ADVANCES IN MATHEMATICS 11, 326–338 (1973)

Combinatorial Structure on Triangulations. I. The Structure of Four Colorings

STEVE FISK

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

This is the first of a series of several papers dealing with various combinatorial properties of triangulations. The problems generally have their origin in combinatorial theory, but are illuminated by viewing them from a more topological viewpoint.

Successive papers will deal with a generalization of four colorings to nonsimply connected surfaces (using discrete fiber bundles); analogs of coloring based on regular polyhedra other than the tetrahedron, triangulations of surfaces whose vertex degrees are all divisible by a fixed integer, homotopy and cobordism problems of the preceeding structures, and generalizations to dimensions greater than 2.

1. DEFINITIONS

A four coloring of a triangulation M is a simplicial map $f: M \to \partial\Delta^3$ with the property that the image of any triangle of M is a triangle of $\partial\Delta^3$. Such a map we call a nondegenerate simplicial map. By $\partial\Delta^3$ we mean the boundary of the tetrahedron, so it has exactly four vertices. We consider two four colorings f and g to be the same if they differ by a permutation of vertices. That is, there is an automorphism σ of $\partial\Delta^3$ such that $\sigma f = g$. The two-manifold M is a sphere, except in Section 5, where it may be any orientable two-manifold. We assume an orientation for M has been fixed.

If M is a triangulation, and p a vertex of M, the degree of p, written $\rho(p)$, is the number of triangles of M containing p. A vertex is odd (even) in M iff its degree is odd (even). A three coloring of M is a simplicial map $f: M \to \Delta^2$ such that every triangle maps to a triangle. (Here Δ^2 is the

2 simplex, or triangle.) It is easy to see that a necessary condition for M to have a three coloring is that every interior vertex be even. If M is simply connected, it is also sufficient [3].

The subgraph of M consisting of all the odd vertices, and the edges joining them, is denoted $OD[M]$. We shall see that the structure of $OD[M]$ is reflected in the structure of all four colorings of M.

2. The Degree of a Coloring

If $f : K \to \partial\Delta^3$ is a coloring of K, then f is a map between two spheres. Consequently, there is an integer assigned to f, its degree, determined as follows: Let D be any triangle of $\partial\Delta^3$. Let $p(n)$ be the number of triangles of K mapping to D which preserve (reverse) orientation. Then the degree of f, $\deg(f)$, is $p - n$. Note this is independent of D. If we compute $\deg(f)$ for each of the four triangles of $\partial\Delta^3$, we get the result that $\deg(f)$ is $1/4$ (number of triangles of K with positive orientation minus those of negative orientation).

If we are only interested in the degree mod 2, then we have the following formula (see Tutte [1])

$$\deg(f) = \sum_{f(x)=c} \rho(x) \qquad (\text{mod } 2). \tag{*}$$

In this formula, c is a vertex of $\partial\Delta^3$ (a "color").

To prove the formula we observe that, mod 2, $\deg(f)$ is just the number of triangles mapping to a fixed triangle of $\partial\Delta^3$. If the three triangles of $\partial\Delta^3$ around c are A, B, and C, then the number of triangles, mapping to A, B, or C is just the number of triangles of K which have a vertex colored c. Thus, $\deg(f) = A + B + C =$ the right side of (*). This is one of the places where we use the nondegeneracy of f.

From this simple fact we deduce some nonobvious results. If, for instance, all the vertices of K were even, then we see every four coloring has even degree. If there are exactly two odd vertices, then by applying the formula to a color not occuring on either of the odd vertices, we see every four coloring has even degree. Furthermore, the two odd vertices are colored the same. If $OD(K) =$  then we can see that every coloring has odd degree.

Surprisingly, even more is true.

THEOREM 1. *If K is an even triangulation of the sphere, and f a coloring, then* $\deg(f) = 0$.

