Fundamental Study

Stable trace automata vs. full trace automata

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Abstract

Trace automata provide a well-studied model for systems with concurrent behavior, which is usually given by associated domains of finite or infinite computation systems. Several authors in the literature have characterized order-theoretically these domains which are typically particular Scott domains. In most of these investigations, the question remain open when such a domain can be obtained from some finite automaton. In this paper it is shown that finite stable trace automata and finite full trace automata give rise to the same class of coherent dI-domains. The proof of this result involves combinatorial means from graph theory (colorings) and Ramsey's theorem. © 1998—Elsevier Science B.V. All rights reserved

Keywords: Finite trace automata; Expressiveness; Coherent dI-domains

Contents

1. Introduction ............................................................... 46
2. Stable trace automata theory ........................................... 48
2.1. Automata ................................................................. 49
2.2. Concurrent semantics ............................................... 50
2.3. Domains ................................................................. 52
2.4. Domains as semantics ............................................... 54
2.5. Folding morphisms ................................................. 54
2.6. Event structures ..................................................... 57
2.7. Labellings of domains ............................................. 58
2.8. Lemmas ................................................................. 60
3. Semantics of finite automata ......................................... 63
3.1. Regularity .............................................................. 63
3.2. Stable trace vs. full trace automata ............................. 67

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1. Introduction

In order to motivate the results presented here, it seems necessary to give a brief overview of concurrent automata theory and its connections with domains and event structures. Concurrent automata are abstract models of concurrent machines which were introduced in [9]. They are a good generalization of classical deterministic automata to the concurrent case, in the sense that the essential notions of automata theory (simulation, minimal deterministic automaton realizing some language, residuals) are generalized to concurrent automata. Concurrent automata are deterministic automata with independency predicates on pairs of events which are both enabled at some state. These predicates specify at each state the concurrent behaviour. Given some concurrent automaton an equivalence can be defined on accepted words by permutation of events. Its classes are called here “traces” by reference to Mazurkiewicz’s theory [13]. The independency predicates of concurrent automata should satisfy a coherence axiom called “cube axiom”. This axiom is inherited from the residual calculus [12], and states precisely that independency predicates may be extended to traces. The main consequence of the latter fact is left-cancelation of concatenation of traces. This property is a crucial point of the theory since left-cancelation on traces implies that partial order structures on traces reflect concurrency of automata and therefore constitute correct concurrent semantics. This is a generalization of the sequential case since languages are commonly considered as the semantics of deterministic automata. The generalization is even deeper, since one may retrieve the usual categorical setting of automata theory [2,6]. Concurrent automata are objects of a category with “folding” morphisms as arrows. These morphisms consist of a generalization of the simulation morphisms of deterministic automata. The partial order of traces of some concurrent automaton \( \mathcal{A} \) may be seen as its unfolding \( U(\mathcal{A}) \) which is the greatest automaton “folding onto” or “covering” \( \mathcal{A} \). This unfolding process is a closure operation since unfolded objects – objects isomorphic to their unfoldings, form a coreflective subcategory.

In order-theoretic terms partial orders of traces are finitary Scott domains, and the unfolding domains of particular classes of automata are known to admit representation by means of event structures. Event structures have many interests. One is to supply a dual viewpoint for the semantics of concurrent automata: the stress is not put any more on operational aspects proper to domains but rather on coherence or causality between events. From a combinatorial viewpoint, it is also convenient to consider traces as sets of events. Historically, many kinds of concurrent automata have been defined. Full trace automata were introduced first [17,12] (see also [15]) for the study of
marking graphs of elementary nets. The independency predicates of full trace automata do not depend on states as they mimic marking graphs — recall that events of elementary nets are independent when they have no common conditions, hence their independency is independent of markings. Full trace automata were shown to unfold into coherent dl-domains which are the domains of configurations of conflict prime event structures. The correspondence between elementary nets and conflict prime event structures was enlightened in [14] where a direct translation from nets to event structures was given. Nevertheless, up to now, conflict prime event structures have not been known to be unfolded objects of any category. For this reason, it seems quite clearer to define the latter connection between nets and event structures via unfoldings of automata, i.e. as the result of the following correspondences: nets/automata, automata/domains, and domains/event structures. Note that the notion of unfolding of automata does not refer explicitly to that of events which is proper both to C/E nets and to event structures. An unfolding notion relying on events is certainly worth studying. Trace automata were introduced in [18] as an automaton version of concurrent transition systems. They are more general than the full trace automata but their independency predicates are still independent of states. This has the effect that their unfoldings are conflict event domains which are generally not distributive and correspond to conflict event structures. Along the same line, concurrent automata were finally introduced in [9]. They are the correct generalization of trace automata enabling the independency on events to be independent of states. They unfold into CR-domains which are strictly more general objects than the domains of configuration of general event structures. CR-structures provide an event-structure-like representation for CR-domains [16]. Finally, the stability condition for concurrent automata was shown to be necessary and sufficient in order for their unfolding domains to be distributive [10,11]. Recall that distributive event domains are called dl-domains and are also the domains of configurations of prime event structures. General event structures are a natural generalization of prime event structures and conflict event structures. Their domains of configurations, the so-called event domains, are characterized in [8]. They are particular CR-domains, but they do not occur as unfoldings of any well axiomatized class of concurrent automaton.

