Curvature from quantum deformations

Angel Ballesteros [a], Francisco J. Herranz [a], Orlando Ragnisco [b]

[a] *Departamento de Física, Universidad de Burgos, Pza. Misael Bañuelos s.n., E-09001 Burgos, Spain*
[b] *Dipartimento di Fisica, Università di Roma Tre and Instituto Nazionale di Fisica Nucleare Sezione di Roma Tre, Via Vasca Navale 84, I-00146 Roma, Italy*

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Abstract

A Poisson coalgebra analogue of a (non-standard) quantum deformation of $sl(2)$ is shown to generate an integrable geodesic dynamics on certain 2D spaces of non-constant curvature. Such a curvature depends on the quantum deformation parameter z and the flat case is recovered in the limit $z \to 0$. A superintegrable geodesic dynamics can also be defined in the same framework, and the corresponding spaces turn out to be either Riemannian or relativistic spacetimes (AdS and dS) with constant curvature equal to z. The underlying coalgebra symmetry of this approach ensures the existence of its generalization to arbitrary dimension.

1. Introduction

Both theory and applications of quantum groups have certainly motivated an intensive effort aimed at understanding the role played by Hopf algebra deformations from many different viewpoints (see, for instance, [1–4]). In particular, Poisson–Hopf algebras

E-mail addresses: angelb@ubu.es (A. Ballesteros),
fjherranz@ubu.es (F.J. Herranz), ragnisco@fis.uniroma3.it
(O. Ragnisco).

(namely, Poisson–Lie groups and their associated Lie bialgebras and Drinfel'd doubles [5]) are just the Poisson counterparts of quantum groups and algebras. Recently, a systematic approach to the construction of integrable and superintegrable Hamiltonian systems from Poisson coalgebras has been introduced (see [6–8] and references therein). In this context, Poisson coalgebras associated to quantum groups can be understood as the dynamical symmetries that generate integrable deformations of well-known dynamical systems with an arbitrary number of degrees of freedom, but a clear geometrical interpretation of this

integrability-preserving deformation procedure was still lacking.

The aim of this Letter is to show a neat connection between two-dimensional (2D) spaces with non-constant curvature and Poisson coalgebra deformations. In particular, we will show that a certain class of q-Poisson coalgebras ($q = e^z$) generates in a very natural way a family of integrable geodesic motions on 2D manifolds with a (in general, non-constant) curvature K that turns out to be a function of the deformation parameter z. Moreover, such a curvature is directly generated by the "twisted" coproduct map of the deformed coalgebra. We also stress that, as a consequence of coalgebra symmetry, a straightforward generalization of this construction to arbitrary dimensions can be obtained, whose complete description will be presented elsewhere [9].

Let us briefly recall the basics of the construction of Hamiltonian systems with coalgebra symmetry by using the non-deformed Poisson coalgebra $(sl(2), \Delta)$, which is defined by the following Poisson brackets and coproduct map Δ:

$$\{J_3, J_+\} = 2J_+, \qquad \{J_3, J_-\} = -2J_-,$$
$$\{J_-, J_+\} = 4J_3, \tag{1.1}$$
$$\Delta(J_i) = J_i \otimes 1 + 1 \otimes J_i, \quad i = +, -, 3. \tag{1.2}$$

The Casimir function for this Poisson coalgebra is $\mathcal{C} = J_- J_+ - J_3^2$. A one-particle symplectic realization of (1.1) is given by

$$J_-^{(1)} = q_1^2, \qquad J_+^{(1)} = p_1^2 + \frac{b_1}{q_1^2}, \qquad J_3^{(1)} = q_1 p_1, \tag{1.3}$$

where b_1 is just the constant that labels the phase space realization: $\mathcal{C}^{(1)} = b_1$. The corresponding two-particle realization is obtained through the coproduct (1.2) by using one symplectic realization for each lattice site:

$$J_-^{(2)} = q_1^2 + q_2^2, \qquad J_+^{(2)} = p_1^2 + p_2^2 + \frac{b_1}{q_1^2} + \frac{b_2}{q_2^2},$$
$$J_3^{(2)} = q_1 p_1 + q_2 p_2. \tag{1.4}$$