Proof. We begin by recalling that K has a three coloring. We will view this as a nondegenerate map $e\colon K \to \Delta^2$. Now f is a nondegenerate map $K \to \partial\Delta^2$. We shall construct the product of Δ^2 and $\partial\Delta^3$ in the category of complexes and nondegenerate maps. $\Delta^2 \times \partial\Delta^3$ has 12 vertices, consisting of ordered pairs (c, b) with c a vertex of Δ^2 and b a vertex of $\partial\Delta^3$. Three pairs form a triangle of $\Delta^2 \times \partial\Delta^3$ if each projection is a triangle. The resulting triangulation is shown in Fig. 1. The relevant fact is that $\Delta^2 \times \partial\Delta^3$ is a torus. We have the following diagram:

$$\begin{tikzcd}
K \arrow[rr, "e"] \arrow[d, "f"'] \arrow[drr, "exf"] & & \Delta^2 \arrow[d, dash, "P_2"] \\
\partial\Delta^3 & & \Delta^2 \times \partial\Delta^3 \arrow[ll, "P_3"]
\end{tikzcd}$$

Here exf is the product map, and $P_2(P_3)$ is the projection to the first (second) factor. Now we have $(\deg f) = \deg(exf) \cdot \deg(P_3)$. From topology [5] we have that the degree of any map from the sphere to the torus is 0. $\deg(f) = 0 \cdot \deg P_3 = 0$.

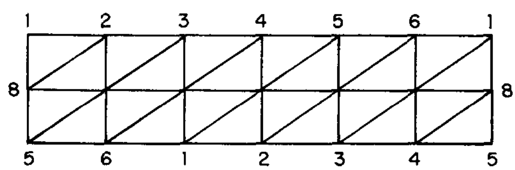


FIGURE 1

COROLLARY 2. *If $OD(K) = \cdot\cdot$ then the degree of any coloring is* 0.

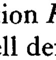
Proof. There exists a ramified covering $\hat{K}$ of K which is ramified only at the odd vertices with degree 2. Triangulate $\hat{K}$ using the triangulation of K. (For a direct construction of $\hat{K}$ see [4].) We have the following diagram:

$$\begin{tikzcd}
\hat{K} \arrow[d, "p"'] \arrow[dr, "\hat{f} = f \circ p"] & \\
K \arrow[r, "f"'] & \partial \Delta^3
\end{tikzcd}$$

Now $\hat{K}$ is still a sphere, and $\hat{f}$ a coloring; hence, $\deg \hat{f} = 0$ since all vertices of $\hat{K}$ are even. $\deg p$ is seen to be 2, so $\deg f = 0$.

COROLLARY 3. (a) *If* $OD(K) =$ , *f a coloring of K, then* $\deg f = \pm 1$.

(b) *If* $OD(K) =$ , *f a coloring of K, then* $\deg f = 0, 1, -1$.

Proof. Suppose $OD(K) =$  . If we remove edge 23 and add an edge 14 we get a new triangulation K' which is even. From formula (*) we see that $f(1) \neq f(4)$, so f is well defined on K'. Let triangle 124 of K' map to D. Then on K there is one less triangle mapping to D, so $\deg f$ is ± 1. A proof for case b is similar. We can formulate the following general result.

THEOREM 4. *Consider the set $\mathfrak{A}$ of all triangulations of the sphere such that if K, $K' \in \mathfrak{A}$, then $OD(K) \approx OD(K')$. If $OD(K')$ is connected, then there is an integer N such that if f is a coloring of any member of $\mathfrak{A}$, then* $|\deg(f)| < N$.

Proof. The hypothesis that $OD(K)$ is connected implies that $K \backslash OD(K)$ is the union of disks. Let f be a coloring of K. Then we shall show that $\deg(f)$ is determined by $f \mid OD(K)$. Since there are only a finite number of maps $OD(K) \to \partial \Delta^3$ we will be done.

