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THE CONVERGENCE OF CESARO AVERAGES FOR CERTAIN NONSTATIONARY MARKOV CHAINS

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If P is a stochastic matrix corresponding to a stationary, irreducible, positive persistent Markov chain of period $d>1$, the powers P^n will not converge as $n\to\infty$. However, the subsequences P^{nd+k} for $k=0,1,\dots,d-1$, and hence Cesaro averages $\sum_{k=1}^{n} P^k/n$, will converge. In this paper we determine classes of nonstationary Markov chains for which the analogous subsequences and/or Cesaro averages converge and consider the rates of convergence. The results obtained are then applied to the analysis of expected average cost.

periodic
strongly ergodic
nonstationary Markov chain
rates of convergence
expected average cost

1. Notation, definitions, and the stationary case

If $A=[a_{ij}]$ is a real matrix defined on $S\times S$ where $S=\{1,2,\dots\}$, we define the norm $\|\cdot\|$ of A as follows:

$$\|A\| = \sup_{i\in S}\sum_{j\in S}|a_{ij}|.$$

The convergence results in this paper will be given in terms of $\|\cdot\|$, which, in addition to being a norm, satisfies the following two conditions (see [3]):

(a) $\|AB\|\leqslant\|A\|\|B\|$ for all matrices A and B;

(b) $\|P\|=1$ for any stochastic matrix P.

These two properties will be used repeatedly in this paper.

Let $\{X_t\}_{t=1}^{\infty}$ be a Markov chain with transition matrices $\{P_t\}_{t=1}^{\infty}$ defined on the countable state space $S=\{1,2,\dots\}$. We will talk interchangeably about $\{X_t\}_{t=1}^{\infty}$ and $\{P_t\}_{t=1}^{\infty}$, where $\{P_t\}_{t=1}^{\infty}$ will be denoted by simply P when the Markov chain is stationary. For $m\geqslant 0$ define the product $P_{m+1}P_{m+2}\cdots P_{m+n}$ by $P^{m,m+n}$ (or by P^n when the Markov chain is stationary). Next, define a "constant" matrix Q to be a matrix each row of which is the same. Then the sequence $\{P_t\}_{t=1}^{\infty}$ is said to be strongly ergodic (with constant stochastic matrix Q) if for every $m\geqslant 0$:

$$\lim_{n\to\infty}\|P^{m,m+n}-Q\|=0.$$

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In [2] is studied the rate of convergence of $P^{m,m+n}$ to Q for certain stationary and nonstationary Markov chains. In this paper we will consider the convergence and rates of convergence of subsequences $P^{m,m+nd+k}$ and Cesaro averages $\sum_{t=1}^{n} P^{m,m+t}/n$ for certain Markov chains for which $P^{m,m+n}$ does not necessarily converge. In the remainder of this section, we will consider the stationary case. In section 2 and section 3 we will consider certain nonstationary Markov chains. Finally, in section 4 we will give applications to the analysis of expected average cost.

In order to establish notation, recall that an irreducible stochastic matrix, P, of period d $(d \geq 1)$ partitions the state space S into d disjoint subspaces $C_0, C_1, \ldots, C_{d-1}$ and that the matrix P^d yields d stochastic matrices $\{T_l\}_{l=0}^{d-1}$, where T_l is defined on C_l. If the irreducible periodic matrix P is finite, then each T_l is automatically strongly ergodic, but if P is infinite the strong ergodicity of each T_l is not guaranteed. In this paper we will deal only with an irreducible stochastic matrix, P, of period d in which T_l is strongly ergodic for $l = 0, 1, \ldots, d-1$; we henceforth refer to this type of stochastic matrix as a periodic strongly ergodic (or simply "PSE") matrix.