The whole situation is depicted in the following array:

<table>
<thead>
<tr>
<th>Automata</th>
<th>Domains</th>
<th>Structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>concurrent</td>
<td>CR-domain</td>
<td>CR-structure (general) event structure</td>
</tr>
<tr>
<td>stable</td>
<td>dl-domain</td>
<td>conflict e.s.</td>
</tr>
<tr>
<td>stable trace or full trace</td>
<td>coherent dl-domain</td>
<td>conflict prime e.s.</td>
</tr>
</tbody>
</table>

It is worth noting that the classification above does not take into account any finiteness conditions on automata. Having in mind great lines of language theory, it seems
clear that the characterization and the classification of "recognizable" domains – i.e. unfoldings of finite automata, are fundamental for the study of the semantics of concurrency. The purpose is to define and compare robust recognizability notions associated with various kinds of machines. In this way one may hope to design the most relevant automata together with their semantics for modelizing concurrency. Combinatorial problems lie in comparing the semantics of finite machines. Up to now a little attention has been paid to finiteness of machines. A sufficient condition for domains to be recognizable is given in [3]. Let us mention also fundamental works about finite labellings of event structures [1]. In this paper, the following result is shown. Since stable trace automata and full trace automata are known to unfold both into coherent dl-domains, a question is whether unfoldings of finite stable trace automata and those of finite full trace automata coincide. The answer is yes. Precisely, any finite stable trace automaton is covered by some finite full trace automaton (the reverse being obvious since full trace automata are stable). Let us mention that the proof of this result presented here does not provide any algorithm of "translation" from stable trace automata to full trace ones.

This paper is organized as follows. Required existing elements of stable trace automata theory are recalled and notations are introduced in Section 2. Automata are introduced in Section 2.1, their semantics are in Section 2.2, coherent dl-domains are in Section 2.3 and, connections between these domains and automata's semantics are in Section 2.4. The representation theorem of coherent dl-domains by event structures is recalled in Section 2.6 since the event structure formalism appears crucial in proofs. Various labellings of domains unfoldings of stable trace automata are introduced in Section 2.7. A few technical and useful lemmas are grouped in Section 2.8. Then, Section 3 focuses on the case of domain unfoldings of finite stable trace automata. The latter domains are shown to be the regular coherent dl-domains admitting particular finite regular labellings. These regularity notions are studied in Section 3.1. Finally, Section 3.2 is devoted to the proof of our claimed result. A short conclusion follows in Section 4. An appendix contains necessary recalls about graphs and Ramsey's theorem. Many results presented in Section 2 are inherited from concurrent automata theory. Nevertheless, we do not introduce these automata which would be out of the scope of this paper. Along the same line, the only domains that we present are the coherent dl-domains. Most of the results about labellings and regularity may be established for more general domains (dl-domains, event domains or, CR-domains).

2. Stable trace automata theory

This section is devoted to recall required existing elements of stable trace automata theory and to introduce notations. Only a few new results are presented in Sections 2.7 and 2.8.
2.1. Automata

**Definition 2.1.** An automaton is a quadruple \( \mathcal{A} = (E, Q, T, \star) \) where
- \( E \) and \( Q \) are at most denumerable sets, respectively, of events and of states,
- \( \star \) is a distinguished state called initial state;
- \( T \subseteq Q \times E \times Q \), is the set of transitions.

