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The Meta-elliptical Distributions with Given Marginals[1]

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Based on an analysis of copulas of elliptically contoured distributions, joint densities of continuous variables with given strictly increasing marginal distributions

distribution. The new class of distributions is then called meta-elliptical distributions. The corresponding analytic forms of the density, conditional distribution functions, and dependence properties are derived. This new class of distributions has the same Kendall's rank correlation coefficient as meta-Gaussian distributions. As an extension of elliptically contoured distributions, some new classes of distributions are also obtained.

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1. INTRODUCTION

It is often required in multivariate analysis to construct a multivariate distribution from specified marginal distributions with a given dependence structure. Multivariate distributions with given marginals arise not only from statistical theory, but quite often from practical problems, especially in the context of risk and decision analysis for complex multivariate systems. Jouini and Clemen (1996) studied this kind of models for

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aggregating expert opinions and Frees *et al.* (1996) discussed an actuarial model for insurance applications. In recent years, many statisticians have studied multivariate distributions with specified structures (see, e.g., Dall'Aglio *et al.* (1991)).

Krzysztofowicz and Kelly (1996) constructed the so-called meta-Gaussian distributions using the embedding method. The construction of the meta-Gaussian distributions is based on the *copula* technique. Copulas are functions which join univariate uniform distribution functions to form multivariate distributions. They were initially introduced by Sklar (1959). A historical review and major developments can be found in Dall'Aglio (1991), Schweizer (1991), and Kotz (1997) among others. Utilizing copulas, Kotz and Seeger (1991) observed that the density function of a multivariate distribution can be decomposed into the *density weighting function* and the product of marginal densities.

In this paper, a wider class of continuous distributions with given marginals is discussed. In place of the normal distribution, elliptically contoured distributions are considered as the basic framework and meta-Gaussian distributions are extended to **meta-elliptical distributions**.

Definition 1.1. A n-dimensional random vector $\mathbf{z}$ is said to have an elliptically contoured distribution (or simply called elliptical distribution or ECD) with parameters $\boldsymbol{\mu}$ $(n \times 1)$ and $\boldsymbol{\Sigma}$ $(n \times n)$ if it has the stochastic representation

$$\mathbf{z} \stackrel{d}{=} \boldsymbol{\mu} + r\mathbf{A}\mathbf{u}, \tag{1.1}$$

where $r \geqslant 0$ is a random variable, $\mathbf{u}$ is uniformly distributed on the unit sphere in $\mathbb{R}^n$ and is independent of r, $\mathbf{A}$ is an $n \times n$ constant matrix such that $\mathbf{A}\mathbf{A}' = \boldsymbol{\Sigma}$ and the sign $\stackrel{d}{=}$ means that both sides of the equality have the same distribution. In particular, if r has a density, then the density of $\mathbf{z}$ is of the form

$$|\boldsymbol{\Sigma}|^{-\frac{1}{2}}\, g((\mathbf{z}-\boldsymbol{\mu})'\, \boldsymbol{\Sigma}^{-1}(\mathbf{z}-\boldsymbol{\mu})), \tag{1.2}$$

where $g(\cdot)$ is a scale function uniquely determined by the distribution of r. We shall use the notation $\mathbf{z} \sim EC_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$. When $g(x) = (2\pi)^{-n/2} \exp\{-x/2\}$, $\mathbf{z}$ has an n-dimensional normal distribution.

Throughout this paper, without the loss of generality, we shall consider only the case $\mathbf{z} \sim EC_n(\mathbf{0}, \mathbf{R}, g)$, where

$$\mathbf{R} = \{\rho_{ij}\colon \rho_{ii} = 1, -1 < \rho_{ij} < 1 \text{ for } i \neq j, \rho_{ij} = \rho_{ji};\, i, j = 1, \ldots, n\}. \tag{1.3}$$

In this case all the marginal distributions of $\mathbf{z}$ are identical with pdf

$$q_g(x) = \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_{x^2}^{\infty} (y - x^2)^{(n-1)/2-1} \, g(y) \, dy, \tag{1.4}$$

and the cdf

$$Q_g(x) = \frac{1}{2} + \frac{\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_0^x \int_{u^2}^{\infty} (y - u^2)^{(n-1)/2-1} \, g(y) \, dy \, du. \tag{1.5}$$

Let $\mathbf{x} = (X_1, X_2, ..., X_n)'$ be a random vector with each component X_i having a given continuous density $f_i(x_i)$ and the cumulative distribution $F_i(x_i)$. Let the random vector $\mathbf{z} = (Z_1, Z_2, ..., Z_n)' \sim EC_n(\mathbf{0}, \mathbf{R}, g)$. Suppose that

$$Z_i = Q_g^{-1}(F_i(X_i)), \qquad i = 1, 2, ..., n, \tag{1.6}$$

where Q_g^{-1} is the inverse of Q_g. Then, the Jacobian of the transformation is

$$J\{(z_1, ..., z_n)' \to (x_1, ..., x_n)'\} = \prod_{i=1}^{n} \frac{dz_i}{dx_i} = \prod_{i=1}^{n} \frac{f_i(x_i)}{q_g(Q_g^{-1}(F_i(x_i)))}, \tag{1.7}$$

and the density function of $\mathbf{x}$ is given by

$$h(x_1, ..., x_n) = \phi(Q_g^{-1}(F_1(x_1)), ..., Q_g^{-1}(F_n(x_n)) \prod_{i=1}^{n} f_i(x_i), \tag{1.8}$$

where ϕ is the n-variate density weighting function

$$\phi(z_1, ..., z_n) = |\mathbf{R}|^{-\frac{1}{2}} \, g(\mathbf{z}'\mathbf{\Sigma}^{-1}\mathbf{z}) \Big/ \prod_{i=1}^{n} q_g(z_i) \tag{1.9}$$

(see below).