We note that, if any one of $T_0, T_1, \ldots, T_{d-1}$ is strongly ergodic, then the others are also strongly ergodic. This can be shown as follows. The matrices T_l have the cyclic structure

$$T_0 = P_0P_1 \cdots P_{d-1}, \qquad T_1 = P_1P_2 \cdots P_{d-1}P_0,$$

$$T_2 = P_2 \cdots P_{d-1}P_0P_1, \ldots, T_{d-1} = P_{d-1}P_0P_1 \cdots P_{d-2},$$

where each P_j is a possibly non-square stochastic matrix and where P_j gives one step transition probabilities from C_j to C_{j+1} for $j = 0, 1, \ldots, d-2$, and P_{d-1} gives one step transition probabilities from C_{d-1} to C_0. Next, if $R = [r_{ij}]$ is a possibly non-square stochastic matrix, define $\delta(R)$ by

$$\delta(R) = \tfrac{1}{2} \sup_{i,k} \sum_{j \in S} | r_{ij} - r_{kj} |.$$

By [3] it follows that

(a) $\delta(AB) \leq \delta(A)\delta(B)$ and $\delta(A) \leq 1$ for possibly non-square stochastic matrices A and B;

(b) a stationary Markov chain P is strongly ergodic if and only if there exists a finite integer n such that $\delta(P^n) < 1$.

Assuming, then, that $T_0 = P_0P_1 \cdots P_{d-1}$ is strongly ergodic, there exists an n_0 such that $\delta((P_0P_1 \cdots P_{d-1})^{n_0}) < 1$. Hence, for $j = 1, 2, \ldots, d-1$,

$$\begin{aligned} \delta(T_j^{n_0+1}) &= \delta((P_j \cdots P_{j-1})^{n_0+1}) \\ &= \delta\left(\left(\prod_{i=j}^{d-1} P_i\right)(P_0P_1 \cdots P_{d-1})^{n_0}\left(\prod_{i=0}^{j-1} P_i\right)\right) \\ &\leq \prod_{i=j}^{d-1} \delta(P_i)\delta((P_0P_1 \cdots P_{d-1})^{n_0}) \prod_{i=0}^{j-1} \delta(P_i) < 1, \end{aligned}$$

which implies that T_j is strongly ergodic.

Lemma 1.1. *Let P be a PSE matrix of period d, and let Q be the matrix each row of which is the eigenvector $\psi = (\psi_1, \psi_2, \dots)$ of P, which solves uniquely the system of equations $\psi P = \psi$ and $\sum_{i \in S} \psi_i = 1$. Then,*

(a) *for $k = 0, 1, \dots, d-1$ there exist finite constants C_0 and β_0 $(0 < \beta_0 < 1)$ and a matrix Q_k, with $\sum_k Q_k = dQ$, such that*

$$\|P^{nd+k} - Q_k\| \leq C_0 \beta_0^n \quad \text{for } n \geq 1;$$

(b) $\lim_{n \to \infty} \|\sum_{t=1}^{n} P^t/n - Q\| = 0.$

Proof. Since P is a PSE matrix, for $l = 0, 1, \dots, d-1$, T_l is strongly ergodic, and hence by [2] there exist constants a_l and δ_l $(0 < \delta_l < 1)$ and a constant matrix L_l such that:

$$\|T_l^n - L_l\| \leq a_l \delta_l^n \quad \text{for } n \geq 1. \tag{1.1}$$

Since the non-zero elements of P^{nd} can be grouped into the blocks T_l^n, part (a) follows for $k = 0$. Moreover, part (a) also follows with $Q_k = Q_0 P^k$, since

$$\|P^{nd+k} - Q_0 P^k\| \leq \|P^{nd} - Q_0\| \|P^k\|$$
$$\leq C_0 \beta_0^n \quad \text{for } n \geq 1.$$

By using properties of the norm $\|\cdot\|$, part (b) easily follows from part (a). For the details see [1].

Whereas the rate of convergence in part (a) of Lemma 1.1 is geometric, the rate of the convergence in part (b) is only guaranteed to be "$1/n$", as the following example shows.

Example 1.1. Let $P = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. In this case,

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \left\| \sum_{t=1}^{n} P^t/n - Q \right\| = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1/n & \text{if } n \text{ is odd.} \end{cases}$$