\( \mathcal{A} \) is deterministic when it satisfies the property:

\[ ((q, e, p) \in T) \land ((q, e, r) \in T) \Rightarrow p = r. \]

An event \( e \) is enabled at \( q \) when there is a transition of the form \( (q, e, p) \). When the automaton is deterministic, \( q^+ \) denotes the set of events enabled at state \( q \) and if \( e \) is enabled at \( q \) then \( q.e \) denotes the state \( p \) with \( (q, e, p) \in T \).

Let \( \mathcal{A} \) be an automaton. A path is either a sequence \( (t_i)_{i \in \mathbb{N} \land 1 \leq i \leq l} \) where \( 1 \leq l \leq \omega \) is its length and the \( t_i \)'s are transitions of respective forms \( (q_{i-1}, e_i, q_i) \), or it is some empty path \( \varepsilon_q \) associated with state \( q \). By convention empty paths have length zero. Finite paths are those of finite length. Given a non-empty path \( \alpha = ((q_{i-1}, e_i, q_i))_{1 \leq i \leq l} \), its domain, \( \text{dom}(\alpha) \) is the state \( q_0 \), if \( \alpha \) is finite its codomain, \( \text{cod}(\alpha) \) is \( q_l \). Any state is both the domain and the codomain of its associated empty path. If \( t = (q, e, q') \) is a transition then \( \text{ev}(t) = e \). Two paths are coinital (respectively coinitial) when they have the same domain (respectively codomain). Given two paths \( \alpha \) and \( \beta \), \( \alpha \) is a prefix of \( \beta \) when there is \( \gamma \) with \( \beta = \alpha \gamma \), \( \alpha \gamma \) denoting the concatenation "\( \alpha \) then \( \gamma \". A path is initial when its domain is the initial state. The automaton \( \mathcal{A} \) is accessible when any of its states is the codomain of some initial path. Further on, \( C(\mathcal{A}) \) denotes the set of paths of \( \mathcal{A} \) and \( C^0(\mathcal{A}) \) that of finite ones. \( C_*(\mathcal{A}) \) and \( C^0_*(\mathcal{A}) \) are the sets respectively made of initial paths and of finite initial paths.

For brevity, we define trace automata as some particular accessible automata.

**Definition 2.2 (Trace automata).** A trace automata \( \mathcal{A} \) consists in a quintuple \( (E, Q, T, \star, ||) \) as follows:
- \( (E, Q, T, \star) \) is a deterministic accessible automaton;
- The independency predicate \( || \) is a partial predicate on pairs of distinct events. \( a \parallel b \) (resp. \( a \parallel b \)) denotes the fact that \( || \) is defined in \( \{a, b\} \) and equals 1 (respectively 0);
- \( || \) is defined in \( \{a, b\} \) if and only there is some state \( q \) where \( a \) and \( b \) are both enabled;
- The following axiom is satisfied:
  - \( (Tr) \): if \( a \parallel b \) and \( a \) and \( b \) are both enabled at \( q \) then \( b \) is enabled at \( q.a, a \) is enabled at \( q.b \) and, \( q.a.b = q.b.a \).

**Definition 2.3 (Stable trace automata).** A trace automaton \( \mathcal{A} = (E, Q, T, \star, ||) \) is stable when it satisfies the inverse cube axiom (see Fig. 1):
- \( (Icu) \): If \( a \) and \( c \) are enabled at \( q \), \( b \) is enabled at \( q.a \) and at \( q.c \) and, \( a \parallel c \land b \parallel c \land b \parallel a \) then \( b \) is enabled at \( q \).
Definition 2.4 (Full trace automaton [17, 4]). A trace automaton $\mathcal{A} = (E, Q, T, \star, ||)$ is full when it satisfies the axiom:

$(As)$ If $a||b$ and $a$ is enabled at $q$ and $b$ is enabled at $q.a$ then $b$ is enabled at $q$.

Full trace automata are the so-called "asynchronous transition systems" of [4].

Remark 2.5. Any full trace automaton is stable.

The following stable trace automaton is not full: it does not satisfy $(As)$.

Example 2.6. $\mathcal{A}$ is the automaton whose transition graphs is depicted in Fig. 2, with initial state $\star$, with set of events \{a, b, c, d, e, f\} and independency predicate $||$: $a||b$, $a||c$, $a||d$, $b||e$, $b||f$, $c||d$.

