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Constructions of uniform designs by using resolvable packings and coverings☆

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Abstract

Uniform designs have been widely used in computer experiments, as well as in industrial experiments when the underlying model is unknown. Based on the discrete discrepancy, the link between uniform designs, and resolvable packings and coverings in combinatorial theory is developed. Through resolvable packings and coverings without identical parallel classes, many infinite classes of new uniform designs are then produced.

Keywords: Discrete discrepancy; Resolvable packings/coverings; Uniform designs; Constructions

1. Introduction

The study of uniform designs (UDs) was motivated by projects in system engineering in 1978 (see, [5,24]). In the past 20 years, uniform designs have been widely used in computer experiments, as well as in industrial experiments when the underlying model is unknown.

To establish a uniform design, as one of space filling designs, one needs to find suitable design points so that they are scattered uniformly on its experimental domain. A number of criteria such as star L_p-discrepancy, centered L_p-discrepancy and wrap-around L_p-discrepancy, which serve as a benchmark of uniformity, have been

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proposed in the literature (see, for example, [15]). A design is referred to as a uniform design if it is optimal under a certain criterion. If we restrict the experimental domain of a uniform design to certain lattice points, then it can be thought of as one kind of fractional factorial designs under a nonparametric regression model [8]. We use notation $U_n(q^m)$ to denote a uniform design with n runs and m q-level factors.

Generating a uniform design is generally difficult. Several known methods such as the *good lattice method* in quasi-Monte Carlo methods, the *Latin square method*, and *optimization heuristic threshold accepting method* (see, for example, [7]) involve a heavy computing search. In fact, it is an NP hard problem to search a $U_n(q^m)$ when (n,q,m) increase.

Recently, Hickernell and Liu [16] proposed a criterion called *discrete discrepancy* (DD for short). Under the discrete discrepancy, Fang et al. [6] constructed many infinite classes of uniform designs via combinatorial configurations. The designs they obtained are all *U-type*. The number of runs in such a design is a multiple of the number of levels of each factor where each factor takes q levels equally often. In this paper the notion of a *nearly U-type design* is introduced and used to construct UDs whose number of runs are not necessarily a multiple of their number of levels.

Given the parameters n, m and q, a fractional factorial design with n runs and m q-level factors corresponds to a $n \times m$ matrix $\boldsymbol{X}$. If we regard its rows as points in Q^m, where $Q = \{1, 2, \ldots, q\}$, then to search a uniform design $U_n(q^m)$ is equivalent to finding n points in the domain Q^m as even as possible. A nature way is to first select these points uniformly distributed in every one dimension. When n is a multiple of q, it is a U-type design. In this case, each column of $\boldsymbol{X}$ takes values from Q equally often. When n is not divisible by q, let $n = qt + r$ $(0 < r < q)$. In this case, we arrange the design $\boldsymbol{X}$ so that $q - r$ values of Q occur t times, while the remaining r values occur $t + 1$ times in every column of $\boldsymbol{X}$. This guarantees that every value in Q appears in each column of $\boldsymbol{X}$ as equally as possible. We will call such a design a nearly U-type design and denote it by $\mathrm{NU}(n, q^m)$. For completeness, we admit $r = 0$. An $\mathrm{NU}(n, q^m)$ with $n = qt$ is nothing else than a U-type design.

The remainder of this paper is organized as follows. In Section 2, we will give a detailed discussion about the discrete discrepancy. In Section 3, the connection between uniform designs in the sense of discrete discrepancy and resolvable packings/coverings in combinatorial theory is developed. Through resolvable packings and coverings without identical parallel classes, many infinite classes of new uniform designs are then produced. These are presented in Section 4. Finally Section 5 contains some remarks.

2. The discrete discrepancy

According to Hickernell [15], the discrete discrepancy is defined by way of a kernel function. Let $\mathscr{X}$ be a measurable subset of R^m. A kernel function $\mathscr{K}(\boldsymbol{x}, \boldsymbol{w})$ is a symmetric, non-negative definite and real-valued function defined on $\mathscr{X} \times \mathscr{X}$, i.e.,

$$\begin{cases} \mathscr{K}(\boldsymbol{x}, \boldsymbol{w}) = \mathscr{K}(\boldsymbol{w}, \boldsymbol{x}) & \text{for any } \boldsymbol{x}, \boldsymbol{w} \in \mathscr{X}, \\ \displaystyle\sum_{i,j=1}^{n} a_i a_j \mathscr{K}(\boldsymbol{x}^i, \boldsymbol{x}^j) \geqslant 0 & \text{for any } a_i, a_j \in R, \boldsymbol{x}^i, \boldsymbol{x}^j \in \mathscr{X}. \end{cases} \tag{1}$$

Let F_* denote the uniform distribution function on $\mathscr{X}$, $P \subseteq \mathscr{X}$ be a set of design points and F_n denote the empirical distribution of P, where

$$F_n(\boldsymbol{x}) = \frac{1}{n}\sum_{z \in P} 1_{\{z \leqslant x\}}.$$

Here $\boldsymbol{z} = (z_1, \ldots, z_m) \leqslant \boldsymbol{x} = (x_1, \ldots, x_m)$ means that $z_j \leqslant x_j$ for all j and 1_A is the indicator function of A. Then given a kernel function $\mathscr{K}(\boldsymbol{x}, \boldsymbol{w})$, the discrepancy of P [15] is defined by

$$\begin{aligned} D(P;\mathscr{K}) &= \left\{ \int_{\mathscr{X}^2} \mathscr{K}(\boldsymbol{x},\boldsymbol{w})\, \mathrm{d}[F_*(\boldsymbol{x}) - F_n(\boldsymbol{x})]\, \mathrm{d}[F_*(\boldsymbol{w}) - F_n(\boldsymbol{w})] \right\}^{1/2} \\ &= \left\{ \int_{\mathscr{X}^2} \mathscr{K}(\boldsymbol{x},\boldsymbol{w})\, \mathrm{d}F_*(\boldsymbol{x})\, \mathrm{d}F_*(\boldsymbol{w}) - \frac{2}{n}\sum_{z\in P} \int_{\mathscr{X}} \mathscr{K}(\boldsymbol{x},\boldsymbol{z})\, \mathrm{d}F_*(\boldsymbol{x}) \right. \\ &\quad \left. + \frac{1}{n^2}\sum_{z,z' \in P} \mathscr{K}(\boldsymbol{z},\boldsymbol{z}') \right\}^{1/2}. \end{aligned} \tag{2}$$

From the above definition, it is clear that the discrepancy measures how far apart the empirical distribution F_n is from F_*. The lower the discrepancy, the better the uniformity of the design is.