Given any Hamiltonian function $\mathcal{H}$ on the generators of $(sl(2), \Delta)$, the coalgebra symmetry ensures that the associated two-body Hamiltonian $\mathcal{H}^{(2)} := \Delta(\mathcal{H}) = \mathcal{H}(J_-^{(2)}, J_+^{(2)}, J_3^{(2)})$ is integrable since the two-particle

Casimir

$$\mathcal{C}^{(2)} = \Delta(\mathcal{C})$$
$$= (q_1 p_2 - q_2 p_1)^2 + \left(b_1 \frac{q_2^2}{q_1^2} + b_2 \frac{q_1^2}{q_2^2}\right) + b_1 + b_2, \tag{1.5}$$

Poisson commutes with $\mathcal{H}^{(2)}$ with respect to the bracket $\{f, g\} = \sum_{i=1}^2 \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i}\right)$.

Some well-known (super)integrable Hamiltonian systems can be recovered as specific choices for $\mathcal{H}^{(2)}$. In particular, if we set

$$\mathcal{H} = \frac{1}{2} J_+ + \mathcal{F}(J_-), \tag{1.6}$$

where $\mathcal{F}$ is an arbitrary smooth function, we find the following family of integrable systems defined on the 2D Euclidean space $\mathbf{E}^2$:

$$\mathcal{H}^{(2)} = \frac{1}{2}\left(p_1^2 + p_2^2\right) + \frac{b_1}{2q_1^2} + \frac{b_2}{2q_2^2} + \mathcal{F}\left(q_1^2 + q_2^2\right). \tag{1.7}$$

The case $\mathcal{F}(J_-) = \omega^2 J_- = \omega^2(q_1^2 + q_2^2)$ gives rise to the 2D Smorodinsky–Winternitz system [10–12]. Obviously, the free motion on $\mathbf{E}^2$ is described by $\mathcal{H} = \frac{1}{2} J_+$. The N-body generalization of this construction follows from iteration of the coproduct map [7].

The very same coalgebra approach [6] leads to integrable deformations of (1.7) by considering deformations of $sl(2)$ coalgebras. This was the procedure applied in [7] through the Poisson analogue $(sl_z(2), \Delta_z)$ of the non-standard quantum deformation of $sl(2)$ [13]:

$$\{J_3, J_+\} = 2J_+ \cosh z J_-,$$
$$\{J_3, J_-\} = -2 \frac{\sinh z J_-}{z},$$
$$\{J_-, J_+\} = 4J_3, \tag{1.8}$$

$$\Delta_z(J_-) = J_- \otimes 1 + 1 \otimes J_-,$$
$$\Delta_z(J_i) = J_i \otimes e^{z J_-} + e^{-z J_-} \otimes J_i, \quad i = +, 3, \tag{1.9}$$

where $z = \ln q$ is the deformation parameter. Hereafter we shall assume that $z \in \mathbb{R}$. The deformed Casimir function reads

$$\mathcal{C}_z = \frac{\sinh z J_-}{z} J_+ - J_3^2. \tag{1.10}$$

Since in the following we shall consider only free motion, we restrict to the $b_i = 0$ case. Then, a one-particle symplectic realization of $sl_z(2)$ turns out to be [7]

$$J_-^{(1)} = q_1^2, \qquad J_+^{(1)} = \frac{\sinh zq_1^2}{zq_1^2} p_1^2,$$

$$J_3^{(1)} = \frac{\sinh zq_1^2}{zq_1^2} q_1 p_1, \qquad (1.11)$$

where $\mathcal{C}_z^{(1)} = 0$. Hence dimensions of z are $[z] = [q_1]^{-2} = [J_-]^{-1}$. Next, the deformed coproduct Δ_z provides the following two-particle symplectic realization of (1.8):

$$J_-^{(2)} = q_1^2 + q_2^2,$$

$$J_+^{(2)} = \frac{\sinh zq_1^2}{zq_1^2} e^{zq_2^2} p_1^2 + \frac{\sinh zq_2^2}{zq_2^2} e^{-zq_1^2} p_2^2,$$

$$J_3^{(2)} = \frac{\sinh zq_1^2}{zq_1^2} e^{zq_2^2} q_1 p_1 + \frac{\sinh zq_2^2}{zq_2^2} e^{-zq_1^2} q_2 p_2. \qquad (1.12)$$

Consequently, the two-particle Casimir given by

$$\mathcal{C}_z^{(2)} = \Delta_z(\mathcal{C}_z)$$
$$= \frac{\sinh zq_1^2}{zq_1^2} \frac{\sinh zq_2^2}{zq_2^2} e^{-zq_1^2} e^{zq_2^2} (q_1 p_2 - q_2 p_1)^2, \qquad (1.13)$$

is, by construction, a constant of the motion for any Hamiltonian $\mathcal{H}_z^{(2)} = \Delta_z(\mathcal{H}) = \mathcal{H}(J_-^{(2)}, J_+^{(2)}, J_3^{(2)})$.