Let f' be a coloring of K' such that $f \mid OD(K) \approx f' \mid OD(K')$. Fix a triangle D of $\partial \Delta^3$. If R (R') is a region of K (K'), let $\#(R)$ $(\#(R'))$ be the number of triangles of $R\,(R')$ mapping to D under f (f') with positive orientation minus those mapping with negative orientation.

Let the disks of $K \backslash OD(K)$ be $E_1 \cdots E_n$ and those of $K' \backslash OD(K')$ be $E_1{}^1 \cdots E_n{}^1$ arranged so that $\partial \bar{E}_i \approx \partial E_i{}^1$. Then $\deg f = \sum \#(E_i)$ and $\deg f' = \sum \#(E_i{}^1)$. Thus, it suffices to show $\#(E_i) = \#(E_i{}^1)$.

Consider the sphere S obtained by identifying the boundaries of E_i

and E_i^1. By hypothesis, S is even. Moreover, since $f \cup f' = f_S$ is well defined on S, and orientation of one of E_i and E_i^1 was reversed to form S, $0 = \deg f_S = (\# E_i) - \#(E_i^1)$.

Thus, we are done.

If we remove the restriction that $OD(K)$ is connected, the result is false. We give an example where $OD(K)$ consists of two circles. Let Y be the 24 triangles of Fig. 1 joined along the short (2 edge) sides. Join n copies of Y together to get a long cylinder. Add two new vertices, one joined to all vertices of one of the ends of the cylinder, the other to those of the other. We define a map f on our triangulation by taking it on the union of the Y's to be the projection map to the second factor $\Delta^2 \times \partial\Delta^3 \to \partial\Delta^3$. On the two new points it may be defined arbitrarily. Then it is easy to see the degree is $6n$, and $OD(\cdot)$ is just two circles (hexagons).

3. Kempe Components

If we are given a coloring f of K, the following way is known to obtain another coloring of K. Suppose there is a region R whose boundary is two colored by f. ($f \mid \partial R$ has only 2 vertices of $\partial\Delta^3$ in its image.) Then we can construct a coloring g of K by putting $g = f$ outside R, and $g = \sigma \circ f$ in R where σ is the automorphism of order two of $\partial\Delta^3$ which fixes the two colors of ∂R, and interchanges the other two. We call such a region R a Kempe region for f.

Extending this relation to an equivalence relation, we divide the colorings of K into Kempe classes. Some empirical facts about the number of Kempe classes are the following:

(1) If $OD(K) = \varnothing$, there is exactly one class.

(2) If $OD(K) = \cdot\cdot$ there is exactly one class.

(3) If $OD(K) =$  there are two classes. One consists of all colorings of even degree, and the other of odd degree.

(4) If $OD(K) =$  then there is one class.

Some triangulations have more than just two Kempe classes. For instance, the icosahedron has ten colorings, and ten classes, all colorings of degree 1.

Presently, we are able to prove (1) and half of (3). To do this we need some simple results on three colorings of the circle.

LEMMA 5. *Let S^1 have n vertices, α: $S^1 \to \partial\Delta^2$ a three coloring. Let N be the set of nonsingular vertices of S^1. Then*

(1) $\deg(\alpha) \equiv n \pmod 2$,

(2) *For p a vertex of $\partial\Delta^2$, the number of vertices of $\alpha^{-1}(p) \cap N$ is congruent to* $\deg \alpha \pmod 2$.

Proof. For part 1, we recall that, mod 2, $\deg \alpha$ is just the number of edges of S^1 which map to a fixed edge of $\partial\Delta^2$. Hence, $3 \deg \alpha$ is the total number of edges, which is just n.

For (2), a vertex X of S^1 is nonsingular iff the two vertices adjacent to it map to different vertices of $\partial\Delta^2$. Let S (the singular vertices) be the rest of the vertices of S^1. Let e be an edge of $\partial\Delta^2$ containing p. $\deg(\alpha) \equiv$ the number of edges to $\alpha^{-1}(e)$. Each vertex of $\alpha^{-1}(p) \cap N$ contributes one edge to $\alpha^{-1}(e)$, while those in S contribute 0 or 2. Each edge mapping to e comes from one of these two types.