DEFINITION 1.2. The n-dimensional random vector $\mathbf{x}$ is said to have a *meta-elliptical distribution*, if its density function is given by (1.8). Denote $\mathbf{x} \sim ME_n(\mathbf{0}, \mathbf{R}, g; F_1, ..., F_n)$. The function $\phi(Q_g^{-1}(F_1(x_1)), ..., Q_g^{-1}(F_n(x_n)))$ is referred to as the *density weighting function.*

The class of meta-elliptical distributions includes various distributions, such as elliptically contoured distributions, the meta-Gaussian distributions and various asymmetric distributions. The marginal distributions $F_i(\cdot)$ can be arbitrarily chosen. As it will be shown in Example 4.1 below, a class of new distributions which is so-called a *multivariate asymmetric t-distribution* can be constructed by setting the marginals of multivariate t-distribution to be univariate t-distributions with different degrees of freedom. Multivariate asymmetric t-distributions enjoy certain symmetry but the marginal degrees of freedom are different.

The remainder of this paper is organized as follows. Copulas of ECDs are first discussed in the next section. The pdf of a meta-elliptical distribution is determined by the marginal distributions and the density weighting function $\phi(Q_g^{-1}(x_1), \ldots, Q_g^{-1}(x_n))$, the latter being the copula density of an ECD. In Section 3, the structure and some relevant properties of meta-elliptical distributions are presented. Some applications of these new distributions are discussed in Section 4.

2. COPULAS OF BIVARIATE ELLIPTICAL DISTRIBUTIONS

In this section, we shall discuss copulas of elliptical distributions. For simplicity, we shall consider only the two-dimensional case. Suppose $\mathbf{z} = (Z_1, Z_2)' \sim EC_2(\mathbf{0}, \mathbf{R}, g)$, where

$$\mathbf{R} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \qquad -1 < \rho < 1, \tag{2.1}$$

and $\mathbf{z}$ has the density function

$$f(z_1, z_2; \rho) = \frac{1}{\sqrt{1-\rho^2}}\, g\left(\frac{z_1^2 + z_2^2 - 2\rho z_1 z_2}{\sqrt{1-\rho^2}}\right). \tag{2.2}$$

The marginal pdf and cdf of $\mathbf{z}$ are

$$\begin{aligned} q_g(z) &= \int_{z^2}^{\infty} (y - z^2)^{-1/2}\, g(y)\, dy \qquad \text{and} \\ Q_g(z) &= \frac{1}{2} + \int_{z^2}^{\infty} \arcsin\left(\frac{z}{\sqrt{y}}\right) g(y)\, dy, \end{aligned} \tag{2.3}$$

respectively. In view of the Sklar theorem (see Sklar (1959)), we have

THEOREM 2.1. *Let the bivariate vector $\mathbf{z} = (Z_1, Z_2)'$ possess an elliptically contoured distribution with the density function* (2.2). *Then the copula of Z_1 and Z_2 is given by*

$$C_{\mathbf{z}}(u, v) = \frac{1}{\sqrt{1-\rho^2}} \int_{-\infty}^{Q_g^{-1}(u)} \int_{-\infty}^{Q_g^{-1}(v)} g\left(\frac{x^2 + y^2 - 2\rho x y}{\sqrt{1-\rho^2}}\right) dx\, dy, \tag{2.4}$$

with the density

$$c_z(u, v) = \phi(Q_g^{-1}(u), Q_g^{-1}(v); \rho), \tag{2.5}$$

where

$$\phi(x, y; \rho) = f(x, y; \rho)/[q_g(x)\, q_g(y)]. \tag{2.6}$$

A proof follows directly from the definitions. To assess the correlation between two random variables, Spearman's correlation coefficient ρ_s, a measure of the average positive (and negative) quadrant dependence, is often utilized (see, e.g., Lovie (1995)).