For a factorial plan of m q-factors, the domain $\mathscr{X} = \{1, 2, \ldots, q\}^m$ is formed by all possible level combinations of the m factor, F_* is the discrete uniform distribution on $\mathscr{X}$. Let

$$\tilde{\mathscr{K}}_j(x,w) = \begin{cases} a & \text{if } x = w \\ b & \text{if } x \neq w \end{cases} \qquad \text{for } x, w \in \{1, \ldots, q_j\},\ a > b > 0$$

and

$$\mathscr{K}(\boldsymbol{x},\boldsymbol{w}) = \prod_{j=1}^{m} \tilde{\mathscr{K}}_j(x_j, w_j), \quad \text{for any } \boldsymbol{x}, \boldsymbol{w} \in \mathscr{X}, \tag{3}$$

then $\mathscr{K}(\boldsymbol{x},\boldsymbol{w})$ is a kernel function and satisfies conditions (1). And the corresponding discrete discrepancy, denoted by $D(\boldsymbol{X}; a, b)$, can be used for measuring the uniformity of design points [16].

$$D^2(\boldsymbol{X}; a, b) = -\prod_{j=1}^{m}\left[\frac{a + (q-1)b}{q}\right] + \frac{1}{n^2}\sum_{\alpha,\beta=1}^{n}\prod_{j=1}^{m}\tilde{\mathscr{K}}_j(x_{\alpha j}, x_{\beta j}). \tag{4}$$

Let $\boldsymbol{X} = (x_1, x_2, \ldots, x_m)$ be a nearly U-type design NU$(n; q^m)$ defined in Section 1. Define a matrix

$$\boldsymbol{Z} = (Z^{(1)}, Z^{(2)}, \ldots, Z^{(m)}),$$

where $Z^{(j)} = (z_{l\alpha}^{(j)})$ is an $n \times q$ sub-matrix with

$$z_{l\alpha}^{(j)} = \begin{cases} 1 & \text{if factor } x_j \text{ takes level } \alpha \text{ in run } l; \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

$\boldsymbol{Z}$ is called the *induced matrix* of $\boldsymbol{X}$. Suppose $(\lambda_{ij}) = \boldsymbol{Z}\boldsymbol{Z}'$. Then $\lambda_{ij}(i \neq j)$ represents the number of coincidences between any two distinct rows of $\boldsymbol{X}$.

Fang et al. [9] gave an analytical expression and the lower bound of the discrete discrepancy in the case of U-type designs. A similar lower bound for nearly U-type designs can also be obtained. To present this, we first give the following lemma.

Lemma 2.1. *Let $a > 1$ be a constant number and c be a positive integer. Let $\boldsymbol{Y}$ be a set of s-dimension vectors of non-negative integers. Suppose for any $\boldsymbol{y} = (y_1, y_2, \ldots, y_s) \in \boldsymbol{Y}$, $\sum_{l=1}^{s} y_l = c$ and*

$$f(\boldsymbol{y}) = \sum_{l=1}^{s} a^{y_l}$$

is a function over $\boldsymbol{Y}$. Then for any $\boldsymbol{y} \in \boldsymbol{Y}$, $f(\boldsymbol{y}) \geqslant (s + st - c)a^t + (c - st)a^{t+1}$, and the minimum value of $f(\boldsymbol{y})$ over $\boldsymbol{Y}$ can be achieved at $\hat{\boldsymbol{y}}$ if and only if $s + st - c$ coordinates of $\hat{\boldsymbol{y}}$ take the value t and $c - st$ coordinates of $\hat{\boldsymbol{y}}$ take the value $t + 1$, where $t = \lfloor c/s \rfloor$.

Proof. By definition, $\boldsymbol{Y}$ is a finite set. So the minimum of f over $\boldsymbol{Y}$ can be achieved at some $\hat{\boldsymbol{y}} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_s)$. We claim that

$$|\hat{y}_i - \hat{y}_j| \leqslant 1 \text{ for all } i \neq j. \tag{6}$$

Otherwise, there exist some $i \neq j$ such that $\hat{y}_i - \hat{y}_j > 1$. Let $\boldsymbol{y}^* = (y_1^*, y_2^*, \ldots, y_s^*)$ be obtained from $\hat{\boldsymbol{y}}$, where $y_i^* = \hat{y}_i - 1$, $y_j^* = \hat{y}_j + 1$, and $y_l^* = \hat{y}_l$, for $l \neq i$ and j. Obviously, the coordinates of $\boldsymbol{y}^*$ are all non-negative integers and satisfy $\sum_{l=1}^{s} y_l^* = c$. But $f(\boldsymbol{y}^*) < f(\hat{\boldsymbol{y}})$, which is a contradiction.

Since $\sum_{l=1}^{s} \hat{y}_l = c$, (6) guarantees that there are $s+st-c$ coordinates of $\hat{\boldsymbol{y}}$ taking value t and $c-st$ coordinates of $\hat{\boldsymbol{y}}$ taking value $t+1$. So $f(\hat{\boldsymbol{y}}) = (s+st-c)a^t + (c-st)a^{t+1}$. The proof is then complete. □

Now we can prove that

Theorem 2.2. *Let n, m and q be positive integers and $n = qt + r$, $0 \leqslant r \leqslant q-1$. Let $\boldsymbol{X}$ be a fractional factorial design of n runs and m q-level factors, and $\boldsymbol{Z}$ be its induced matrix. Suppose $(\lambda_{ij}) = \boldsymbol{Z}\boldsymbol{Z}'$, $\mu = mt(n-q+r)/n(n-1)$ and $\gamma = \lfloor \mu \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. Then*

$$D^2(\boldsymbol{X}; a, b) = -\prod_{j=1}^{m} \left[\frac{a + (q-1)b}{q}\right] + \frac{a^m}{n} + \frac{b^m}{n^2} \sum_{i,j=1, i \neq j}^{n} \left(\frac{a}{b}\right)^{\lambda_{ij}}, \tag{7}$$

$$D^2(\boldsymbol{X}; a, b) \geqslant -\prod_{j=1}^{m} \left[\frac{a + (q-1)b}{q}\right] + \frac{a^m}{n} + \frac{b^m(n-1)}{n} \times \left[(\gamma + 1 - \mu)\left(\frac{a}{b}\right)^{\gamma} + (\mu - \gamma)\left(\frac{a}{b}\right)^{\gamma+1}\right] \tag{8}$$

and the lower bound of $D^2(\boldsymbol{X}; a, b)$ on the right-hand side of (8) *can be achieved if and only if all the off-diagonal entries of $\boldsymbol{ZZ}'$ take the same value γ, or take only two values γ and $\gamma + 1$.*

Proof. The conclusion comes from (4) and Lemma 2.1. Note that $\boldsymbol{Z}$ is the induced matrix of $\boldsymbol{X}$ and $(\lambda_{ij})=\boldsymbol{ZZ}'$, so λ_{ij}'s are all non-negative integers and satisfy $\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\lambda_{ij} = mt(n-q+r)$. □

From Theorem 2.2, an $\mathrm{NU}(n; q^m)$ is a uniform design if the lower bound in (8) is achieved. As in Section 1, we write $U_n(q^m)$ for an $\mathrm{NU}(n; q^m)$ which is a uniform design under the discrete discrepancy, in which n is not necessarily a multiple of q.