If f is a four coloring of K, then we distinguish two types of edges. If the two triangles containing an edge both map to the same triangle, we call the edge singular. Otherwise it is nonsingular. Let $N(f)$ be the set of nonsingular edges of f.

If g is obtained from f by changing f in a Kempe region R, how is $N(f)$ related to $N(g)$? If we consider the sets of edges $N(f)$, $N(g)$, and ∂R mod 2, then we have the simple relation

$$N(f) + N(g) = \partial R.$$

In other words, if an edge is not on the boundary, it does not change from singular to nonsingular or vice versa as f changes to g. However, every edge of the boundary does change.

Now let us suppose that $OD(K) = \varnothing$. If φ is the three coloring of K, then $N(\varphi) = \varnothing$. Moreover, φ is the only coloring with this property. We shall show every f is related to a g with $N(g) \subsetneq N(f)$. By induction, f is related to φ so there is only one Kempe class. Let e be a nonsingular edge of f, colored 1, 2 say. Consider the set T of all edges in $N(f)$ colored 1, 2. By the lemma, there are an even number of edges at every vertex of T. Therefore, there is a simple closed curve contained in T which is two colored. This curve bounds a region R. Since $\partial R \subset N(f)$, we get $N(g) \subsetneq N(f)$ where we use R to obtain g from f. Thus, we are done.

It is possible to extend this to manifolds with boundary as follows: Call M admissible if

(1) topologically M is the two sphere minus several disks;

(2) there is a three coloring φ: $M \to \Delta^2$.

A coloring f of M is admissible if for each component C of ∂M, there is an automorphism σ of $\partial \Delta^3$ (depending on f and c) such that $\sigma \circ f \mid c = \varphi \mid c$.

A Kempe region for f is a region R such that for each component c of ∂R either

(1) $f \mid c$ is a two coloring or

(2) c is a component of ∂M.

With these definitions we prove the following.

THEOREM 6. *Any two admissible colorings of an admissible triangulation are Kempe related.*

Proof. From conditions (1) and (2) it follows that there is an M', an even triangulation of the sphere such that M is a subcomplex (see [4]). Given an admissible coloring f, we can extend to an f' on M' such that on each component of $M' \backslash M$, f' is a three coloring. Thus, $N(f') \subseteq N(f) \cup \partial M$. A Kempe region R for f is contained in a Kempe region for f'. The old proof now applies. In the case of a disk, we make the following conjecture.

Conjecture. Let M be an even disk, and f be any coloring. Let S be the set of all colorings g such that $g \mid \partial M = f \mid \partial M$. Then all elements of S are Kempe related.

Our last theorem contained the case where f is the three coloring of M. A consequence of the conjecture is that if $OD(M)$ is connected, $\mathfrak{A}$ is the set of all K such that $OD(M) = OD(K)$, then there is an N such that $K \in \mathfrak{A}$ implies K has less than N Kempe classes. We can establish another special case.

THEOREM 7. *Let M be a disk, whose interior has all vertices of even degree, and $\partial M = \square$. The degrees of the boundary are even, odd, even, odd. The set S of all colorings of M such that all four colors are used on ∂M forms one Kempe class.*

Proof. Join the two odd vertices by a new edge e forming an even triangulation T. A coloring f in M extends to $\hat{f}$ on T. We notice that $e \in N(\hat{f})$. We begin to decrease $N(\hat{f})$ as before, but we never use any region containing e in its boundary. If $\hat{f}$ is related to $\hat{g}$, then f is related to g. We finally stop where there is exactly one cycle in $N(\hat{f})$ containing e. Suppose we have another coloring $h \in S$ such that $N(\hat{h})$ is just one cycle with $e \in N(\hat{h})$. It remains to show h is related to f. Call the two cycles F and H. We claim that the set of points of $F \cup H$ is two colored by f. Let the end points of e be p and q. We want to show that H is colored with $f(p)$ and $f(q)$. Let Γ be the first point of H after p which lies on F (Γ might be q, if F and H have only two points in common). Then the segments of F and H from p to Γ determine an even region K. We claim that on K f two colors $F \cup H$. If we can show this we will be done by induction along H. If we remove the vertices Γ and p, we see that the two arcs of ∂K determined are two colored by f. Thus, we must study the neighborhood of p. We reduce the proof of the theorem to a lemma, whose proof is easy.