Thus, by taking into account the explicit expressions (1.12), we find that the most general *integrable* (and quadratic in the momenta) deformation of the free motion on $\mathbf{E}^2$ with $(sl_z(2), \Delta_z)$-symmetry reads

$$\mathcal{H} = \frac{1}{2} J_+ f(zJ_-), \qquad (1.14)$$

where f is any smooth function such that $\lim_{z\to 0} f(zJ_-) = 1$ (note that $\lim_{z\to 0} J_+ = p_1^2 + p_2^2$). The simplest choice of (1.14) corresponds to taking $\mathcal{H} = \frac{1}{2} J_+$, namely

$$\mathcal{H}_z^{\mathrm{I}} = \frac{1}{2} \left(\frac{\sinh zq_1^2}{zq_1^2} e^{zq_2^2} p_1^2 + \frac{\sinh zq_2^2}{zq_2^2} e^{-zq_1^2} p_2^2 \right). \qquad (1.15)$$

On the other hand, a further analysis of (1.14) leads to a *superintegrable* deformation of the free Euclidean motion which is provided by $\mathcal{H} = \frac{1}{2} J_+ e^{zJ_-}$, that is,

$$\mathcal{H}_z^{\mathrm{S}} = \frac{1}{2} \left(\frac{\sinh zq_1^2}{zq_1^2} e^{zq_1^2} e^{2zq_2^2} p_1^2 + \frac{\sinh zq_2^2}{zq_2^2} e^{zq_2^2} p_2^2 \right). \qquad (1.16)$$

In this case, besides (1.13), there exists the additional constant of the motion [7]:

$$\mathcal{I}_z = \frac{\sinh zq_1^2}{2zq_1^2} e^{zq_1^2} p_1^2. \qquad (1.17)$$

As $\mathcal{C}_z^{(2)}$, $\mathcal{I}_z$ and $\mathcal{H}_z^{\mathrm{S}}$ are functionally independent functions, the latter is a (maximally) superintegrable Hamiltonian.

It becomes clear that (1.14) and, consequently, both Hamiltonians $\mathcal{H}_z^{\mathrm{I}}$ and $\mathcal{H}_z^{\mathrm{S}}$, can be interpreted as a deformed kinetic energy $\mathcal{T}_z(q_i, p_i)$ in such a way that a "deformation" of $\mathbf{E}^2$ arises from the dynamics. In fact, $\mathcal{T}_z(q_i, p_i) \to \mathcal{T}(p_i) = \frac{1}{2}(p_1^2 + p_2^2)$ under the limit $z \to 0$. In the sequel we shall unveil the "hidden" supporting spaces related to $\mathcal{T}_z$ coming from (1.15) and (1.16). In particular, in Section 2 we shall show that the Hamiltonian $\mathcal{H}_z^{\mathrm{I}}$ will give rise to integrable geodesic motions on 2D Riemannian spaces and $(1 + 1)$D relativistic spacetimes, all of them with a non-constant curvature governed by the deformation parameter z. Section 3 will be devoted to the Hamiltonian $\mathcal{H}_z^{\mathrm{S}}$, which will provide superintegrable geodesics on the sphere and hyperbolic spaces as well as on the (anti-) de Sitter and Minkowskian spacetimes with a curvature exactly equal to z. Some comments concerning other possible particular Hamiltonians contained in the family (1.14) (including the general expression for the associated curvature) and several open problems close the Letter.

