P_i A(t) = A(t) P_i \) for each \( t \) in \( (-\infty, \infty) \), then the condition (7) is not needed for the conclusions of Theorem 1 to be valid (take \( \mathcal{A}_i \) to be the range of \( P_i \)). However, if \( P_i \) does not commute with \( A(t) \), the conclusions of Theorem 1 are not necessarily valid without (7), as can be seen by the following simple example.

**Example 2.** Let

\[ n = 2, \quad P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad A(t) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \]

for all \( t \in (-\infty, \infty) \). If \( |(\xi_1, \xi_2)| = \max\{|\xi_1|, |\xi_2|\} \) for each \( (\xi_1, \xi_2) \in \mathcal{X}^2 \), then \(-\mu_1[-A(t)] = -1, \mu_1[A(t)] = 1, -\mu_2[-A(t)] = -4, \) and \( \mu_2[A(t)] = 4 \).

If \( u(t) = (u_1(t), u_2(t)) \) is a nontrivial solution to (LDE), then there are constants \( c_1 \) and \( c_2 \) such that \( c_1 \) and \( c_2 \) are not both zero, \( u_1(t) = c_1 e^{2t} + c_2 e^{-2t} \), and \( u_2(t) = u_1'(t) \). Hence, \( |u(t)|_1 = |u_1(t)|_1 \) cannot satisfy the inequality \( |u_1(0)| e^t \leq |u_1(t)|_1 \leq |u_1(0)| e^t \) for all \( t \in (-\infty, \infty) \) (or for all \( t \in [0, \infty) \)). Thus, the conclusions of Theorem 1 are not valid for \( i = 1 \).

One can also compute \( \|A\|_i \) and \( \mu_i[A] \) for other norms on \( \mathcal{X}^n \). For example, if each of the suppositions of Proposition 3 are fulfilled, except that for each \( x = (\xi_j)_{j \in \mathbb{N}} \in \mathcal{X}^n \) we let \( |x| = \sum_{j=1}^{n} |\xi_j| \), then it can be shown that

\[ |x|_i = \sum_{j \in \mathcal{A}_i} |\xi_j|; \quad \|A\|_i = \sum_{i=1}^{m} \max \left\{ \sum_{j \in \mathcal{A}_i} |a_{jk}|; k \in \Gamma_i \right\}; \]

and

\[ \mu_i[A] = \max \left\{ \text{Re}(a_{kk}) + \sum_{j \neq k} |a_{jk}|; k \in \Gamma_i \right\} + \sum_{i=1}^{m} \max \left\{ \sum_{j \in \mathcal{A}_i} |a_{jk}|; k \in \Gamma_i \right\}. \]

Note that in this case, if \( |x|_i \geq |x|_j \) for all \( j \in \{1, \ldots, m\} \), it does not necessarily follow that \( |x|_i = |x|_j \), although it is always true that \( |x|_i \leq |x|_j \).
Let us now point out how one may split a matrix $A$ into blocks in order to estimate $\| A \|_i$ and $\mu_i[A]$. Let $\{P_i : i = 1, \ldots, m\}$ be any family of supplementary projections, and let $\mathcal{H}_i$ denote the range of $P_i$ for each $i$ in $\{1, \ldots, m\}$. Now, for each $i$ in $\{1, \ldots, m\}$, let $\| \cdot \|_i^*$ denote the norm on $\mathcal{H}_i$ induced by the norm $\| \cdot \|$ on $\mathbb{R}^n$ (i.e., $\|x\|_i^* = \|x\|$ for all $x \in \mathcal{H}_i$). Since $P_i x = x$ for all $x \in \mathcal{H}_i$, we have that $\|x\|_i^* = \|x\|_i$ for all $x \in \mathcal{H}_i$. For each $i$ and $j$ in $\{1, \ldots, m\}$, the mapping $P_i A P_j$ maps $\mathcal{H}_j$ into $\mathcal{H}_i$. Define

$$\| P_i A P_j \|_{i,j}^* = \max \{\| P_i A P_j x \|_i^* : x \in \mathcal{H}_j \text{ and } \|x\|_j^* = 1\}.$$ 

Since $P_i$ is the identity mapping on $\mathcal{H}_i$, define

$$\mu_i^*[P_i A P_i] = \lim_{h \to 0^+} (\| P_i + hP_i A P_i \|_{i,i}^* - 1)/h.$$ 

**Proposition 4.** With the notations of the above paragraph, we have that, for each $i$ in $\{1, \ldots, m\}$,

(i) $\| A \|_i \leq \sum_{j=1}^m \| P_i A P_j \|_{i,j}^*$;

(ii) $\mu_i[A] \leq \mu_i^*[A] + \sum_{j \neq i} \| P_i A P_j \|_{i,j}^*$, and

(iii) $-\mu_i[-A] \geq -\mu_i^*[-A] - \sum_{j \neq i} \| P_i A P_j \|_{i,j}^*$.