LEMMA 8. *Let f and f' be three colorings of S^1, $N(f)$ and $N(f')$ each contain two members: a, b; a', b'. Then either*

(1) $f(a) = f(b) = f(a') = f'(b)'$ *or*

(2) $f(a) = f(b)$ *and* $f(a) \neq f(a'), f(a) \neq f(b')$.

COROLLARY. *If M is a sphere, $OD(M) = \square$, then all the colorings of odd degree are related.*

Proof. From formula (*), we deduce that if f is an odd coloring, $f \mid OD(M)$ is just as prescribed in the last theorem. It can be shown [4] that the degrees of $OD[M]$ must be as in the theorem, so we apply the last theorem to each of the two components of $M \backslash OD(M)$.

We next give another proof of a relation between degrees and Kempe classes.

THEOREM 9. *If f and g are in the same Kempe class*

$$\deg f \equiv \deg g \pmod 2.$$

Proof. It suffices to show this for g obtained from f by permuting in a Kempe region R. Let $\#R$ be the number of triangles of R mapping positively to some triangle D of $\partial \Delta^3$ less those mapping negatively. We have $\deg(f) = \#(M \backslash R) + \#R$ and $\deg(g) = \#(M \backslash R) - \#R$, so we are done.

One can show that $\#R$ is independent of the choice of D.

The degree function is not a very fine invariant of Kempe classes. We will give an example, based on formula (*) that distinguishes between Kempe classes of colorings of odd degree. Consider the triangulations K with $OD(K)$ equaling a hexagon. By formula (*), there are exactly two kinds of colorings of $OD(K)$ such that they could be induced by four colorings of K of odd degree. See Fig. 2. We claim that a coloring f of K such that $f \mid OD(K)$ is of type (1) is not related to any coloring g with $g \mid OD(K)$ of type 2. This is clear, for if two colorings are related by a Kempe region R, only two colors can be permuted. We cannot permute two colors of (1) and get (2), so we get our result.

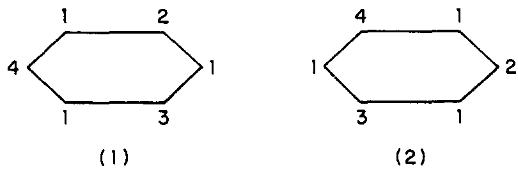


FIGURE 2

A more subtle way of distinguishing classes of colorings uses the even structure. If C is a cycle of $OD(K)$, then there is a map from C to Δ^2. Combining this map with a coloring restricted to C, we can say for instance: Suppose C is a square and the boundary maps to Δ^2 as 2, 1, 3, 1. Then there is no four coloring f of K such that $f \mid C$ is given as 1412. For the proof, we look at the map from C to $\Delta^2 \times \partial\Delta^3$. In our case, one sees that the image of C is not homotopic to a point. Since f is defined over the disk that C bounds, the combined map is from a disk to the torus, and so is homotopic to a point.

It seems that any formulation of the structure of colorings must take this extra information given by the even structure, but how to do so is not apparent.

4. Number of Colorings

Determining how many colorings a triangulation has seems to be a very hard question. In this section we give a few facts and several striking observations.

Suppose K and M are two triangulations, with triangles k of K and m of M given. If we remove triangle k from K, and m from M, and join the two resulting disks along their boundary, we get a connected sum

$K \# M$ of K and M. In case M is a tetrahedron, a connected sum is uniquely determined by the triangle of K. The sum $K \# \partial\Delta^3$ is just the old triangulation with a vertex of degree three in the triangle k.