THEOREM 2.2. *Suppose* $\mathbf{z} = (Z_1, Z_2)' \sim EC_2(\mathbf{0}, \mathbf{R}, g)$ *with pdf* (2.2). *Spearman's* ρ_s *between* Z_1 *and* Z_2 *is given by*

$$\rho_s = 12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_g(x)\, Q_g(y)\, f(x, y)\, dx\, dy - 3. \tag{2.7}$$

Proof. Since by definition

$$\rho_s = 12 \int_{[0,1]^2} \{C(u, v) - uv\}\, du\, dv$$

and from Nelsen (1998) or by direct calculation

$$\int_{[0,1]^2} C(u, v)\, du\, dv = \int_{[0,1]^2} uv\, dC(u, v),$$

(2.7) follows (2.5) and some elementary manipulations. ∎

Thus, Spearman's ρ_s in (2.7) depends not only on ρ, but also on the marginal cdf of $\mathbf{z}$. Since the marginal distribution functions of many elliptically contoured distributions can not be expressed in closed forms, ρ_s is usually obtained by means of numerical calculations. However, for a bivariate normal variable $\mathbf{z}$, Kruskal (1958) obtained a simple expression for Spearman's correlation coefficient: $\rho_s = (6/\pi) \arcsin(\rho/2)$.

Examples below provide copulas for some subclasses of elliptical distributions (see, e.g., Fang *et al.* (1990)). Figure 1 exhibits the form of different copula densities of these subclasses.

EXAMPLE 2.1. *Symmetric Kotz type distributions.* Let $\mathbf{z}$ be distributed according to a bivariate symmetric Kotz type distribution with density

$$g_1(x_1, x_2) = \frac{s r^{N/s} (x_1^2 + s_2^2 - 2\rho x_1 x_2)^{N-1}}{\pi \Gamma(N/s)(1-\rho^2)^{N-1/2}} \exp\left\{ -r \left(\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{1-\rho^2} \right)^s \right\}, \tag{2.8}$$

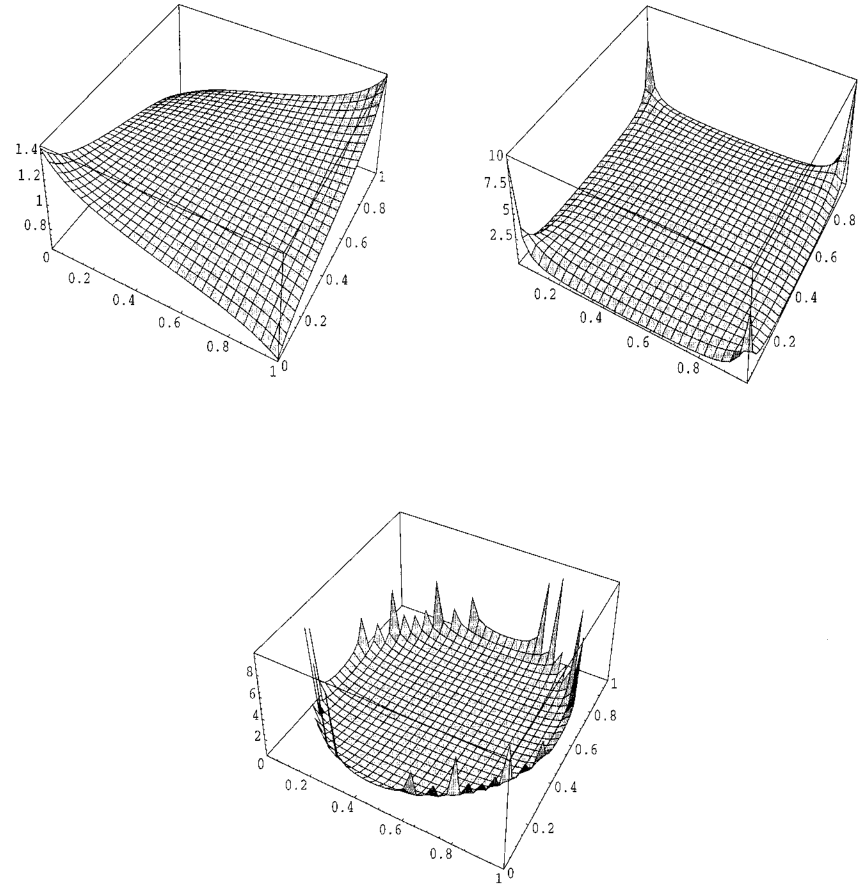

FIG. 1. Copula densities of elliptical distributions for $\rho = 0.1$. (a) Normal distribution. (b) t-distribution with $m = 2$. (c) Pearson type II distribution with $m = -\frac{1}{2}$.

where $r > 0$, $s > 0$, and $N > 0$ are parameters. The marginal pdf is

$$q_1(x) = \frac{2sr^{N/s}}{\pi \Gamma(N/s)} \int_0^\infty (t^2 + x^2)^{N-1} \exp\{-r(t^2+x^2)^s\}\, dt.$$

Denote the corresponding cdf by $Q_1(x)$. Then the density of copula is

$$c_1(u, v) = \frac{g_1(Q_1^{-1}(u), Q_1^{-1}(v))}{q_1(Q_1^{-1}(u))\, q_1(Q_1^{-1}(v))}. \tag{2.9}$$

When $N = s = 1$ and $r = \frac{1}{2}$, $\mathbf{z}$ possesses the bivariate normal distribution $N_2(0, \mathbf{R})$ and its copula density is given by Krzysztofowicz and Kelly (1996). ∎