2.2. Concurrent semantics

Partial orders of traces of trace automata are defined. We explain briefly why they constitute correct concurrent semantics.

Let $\mathcal{A}$ be a trace automaton. $\sim$ is the equivalence on $C(\mathcal{A})$ defined as follows. Empty paths are equivalent if and only if they are associated to a same state. Two non-empty paths $xtt'y$ and $xuu'y$ where $x, y \in C^0(\mathcal{A})$, $t, t', u, u' \in T$, are close when $tt'$ and $uu'$ have respective forms $(q, a, r)(r, b, p)$ and $(q, b, s)(s, a, p)$ with $a||b$. The equivalence $\sim$ is generated by the previous relation of closeness, a trace is a class for $\sim$.

The preorder $\subseteq$ on $C^0(\mathcal{A})$ is defined by $\alpha \subseteq \beta$ if and only if $\beta \sim \alpha \gamma$ for some $\gamma$. Then for any $\alpha$ and $\beta$ in $C^0(\mathcal{A})$, $\alpha \sim \beta$ if and only if $\alpha \subseteq \beta$ and $\beta \subseteq \alpha$. This preorder is extended to $C(\mathcal{A})$ in the following way: $\alpha \subseteq \beta$ if and only if for any finite prefix $\alpha'$. 

![Fig. 1. Inverse cube axiom.](image)
of \( \alpha \) there is some finite prefix \( \beta' \) of \( \beta \) with \( \alpha' \subseteq \beta' \). \( \sim \) is also extented to \( C(\mathcal{A}) \) by: \( \alpha \sim \beta \) if and only if \( \alpha \subseteq \beta \) and \( \beta \subseteq \alpha \). Finally, \( C(\mathcal{A})/\sim \) is ordered in the following way. \([\alpha]\) denoting the trace of \( \alpha \), \([\alpha] \leq [\beta] \) if and only \( \alpha \subseteq \beta \). \( D(\mathcal{A}) \) and \( P(\mathcal{A}) \) denote the respective sets \( C_a(\mathcal{A})/\sim \) and \( C_u(\mathcal{A})/\sim \) ordered by the restriction of the latter partial order \( \leq \). \( \mathcal{A} \) generates some partial order when the latter is isomorphic to \( D(\mathcal{A}) \).

Paths in a same trace having the same length, the same domain and when they are finite the same codomain, length, domain and codomain are naturally defined for traces. Finite traces are those of finite length, initial traces are those with initial state as domain. Two traces are coinitial (respectively cofinal) when they have the same domain (respectively codomain). The two fundamental properties of \( C(\mathcal{A})/\sim \) are the following.

**Proposition 2.7** (Stark [18] and Droste [9]). If \( A \) is a trace automaton then

1. The concatenation in \( C(\mathcal{A})/\sim \) is left-cancelable.
2. The order on \( C(\mathcal{A})/\sim \) is upper semi-modular.

The definition of semimodularity for partial orders is recalled next section.

**Remark 2.8.** If \( \mathcal{A} \) is a trace automaton then

1. Given traces \( x \) and \( x' \), \( x \sim x' \) if and only if \( x' = x[t] \) where \( t \) denotes the class of the path made of the unique transition \( t \). According to the left-cancelation rule, if \( x \sim x' \) then \( x' = x[t] \) for some unique transition \( t \).
2. If \( x[uu'] = x[tt'] \) for some trace \( x \) and transitions \( u, u', t \) and \( t' \) of \( \mathcal{A} \) by left-cancelation, one obtains \( uu' \sim tt' \).
Some considerations related to computer science are in order. As one wants to consider ordered sets of traces as concurrent semantics for trace automata, one should expect to retrieve the independency of some automaton from the partial order of its traces. This means the following. A diamond in some p.o. of traces, which is a quadruple of traces $x, y, y', z$ with $x \prec y$, $x \prec y'$, $y \neq y'$, $y \not\sim z$ and, $y' \not\sim z$, should correspond to independent events i.e. $y$ and $y'$ should have the respective forms $x.(q, e_1, q \cdot e_1)$ and, $x.(q, e_2, q \cdot e_2)$ with $e_1 \parallel e_2$. According to Remark 2.8(2), this property holds for p.o of traces of trace automata. This property is also required in order to define a folding morphism from $\mathcal{U}(\mathcal{A})$ to $\mathcal{A}$ (see Section 2.17 below).