2. The convergent case

In view of the results given in Lemma 1.1 for a stationary Markov chain described by a PSE matrix, we will consider in this section a nonstationary Markov chain $\{P_t\}_{t=1}^{\infty}$ such that $\lim_{t \to \infty} \|P_t - P\| = 0$, where P is a PSE matrix. For such a nonstationary chain it is natural to first consider the convergence of subsequences of the form $P^{m,m+nd+k} = P_{m+1} P_{m+2} \cdots P_{m+nd+k}$ as $n \to \infty$. If these subsequences converge for $k = 0, 1, \dots, d-1$, then Cesaro averages $\sum_{t=1}^{n} P^{m,m+t}/n$ also converge. If we make the restrictive assumption that the matrices of the sequence $\{P_t\}_{t=1}^{\infty}$ all have 0's in the same positions for $t \geq T^* < \infty$, then under the assumption that $\lim_{t \to \infty} \|P_t - P\| = 0$, P a PSE matrix, it can be shown (see [1]) that the subsequences $\{P^{m,m+nd+l}\}_{n=1}^{\infty}$ converge uniformly in m for $l = 0, 1, \dots, d-1$ (and by part (a) of

Lemma 1.1 it also follows by applying a result in [2] that the rate at which subsequences converge is the slower of a certain geometric rate and the rate at which P_t converges to P). However, if we do not make the above restrictive assumption, subsequences may not converge, as the following example shows.

Example 2.1. Construct a sequence of stochastic matrices $\{P_t\}_{t=1}^{\infty}$ using the matrices

$$Q_t = \begin{bmatrix} \frac{1}{t} & 1-\frac{1}{t} \\ 1 & 0 \end{bmatrix}, \qquad R_t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad S_t = \begin{bmatrix} 0 & 1 \\ 1-\frac{1}{t} & \frac{1}{t} \end{bmatrix}.$$

Noting that

$$Q_tR_t = \begin{bmatrix} 1-\frac{1}{t} & \frac{1}{t} \\ 0 & 1 \end{bmatrix},$$

it can be shown that $\{Q_tR_t\}_{t=1}^{\infty}$ is strongly ergodic with constant matrix $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Noting that

$$S_tR_t = \begin{bmatrix} 1 & 0 \\ \frac{1}{t} & 1-\frac{1}{t} \end{bmatrix},$$

it can similarly be shown that $\{S_tR_t\}_{t=1}^{\infty}$ is strongly ergodic with constant matrix $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. Hence, by letting $P_t = R_t$ for t even and by appropriate selection of P_t for t odd, the sequence $(P_1P_2)(P_3P_4)\cdots(P_{2n-1}P_{2n})$ can be made to oscillate from near $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ to near $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ as $n \to \infty$. Therefore, even though $\lim_{t\to\infty} \|P_t - \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\| = 0$, the sequence $P_1P_2\cdots P_{2n}$ can be made not to converge as $n \to \infty$.

Fortunately, in spite of the fact that subsequences do not in general converge, Cesaro averages do in general converge. Theorem 2.1 yields not only the convergence of the Cesaro averages but also an upper bound on the rate at which they converge. In proving Theorem 2.1 we will need the following lemma.

Lemma 2.1. *Let $\{P_t\}_{t=1}^{\infty}$ be a sequence of stochastic matrices. Then, if k is a fixed integer, for $n \geq 0$,*

$$\|P^{n,n+k} - P^k\| \leq \sum_{r=1}^{k} \|P_{n+r} - P\|.$$

Proof. The proof follows by induction and using properties of $\|\cdot\|$. In particular, for $k = 2$,

$$\begin{aligned} \|P_{n+1}P_{n+2} - P^2\| &\leq \|P_{n+1}P_{n+2} - P_{n+1}P\| + \|P_{n+1}P - P^2\| \\ &\leq \|P_{n+2} - P\| + \|P_{n+1} - P\|. \end{aligned}$$

Theorem 2.1. *Let $\{P_t\}_{t=1}^{\infty}$ be a nonstationary Markov chain and let P be a PSE matrix with left eigenvector ψ, and Q the matrix each row of which is ψ.*

(a) *If* $\lim_{t\to\infty} \|P_t - P\| = 0$, *then*

$$\lim_{n\to\infty} \sup_{m\geq 0} \left\| \sum_{t=1}^{n} P^{m,m+t}/n - Q \right\| = 0.$$

(b) *If there exists a finite constant G such that*

$$\|P_t - P\| \leq G/t^{\alpha} \quad \text{for } t \geq 1, \tag{2.1}$$

then:

(1) *If $\alpha > 1$ there is, for any $\varepsilon > 0$, a $D(\varepsilon)$ such that*

$$\sup_{m\geq 0} \left\| \sum_{t=1}^{n} P^{m,m+t}/n - Q \right\| \leq \frac{D(\varepsilon)}{n^{1-\varepsilon}} \quad \text{for } n \geq 1,$$