EXAMPLE 2.2. *Symmetric bivariate Pearson type* VII *distributions.* If $\mathbf{z}$ has a symmetric bivariate Pearson type VII distribution, its density function is given by

$$g_2(x, y) = \frac{N-1}{\pi m \sqrt{1-\rho^2}} \left\{1 + \frac{1}{m(1-\rho^2)} (x^2 + y^2 - 2\rho xy)\right\}^{-N}, \quad (2.10)$$

where $N > 1$, and $m > 0$ are parameters. When $N = m/2 + 1$, $\mathbf{z} \sim Mt_2(m, \mathbf{0}, \mathbf{R})$, the bivariate t-distribution with m degrees of freedom. It can be shown directly by integration that the marginal cdf of $\mathbf{z}$ is

$$Q_2(x) = \frac{\Gamma(N-1/2)}{\sqrt{\pi m}\, \Gamma(N-1)} \int_{-\infty}^{x} (1 + y^2/m)^{-(N-1/2)}\, dy.$$

The density of copula is then $c_2(u, v) = b_2(Q_2^{-1}(u), Q_2^{-1}(v))$, where

$$b_2(x, y) = \frac{\Gamma(N-1)\, \Gamma(N)}{[\Gamma(N-1/2)]^2} \frac{(1 + x^2/m)^{(N-1/2)} (1 + y^2/m)^{(N-1/2)}}{\sqrt{1-\rho^2}\, \{1 + 1/(m(1-\rho^2))(x^2 + y^2 - 2\rho xy)\}^N}. \quad (2.11)$$

EXAMPLE 2.3. *Symmetric bivariate Pearson type* II *distributions.* If $\mathbf{z}$ has a symmetric bivariate Pearson type II distribution, its density is given by

$$g_3(x, y) = \begin{cases} \dfrac{(m+1)}{\pi \sqrt{1-\rho^2}} \left[1 - \dfrac{x^2 + y^2 - 2\rho xy}{(1-\rho^2)}\right]^m, & \text{for} \quad (x, y)\, \mathbf{R}^{-1}(x, y)' \leqslant 1, \\ 0, & \text{otherwise}, \end{cases} \quad (2.12)$$

where $m > -1$. The marginal cdf of $\mathbf{z}$ is

$$Q_3(x) = \frac{\Gamma(m+2)}{\sqrt{\pi}\, \Gamma(m+3/2)} \int_{-1}^{x} (1 - y^2)^{m+1/2}\, dy, \qquad \text{for} \quad |x| \leqslant 1.$$

The density of copula is then $c_3(u, v) = b_3(Q_3^{-1}(u), Q_3^{-1}(v))$, where

$$b_3(x, y) = \frac{(m+1)[\Gamma(m+3/2)]^2}{[\Gamma(m+2)]^2} \frac{[1 - (1/(1-\rho^2))(x^2 + y^2 - 2\rho xy)]^m}{\sqrt{1-\rho^2}\, (1 - x^2)^{m+1/2} (1 - y^2)^{m+1/2}}. \quad (2.13)$$

3. META-ELLIPTICAL DISTRIBUTIONS

Meta-elliptical distributions have been defined in Section 1. In this section we will discuss some of their properties. First, two-dimensional meta-elliptical distributions will be studied. Some properties can straighforwardly be extended to the multivariate case. Let $\mathbf{x} = (X_1, X_2)' \sim ME_2(\mathbf{0}, \mathbf{R}, g; F_1, F_2)$ with density

$$h(x_1, x_2; \rho) = \phi(Q_g^{-1}(F_1(x_1)), Q_g^{-1}(F_2(x_2)); \rho)\, f_1(x_1)\, f_2(x_2), \quad (3.1)$$

where ϕ is as defined in (2.6), and F_i and f_i are the cdf and pdf of X_i, respectively, $i = 1, 2$.

3.1. *Kendall's Correlation Coefficient*

Since the expression of Spearman's correlation coefficient ρ_s is somewhat complicated for elliptically contoured distributions, it is more expedient to consider Kendall's correlation coefficient τ. Indeed, Kendall's τ is a measure of average total positivity (and reverse regularity) of order two (see, e.g., Nelsen (1992, 1998)). It is the difference between the probabilities of concordance and discordance for two independent and identically distributed pairs of random variables. Let $\mathbf{x}_i = (X_{1i}, X_{2i})'$ and $\mathbf{x}_j = (X_{1j}, X_{2j})'$ be two independent and identically distributed pairs of random vectors. Kendall's τ of X_1 and X_2 can be written as (see, e.g., Kruskal (1958))

$$\begin{aligned} \tau = & P[(X_{1i} < X_{1j}, X_{2i} < X_{2j}) \text{ or } (X_{1i} > X_{1j}, X_{2i} > X_{2j})] \\ & - P[(X_{1i} < X_{1j}, X_{2i} > X_{2j}) \text{ or } (X_{1i} > X_{1j}, X_{2i} < X_{2j})]. \end{aligned} \quad (3.2)$$

THEOREM 3.1. *Let $\mathbf{x} = (X_1, X_2)$ have a meta-elliptical distribution with the density given in* (3.1). *Then Kendall's τ of X_1 and X_2 is given by*

$$\tau = \frac{2}{\pi} \arcsin(\rho), \quad (3.3)$$

which depends only on ρ, and is invariant in the class of meta-elliptical distributions $ME_2(\mathbf{0}, \mathbf{R}, g; F_1, F_2)$.