2.3. Domains

Elements of domain theory are recalled. We introduce coherent dl-domains which will be later shown to coincide with partial orders of traces of stable trace automata.

Recall that a Scott-domain $D$ is distributive when it satisfies

$$(D_1) \forall x, y, z \in D, x \uparrow y \Rightarrow z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y).$$

Let $\downarrow x$ denote the principal ideal generated by $x$. Axiom $(D_1)$ for a Scott-domain $D$ is also equivalent to

$$(D_2) \text{ for any } x \in D, \downarrow x \text{ is a distributive lattice.}$$

Finitary distributive Scott domains are commonly called dl-domains. A well-known fact is that they coincide with finitary prime algebraic domains [19]. Recall that a domain is prime algebraic when any of its elements is least upper bound of the complete prime elements that it dominates. Now, a domain is coherent when any of its subset is upper bounded as soon as any pair in it is upper bounded.

There is an alternative way to define coherent dl-domains. In order to present it, we need some terminology. We will be concerned with partial orders satisfying the following chain condition:

$$(F) \text{ Any upper and lower bounded chain is finite.}$$

For such partial orders, we use the following notations. If $x \leq y$, $[x, y]$ denotes the interval from $x$ to $y$, which is the set $\{z \in D \mid (x \leq z) \wedge (z \leq y)\}$. $[x, y]$ is prime when $x \prec y$ - which means that $y$ covers $x$, i.e. there is no $z$ strictly between $x$ and $y$. Let $[x, x'] \prec [y, y']$ if and only if $x \prec y$ and $x' \prec y'$. The projectivity relation on prime intervals, $\prec_\prec$, is the equivalence generated by $\prec$. A partial order is semi-modular (respectively lower semi-modular) when it satisfies Condition $(F)$ above and the following axiom $(C)$ (respectively $(C')$) - where $x \uparrow y$ (respectively $x \downarrow y$) means that $x$ and $y$ have a common least upper bound (respectively a greatest lower bound):

$(C) \text{ if } x \neq y, x \uparrow y \text{ and, } x, y \text{ both cover a same element, then } x \text{ and } y \text{ have a least upper bound which covers } x \text{ and } y;$

$(C') \text{ if } x \neq y, x \downarrow y \text{ and, both } x \text{ and } y \text{ are covered by some } z \text{ then } x \text{ and } y \text{ have a greatest lower bound which is covered by } x \text{ and } y.$
The following axioms concern partial orders satisfying \((F)\):

\((R)\): if \(x \prec y\), \(x \prec z\) and \([x, y] \gg [x, z]\) then \(y \equiv z\);  

\((V)\): if \([x, x'] \gg [y, y']\) and \([x, x''] \gg [y, y'']\) then \((x' \uparrow x''\text{ if and only if } y' \uparrow y'')\).

**Definition 2.9 (Coherent dl-domains).** Coherent dl-domains are the finitary \(\omega\)-algebraic Scott domains whose ordered sets of compacts satisfy the following Axioms \\{(C), (R), (C'), (V)\}.

Without going into details which would be out of the scope of this paper, let us mention the following. Axiom \((R)\) is deeply linked with the determinism of automata. Axiom \((V)\) is related to trace automata and the fact that their independency predicates are independent from states. Further we call events the projectivity classes of coherent dl-domains. Elements of a coherent dl-domain may be represented by set of events, in this way the order corresponds to the inclusion. This fact is recalled now – see Proposition 2.11 below.

Consider a partial order satisfying \((F)\). Given \(x, y\) with \(x \leq y\), there is a covering chain between them, which is a finite sequence of the form \(x = x_0 \prec x_1 \prec \cdots \prec x_n = y\). Two covering chains \(x_0 \prec x_1 \prec \cdots \prec x_n\) and \(y_0 \prec y_1 \prec \cdots \prec y_m\) are said close when \(m = n\) and there is some \(i\) with \(1 < i < n\) and \(x_j = y_j \iff i \neq j\). Further on, the equivalence on covering chains is the equivalence generated by the latter relation of closeness. The following facts concerning Axioms \((C), (R)\) and \((V)\) belong to folklore \([7,8]\).

**Proposition 2.10.** Let \(P\) be a partial order satisfying the chain condition \((F)\). \(\text{Proj}(P)\) standing for its set of projectivity classes of prime intervals, \(P\) is associated with a map \(s_P : P \to \phi(\text{Proj}(P))\) – denoted \(s\) when there is no ambiguity – sending \(x\) on \([\{y, y'\} \gg \{y' \leq x\}\).