2. Integrable deformation and non-constant curvature

The kinetic energy $\mathcal{T}_z^{\mathrm{I}}(q_i, p_i)$ coming from (1.15) can be rewritten as the free Lagrangian

$$\mathcal{T}_z^{\mathrm{I}}(q_i, \dot{q}_i) = \frac{1}{2} \left(\frac{zq_1^2}{\sinh zq_1^2} e^{-zq_2^2} \dot{q}_1^2 + \frac{zq_2^2}{\sinh zq_2^2} e^{zq_1^2} \dot{q}_2^2 \right), \qquad (2.1)$$

that defines a geodesic flow on a 2D Riemannian space with a definite positive metric with signature

diag$(+, +)$ given, up to a constant factor, by

$$ds^2 = \frac{2zq_1^2}{\sinh zq_1^2}e^{-zq_2^2}\,dq_1^2 + \frac{2zq_2^2}{\sinh zq_2^2}e^{zq_1^2}\,dq_2^2. \quad (2.2)$$

If we write the metric as $ds^2 = g_{11}(q_1, q_2)\,dq_1^2 + g_{22}(q_1, q_2)\,dq_2^2$, the Gaussian curvature K can be directly computed by using the formula [14]

$$K = \frac{-1}{\sqrt{g_{11}g_{22}}}\left\{\frac{\partial}{\partial q_1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial\sqrt{g_{22}}}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{11}}}{\partial q_2}\right)\right\}, \quad (2.3)$$

which gives a non-constant and *negative* curvature

$$K(q_1, q_2; z) = -z\sinh\left(z(q_1^2 + q_2^2)\right). \quad (2.4)$$

Thus the underlying 2D space is of hyperbolic type and endowed with a radial symmetry. We stress that the exponentials in the metric (2.2) are essential in order to obtain a non-vanishing curvature K. Such exponentials are indeed the ones appearing in the deformed coproduct (1.9), thus showing the direct connection between coproduct deformation and curved spaces.

Let us introduce a change of coordinates that incorporate, besides z, another parameter $\lambda_2 \neq 0$. In particular, we write $z = \lambda_1^2$ and consider a pair of new coordinates (ρ, θ) defined through the expressions

$$\cosh(\lambda_1\rho) = \exp\left\{z(q_1^2 + q_2^2)\right\},$$
$$\sin^2(\lambda_2\theta) = \frac{\exp\{2zq_1^2\} - 1}{\exp\{2z(q_1^2 + q_2^2)\} - 1}, \quad (2.5)$$

where both $\lambda_1 = \sqrt{z}$ and λ_2 can take either a real or a pure imaginary value. In this way, we will be able to rewrite the initial metric (2.2) as a family of six metrics on spaces with different signature and curvature. The geometrical meaning of (ρ, θ) can be appreciated by taking the first-order terms of the expansion in z of (2.5):

$$\rho^2 \simeq 2(q_1^2 + q_2^2), \qquad \sin^2(\lambda_2\theta) \simeq \frac{q_1^2}{q_1^2 + q_2^2}. \quad (2.6)$$

Therefore ρ can be interpreted as a radial coordinate and θ is a circular or hyperbolic angle for either a real or an imaginary λ_2, respectively. At this first-order level, the "Cartesian" coordinates would be $(x, y) = \sqrt{2}(q_2, q_1)$.

Under the transformation (2.5), the metric (2.2) takes a simpler form:

$$ds^2 = \frac{1}{\cosh(\lambda_1\rho)}\left(d\rho^2 + \lambda_2^2\frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2}d\theta^2\right)$$
$$= \frac{1}{\cosh(\lambda_1\rho)}ds_0^2. \quad (2.7)$$

Now, if we recall the description of the 2D Cayley–Klein (CK) spaces in terms of geodesic polar coordinates [15,16] (these spaces are parametrized by their constant curvature κ_1 and a second real parameter κ_2), we realize that ds_0^2 is just the metric of the CK spaces provided that we identify $z = \lambda_1^2 \equiv -\kappa_1$ and $\lambda_2^2 \equiv \kappa_2$. In particular, from (2.7) and by taking into account the admissible specializations of z and λ_2, we find the following underlying spaces:

- When λ_2 is real, we get a 2D deformed sphere $\mathbf{S}_z^2$ ($z < 0$), and a deformed hyperbolic or Lobachewski space $\mathbf{H}_z^2$ ($z > 0$).
- When λ_2 is imaginary, we obtain a deformation of the $(1 + 1)$D anti-de Sitter spacetime $\mathbf{AdS}_z^{1+1}$ ($z < 0$) and of the de Sitter one $\mathbf{dS}_z^{1+1}$ ($z > 0$).
- In the non-deformed case $z \to 0$, the Euclidean space $\mathbf{E}^2$ (λ_2 real) and Minkowskian spacetime $\mathbf{M}^{1+1}$ (λ_2 imaginary) are recovered.