3. The connection between uniform designs and resolvable packings and coverings

Let n and λ be positive integers. A packing (resp. covering) of pairs of points is an ordered pair $(V, \mathscr{B})$ where V is an n-set (of points), and $\mathscr{B}$ is a collection of subsets of V, called blocks, such that each pair of points of V occurs at the most (resp. at least) λ times in the blocks. If $|B| \in K$ for any $B \in \mathscr{B}$, then the packing (resp. covering) is often written as a $P(K, \lambda; n)$ (resp. $C(K, \lambda; n)$).

From the graph theoretic terms, a $P(K, \lambda; n)$ (resp. $C(K, \lambda; n)$) is a decomposition of the complete multigraph λK_n, each of its edges having multiplicity λ, into its cliques (blocks) of order k ($k \in K$), in which some of its edges are allowed to be used less than (resp. more than) λ times. For any edge $e = \{x, y\}$ of λK_n, let $m(e)$ be the number of cliques containing e. The leave (excess) of the $P(K, \lambda; n)$ (resp. $C(K, \lambda; n)$) is the multigraph spanned by all pairs e of points with multiplicity $\lambda - m(e)$ (resp. $m(e) - \lambda$).

A packing or a covering is called resolvable if its block set admits a partition into parallel classes, each parallel class being a partition of its point set V. In what follows, the notation $\mathrm{RP}(K, \lambda; n, m)$ (resp. $\mathrm{RC}(K, \lambda; n, m)$) is adopted for a resolvable $P(K, \lambda; n)$ (resp. $C(K, \lambda; n)$) with m parallel classes. Whenever $K = \{k\}$, we omit the braces.

Resolvable packings and coverings arise in the study of resolvable pairwise balanced designs (RPBDs). An RPBD, $\mathrm{RB}(K, \lambda; n)$, is a pair $(V, \mathscr{B})$ where V is an n-set (of points), and $\mathscr{B}$ is a collection of subsets (called blocks) of V. Each block has size $k \in K$ and each pair of distinct points of V occurs in exactly λ blocks of $\mathscr{B}$. And $\mathscr{B}$ admits a partition into parallel classes. Thus, an $\mathrm{RB}(K, \lambda; n)$ is both an $\mathrm{RP}(K, \lambda; n)$ and $\mathrm{RC}(K, \lambda; n)$. However, the converse is clearly not true. Therefore, we can think of resolvable packings and coverings as a generalization of RPBDs. It has been proved (see [6]) that RPBDs are very useful in construction of optimal factorial designs under various criteria. To ensure that the derived factorial designs from RPBDs contain no fully aliased column, that is, each of its column cannot be obtained from another column by a permutation of levels, it is natural to require the RPBDs used to contain no identical parallel classes.

Resolvable packings and coverings without identical parallel classes can be used directly to obtain uniform designs, which are nearest to U-type designs. For simplicity, here we develop the relationship between $\mathrm{RP}(K, \lambda; n, m)$'s/$\mathrm{RC}(K, \lambda; n, m)$'s and $U_n(q^m)$'s for the case $n = qt + r$ with $r \in \{0, 1, q-1\}$. To this end, the resolvable packings and coverings under our consideration are mainly for the case where the block sizes are restricted to be in $K = \{k-1, k, k+1\}$. Throughout the remainder of this paper, we use notation $\tilde{\mathrm{R}}\mathrm{MP}(k, \lambda; n, m)$ (resp. $\tilde{\mathrm{R}}\mathrm{MC}(k, \lambda; n, m)$) to indicate an $\mathrm{RP}(\{k-1, k, k+1\}, \lambda; n, m)$ (resp. $\mathrm{RC}(\{k-1, k, k+1\}, \lambda; n, m)$) which satisfies the following properties:

(1) $n \equiv k-1, 0$ or $1 \pmod k$;
(2) it contains the maximum (resp. minimum) possible number m of parallel classes, which are mutually distinct;
(3) each parallel class consists of $\lfloor (n-k+1)/k \rfloor$ blocks of size k and one block of size $n - k\lfloor (n-k+1)/k \rfloor$;
(4) its leave (resp. excess) is a simple graph, that is, $\lambda - m(e) \leqslant 1$ (resp. $m(e) - \lambda \leqslant 1$) for any pair e of distinct points.

In the particular case where the $\tilde{\mathrm{R}}\mathrm{MP}$ and $\tilde{\mathrm{R}}\mathrm{MC}$ are exact, namely, an $\mathrm{RB}(\{k-1, k, k+1\}, \lambda; n)$ satisfying the above properties exists, we simply write $\tilde{\mathrm{R}}\mathrm{B}(k, \lambda; n, m)$ for both $\tilde{\mathrm{R}}\mathrm{MP}(k, \lambda; n, m)$ and $\tilde{\mathrm{R}}\mathrm{MC}(k, \lambda; n, m)$.

It is easily calculated that the number m of parallel classes in an $\tilde{\mathrm{R}}\mathrm{MP}(k, \lambda; n, m)$ is upper bounded by

$$\left\lfloor \frac{\lambda n(n-1)}{k(k-1)t + (n-kt)(n-kt-1)} \right\rfloor,$$

while the number m of parallel classes in an $\tilde{\mathrm{R}}\mathrm{MC}(k, \lambda; n, m)$ is lower bounded by

$$\left\lfloor \frac{\lambda n(n-1) - 1}{k(k-1)t + (n-kt)(n-kt-1)} \right\rfloor + 1,$$

where $t = \lfloor (n-k+1)/k \rfloor$.

In the literature, an $\tilde{\mathrm{R}}\mathrm{MP}(3, 1; n, m)$ is called a Kirkman school project design (KSPD) introduced in a paper by Černý et al. [3]. Further investigation into KSPDs was done by a number of authors (see, for example, [4,1,20]

Now given an $\tilde{\mathrm{R}}\mathrm{MP}(k, \lambda; n, m)$ (or $\tilde{\mathrm{R}}\mathrm{MC}(k, \lambda; n, m)$), $(V, \mathscr{B})$. Without loss of generality we assume that $V = \{1, 2, \ldots, n\}$ and $\mathscr{B} = \bigcup_{j=1}^{m} \mathscr{P}_j$, where each $\mathscr{P}_j$ stands for a parallel class of blocks which contains $\lfloor (n-k+1)/k \rfloor$ blocks of size k and one block of size $n - k\lfloor (n-k+1)/k \rfloor$. Then we can construct a factorial design as follows.