(2) *if $\alpha \in (0,1]$, there is, for every $\varepsilon > 0$, a $D(\varepsilon)$ such that*

$$\sup_{m\geq 0} \left\| \sum_{t=1}^{n} P^{m,m+t}/n - Q \right\| \leq \frac{D(\varepsilon)}{n^{\alpha-\varepsilon}} \quad \text{for } n \geq 1.$$

Proof. We first establish an inequality important for proving both parts (a) and (b). To this end let $L = L(n) = [n/d]$ (i.e., $n = Ld + r$, $0 \leq r < d$), and write

$$\sum_{t=1}^{n} P^{m,m+t} - nQ = \sum_{t=1}^{Jd+r} (P^{m,m+t} - Q) \tag{2.2a}$$

$$+ \sum_{t=Jd+r+1}^{n} (P^{m,m+t} - Q), \tag{2.2b}$$

the three useful features of this partition being that J is to be small enough for (2.2a) to contain substantially fewer terms than (2.2b), that (2.2b) contains a number of terms that is a multiple of d, and that all terms $P^{m,m+t}$ of (2.2b) contain enough factors (i.e., at least $Jd + r + 1$) to allow factoring out the matrices S_j below.

In view of (2.2), we now write

$$\left\| \sum_{t=1}^{n} P^{m,m+t} - nQ \right\| \leq \left\| \sum_{t=1}^{Jd+r} (P^{m,m+t} - Q) \right\| \tag{2.3a}$$

$$+ \sum_{j=J}^{L-1} \left\| \sum_{k=1}^{d} P^{m,m+jd+k+r} - dQ \right\| \tag{2.3b}$$

and proceed to cut down the unmanageably large number of factors of the terms $P^{m,m+jd+k+r}$ of (2.3b) to no more than $M + d - 1$, where M is eventually to be chosen appropriately under the restriction that $M \leq Jd$. This cutting down is done by writing $Q = S_j Q$ in the jth term of (2.3b), where S_j is the stochastic matrix $P^{m,m+jd+1+r-M}$, and then factoring out S_j. Using the properties of $\|\cdot\|$ cited in section 1, we are thereby led to the assertion that (2.2b) is no greater than

$$\sum_{j=J}^{L-1} \left\| \sum_{k=1}^{d} P^{m+jd+1+r-M,m+jd+k+r} - dQ \right\|. \tag{2.4}$$

The terms $P^{m+jd+1+r-M,m+jd+k+r}$ of (2.4) now contain $M+k-1$ factors, and we add and subtract $\sum_{k=1}^{d} P^{M+k-1}$ in order to create two terms intended, respectively, to exploit our assumptions about the behavior of $\|P_t - P\|$ and Lemma 1.1, thus bounding (2.4) by

$$\sum_{j=J}^{L-1} \left\| \sum_{k=1}^{d} (P^{m+jd+1+r-M,m+jd+k+r} - P^{M+k-1}) \right\| + \\ + \sum_{j=J}^{L-1} \left\| \sum_{k=1}^{d} P^{M+k-1} - dQ \right\| \tag{2.5}$$

$$\leq d \sum_{j=J}^{L-1} \sum_{v=1}^{M+d-1} \|P_{m+jd+1+r-M+v} - P\| \\ + (L-J) \left\| \sum_{k=1}^{d} P^{M+k-1} - dQ \right\|, \tag{2.6}$$

where we have used Lemma 2.1 in going from (2.5) to (2.6).

Finally, since (2.6) bounds (2.3b),

$$\left\| \sum_{t=1}^{n} P^{m,m+t}/n - Q \right\| \leq \frac{1}{n} \left\| \sum_{t=1}^{Jd+r} (P^{m,m+t} - Q) \right\| \tag{2.7a}$$

$$+ \frac{d}{n} \sum_{j=J}^{L-1} \sum_{v=1}^{M+d-1} \|P_{m+jd+1+r-M+v} - P\| \tag{2.7b}$$

$$+ \left(\frac{L-J}{n}\right) \cdot \left\| \sum_{k=1}^{d} P^{M+k-1} - dQ \right\|. \tag{2.7c}$$

To prove part (a), let $\eta > 0$ be given. We first choose, by part (a) of Lemma 1.1, an M large enough so that $\|\sum_{k=1}^{d} P^{M+k-1} - dQ\| \leq \eta$. Then choose J large enough so that $Jd > M$ and $\|P_t - P\| \leq \eta/M + d - 1$ for $t > Jd - M$, which we can do since we are assuming that $\lim_{t\to\infty} \|P_t - P\| = 0$. Finally, choose N large enough that $2(J+1)d/N \leq \eta$. Then, for $n > N$, each of the three terms of (2.7) do not exceed η, establishing part (a).