Proof. Denote $\tau(\mathbf{x})$ the right side in (3.2). Then $\tau(\alpha\mathbf{x}) = \tau(\mathbf{x})$ for all $\alpha > 0$. Since $\tau(\mathbf{x})$ is determined by the copula of $\mathbf{x}$, we can assume that $\mathbf{x} \in EC_2(\mathbf{R})$. By Theorem 2.22 in Fang *et al.* (1990) the assertion follows from Kruskal (1958). ∎

For the given marginal distributions $F_1(x_1)$ and $F_2(x_2)$, X_1 and X_2 result in perfect positive dependence or perfect negative dependence if we take $\rho \to +1$ or $\rho \to -1$, respectively. As $\rho \to 1$, the total probability mass under ϕ is concentrated on the main diagonal line $z_1 = z_2$; hence $H(x_1, x_2)$, the joint distribution of X_1 and X_2, attains the upper Fréchet bound $H^*(x_1, x_2) = \min\{F_1(x_1), F_2(x_2)\}$. As $\rho \to -1$, the total probability mass under ϕ becomes concentrated on the line $z_1 = -z_2$; hence $H(x_1, x_2)$ attains the lower Fréchet bound $H_*(x_1, x_2) = \max\{0, F_1(x_1) + F_2(x_2) - 1\}$.

3.2. *Conditional Distribution*

The conditional density of X_1 given $X_2 = x_2$, denoted by $k(x_1 \mid x_2)$, follows directly from (3.1):

$$k(x_1 \mid x_2) = \phi(Q_g^{-1}(F_1(x_1)), Q_g^{-1}(F_2(x_2)); \rho)\, f_1(x_1). \tag{3.4}$$

Hence, the corresponding conditional distribution $K(x_1 \mid x_2)$ is given by

$$K(x_1 \mid x_2) = \frac{1}{2} + \xi\left(Q_g^{-1}(F_2(x_2)), \frac{Q_g^{-1}(F_1(x_1)) - \rho Q_g^{-1}(F_2(x_2))}{\sqrt{1-\rho^2}}\right), \tag{3.5}$$

where

$$\xi(a, b) = \frac{\int_0^b g(y^2 + a^2)\, dy}{\int_0^\infty g(y^2 + a^2)\, dy}.$$

Denote by $x_1(p \mid x_2)$ the conditional p-quantile of X_1 given $X_2 = x_2$. Explicitly, $K(x_1(p \mid x_2) \mid x_2) = p$. For $p = 0.5$, we have from (3.5)

$$\xi\left(Q_g^{-1}(F_2(x_2)), \frac{Q_g^{-1}(F_1(x_1)) - \rho Q_g^{-1}(F_2(x_2))}{\sqrt{1-\rho^2}}\right) = 0.$$

Since $\xi(a, b) = 0$ implies that $b = 0$, it follows that

$$Q_g^{-1}(F_1(x_1(0.5 \mid x_2))) - \rho Q_g^{-1}(F_2(x_2)) = 0,$$

i.e.,

$$x_1(0.5 \mid x_2) = F_1^{-1}\{Q_g[\rho Q_g^{-1}(F_2(x_2))]\}. \tag{3.6}$$

The plot of $x_1(0.5 \mid x_2)$ versus x_2 defines the conditional median of X_1, given $X_2 = x_2$. The conditional median $x_1(0.5 \mid x_2)$ is a monotone increasing function of x_2 if $\rho > 0$, and it is a monotone decreasing function of x_2 if $\rho < 0$. Moreover, we have the following Theorem 3.2 which is similar to one in the case of meta-Gaussian distributions (see Krzysztofowicz and Kelly (1996)).

THEOREM 3.2. *Let (X_1, X_2) follow a meta-elliptical distribution with the marginal distributions $F_1(x_1)$ and $F_2(x_2)$. Then the graph $\{(x_1(0.5 \mid x_2), x_2)\}$ possesses the following properties:*

(i) *For $\rho \geqslant 0$, the graph is strictly increasing and lies within the region:*

$$\begin{aligned} F_1^{-1}(F_2(x_2)) < x_1(0.5 \mid x_2) < F_1^{-1}(0.5) &\quad \text{for} \quad x_2 < F_2^{-1}(0.5), \\ x_1(0.5 \mid x_2) = F_1^{-1}(0.5) &\quad \text{for} \quad x_2 = F_2^{-1}(0.5), \\ F_1^{-1}(0.5) < x_1(0.5 \mid x_2) < F_1^{-1}(F_2(x_2)) &\quad \text{for} \quad x_2 > F_2^{-1}(0.5). \end{aligned} \tag{3.7}$$

(ii) *For $\rho \leqslant 0$, the graph is strictly increasing and lies within the region:*

$$\begin{aligned} F_1^{-1}(0.5) < x_1(0.5 \mid x_2) < F_1^{-1}(1 - F_2(x_2)) &\quad \text{for} \quad x_2 < F_2^{-1}(0.5). \\ x_1(0.5 \mid x_2) = F_1^{-1}(0.5) &\quad \text{for} \quad x_2 = F_2^{-1}(0.5), \\ F_1^{-1}(1 - F_2(x_2)) < x_1(0.5 \mid x_2) < F_1^{-1}(0.5) &\quad \text{for} \quad x_2 > F_2^{-1}(0.5). \end{aligned} \tag{3.8}$$

A "judgmental" estimation procedure of ρ can be developed based on Theorem 3.2. For details the reader is referred to Krzysztofowicz and Kelly (1996).