1. Let $q = \lfloor (n-k+1)/k \rfloor + 1$. Give a natural order $1, 2, \ldots, q$ to the q blocks in each parallel class $\mathscr{P}_j (j = 1, 2, \ldots, m)$;
2. For each $\mathscr{P}_j$, construct a q-level column $d^j = (d_{ij})$ such that $d_{ij} = u$, if point i is contained in the uth block of $\mathscr{P}_j$ of $\mathscr{B} (j = 1, 2, \ldots, m)$.

The m columns constructed from $\mathscr{P}_j$ of $\mathscr{B} (j = 1, 2, \ldots, m)$ form a factorial design with n runs and m factors of q-level.

Further, we can prove that

Theorem 3.1. *Suppose n, k, λ and m are positive integers, and $n \equiv r \pmod{k}$ where $r \in \{0, 1, k-1\}$. Then the factorial design derived from an $\tilde{\mathrm{R}}\mathrm{MP}(k,\lambda;n,m)$ (or $\tilde{\mathrm{R}}\mathrm{MC}(k,\lambda;n,m)$) as above is a uniform design $U_n(q^m)$, where $q = \lfloor (n-k+1)/k \rfloor + 1$.*

Proof. Let $\boldsymbol{X}$ be the factorial design derived from the given $\tilde{\mathrm{R}}\mathrm{MP}(k,\lambda;n,m)$ (or $\tilde{\mathrm{R}}\mathrm{MC}(k,\lambda;\ n,m)$) and $\boldsymbol{Z}$ is the induced matrix of $\boldsymbol{X}$. Since each parallel class in the $\tilde{\mathrm{R}}\mathrm{MP}(k,\lambda;n,m)$ (or $\tilde{\mathrm{R}}\mathrm{MC}(k,\lambda;n,m)$) has $\lfloor (n-k+1)/k \rfloor$ blocks of size k and one block of size $n - k\lfloor (n-k+1)/k \rfloor$, in any column of design $\boldsymbol{X}$ $\lfloor (n-k+1)/k \rfloor$ levels appear k times and one level appears $k-1, k$ or $k+1$ times depending on $n \equiv k-1, 0$ or $1 \pmod{k}$. Hence, it is a nearly U-type design. Now suppose $(\lambda_{ij}) = \boldsymbol{ZZ}'$; by Theorem 2.2, we need to show that all the off-diagonal entries of $\boldsymbol{ZZ}'$ take the same value γ, or take only two values γ and $\gamma+1$, i.e. the numbers of coincidences between any two distinct rows of $\boldsymbol{X}$ can only take two values, whose differences do not exceed 1. In fact, the elements in rows r_i and r_j in $\boldsymbol{X}$ are coincident if and only if the pair $e = (i,j)$ is contained in the same block of the $\tilde{\mathrm{R}}\mathrm{MP}(k,\lambda;n,m)$ (or $\tilde{\mathrm{R}}\mathrm{MC}(k,\lambda;n,m)$). But by definition, in an $\tilde{\mathrm{R}}\mathrm{MP}(k,\lambda;n,m)$, the pair (i,j) is contained in either λ or $\lambda-1$ blocks; in an $\tilde{\mathrm{R}}\mathrm{MC}(k,\lambda;n,m)$, the pair (i,j) is contained in either λ or $\lambda+1$ blocks. Therefore, the uniform property of the design follows from Theorem 2.2.

Finally, the parallel classes in the $\tilde{\mathrm{R}}\mathrm{MP}(k,\lambda;n,m)$ (resp. $\tilde{\mathrm{R}}\mathrm{MC}(k,\lambda;n,m)$) are all mutually distinct, which guarantees that there are no fully aliased columns in $\boldsymbol{X}$. □

4. New uniform designs

Theorem 3.1 enables us to obtain uniform designs from $\tilde{\mathrm{R}}$MPs and $\tilde{\mathrm{R}}$MCs. The latter has been studied thoroughly for certain cases in combinatorial design theory. In particular, we have the following known results.

Lemma 4.1. *There exist*:

1. *an* $\tilde{\mathrm{R}}\mathrm{B}(3,1;n,(n-1)/2)$ *when* $n \equiv 3 \pmod{6}$ [21];
2. *an* $\tilde{\mathrm{R}}\mathrm{B}(3,2;n,n-1)$ *when* $n \equiv 0 \pmod{3}$ *and* $n \geqslant 9$ [23];
3. *an* $\tilde{\mathrm{R}}\mathrm{MP}(3,1;n,(n-2)/2)$ *and an* $\tilde{\mathrm{R}}\mathrm{MC}(3,1;n,n/2)$ *when* $n \equiv 0 \pmod{6}$ *and* $n \neq 6, 12$ [22,17];
4. *an* $\tilde{\mathrm{R}}\mathrm{MP}(3,1;n,(n-4)/2)$ *when* $n \equiv 4 \pmod{6}$ *and* $n \neq 4, 10$ [4];
5. *an* $\tilde{\mathrm{R}}\mathrm{MP}(3,1;n,(n-3)/2)$ *for* $n \equiv 1 \pmod{6}$ *and except for* $n \in \{1,7,12\}$ *and possibly except for* $n \in \{19,55,61,67,73,85,109\}$ [4,20];
6. *an* $\tilde{\mathrm{R}}\mathrm{MC}(3,1;n,\lfloor (n-1)/2 \rfloor)$ *when* $n \equiv 1 \pmod{3}$ *and* $n \notin \{13,16,67\}$ [20];
7. *an* $\tilde{\mathrm{R}}\mathrm{MP}(3,1;n,\lfloor n/2 \rfloor)$ *when* $n \equiv 2 \pmod{3}$ *and* $n \neq 5, 11$ [2];
8. *an* $\tilde{\mathrm{R}}\mathrm{MC}(3,1;n,\lfloor (n+1)/2 \rfloor)$ *if* $n \equiv 2 \pmod{3}$ *and* $n \notin \{5,11\}$ [20];
9. *an* $\tilde{\mathrm{R}}\mathrm{B}(4,1;n,(n-1)/3)$ *when* $n \equiv 4 \pmod{12}$ [14].
10. *an* $\tilde{\mathrm{R}}\mathrm{MP}(4,1;n,(n-3)/3)$ *when* $n \equiv 0 \pmod{12}$ *and* $n \notin \{264,372\}$ [11];

11. *an* $\tilde{\mathrm{R}}\mathrm{MC}(4,1;n,\lfloor (n+1)/3\rfloor)$ *for* $n \equiv 0,8 \pmod{12}$ *and except for* $n=12$ *and possibly except for* $n \in \{104,108,116,132,156,164,204,212,228,276\}$ [18].