To prove (2) of part (b), we note from part (a) of Lemma 1.1 that there are constants C and β, $C > 0$ and $0 < \beta < 1$, such that

$$\left\| \sum_{k=1}^{d} P^{M+k-1} - dQ \right\| \leq C\beta^{M},$$

so that, when $M = M(n) = [-\alpha \ln n / \ln \beta] + 1$,

$$\left\| \sum_{k=1}^{d} P^{M+k-1} - dQ \right\| \leq Cn^{-\alpha},$$

and (2.7c) is of order no greater than $n^{-\alpha}$. Next, staying with $M(n)$, and also letting $J = J(n) = M(n) + 1$, we have, for any $\varepsilon > 0$, that, for large enough n,

$$\frac{d}{n} \sum_{j=J}^{L-1} \sum_{v=1}^{M+d-1} \|P_{m+jd+1-r-M+v} - P\| \leq$$

$$\frac{d}{n}\sum_{j=J}^{L-1}(M+d)(G)(jd-M)^{-\alpha} \leq \frac{Gd(M+d)}{n}\cdot\sum_{j=1}^{n} j^{-\alpha}$$

in view of assumption (2.1).

But $M+d$ is of order $\ln n$ and $\sum_{j=1}^{n} j^{-\alpha}$ is of order $n^{1-\alpha}$, and thus (2.7b) is of order less than $n^{-\alpha+\varepsilon}$.

Finally, since J is of order $\ln n$, (2.7a) is of order no greater than $n^{-1+\varepsilon}$.

Our three order assertions concerning (2.7) establish (2) of part (b). (1) follows in similar fashion.

If the results in (a) or (b) of Theorem 2.1 hold for a certain Markov chain, we thus recognize that the left eigenvector $\psi=(\psi_1,\psi_2,\dots)$ of P serves as the "Cesaro long-run distribution" of the Markov chain, so Theorem 2.1 can be interpreted as a statement about robustness. It says that if the transition matrices $\{P_t\}_{t=1}^{\infty}$ of a nonstationary Markov chain $\{N_t, t=0,1,\dots\}$ converge as $t\to\infty$ to a PSE matrix P describing some stationary Markov chain $\{S_t, t=0,1,\dots\}$, then $\{N_t, t=0,1,\dots\}$ has the same Cesaro long run distribution as $\{S_t, t=0,1,\dots\}$.

3. The cyclic case

Another nonstationary Markov chain $\{P_t\}_{t=1}^{\infty}$ for which subsequences converge even though the chain is not strongly ergodic is the chain in which the P_t's repeat themselves in a cyclic fashion, that is, $P_{td+l}=P_l$ for $l=1,\dots,d$ and $t=0,1,2,\dots$. This situation is studied in Theorem 3.1.

Theorem 3.1. *Let $\{P_t\}_{t=1}^{\infty}$ be a nonstationary Markov chain such that $P_{td+l}=P_l$ for $l=1,\dots,d$ and $t=0,1,2,\dots$. Assume that $R_1=P_1P_2\cdots P_d$ is strongly ergodic with constant matrix Q_1. Moreover, for $j=2,3,\dots,d$ let $Q_j=Q_1\prod_{i=1}^{j-1}P_i$ and define $\bar{Q}=(1/d)\sum_{l=1}^{d}Q_l$. Then*

(a) *if $R_2=P_2P_3\cdots P_dP_1,\dots,R_d=P_dP_1\cdots P_{d-1}$, there exist finite constants C and β $(0<\beta<1)$ such that for $n\geq 2$:*

$$\|R_1^n-Q_1\|\leq C\beta^n, \tag{3.1}$$

$$\|R_j^n-Q_j\|\leq C\beta^{n-1} \quad \text{for } j=2,\dots,d; \tag{3.2}$$

(b) *for $1\leq l\leq k$: $\sup_{m\geq 0}\|P^{m,m+nd+l}-Q_{k(l,m)}\|\leq D\lambda^n$ for $n\geq 1$; where $k(l,m)$ is appropriately chosen from $\{1,2,\dots,d\}$ and $0<\lambda<1$.*