3.3. *Forms of Dependence*

Some stronger forms of dependence than the classical orthant dependence (Lehmann (1966)) are regression dependence and likelihood ratio dependence (see, e.g., Barlow and Proschan (1975), Krzysztofowicz and Kelly (1996)).

THEOREM 3.3. *Let $(X_1, X_2)' \sim ME_2(\mathbf{0}, \mathbf{R}, g; F_1, F_2)$. Then, X_1 and X_2 are positively likelihood ratio dependent if $\rho \geqslant 0$ and are negatively likelihood ratio dependent if $\rho \leqslant 0$.*

Proof. Let the density of (X_1, X_2) be $h(x_1, x_2)$. If for all $x_1 < x_1'$ and $x_2 < x_2'$,

$$h(x_1', x_2)\, h(x_1, x_2') \leqslant h(x_1, x_2)\, h(x_1', x_2')$$

(namely, the joint density h is totally positive of order two), then by definition, X_1 and X_2 are positively likelihood ratio dependent. Since any strictly increasing transformation of the variates does not affect the positive (negative) likelihood ratio dependence, the theorem follows directly from the properties of elliptical distributions. ∎

THEOREM 3.4. *Let $(X_1, X_2)' \sim ME_2(\mathbf{0}, \mathbf{R}, g; F_1, F_2)$. X_1 is positively (negatively) regression dependent on X_2 if and only if the conditional quantile $x_1(p \mid x_2)$ is non-decreasing (non-increasing) in x_2 for all p, $0 < p < 1$.*

The proof of the theorem is similar to that of Theorem 2 of Krzysztofowicz and Kelly (1996).

3.4. *Multivariate Case*

Let $\mathbf{x} = (X_1, X_2, ..., X_n)'$ follow a meta-elliptical distribution with the density given by (1.8). Following Theorem 3.1, the Kendall's correlation coefficient between X_i and X_j is

$$\tau_{ij} = \frac{2}{\pi} \arcsin(\rho_{ij}), \qquad \text{for} \quad i \neq j, \qquad i, j = 1, 2, ..., n.$$

Thus, for fixed values of all the variates except X_i and X_j, $i \neq j$, the variates (X_i, X_j) are positively dependent if $\rho_{ij} \geqslant 0$, and are negatively dependent if $\rho_{ij} \leqslant 0$.

Denote by $k_i(x_i \mid x_1, ..., x_{i-1})$ the conditional density of X_i given $X_1, ..., X_{i-1}$, and by $k_1(x_1 \mid x_0) = f_1(x_1)$ the marginal density of X_1. Then the joint density of $\mathbf{x}$ can be written as

$$h(x_1, ..., x_n) = \prod_{i=1}^{n} k_i(x_i \mid x_1, ..., x_{i-1}). \tag{3.9}$$

Suppose that $\mathbf{z} \sim EC_n(\mathbf{0}, \mathbf{R}, g)$ and its i-dimensional marginal density is $g_i(z_1, ..., z_i)$. Set

$$\phi_i(z_1, ..., z_i) = g_i(z_1, ..., z_i) \Bigg/ \prod_{j=1}^{i} q_g(z_j).$$

We then have

$$k_i(x_i \mid x_1, ..., x_{i-1}) = \phi_i(Q_g^{-1}(F_1(x_1)), ..., Q_g^{-1}(F_i(x_i)))\, f_i(x_i). \tag{3.10}$$

Since the i-dimensional marginals are also elliptically contoured distributions, $\phi_i(Q_g^{-1}(F_1(x_1)), ..., Q_g^{-1}(F_i(x_i)))$ is a density weighting function for i-dimensional meta-elliptical distributions.

4. APPLICATIONS

Being an extension of elliptical distributions, meta-elliptical distributions can widely be applied in various fields. In particular, many new classes of multivariate distributions can be constructed. In this section, we propose one as an indicative example. A Monte Carlo generator of meta-elliptical distributions is also suggested in this section.

4.1. *Various Alternative Hypotheses*

For a given set of marginal distributions, we can construct various meta-elliptical distributions. In particular, some new classes of distributions can be obtained by changing the marginal parameters of elliptical distributions. Since parameters in marginal distributions govern the behavior of the tail probabilities of marginal distributions, the tail probabilities in different directions for the new distribution may be different. In Example 4.1 below, a multivariate asymmetric t-distribution demonstrates this property. The new distributions can be utilized to provide various alternative hypotheses for testing elliptical distributions.