As an immediate consequence of Theorem 3.1 and Lemma 4.1, we have

Theorem 4.2. *A uniform design* $U_n((t+1)^m)$ *exists if the parameters satisfy one of the following*:

1. $n \equiv 0 \pmod 6$, $t=(n-3)/3$, $m \in \{\lfloor (n-1)/2\rfloor, \lfloor n/2\rfloor, n-1\}$ *and* $(n,m) \notin \{(6,2),(12,5),(6,3),(12,6),(3,2),(6,5)\}$;
2. $n \equiv 1 \pmod 3$, $t=(n-4)/3$, $m \in \{\lfloor (n-3)/2\rfloor, \lfloor (n-1)/2\rfloor\}$ *and* $(n,m) \notin \{(10,3),(7,2),(13,5),(19,8),(55,26),(61,29),(67,32),(73,35),(85,41),(109,53),(13,6),(16,7),(67,33)\}$;
3. $n \equiv 2 \pmod 3$, $t=(n-2)/3$, $m \in \{\lfloor n/2\rfloor, \lfloor (n+1)/2\rfloor\}$ *and* $(n,m) \notin \{(5,2),(11,5),(5,3),(11,6)\}$;
4. $n \equiv 0 \pmod{12}$, $t=(n-4)/4$, $m \in \{(n-3)/3, n/3\}$ *and* $(n,m) \notin \{(264,87),(372,123),(12,4),(108,36),(132,44),(156,52),(204,68),(228,76),(276,92)\}$;
5. $n \equiv 4 \pmod{12}$, $t=(n-4)/4$, $m=(n-1)/3$;
6. $n \equiv 8 \pmod{12}$, $t=(n-4)/4$, $m=(n+1)/3$ *and* $(n,m) \notin \{(104,35),(116,39),(164,55),(212,71)\}$.

Now we present some more new infinite classes of $\tilde{\mathrm{R}}$MPs and $\tilde{\mathrm{R}}$MCs. To begin with, we state some terminology and related results from combinatorial design theory, which will be used later.

Definition 4.3. Let n be a positive integer, K be a set of positive integers. A group divisible design of index λ, denoted by (K,λ)-GDD is a triple $(V,\mathscr{G},\mathscr{B})$, which satisfies the following properties:

1. V is a set of n points;
2. $\mathscr{G}$ is a partition of set V into subsets (called groups);
3. $\mathscr{B}$ is a collection of subsets of V with sizes from K (called blocks), such that a group and a block contain at most one common point;
4. every pair of points from distinct groups occurs in exactly λ blocks.

The *group-type* (or *type*) of a GDD is the multiset $\{|G| : G \in \mathscr{G}\}$. Usually, an "exponential notation" is used to describe the type of a GDD: A GDD of type $t_1^{u_1} t_2^{u_2} \cdots t_l^{u_l}$ is a GDD where there are u_i groups of size t_i for $1 \leqslant i \leqslant l$. When $K=\{k\}$, the notation (k,λ)-GDD is used. A *transversal design* $\mathrm{TD}(k,\lambda,n)$ is a (k,λ)-GDD of type n^k. If for all $i=1,2,\ldots,l$, $t_i=1$, then a (K,λ)-GDD of type 1^n is called a *pairwise balanced design*, or a $B(K,\lambda;n)$.

For the existence of pairwise balanced designs, the following result is known (see [12,2]).

Lemma 4.4. *Let* $K=\{5,9,13,17,29\}$. *Then for any* $n \equiv 1 \pmod 4$ *and* $n \neq 33$, *there exists a* $B(K,1;n)$.

Definition 4.5. Let k and λ be positive integers. A $(k;\lambda)$-frame is a triple $(V,\mathscr{G},\mathscr{B})$, where V is a set of cardinality n, $\mathscr{G}$ is a partition of V into subsets (called groups), and $\mathscr{B}$ is a collection of k-subsets of V (called blocks), which satisfies the following properties:

1. $\mathscr{B}$ can be partitioned into partial parallel classes, where each partial parallel class forms a partition of $V \setminus G$ for some $G \in \mathscr{G}$;
2. each unordered pair $\{x,y\}$ of V which does not lie in some group G of $\mathscr{G}$ occurs in precisely λ blocks of $\mathscr{B}$;
3. no unordered pair $\{x,y\}$ of elements of V which lies in some group G of $\mathscr{G}$ also lies in a block of $\mathscr{B}$.

The *type* of the $(k;\lambda)$-frame is the multiset $\{|G| : G \in \mathscr{G}\}$. As with GDDs, if $\mathscr{G}$ contains a_1 groups of size g_1, a_2 groups of size g_2, etc., where $n=a_1g_1+a_2g_2+\cdots+a_sg_s$, then the exponential notation $g_1^{a_1}g_2^{a_2}\cdots g_s^{a_s}$ is used.

The following recursive method is frequently used in constructions of frames. Its proof can be found in Furino et al. [10].

Lemma 4.6. *Let m be a positive integer. If there exist a $(K,1)$-GDD of type $t_1^{u_1}t_2^{u_2}\cdots t_s^{u_s}$ and an "ingredient" (k,λ)-frame of type m^g for any $g \in K$, then there exists a (k,λ)-frame of type $(mt_1)^{u_1}(mt_2)^{u_2}\cdots(mt_s)^{u_s}$.*

Now we employ both direct and recursive constructions to establish a number of new infinity classes of $\tilde{\mathrm{R}}$MPs and $\tilde{\mathrm{R}}$MCs.

Theorem 4.7. *Suppose $n \equiv 4 \pmod 6 \geqslant 16$ and $n \notin \{28,34,40,46,58,70,82,94,142\}$. Then both an $\tilde{\mathrm{R}}\mathrm{MP}(3,2;n,n-3)$ and an $\tilde{\mathrm{R}}\mathrm{MC}(3,2;n,n-2)$ exist.*

Proof. The proof splits into two cases depending on the values of n modulo 12.