(c) *there exists a finite constant g such that*

$$\sup_{m\geq 0}\left\|\sum_{t=1}^{n}P^{m,m+t}/n-\bar{Q}\right\|\leq g/n \quad \text{for } n\geq 1.$$

Proof. First, since R_1 is strongly ergodic, (3.1) follows by a result in [2]. Next, (3.2) follows since for $j=2,\dots,d$,

$$\|R_j^n - Q_j\| = \left\|\left(\prod_{i=j}^{d} P_i\right) R_1^{n-1} \left(\prod_{i=1}^{j-1} P_i\right) - \left(\prod_{i=j}^{d} P_i\right) Q_1 \left(\prod_{i=1}^{j-1} P_i\right)\right\|$$
$$\leq \|R_1^{n-1} - Q_1\| \leq C\beta^{n-1} \quad \text{for } n \geq 2.$$

Next, part (b) follows from part (a) by consideration of the cyclic order in which the (finite number of) matrices $P_1, P_2, \ldots, P_d$ are assumed by $\{P_t\}_{t=1}^{\infty}$. Finally, part (c) follows from part (b).

Before leaving this section, let us consider modifying the situation in Theorem 3.1 by assuming, instead of $P_{td+l} = P_l$, that $\lim_{t\to\infty} \|P_{td+l} - P_l\| = 0$ for $l = 1, \ldots, d$. In this case, the quantities in parts (b) and (c) still converge (see [1]). Moreover, by a result in [2], the rate of convergence of the quantities in (b) can be shown to be the slower of a certain geometric rate and a function of the rate that P_{td+l} converges to P_l. Rate results of the type obtained in Theorem 2.1 also apply to the quantity in part (c).

4. Applications to analysis of the expected average cost

The following theorem considers the "expected average cost" of a Markov chain which incurs a cost $C_t(j)$ if the state of the Markov chain at time t is j.

Theorem 4.1. *Consider a Markov chain $\{X_t, t = 0, 1, \ldots\}$ having transition matrices $\{P_t\}_{t=1}^{\infty}$ defined on a countable state space $S = \{1, 2, \ldots\}$ where there exists a constant matrix Q, each row of which we denote by $\psi = (\psi_1, \psi_2, \ldots)$, such that*

$$\lim_{n\to\infty} \sup_{m\geq 0} \left\| \sum_{t=1}^{n} P^{m,m+t}/n - Q \right\| = 0. \tag{4.1}$$

Moreover, assume that for each $j \in S$ there exists a sequence $\{C_t(j), t \geq 0\}$ of uniformly bounded numbers and a constant $C(j)$ such that

$$\lim_{t\to\infty} \|C_t - C\| = 0, \tag{4.2}$$

where C_t and C are column vectors with jth elements of, respectively, $C_t(j)$ and $C(j)$. Then, letting

$$\phi^{m,m+n}(i) = \mathrm{E}\left\{ \sum_{t=m+1}^{m+n} C_t(X_t)/n \mid X_m = i \right\} \quad \textit{for each } i \in S,$$

$\phi^{m,m+n}$ be a column vector with ith element $\phi^{m,m+n}(i)$, and ϕ be a column vector with elements all equal to $\sum_{j\in S} C(j)\psi_j$, the following results hold.

(a) $\lim_{n\to\infty} \sup_{m\geq 0} \|\phi^{m,m+n} - \phi\| = 0$.