Thus the "additional" parameter λ_2 governs the signature of the metric. The case $\lambda_2 = 0$, that we do not consider here, corresponds to Newtonian spacetimes endowed with a degenerate metric (see (2.7)).

In the new coordinates, the sectional (Gaussian) curvature reads

$$K(\rho) = -\frac{1}{2}\lambda_1^2\frac{\sinh^2(\lambda_1\rho)}{\cosh(\lambda_1\rho)}, \quad (2.8)$$

and the scalar curvature is just $2K(\rho)$. Hence the behaviour of the function $K(\rho)$ has a non-trivial dependence on the sign of the deformation parameter:

- If z is positive then λ_1 is a real number and $K(\rho)$ is always an increasing *negative* function that goes from $K = 0$ at the origin $\rho = 0$ up to $K \to -\infty$ when $\rho \to +\infty$.
- If z is negative then λ_1 is a pure imaginary number, and $K(\rho)$ is a periodic function with single poles at the points $|\lambda_1|\rho = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$, and

Table 1
Metric and sectional curvature of the underlying spaces for different values of the deformation parameter $z = \lambda_1^2$ and signature parameter λ_2

2D deformed Riemannian spaces	$(1+1)$D deformed relativistic spacetimes
• Deformed sphere $\mathbf{S}_z^2$	• Deformed anti-de Sitter spacetime $\mathbf{AdS}_z^{1+1}$
$z = -1$; $(\lambda_1, \lambda_2) = (i, 1)$	$z = -1$; $(\lambda_1, \lambda_2) = (i, i)$
$ds^2 = \dfrac{1}{\cos \rho}\big(d\rho^2 + \sin^2 \rho\, d\theta^2\big)$	$ds^2 = \dfrac{1}{\cos \rho}\big(d\rho^2 - \sin^2 \rho\, d\theta^2\big)$
$K = -\dfrac{\sin^2 \rho}{2\cos \rho}$	$K = -\dfrac{\sin^2 \rho}{2\cos \rho}$
• Euclidean space $\mathbf{E}^2$	• Minkowskian spacetime $\mathbf{M}^{1+1}$
$z = 0$; $(\lambda_1, \lambda_2) = (0, 1)$	$z = 0$; $(\lambda_1, \lambda_2) = (0, i)$
$ds^2 = d\rho^2 + \rho^2\, d\theta^2$	$ds^2 = d\rho^2 - \rho^2\, d\theta^2$
$K = 0$	$K = 0$
• Deformed hyperbolic space $\mathbf{H}_z^2$	• Deformed de Sitter spacetime $\mathbf{dS}_z^{1+1}$
$z = 1$; $(\lambda_1, \lambda_2) = (1, 1)$	$z = 1$; $(\lambda_1, \lambda_2) = (1, i)$
$ds^2 = \dfrac{1}{\cosh \rho}\big(d\rho^2 + \sinh^2 \rho\, d\theta^2\big)$	$ds^2 = \dfrac{1}{\cosh \rho}\big(d\rho^2 - \sinh^2 \rho\, d\theta^2\big)$
$K = -\dfrac{\sinh^2 \rho}{2\cosh \rho}$	$K = -\dfrac{\sinh^2 \rho}{2\cosh \rho}$

(double) zeros at $|\lambda_1|\rho = 0, \pi, 2\pi, \dots$. Then $K(\rho)$ is *negative* in the intervals $|\lambda_1|\rho \in (0, \frac{\pi}{2})$, $(\frac{3\pi}{2}, \frac{5\pi}{2}), \dots$ but it is *positive* when $|\lambda_1|\rho \in (\frac{\pi}{2}, \frac{3\pi}{2}), (\frac{5\pi}{2}, \frac{7\pi}{2}), \dots$.