EXAMPLE 4.1. *Multivariate asymmetric t-distributions.* Let $q_m(x)$ and $Q_m(x)$ be the density and distribution functions of the t-distribution

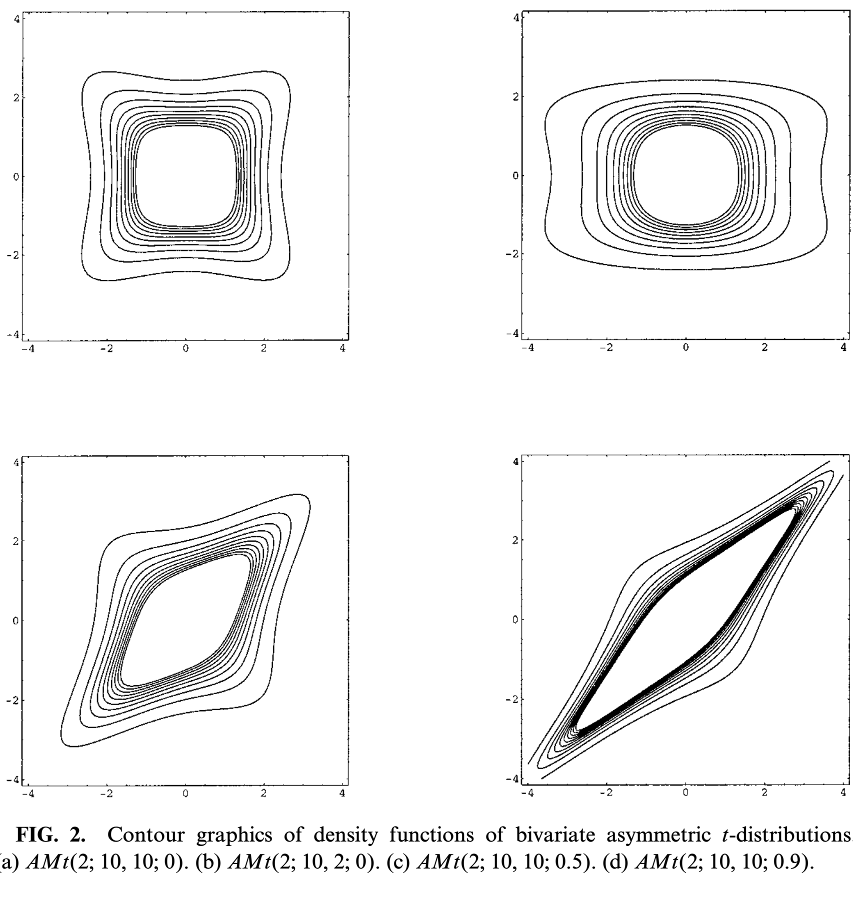

FIG. 2. Contour graphics of density functions of bivariate asymmetric t-distributions. (a) $AMt(2; 10, 10; 0)$. (b) $AMt(2; 10, 2; 0)$. (c) $AMt(2; 10, 10; 0.5)$. (d) $AMt(2; 10, 10; 0.9)$.

with m degrees of freedom, respectively. Let $\mathbf{x} = (X_1, X_2, \ldots, X_n)'$ be an n-dimensional random vector. If $\mathbf{x}$ has the joint density

$$f(x_1, \ldots, x_n) = \phi(Q_m^{-1}(Q_{m_1}(x_1)), \ldots, Q_m^{-1}(Q_{m_1}(x_n)); \mathbf{R}) \cdot \prod_{i=1}^{n} q_{m_i}(x_i), \tag{4.1}$$

where

$$\phi(y_1, \ldots, y_n) = \frac{\Gamma((m+n)/2)[\Gamma(m/2)]^{n-1}}{[\Gamma((m+1)/2)]^n} |\mathbf{R}|^{-1/2} \left(1 + \frac{\mathbf{y}'\mathbf{R}^{-1}\mathbf{y}}{m}\right)^{-\frac{m+n}{2}} \prod_{i=1}^{n} \left(1 + \frac{y_i^2}{m}\right)^{\frac{m+1}{2}},$$

$\mathbf{y} = (y_1, \ldots, y_n)'$ and $\mathbf{R}$ is given by (1.3), we then call $\mathbf{x}$ to have an n-dimensional asymmetric t-distribution with $(m; m_1, m_2, \ldots, m_n)$ degrees of freedom, and use the notation $\mathbf{x} \sim AMt_n(m; m_1, m_2, \ldots, m_n; \mathbf{0}, \mathbf{R})$.

It is of interest to note that $AMt_n(m; m_1, m_2, \ldots, m_n; \mathbf{0}, \mathbf{R})$ and $Mt_n(m; \mathbf{0}, \mathbf{R})$ possess the same copulas. The marginals of $\mathbf{x}$ have t-distributions with different degrees of freedom. When $m = m_i$, $i = 1, 2, \ldots, n$, $\mathbf{x}$ has an n-dimensional t-distribution with m degrees of freedom (see, e.g., Fang *et al.* (1990)). Figure 2 shows the densities of asymmetric t-distributions with different marginal degrees of freedom and correlation coefficients.

EXAMPLE 4.2. *Multimodal distributions.* Some multimodal meta-elliptical distributions can be constructed using the methodology proposed. As an example, consider the meta-elliptical distribution (X, Y) with its copula density given in Example 2.2 for $N = m = 2$.