(b) *If there exist constants M_1 and M_2 such that for any $\varepsilon > 0$,*

$$\sup_{m\geq 0} \left\| \sum_{t=1}^{n} P^{m,m+t}/n - Q \right\| \leq M_1/n^{1-\varepsilon} \quad \textit{for } n \geq 1$$

and

$$\|C_t - C\| \leq M_2/t^{1+\delta} (\delta > 0) \quad \textit{for } t \geq 1,$$

then there exists a constant M_3 *such that*

$$\sup_{m\geq 0} \|\phi^{m,m+n} - \phi\| \leq M_3/n^{1-\varepsilon} \quad \textit{for } n \geq 1.$$

Proof. Letting λ $(<\infty)$ be the uniform bound of the $C_t(j)$'s and $C(j)$'s, we have that:

$$\begin{aligned}\sup_{m\geq 0} \|\phi^{m,m+n} - \phi\| &= \sup_{m\geq 0} \left\| \sum_{t=1}^{n} P^{m,m+t} C_{m+t}/n - QC \right\| \\ &\leq \sup_{m\geq 0} \left\| \sum_{t=1}^{n} P^{m,m+t}(C_{m+t} - C)/n \right\| + \sup_{m\geq 0} \left\| \left(\sum_{t=1}^{n} P^{m,m+t}/n - Q \right) C \right\| \\ &\leq \sup_{m\geq 0} \sum_{t=1}^{n} \|C_{m+t} - C\|/n + \lambda \sup_{m\geq 0} \left\| \sum_{t=1}^{n} P^{m,m+t}/n - Q \right\| \\ &\leq \varepsilon\end{aligned}$$

for any $\varepsilon > 0$ for sufficiently large n, by assumptions (4.1) and (4.2), thus proving (a); or

$$\leq \sup_{m\geq 0} \sum_{t=1}^{n} (M_2/(m+t)^{1+\delta})/n + \lambda M_1/n^{1-\varepsilon} = M_3/n^{1-\varepsilon},$$

for an appropriately chosen constant M_3, under the assumptions in (b), thus proving (b).

Before proceeding to an example, it should be noted that by a more complicated proof than that just given (see [1]) the result in (a) holds under assumption (4.1) and the assumption that for each $i \in S$: $\lim_{t\to\infty} C_t(i) = C(i)$, which is a weakening of the assumption (4.2).

Note also, in connection with the example, that, when $\lim_{n\to\infty} \phi^{0,n}(i)$ exists for each $i \in S$, we call this limit the expected average cost of $\{X_t, t = 0, 1, \dots\}$.

Example 4.1. Assume $\{N_t, t = 0, 1, \dots\}$ is a nonstationary Markov chain with transition matrices $\{P_t = [p_t(i,j)]\}_{t=1}^{\infty}$ defined on a countable state space $S = \{1, 2, \dots\}$, such that $\lim_{t\to\infty} \|P_t - P\| = 0$, where $P = [p_{ij}]$ is a PSE matrix describing the stationary Markov chain $\{S_t, t = 0, 1, \dots\}$. Assume $\{N_t, t = 0, 1, \dots\}$ and $\{S_t, t = 0, 1, \dots\}$ both incur a cost $C(i,j)$ in making a one step transition at any time from state i to state j, where the $C(i,j)$'s are uniformly bounded by λ $(<\infty)$. Then $C_t(i) = \sum_{j\in S} C(i,j) p_t(i,j)$ and $C(i) = \sum_{j\in S} C(i,j) p_{ij}$ are the expected costs incurred by, respectively, $\{N_t, t = 0, 1, \dots\}$ and $\{S_t, t = 0, 1, \dots\}$ landing in state i, and it follows that

$$\|C_t - C\| \leq \lambda \|P_t - P\| \to 0 \quad \text{as } t \to \infty. \tag{4.3}$$

Now, by (4.3) and the fact that (by Theorem 2.1) $\{N_t, t = 0, 1, \dots\}$ and $\{S_t, t = 0, 1, \dots\}$ have the same Cesaro long run distribution it follows by part (a), which holds for $\{N_t, t = 0, 1, \dots\}$ and $\{S_t, t = 0, 1, \dots\}$, that both Markov chains possess the same expected average cost — another robustness conclusion. Moreover, if $\|P_t - P\| \leq G/t^{1+\delta}$ $(\delta > 0)$ for $t \geq 1$, then by (4.3) and Theorem 2.1 the assumptions and hence conclusions of part (b) of Theorem 4.1, in addition to holding for $\{S_t, t = 0, 1, \dots\}$, also hold for $\{N_t, t = 0, 1, \dots\}$.

It should also be noticed that from Theorem 4.1 the expected average cost for both S_t and N_t equals $\sum_{j \in S} C(j)\psi_j$, where $\psi = (\psi_1, \psi_2, \dots)$ is the unique lett eigenvector of P.

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