In Table 1 we display the metric (2.7) and the sectional curvature (2.8) for the six particular spaces that arise according to (λ_1, λ_2) by considering the simplest values $\lambda_i \in \{1, i\}$. In the deformed Riemannian spaces ($\lambda_2 = 1$) the metric is always a positive definite one on $\mathbf{H}_z^2$, while on $\mathbf{S}_z^2$ this can be either a positive or negative definite metric in the intervals with $\rho \in (0, \frac{\pi}{2}), (\frac{3\pi}{2}, \frac{5\pi}{2}), \dots$ and $\rho \in (\frac{\pi}{2}, \frac{3\pi}{2}), (\frac{5\pi}{2}, \frac{7\pi}{2}), \dots$, respectively. Likewise, in the deformed spacetimes ($\lambda_2 = i$) we obtain a Lorentzian metric that keeps its global sign on $\mathbf{dS}_z^{1+1}$ but alternates it on $\mathbf{AdS}_z^{1+1}$ in the same previous intervals. The *contraction* $\lambda_1 \to 0$ ($z \to 0$) in each column of Table 1 gives either the proper Euclidean or the Minkowskian space as the limiting non-deformed/flat case; in the latter the "angle" θ is indeed a rapidity in units $c = 1$.

The metric (2.7) gives rise to the kinetic term of $\mathcal{H}_z^{\mathrm{I}}$ in the new coordinates (ρ, θ):

$$\mathcal{T}_z^{\mathrm{I}}(\rho, \theta; \dot\rho, \dot\theta)$$
$$= \frac{1}{2\cosh(\lambda_1\rho)}\left(\dot\rho^2 + \lambda_2^2 \frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2}\dot\theta^2\right). \quad (2.9)$$

It is a matter of straightforward computation to obtain the new Hamiltonian, that we define as $\tilde{H}_z^{\mathrm{I}}(\rho, \theta; p_\rho, p_\theta) = 2\mathcal{H}_z^{\mathrm{I}}(q_i, p_i)$; this reads

$$\tilde{H}_z^{\mathrm{I}} = \frac{1}{2}\cosh(\lambda_1\rho)\left(p_\rho^2 + \frac{\lambda_1^2}{\lambda_2^2 \sinh^2(\lambda_1\rho)}p_\theta^2\right). \quad (2.10)$$

The corresponding constant of the motion comes from (1.13). If we denote the new integral as $\tilde{C}_z(\rho, \theta; p_\rho, p_\theta) = 4\lambda_2^2\mathcal{C}_z^{(2)}(q_i, p_i)$ it can be shown that $\tilde{C}_z = p_\theta^2$ which, in turn, allows us to perform the usual reduction of (2.10) to the 1D (radial) Hamiltonian given by

$$\tilde{H}_z^{\mathrm{I}} = \frac{1}{2}\cosh(\lambda_1\rho)p_\rho^2 + \frac{\lambda_1^2\cosh(\lambda_1\rho)}{2\lambda_2^2 \sinh^2(\lambda_1\rho)}\tilde{C}_z. \quad (2.11)$$

The integration of the geodesic motion on all these spaces can be explicitly performed in terms of elliptic integrals, and it will be fully described elsewhere [9].

3. Superintegrable deformation and constant curvature

Let us consider now the superintegrable Hamiltonian (1.16). The free Lagrangian $\mathcal{T}_z^{\mathrm{S}}$ turns out to be

$$\mathcal{T}_z^S(q_i, \dot{q}_i) = \frac{1}{2}\left(\frac{zq_1^2}{\sinh zq_1^2} e^{-zq_1^2} e^{-2zq_2^2} \dot{q}_1^2 \right.$$
$$\left. + \frac{zq_2^2}{\sinh zq_2^2} e^{-zq_2^2} \dot{q}_2^2 \right). \tag{3.1}$$

Thus the associated metric is given by

$$ds^2 = \frac{2zq_1^2}{\sinh zq_1^2} e^{-zq_1^2} e^{-2zq_2^2}\, dq_1^2$$
$$+ \frac{2zq_2^2}{\sinh zq_2^2} e^{-zq_2^2}\, dq_2^2. \tag{3.2}$$

Remarkably enough, in this case the Gaussian curvature (obtained by applying (2.3)) is *constant* and coincides with the deformation parameter $K = z$.