(i) Let X be distributed according to a mixture of Weibull distributions with the density given by

$$f(x) = \tfrac{3}{2} x^2 \exp\{-x^3\} I_{\{x>0\}} + \tfrac{5}{2}(x-1)^4 \exp\{-(x-1)^5\} I_{\{x>1\}},$$

and Y be distributed according to the symmetric Beta distribution *Beta*(2, 2). Then (X, Y) has a bimodal meta-elliptical distribution.

(ii) Let X be as in (i) and Y be distributed according to a mixture of Beta distributions with density given by

$$g(y) = 3y(1-y) I_{\{0<y<1\}} + 3(y-1)(2-y) I_{\{1<y<2\}}.$$

Then, (X, Y) has a four-modal meta-elliptical distribution.

Figure 3 exhibits these distributions with $\rho = 0.1$. For absolutely continuous variables, this seems to be the most natural example of a multimodal bivariate distribution available in the literature.

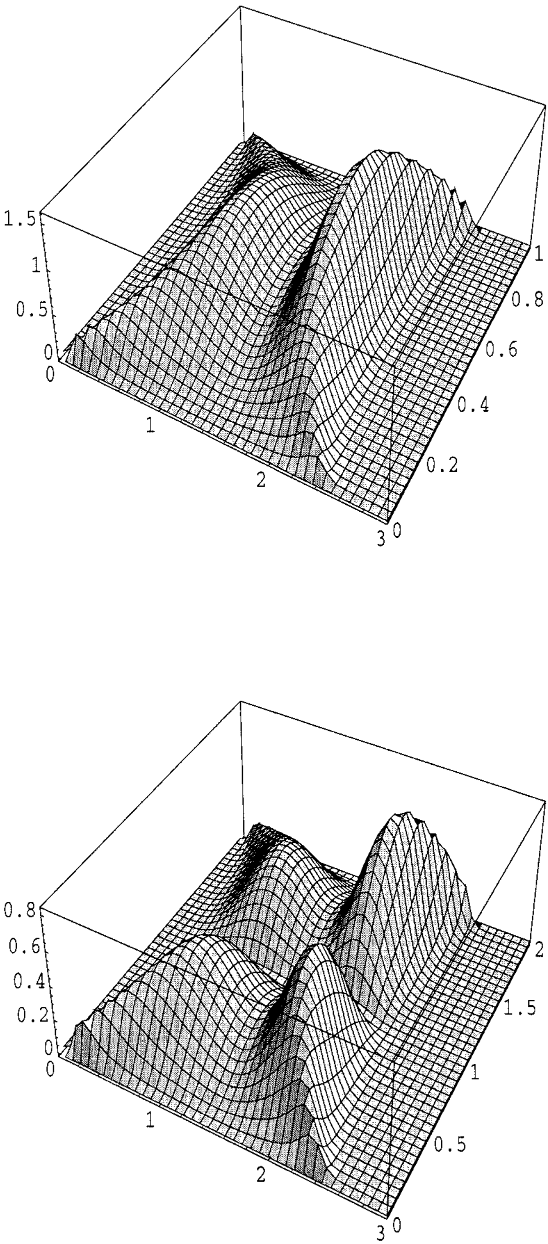

FIG. 3. Bivariate meta-elliptical density (X, Y) with specified marginal densities. The framework is 2-dimensional t-distribution with 2 degrees of freedom with $\rho = 0.1$. (a) $X \sim \frac{1}{2} W(3, 0, 1) + \frac{1}{2} W(5, 1, 1)$, $Y \sim Beta(2, 2)$. (b) $X \sim \frac{1}{2} W(3, 0, 1) + \frac{1}{2} W(5, 1, 1)$, $Y \sim \frac{1}{2} Beta(2, 2; 0) + \frac{1}{2} Beta(2, 2; 1)$.

4.2. *Monte Carlo Simulation*

Johnson (1987) provided a comprehensive discussion of generation of elliptical distributions while Fang and Wang (1994) developed the number theoretical approach to this problem. The following algorithm is based on the approach of Fang and Wang.

Let it be required to generate a meta-elliptical distribution $ME_n(\mathbf{0}, \mathbf{R}, g; F_1, ..., F_n)$. Let $F_R(r)$ be the associated distribution of R with the desity generator g in the $EC_n(\mathbf{0}, \mathbf{R}, g)$. Fang and Wang's algorithm (1994) adapted to our situation involves the following steps:

(i) Generate $\mathbf{u} = (u_1, ..., u_n)$ from the uniform distribution on n-dimensional unit sphere by the algorithm given in Section 4.3 of Fang and Wang (1994) (or any other algorithm);

(ii) Generate $R \sim F_R(r)$;

(iii) Let $w_i = Ru_i$, $i = 1, ..., n$, then $(w_1, ..., w_n)' \sim EC_n(\mathbf{0}, \mathbf{R}, g)$;

(iv) Find $x_i = F_i^{-1}(Q_g(w_i))$, $i = 1, ..., n$. We then deliver $(x_1, x_2, ..., x_n)$ as a sample from the distribution $ME_n(\mathbf{0}, \mathbf{R}, g; F_1, ..., F_n)$.

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