Case 1: $n \equiv 4 \pmod{12}$

In this case, we first note that a $(3,2)$-frame of type 12^u for any integer $u \geqslant 4$ has been constructed by Hanani [13]. Let $(Z_u \times Z_{12},\mathscr{G},\mathscr{A})$ be such a frame, where Z_m stands for the residue ring of integers modulo m, and $\mathscr{G}=\{\{i\}\times Z_{12} : i \in Z_u\}$. For any group $\{i\}\times Z_{12}$ $(i \in Z_u)$, we write $\mathscr{A}_1(i),\mathscr{A}_2(i),\ldots,\mathscr{A}_{12}(i)$ for the 12 partial parallel classes of blocks, each of which partitions $Z_u \times Z_{12} \setminus \{i\}\times Z_{12}$. We then add four infinite points $\{\infty_1,\infty_2,\infty_3,\infty_4\}$ to the group $\{i\}\times Z_n$ for any $i \in Z_u$ and construct a resolvable $P(\{3,4\},2;16)$ over $(\{i\}\times Z_{12})\cup\{\infty_1,\infty_2,\infty_3,\infty_4\}$ with 12 distinct parallel classes. The following 5 blocks form a parallel class of the desired packing:

$$\{\infty_1,\quad (i,6),\quad (i,11)\},$$
$$\{\infty_2,\quad (i,2),\quad (i,7)\},$$
$$\{\infty_3,\quad (i,8),\quad (i,10)\},$$
$$\{\infty_4,\quad (i,4),\quad (i,5)\},$$
$$\{(i,0),\quad (i,1),\quad (i,3),\quad (i,9)\}.$$

Cycling the blocks in the second components modulo 12, under the rule $\infty_j + x = \infty_j$, gives its 12 distinct parallel classes. Obviously, these 12 parallel classes of blocks over $(\{i\} \times Z_{12}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ and $\mathscr{A}_j(i)(j = 1, 2, \ldots, 12)$ can be matched into 12 parallel classes over $(Z_u \times Z_{12}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$.

It is important to point out that the leave of the packing over $\{i\} \times Z_u$ consists of the following 4 vertex-disjoint K_3s (triples) and two identical K_4s (quadruples):

$$\begin{array}{llll}
\{(i,0), & (i,4), & (i,8)\}, & \\
\{(i,1), & (i,5), & (i,9)\}, & \\
\{(i,2), & (i,6), & (i,10)\}, & \\
\{(i,3), & (i,7), & (i,11)\}, & \\
\{\infty_1, & \infty_2, & \infty_3, & \infty_4\} \quad (\textit{twice}).
\end{array}$$

Thus, the desired $\tilde{\mathrm{R}}\mathrm{MP}(\{3,4\}, 2; 12u+4, 12u+1)$ over $(Z_u \times Z_{12}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ can be formed if we take all triples and one quadruple in the leave of the packings over $\{i\} \times Z_{12} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ $(i \in Z_u)$ as one more parallel class. Furthermore, we can obtain an $\tilde{\mathrm{R}}\mathrm{MC}(\{3,4\}, 2; 12u + 4, 12u + 2)$ by adding one more parallel class to the packing as follows:

$$\begin{array}{llll}
\{(i,0), & (i,1), & (i,2)\}, & \\
\{(i,3), & (i,4), & (i,5)\}, & \\
\{(i,6), & (i,7), & (i,8)\}, & \\
\{(i,9), & (i,10), & (i,11)\}, & \\
\{\infty_1, & \infty_2, & \infty_3, & \infty_4\},
\end{array}$$

where i runs over Z_u.

Case 2: $n \equiv 10 \,(\mathrm{mod}\, 12)$

In this case, we will first construct a $(3,2)$-frame of type $12^u 18^1$ with $u \geqslant 7$ and $u \neq 10$. Take a TD$(5,1,u)$, which is known to exist for all $u \geqslant 7$, $u \neq 10$ (see [3]), and delete all but 6 points in one group. Suppose x is a certain deleted point. Then treat the truncated blocks and group originally containing x as groups and treat the untruncated groups of the TD$(5,1,u)$ as blocks to form a $\{4,5,u\}$-GDD of type $4^u 6^1$. Now give weight 3 to every point of the GDD, and apply Lemma 4.6 to get a $(3,2)$-frame of type $12^u 18^1$. Here the ingredient $(3,2)$-frames of types 3^4, 3^5 and 3^u required in Lemma 4.6 all exist (see [13]).

Now let $((Z_u \times Z_{12}) \cup (\{u\} \times Z_{18}), \mathscr{G}, \mathscr{A})$ be a $(3,2)$-frame of type $12^u 18^1$, where $\mathscr{G} = \{\{i\} \times Z_{12} : i \in Z_u\} \cup (\{u\} \times Z_{18})$. Add four infinite points $\{\infty_1, \infty_2, \infty_3, \infty_4\}$ to each group of this frame. For any group $\{i\} \times Z_{12} (i \in Z_u)$, we construct a resolvable $P(\{3,4\}, 2; 16)$ as in case 1 over $(\{i\} \times Z_{12}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. For the group $\{u\} \times Z_{18}$, we construct a resolvable $P(\{3,4\}, 2; 22)$ over $(\{u\} \times Z_{18}) \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$, whose blocks are obtained by cycling the following blocks in the second components modulo 18 under the rule $\infty_j + x = \infty_j$:

$$\{\infty_1, \quad (u,3), \quad (u,15)\},$$
$$\{\infty_2, \quad (u,4), \quad (u,12)\},$$
$$\{\infty_3, \quad (u,8), \quad (u,9)\},$$
$$\{\infty_4, \quad (u,6), \quad (u,10)\},$$
$$\{(u,5), \quad (u,7), \quad (u,14)\},$$
$$\{(u,1), \quad (u,11), \quad (u,16)\},$$
$$\{(u,0), \quad (u,2), \quad (u,13), \quad (u,17)\}.$$

The leave of this packing consists of 6 disjoint K_3s (triples) and two identical K_4s as follows:

$$\{(u,0), \quad (u,6), \quad (u,12)\},$$
$$\{(u,1), \quad (u,7), \quad (u,13)\},$$
$$\{(u,2), \quad (u,8), \quad (u,14)\},$$
$$\{(u,3), \quad (u,9), \quad (u,15)\},$$
$$\{(u,4), \quad (u,10), \quad (u,16)\},$$
$$\{(u,5), \quad (u,11), \quad (u,17)\},$$
$$\{\infty_1, \quad \infty_2, \quad \infty_3, \quad \infty_4\} \quad (\textit{twice}).$$

Thus, we can employ the same procedure as that in case 1 to obtain the desired $\tilde{\mathrm{R}}\mathrm{MP}$ and $\tilde{\mathrm{R}}\mathrm{MC}$.

It remains to prove both an $\tilde{\mathrm{R}}\mathrm{MP}(3,2;n,n-3)$ and an $\tilde{\mathrm{R}}\mathrm{MC}(3,2;n,n-2)$ exist for $n=16$ or 22.

When $n=16$, an $\tilde{\mathrm{R}}\mathrm{MP}(3,2;16,13)$ can follow from the resolvable $P(\{3,4\},2;16)$ over $\{i\}\times Z_{12}$ for some $i\in Z_u$ constructed in case 1 by adding one more parallel class, which consists of all triples in its leave together with a quadruple of four infinite points.