**Proof.** Suppose that $x \in \mathbb{R}^n$ and $\|x\|_i \geq \|x\|_j$ for all $j \in \{1, \ldots, m\}$. Then

$$\| A x \|_i = \| P_i A x \| = \| P_i A \left[ \sum_{j=1}^m P_j \right] x \| \leq \sum_{j=1}^m \| P_i A P_j x \|_i^*$$

$$= \sum_{j=1}^m \| P_i A P_j (P_j x) \| \leq \sum_{j=1}^m \| P_i A P_j \|_{i,j}^* \| P_j x \|_j^*$$

$$= \sum_{j=1}^m \| P_i A P_j \|_{i,j}^* \| x \|_j \leq \left[ \sum_{j=1}^m \| P_i A P_j \|_{i,j}^* \right] \| x \|_i.$$

Part (i) now follows easily. Using part (i), it follows that for each $h > 0$,

$$\| I + hA \|_i - 1 \leq \sum_{j=1}^m \| P_i \cdot I \cdot P_j + hP_i A P_j \|_{i,j}^* - 1$$

$$= \| P_i + hP_i A P_i \|_{i,i}^* - 1 + \sum_{j \neq i}^{m} h \| P_i A P_j \|_{i,j}^*.$$
Part (ii) now follows by dividing each side of the above inequality by $h$ and letting $h \to 0^+$. Part (iii) is immediate from part (ii).

The above proposition shows that the results presented here are improvements to those of Martin [5, Proposition 4]. For an example of where these estimates may be easily applied, see [5, Example 4].

Now suppose that $A$ is in $\mathcal{L}(\mathcal{X}^n)$, $m \geq 2$, and $\mu_{i-1}[A] < -\mu_i[A]$ for each $i$ in $\{1, \ldots, m\}$. Let $\{Q_i: i = 1, \ldots, m\}$ be the family of supplementary projection guaranteed by Theorem 2 with $A(t) = A$ for all $t \in (-\infty, \infty)$. In this case, (LDE) is $\omega$-periodic for each $\omega > 0$, and it follows from Theorem 3 that $Q_i \exp(tA) = \exp(tA)Q_i$ for each $t \geq 0$ and $i \in \{1, \ldots, m\}$. Consequently, $Q_i A = AQ_i$ for each $i \in \{1, \ldots, m\}$. Furthermore, by Theorem 4, each eigenvalue $\lambda$ of $A$ satisfies $-\mu_i[-A] \leq \Re(\lambda) \leq \mu_i[A]$ for some $i \in \{1, \ldots, m\}$. It is easy to see that if $P_i$ commutes with $A$, then $Q_i = P_i$. Hence, we have that $Q_i = P_i$ iff $P_i$ commutes with $A$. However, there is always a connection between $P_i$ and $Q_i$, even when $A$ is time dependent. It follows from part (v) of Theorem 2 and part (i) of Corollary 2 that if $x \in \mathcal{X}^n$ and $Q_i x \neq 0$, then $P_i Q_i x \neq 0$.

Finally, let us note that, with the suppositions of Theorem 4, $Q_i$ computes with $U(t)$ for each $i$ in $\{1, \ldots, m\}$. However, it is not necessarily true that $Q_i$ commutes with $U(t)$ for all $t$ in $(-\infty, \infty)$, as can be seen by the following example.

**Example 3.** Let

\[
n = m = 2, \quad A(t) = \begin{bmatrix} 2 \cos t & 1 \\ 0 & -1 \end{bmatrix} \quad \text{for all } t \in (-\infty, \infty),
\]

\[
P_1 = \text{diag}(0, 1), \quad P_2 = \text{diag}(1, 0),
\]

and let $|x| = \max(|\xi_1|, |\xi_2|)$ for each $x = (\xi_1, \xi_2)$ in $\mathcal{X}^2$ with $\mathcal{X}$ the real field. By Proposition 3,

\[
-\mu_i[-A(t)] = \mu_i[A(t)] = -1, \quad -\mu_i[-A(t)] = 2 - |\cos t|,
\]

and

\[
\mu_i[A(t)] = 2 + |\cos t| \quad \text{for all } t \in (-\infty, \infty).
\]

In this case,

\[
U(t) = \begin{bmatrix} \exp(2t) & 3 \exp(2t)/10 + \exp(-t) (\sin t - 3 \cos t)/10 \\
0 & \exp(-t) \end{bmatrix}
\]

for all $t \in (-\infty, \infty)$, and

\[
Q_1 = \begin{bmatrix} 0 & -3/10 \\
0 & 1 \end{bmatrix} \quad Q_2 = \begin{bmatrix} 1 & 3/10 \\
0 & 0 \end{bmatrix}.
\]
It is easy to check that, for $i = 1$ or $2$, $Q_i U(t) = U(t) Q_i$ iff $t = 2k\pi$, where $k$ is an integer.

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