Under the change of coordinates (2.5), $(q_1, q_2) \rightarrow (\rho, \theta)$, the metric (3.2) becomes

$$ds^2 = \frac{1}{\cosh^2(\lambda_1\rho)}\left(d\rho^2 + \lambda_2^2 \frac{\sinh^2(\lambda_1\rho)}{\lambda_1^2} d\theta^2 \right)$$
$$= \frac{1}{\cosh^2(\lambda_1\rho)} ds_0^2, \tag{3.3}$$

where ds_0^2 is again the metric of the 2D CK spaces. As these spaces are also of constant curvature, a further change of coordinates should allow us to reproduce exactly the CK metric. This can be achieved by introducing a new radial coordinate r as

$$r = \int_0^\rho \frac{dx}{\cosh(\lambda_1 x)}, \tag{3.4}$$

which for $\lambda_1 = 1$ is the Gudermannian function, while for $\lambda_1 = i$ is the lambda function [16,17]. By making use of the functional relations

$$\tanh\left(\lambda_1 \frac{\rho}{2} \right) = \tan\left(\lambda_1 \frac{r}{2} \right),$$
$$\cosh(\lambda_1\rho) = \frac{1}{\cos(\lambda_1 r)},$$
$$\sinh(\lambda_1\rho) = \tan(\lambda_1 r), \tag{3.5}$$

we finally obtain

$$ds^2 = dr^2 + \lambda_2^2 \frac{\sin^2(\lambda_1 r)}{\lambda_1^2} d\theta^2, \tag{3.6}$$

which is just the CK metric written in geodesic polar coordinates (r, θ) and provided that $z = \lambda_1^2 \equiv \kappa_1$ and

$\lambda_2^2 \equiv \kappa_2$ [16]. Note that in this case $z = \kappa_1$, in contrast with the previous section where $z = -\kappa_1$; this is due to the interchange between circular and hyperbolic trigonometric functions (see (3.5)) entailed by the definition (3.4). Notice also that in the limiting case $z \to 0$ the coordinate $\rho \to r$.

The kinetic energy in the new coordinates reads

$$\mathcal{T}_z^S(r, \theta; \dot{r}, \dot{\theta}) = \frac{1}{2}\left(\dot{r}^2 + \lambda_2^2 \frac{\sin^2(\lambda_1 r)}{\lambda_1^2} \dot{\theta}^2 \right). \tag{3.7}$$

The Hamiltonian $\mathcal{H}_z^S$ (1.16) and its constants of the motion $\mathcal{C}_z^{(2)}$ (1.13) and $\mathcal{I}_z$ (1.17) are expressed in canonical geodesic polar coordinates (r, θ) and momenta (p_r, p_θ) as

$$\tilde{H}_z^S = \frac{1}{2}\left(p_r^2 + \frac{\lambda_1^2}{\lambda_2^2 \sin^2(\lambda_1 r)} p_\theta^2 \right), \qquad \tilde{C}_z = p_\theta^2,$$
$$\tilde{I}_z = \left(\lambda_2 \sin(\lambda_2\theta) p_r + \frac{\lambda_1 \cos(\lambda_2\theta)}{\tan(\lambda_1 r)} p_\theta \right)^2, \tag{3.8}$$

where we have denoted $\tilde{H}_z^S = 2\mathcal{H}_z^S$, $\tilde{C}_z = 4\lambda_2^2 \mathcal{C}_z^{(2)}$ and $\tilde{I}_z = 4\lambda_2^2 \mathcal{I}_z$. Then $\tilde{H}_z^S$ is transformed into a "radial" 1D system from which the geodesic curves can be obtained [9]. Namely,

$$\tilde{H}_z^S = \frac{1}{2} p_r^2 + \frac{\lambda_1^2}{2\lambda_2^2 \sin^2(\lambda_1 r)} \tilde{C}_z. \tag{3.9}$$