If it is impossible to write a triangulation K as a connected sum $A \# B$, we say K is irreducible. For K being a sphere, there is an equivalent formulation: K is irreducible iff there do not exist three points p, q, r so that each of the pairs pq, qr, and pr is an edge of K, yet the points p, q, r are not the vertices of a triangle of K.

If a triangulation K has a vertex of degree 3, then it is not irreducible, for we can write $K = K' \# \partial\Delta^3$ for appropriate K'.

THEOREM 10. *Suppose $K = A \# B$ then*

(1) *the number of colorings of K is the product of the number of colorings of A and the number of colorings of B;*

(2) *the number of Kempe equivalence classes of K is the product of the number of classes of A and the number of classes of B.*

Proof. Any four coloring of $A \# B$ restricts to a coloring of A and a coloring of B, and conversely, proving (1). To prove (2), we observe that a coloring f of $A \# B$ is equivalent to a coloring g of $A \# B$ iff $f \mid A$ is equivalent to $g \mid A$ and $f \mid B$ is equivalent to $g \mid B$.

From this result, if we are interested in the number of colorings, then we see we should restrict ourselves to irreducible triangulations. We have been able to prove no results about the number of colorings but are able to present some striking empirical facts.

Observation 1. If $OD[M] = \varnothing$, then the number of colorings is 0 mod 4. That this modulus is best is shown by the octahedron which has exactly four colorings. We can say a little more if we restrict our attention:

(a) If $OD[M] = \varnothing$, M has an even number of vertices, and M irreducible, the number of four colorings is $\equiv 4 \pmod 8$

(b) If $OD[M] = \varnothing$, M has an odd number of vertices, then the number of four colorings is $0 \pmod 8$. In (b) we dropped irreducible. We can justify this by observing that for $M = A \# B$ then M is even iff both A and B are [6].

Observation 2. Suppose $OD[M] = \cdot\cdot$, M irreducible. Then

(a) If M has an odd number of vertices, the number of four colorings is $1 \pmod 4$.

(b) If M has an even number of vertices the number of four colorings is 0 (mod 2).

Observation 3. Suppose $OD(M) =$  , and M irreducible. There are two Kempe classes. The one consisting of all colorings of odd degree has $\equiv 1 \pmod 4$ four colorings. The other component has $\equiv 2 \pmod 4$ four colorings.

One can continue these observations, but the facts become less clear. Why the mod 4 should appear is a mystery.

The data for these observations consists of all irreducible triangulations with less than 13 vertices and all their four colorings. This is approximately 100 triangulations.

5. Vigneron's Theorem

In this section we generalize a theorem of Vigneron [2]. Let $f : M^2 \to \partial\Delta^3$ be a four coloring. Call a sequence of distinct triangles $t(1), \ldots, t(n)$ a closed path of triangles if (1) $t(1) \cap t(n)$ is a single edge and (2) $t(i) \cap t(i+1)$ is a single edge, for $i = 1, \ldots, n-1$. The edges of the form $t(i) \cap t(i+1)$ are called interior edges. Suppose the vertices of $\partial\Delta^3$ are 1, 2, 3, and 4. If we divide the six edges of $\partial\Delta^3$ into three classes: 12 and 34; 13 and 24; 14 and 23, then if the interior edges never lie in one of the classes, we call the path alternating. Indeed, we see that the sequence of interior edges alternates between the other two classes. We define the Vigneron invariant of f, $v(f)$, to be the number of alternating paths. We generalize Vigneron's theorem to a two-manifold as follows.