An $\tilde{\mathrm{R}}\mathrm{MC}(3,2;16,14)$ can be obtained by adding one more parallel class to the $\tilde{\mathrm{R}}\mathrm{MP}(3,2;16,13)$:

$$\{(i,0), \quad (i,1), \quad (i,2)\},$$
$$\{(i,3), \quad (i,4), \quad (i,5)\},$$
$$\{(i,6), \quad (i,7), \quad (i,8)\},$$
$$\{(i,9), \quad (i,10), \quad (i,11)\},$$
$$\{\infty_1, \quad \infty_2, \quad \infty_3, \quad \infty_4\}.$$

Similarly, an $\tilde{\mathrm{R}}\mathrm{MP}(3,2;22,19)$ and $\tilde{\mathrm{R}}\mathrm{MC}(3,2;22,20)$ can be obtained from the resolvable $P(\{3,4\},2;22)$ constructed in Case 2. $\square$

Applying Theorem 3.1 to Theorem 4.7 gives the following new uniform designs.

Table 1
$U_{16}(5^{14})$

Row	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	5	1	3	5	3	2	1	4	4	5	2	5	1	1
1	5	5	1	3	5	3	2	1	4	4	5	2	2	1
2	2	5	5	1	3	5	3	2	1	4	4	5	3	1
3	5	2	5	5	1	3	5	3	2	1	4	4	4	2
4	4	5	2	5	5	1	3	5	3	2	1	4	1	2
5	4	4	5	2	5	5	1	3	5	3	2	1	2	2
6	1	4	4	5	2	5	5	1	3	5	3	2	3	3
7	2	1	4	4	5	2	5	5	1	3	5	3	4	3
8	3	2	1	4	4	5	2	5	5	1	3	5	1	3
9	5	3	2	1	4	4	5	2	5	5	1	3	2	4
10	3	5	3	2	1	4	4	5	2	5	5	1	3	4
11	1	3	5	3	2	1	4	4	5	2	5	5	4	4
∞_1	1	1	1	1	1	1	1	1	1	1	1	1	5	5
∞_2	2	2	2	2	2	2	2	2	2	2	2	2	5	5
∞_3	3	3	3	3	3	3	3	3	3	3	3	3	5	5
∞_4	4	4	4	4	4	4	4	4	4	4	4	4	5	5

Theorem 4.8. *A uniform design* $U_n(((n-1)/3)^m)$ *exists if* $n \equiv 4 \pmod 6 \geqslant 16$, $m \in \{n-3, n-2\}$ *and* $(n,m) \notin \{(28,25),(28,26),(34,31),(34,32),(40,37),(40,38),(46,43),(46,44),(58,55),(58,56),(70,67),(70,68),(82,79),(82,80),(94,91),(94,92),(142,139),(142,140)\}$.

As an illustration of Theorem 4.7, we provide the following example.

Example 4.9. The $\tilde{\mathrm{R}}\mathrm{MC}(3,2;16,14)$ constructed in the proof of Theorem 4.7 gives us a uniform design $U_{16}(5^{14})$ by Theorem 3.1, which is listed in Table 1.

Before we present another construction, we first establish the following lemma.

Lemma 4.10. *For any* $u \in \{5,9,13,17,29\}$, *there exists a* $(4,2)$-*frame of* 3^u, *whose partial parallel classes are mutually distinct.*

Proof. For these frames, our constructions are as follows.

When $u \in \{5,13,17,29\}$, we take point set $V = Z_u \times Z_3$ and group set $\mathscr{G} = \{\{i\} \times Z_3\colon i \in Z_u\}$. For the required blocks, we first develop (mod 3) the $(u-1)/2$ blocks listed below in the second components to form two initial partial parallel classes. Then we develop (mod u) the blocks in these two initial classes in the first components to get $2u$ desired partial parallel classes.

$u = 5$:

$$\{(1,0),\quad (2,2),\quad (3,2),\quad (4,0)\},\qquad \{(1,0),\quad (2,1),\quad (3,1),\quad (4,0)\}.$$

$u=13$:

$$\begin{array}{llllllll}
\{(1,0), & (5,2), & (8,2), & (12,0)\}, & \{(1,0), & (5,1), & (8,1), & (12,0)\},\\
\{(2,0), & (3,2), & (10,2), & (11,0)\}, & \{(2,0), & (3,1), & (10,1), & (11,0)\},\\
\{(4,0), & (6,2), & (7,2), & (9,0)\}, & \{(4,0), & (6,1), & (7,1), & (9,0)\}.
\end{array}$$

$u=17$:

$$\begin{array}{llllllll}
\{(1,0), & (4,2), & (13,2), & (16,0)\}, & \{(1,0), & (4,1), & (13,1), & (16,0)\},\\
\{(2,0), & (8,2), & (9,2), & (15,0)\}, & \{(2,0), & (8,1), & (9,1), & (15,0)\},\\
\{(3,0), & (5,2), & (12,2), & (14,0)\}, & \{(3,0), & (5,1), & (12,1), & (14,0)\},\\
\{(6,0), & (7,2), & (10,2), & (11,0)\}, & \{(6,0), & (7,1), & (10,1), & (11,0)\}.
\end{array}$$

$u=29$:

$$\begin{array}{llllllll}
\{(1,0), & (2,2), & (27,2), & (28,0)\}, & \{(1,0), & (2,1), & (27,1), & (28,0)\},\\
\{(3,0), & (13,2), & (16,2), & (26,0)\}, & \{(3,0), & (13,1), & (16,1), & (26,0)\},\\
\{(4,0), & (8,2), & (21,2), & (25,0)\}, & \{(4,0), & (8,1), & (21,1), & (25,0)\},\\
\{(5,0), & (10,2), & (19,2), & (24,0)\}, & \{(5,0), & (10,1), & (19,1), & (24,0)\},\\
\{(6,0), & (12,2), & (17,2), & (23,0)\}, & \{(6,0), & (12,1), & (17,1), & (23,0)\},\\
\{(7,0), & (14,2), & (15,2), & (22,0)\}, & \{(7,0), & (14,1), & (15,1), & (22,0)\},\\
\{(9,0), & (11,2), & (18,2), & (20,0)\}, & \{(9,0), & (11,1), & (18,1), & (20,0)\}.
\end{array}$$

When $u=9$, we take point set $V=\{(Z_3\times Z_3)\times Z_3\}$ and group set $\mathcal{G}=\{\{(i,j)\}\times Z_3 : (i,j)\in Z_3\times Z_3\}$. As above, we first develop (mod 3) the following 4 blocks in the last components to form 2 initial partial parallel classes. Then we develop them (mod (3, 3)) in the first two components to get 18 partial parallel classes as desired.