4. Concluding remarks

Some short comments and remarks are in order. Firstly, we stress that have worked out the geometric interpretation of two outstanding representatives among the class of Hamiltonians (1.14). However, the general expression (1.14) comprises many other possible choices for a deformed kinetic energy and therefore for the "dynamical" generation of deformed spaces. A preliminary analysis can be performed by considering the general expression of the curvature coming from (1.14), which turns out to be

$$K(x) = \frac{z}{f(x)}\left(f(x)f'(x)\cosh x \right.$$
$$\left. + \left(f(x)f''(x) - f^2(x) - f'^2(x) \right)\sinh x \right), \tag{4.1}$$

where $x \equiv zJ_- = z(q_1^2 + q_2^2) = zq^2$, $f' = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$ and $f'' = \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}$. Now if we take, for instance, $\mathcal{H} = \frac{1}{2}J_+ \mathrm{e}^{z\alpha J_-}$, where α is an extra real parameter, we obtain the following Gaussian curvature

$$K(q^2; z) = z\mathrm{e}^{z\alpha q^2}\big(\alpha \cosh(zq^2) - \sinh(zq^2)\big), \quad (4.2)$$

with power series expansion in z given by

$$\begin{aligned} K(q^2; z) &= \alpha z + (\alpha^2 - 1)q^2 z^2 \\ &\quad + \frac{\alpha}{2}(\alpha^2 - 1)q^4 z^3 + o[z^4]. \end{aligned} \quad (4.3)$$

Thus the cases $\alpha = \pm 1$ are the only ones with constant K ($\alpha = 1$ is just $\mathcal{H}_z^{\mathrm{S}}$) and $\alpha \neq \pm 1$ defines a class of spaces with non-constant curvature that includes $\mathcal{H}_z^{\mathrm{I}}$ for $\alpha = 0$. Another example is provided by $\mathcal{H} = \frac{1}{2}J_+ \cosh(z\beta J_-)$ where β can be either a real or a pure imaginary parameter. The curvature and its power series expansion read

$$\begin{aligned} K(q^2; z) &= z\big(\beta^2 - \cosh^2(z\beta q^2)\big)\frac{\sinh(zq^2)}{\cosh(z\beta q^2)} \\ &\quad + z\beta \cosh(zq^2)\sinh(z\beta q^2) \\ &= (2\beta^2 - 1)q^2 z^2 - \frac{1}{6}(2\beta^4 - \beta^2 + 1)q^6 z^4 \\ &\quad + o[z^6], \end{aligned} \quad (4.4)$$

so that the underlying deformed spaces are always of non-constant curvature (for any β).

We also stress that the generalization to arbitrary dimensions can be readily obtained from the N-degrees of freedom symplectic realization of the $(sl_z(2), \Delta_z)$ coalgebra given in [7]. For example, the explicit form of the N-body kinetic energy arising from the Hamiltonian $\mathcal{H} = \frac{1}{2}J_+$ would be:

$$\begin{aligned} \mathcal{H}_z^{\mathrm{I},(N)} &= \frac{1}{2}\sum_{i=1}^N \frac{\sinh zq_i^2}{zq_i^2}p_i^2 \\ &\quad \times \exp\left(-z\sum_{k=1}^{i-1} q_k^2 + z\sum_{l=i+1}^N q_l^2\right). \end{aligned}$$

The exponentials coming from the coproduct are the objects that generate the non-vanishing sectional curvatures. The geometric characterization of the underlying N-dimensional curved spaces is under investigation [9].

On the other hand, by considering the deformation of the more general symplectic realization (1.3) with $b_i \neq 0$ [7], together with a Hamiltonian of the type (1.6), one could obtain the non-constant curvature analogues of the Smorodinsky–Winternitz potential. We also mention that the study of the free motion of a quantum mechanical particle on the curved spaces here introduced should provide a geometric interpretation of the quantum analogue of the Poisson algebra (1.8), which is just the non-standard quantum deformation of $sl(2)$ [13]. Work on these lines is in progress.

Finally, one could consider the Poisson algebra (1.8) as a Poisson–Lie structure on a dual group G^* with Lie algebra given by the dual of the Lie bialgebra cocommutator associated to the coproduct (1.9) [18]. In other words, (1.8) is a sub-Poisson coalgebra of the full canonical Poisson–Lie structure on the Drinfeld double $D_{ns}(su(2))$ associated to the non-standard quantum deformation of $sl(2)$ (as a real Lie group, $D_{ns}(su(2))$ was proven to be isomorphic to a $(2+1)$D Poincaré group [19]). Since σ-models related by Poisson–Lie T-duality are directly connected to canonical Poisson–Lie structures on Drinfel'd doubles [20–23], the construction of the σ-model associated to $D_{ns}(su(2))$ and its relationship with the results here presented could be worth to be considered.

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