Theorem 11. *Let $f : M^2 \to \partial\Delta^3$ be a four coloring of an arbitrary orientable two-manifold. Then*

$$v(f) \equiv \deg(f) + (\#\text{ of vertices of } M) \qquad (\text{mod } 2).$$

Corollary 12. *On the sphere, $v(f)$ is well defined* mod 2 *on Kempe equivalence classes.*

Proof. If f and g lie in the same Kempe equivalence class, then $\deg(f) = \deg(g)$ mod 2, so the right side is invariant. Thus, $v(f)$ is also invariant.

Proof of Theorem 11. Orient both M and $\partial\Delta^3$. Then f restricted to a triangle of M either preserves or reverses the orientation. We want to change M and f so that every triangle has its orientation preserved. Let M have a triangle whose orientation is reversed. Add a vertex to the center of the triangle and join it to the three vertices, getting a new triangulation M'. The map f has a unique extension f' to M'. We have that $\deg f' = (\deg f) - 1$, the number of points of M' is one plus the number of M, and $v(f) = v(f')$. Thus, the conclusion of the theorem is true for (M, f) iff it is true for (M', f').

We can now assume f preserves the orientation of all the triangles. Every two adjacent triangles are mapped by f to distinct triangles, so f is a branched covering map. That is, on M minus the vertices, f is a covering map. Our approach is to study alternating paths on $\partial\Delta^3$ and use properties of covering maps to get a result on M.

If we consider $\partial\Delta^3$ minus the vertices, it has fundamental group given by four generators σ_1, σ_2, σ_3, and σ_4 subject to $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4 = 1$. If we pick a base point up in triangle 124, we can describe the generators by the sequence of edges that they cross: σ_1 crosses 12, 13, 14; σ_2 crosses 21, 23, 24; σ_3 crosses 24, 32, 31, 34, 24; and σ_4 crosses 42, 43, 41. $\partial\Delta^3$ has a unique four coloring, and there are three alternating paths. The sequences of interior edges are (A): 12, 13, 43, 42; (B): 12, 23, 34, 41; and (C): 13, 23, 24, 14. It is easily seen that every alternating path of M maps to one of the alternating paths A, B, or C. If we represent the paths A, B, and C by curves $\bar{A}$, $\bar{B}$, and $\bar{C}$, then the number of alternating paths of M is just the number of closed curves in the inverse image of the closed curves $\bar{A}$, $\bar{B}$, and $\bar{C}$.

Recall that the fundamental group of $\partial\Delta^3$ minus the vertices acts on $f^{-1}(p)$ as a permutation group. The number of points of $f^{-1}(p)$ is the degree of f. The number of alternating chains lying over A correspond to the various liftings of the curve A to M. This is the number of cycles in the permutation corresponding to the curve $\bar{A}$. An elementary fact about the symmetric group is that the number of cycles of a permutation is congruent mod 2 to the sign of the permutation plus the number of objects being permuted. Thus, we get that $v(f) \equiv$ sign of permutations corresponding to A, B and C + $\deg f$. By observation, we see that $\bar{A}$ is homotopic to $\sigma_3\sigma_2$, $\bar{B}$ to $\sigma_4\sigma_2$, and $\bar{C}$ to $\sigma_2\sigma_1$. Thus, if τ_i is the permutation corresponding to σ_i, we get

$$\begin{aligned} v(f) &\equiv \operatorname{sign}(\tau_3\tau_2) + \operatorname{sign}(\tau_4\tau_2) + \operatorname{sign}(\tau_2\tau_1) + \deg f \\ &= \operatorname{sign}(\tau_2\tau_2\tau_1\tau_2\tau_3\tau_4) + \deg f = \deg f \end{aligned}$$

since $\operatorname{sign}(\tau_2 \cdot \tau_2) = 0$ and $\tau_1\tau_2\tau_3\tau_4 = 1$. It remains to see that for M, the number of vertices is even. Since $f\colon M \to \partial\Delta^3$ maps all triangles with the same orientation, we have that the number of triangles of M is $4 \cdot \deg f$. Since the number of triangles of M is twice the number of vertices minus the Euler characteristic, we conclude that M has an even number of vertices.

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