$$\begin{array}{llll}
\{((0,1),0), & ((1,2),2), & ((2,1),2), & ((0,2),0)\},\\
\{((1,0),0), & ((1,1),2), & ((2,2),2), & ((2,0),0)\},\\
\{((0,1),0), & ((1,2),1), & ((2,1),1), & ((0,2),0)\},\\
\{((1,0),0), & ((1,1),1), & ((2,2),1), & ((2,0),0)\}.
\end{array}$$ □

Theorem 4.11. *If* $n\equiv 4 \pmod{12}$ *and* $n\neq 100$, *then an* $\tilde{\mathrm{R}}\mathrm{B}(4,2;n,(2(n-1))/3)$ *exists.*

Proof. For each stated value of n, we write $n=3u+1$. It is known from Lemma 4.4 that a PBD, $B(\{5,9,13,17,29\},1;u)$, exists whenever $u\equiv 1 \pmod 4$ and $u\neq 33$. Regard such a PBD as a $(\{5,9,13,17,29\},1)$-GDD of type 1^u and apply Lemma 4.6 with $m=3$, where the ingredient frames required are provided in Lemma 4.10. This yields a $(4,2)$-frame of type 3^u.

Now we add one infinite point ∞ to each group G of a $(4,2)$-frame of type 3^u. Take two copies of quadruple $G\cup\{\infty\}$ twice. Then we use them and the two partial parallel classes of this frame which partition $V \setminus G$ to form two parallel classes over $V \cup \{\infty\}$, where V is the point set of this frame. This creates an $\tilde{\mathrm{R}}\mathrm{B}(4,2;3u+1,2u)$. $\square$

Remark. The $\tilde{\mathrm{R}}\mathrm{B}(4,2;3u+2,2u)$ constructed in the proof of Theorem 4.11 does not contain identical parallel classes. Though an $\mathrm{RB}(4,1;n)$ for any $n \equiv 4 \pmod{12}$ is well known to exist, we cannot take its two copies to obtain an $\mathrm{RB}(4,2;n)$, since its derived uniform design will be fully aliased.

Applying Theorems 3.1 to 4.11, we obtain the following new uniform designs.

Theorem 4.12. *A uniform design* $U_n((n/4)^m)$ *exists if* $n \equiv 4 \pmod{12}$, $m = 2(n-1)/3$ *and* $(n,m) \neq (100,66)$.

To obtain further classes, we need the following lemma.

Lemma 4.13. *If there exists an* $\tilde{\mathrm{R}}\mathrm{MP}(k,\lambda;n,m)$ (*resp.* $\tilde{\mathrm{R}}\mathrm{MC}(k,\lambda;n,m)$) *with* $n \equiv 0 \pmod{k}$, *then there exists an* $\tilde{\mathrm{R}}\mathrm{MP}(k,\lambda;n-1,m)$ (*resp.* $\tilde{\mathrm{R}}\mathrm{MC}(k,\lambda;n-1,m)$).

Proof. Suppose that $(V,\mathscr{B})$ is an $\tilde{\mathrm{R}}\mathrm{MP}(k,1;n,m)$ (resp. $\tilde{\mathrm{R}}\mathrm{MC}(k,1;n,m)$) with $n \equiv 0 \pmod{k}$. Since $n \equiv 0 \pmod{k}$, all blocks in $\mathscr{B}$ have the same size k by definition. So we can take $V^\star = V \setminus \{x\}$ and $\mathscr{B}^\star = \{B \setminus \{x\} | B \in \mathscr{B}\}$ to get an $\tilde{\mathrm{R}}\mathrm{MP}(k,1;n-1,m)$ (resp. $\tilde{\mathrm{R}}\mathrm{MC}(k,1;n-1,m)$), $(V^\star,\mathscr{B}^\star)$, where x is an arbitrary fixed point in V. $\square$

Making use of Lemmas 4.13, 4.1 and Theorem 4.11, we have the following result.

Theorem 4.14. *There exist*:

1. *an* $\tilde{\mathrm{R}}\mathrm{B}(3,2;n,n)$ *if* $n \equiv 2 \pmod 3$ *and* $n \geqslant 8$;
2. *an* $\tilde{\mathrm{R}}\mathrm{B}(4,1;n,n/3)$ *if* $n \equiv 3 \pmod{12}$;
3. an $\tilde{\mathrm{R}}\mathrm{MP}(4,1;n,(n-2)/3)$ *if* $n \equiv 11 \pmod{12}$ *and* $n \notin \{263,371\}$;
4. *an* $\tilde{\mathrm{R}}\mathrm{MC}(4,1;n,\lfloor (n+2)/3 \rfloor)$ *if* $n \equiv 7,11 \pmod{12}$ *and* $n \notin \{11,103,107,115,131,155,163,203,211,227,275\}$;
5. *an* $\tilde{\mathrm{R}}\mathrm{MP}(4,2;n,2n/3)$ *if* $n \equiv 3 \pmod{12}$ *and* $n \neq 99$.

Applying Theorem 3.1 to Theorem 4.14, we obtain the following new uniform designs.

Theorem 4.15. *A uniform design* $U_n((t+1)^m)$ *exists if the parameters satisfy one of the following*:

1. $n \equiv 2 \pmod 3$, $t = (n-2)/3$, $m = n$ *and* $(n,m) \neq (5,5)$;
2. $n \equiv 3 \pmod{12}$, $t = (n-3)/4$, $m \in \{n/3, 2n/3\}$ *and* $(n,m) \neq (99,66)$;
3. $n \equiv 7 \pmod{12}$, $t = (n-3)/4$, $m = (n+2)/3$ *and* $(n,m) \notin \{(103,35),(115,39),(163,55),(211,71)\}$;
4. $n \equiv 11 \pmod{12}$, $t = (n-3)/4$, $m \in \{(n-2)/3,(n+1)/3\}$ *and* $(n,m) \notin \{(263,87),(371,123),(11,4),(107,36),(131,44),(155,52),(203,68),(227,76),(275,92)\}$.

5. Concluding remarks

All the existing uniform designs $U_n(q^m)$ are constructed based on U-type designs, which require the number of experimental runs n to be a multiple of the number of factor levels q. In the present paper, we develop a general construction method for uniform designs via resolvable packings and coverings, which works for the case where n is not a multiple of q. Several series of new uniform designs are then given. It is hoped that some more infinite classes of new uniform designs can be produced by means of resolvable packings and coverings.

It is known that (t,m,s)-nets proposed by Niederreiter [19] have a good structure and can be utilized to construct uniform designs. A further study is encouraged. A (t,m,s)-net in base b involves b^m points. When the number of runs $n \neq b^m$, we have to employ other ways to generate uniform designs.

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