If \(P\) satisfies Axioms \((C)\) and \((R)\) then:

\((CR1)\) If \(x \leq y\) then any two covering chains from \(x\) to \(y\) are equivalent.

\((CR2)\) If \(x \leq y \in P\), if \((z_i)_{1 \leq i \leq n}\) and \((z'_j)_{1 \leq j \leq m}\) are covering chains from \(x\) to \(y\), then for any event \(e\), \(|\{i \mid [z_i, z_{i+1}] \in e\}\} = |\{j \mid [z'_j, z'_{j+1}] \in e\}|\).

\((CR3)\) For any \(x, x', y \in P\), if \(x \prec x', x \leq y\), \(x' \uparrow y\) and \(x' \not\leq y\) then \(y \prec x' \uparrow y\) and \([x, x'] \gg [y, y']\).

\((CR4)\) For any \(x, y \in P\), if \(x \uparrow y\) then \(s(x \uparrow y) = s(x) \uparrow s(y)\).

\((CR5)\) For any \(x, x', y, y' \in P\), if \(x \prec x' \leq y \prec y'\) then \(-([x, x'] \gg [y, y'])), especially in any covering chain the number of occurrences of prime intervals belonging to a same event is at most one.

If \(P\) satisfies Axioms \((C), (R), (V)\) then:

\((I)\) If \([x, x'] \gg [y, y']\) and \(x \leq y\) then \(x' \leq y'\).

\((CI1)\) For any \(x, x', y, y' \in P\), if \(x \prec x', y \prec y', x \leq y\) and \([x, x'] \gg [y, y']\) then for any \(z\) such that \(x \leq z \leq y\), there is \(z'\) such that \(z \prec z'\) and \([x, x'] \gg [z, z']\).

\((CI2)\) \(\phi(\text{Proj}(P))\) being ordered by inclusion, \(s\) is an embedding of orders.
For any coherent dI-domains $D$, $\text{Comp}(D)$ standing for its ordered set of compacts, the map $s : \text{Comp}(D) \to \varphi(\text{Proj(Comp}(D))))$ is extended to $D$ by $s(x) = \{[y, y'] \succ | y' \in \text{Comp}(D) \land y' \leq x\}$. Then

**Proposition 2.11.** If $D$ is a coherent dI-domain then the following assertions holds:

1. For $x, y \in D$, if $x \uparrow y$, $s(x \lor y) = s(x) \cup s(y)$.
2. For any $x, y \in D$, $s(x \land y) = s(x) \cap s(y)$.
3. $s$ is an embedding of orders.

**Proof.** See [7] for the proofs of (1) and (3) and for example [16] for a proof of (2). $\square$

2.4. Domains as semantics

Links between stable automata trace and coherent dI-domains are recalled now.

For any stable trace automaton $\mathcal{A}$, $D(\mathcal{A})$ is a coherent dI-domain whose least element is $[E^*]$ and whose set of compacts is $P(\mathcal{A})$. If $D$ is a coherent dI-domain, it may be associated with a full trace automaton $S'(D)$ such that $D$ is isomorphic to $D(D(\mathcal{A}(D))$. $\mathcal{A}(D)$ is as follows:

- states are compacts of $D$;
- the initial state is $\bot$;
- events are projectivity classes of prime intervals;
- transitions are triples of the form $(x, [x, y] \succ, y)$ where $x, y$ are compact with $x \prec y$;
- the independency predicate is defined in $\{e, f\}$ if and only if there are compacts $x, y, z$ such that $x \prec y, x \prec z, y \neq z, [x, y] \in e$ and $[x, z] \in f$, when it is the case, $e \| f$ if and only if $y \uparrow z$.

Actually, the latter assertion implies only the easy parts of the following theorems.

**Theorem 2.12** (Kuske [10, 11]). Coherent dI-domains are exactly domains generated by stable trace automata. Any trace automata generating a coherent dI-domain is stable.

**Theorem 2.13** (Bednarczyk [4]). Coherent dI-domains are exactly the domains generated by full trace automata.

2.5. Folding morphisms

Stable trace automata theory admits a categorical setting which enables to define the unfolding of automata by means of a coreflection. The folding notion for stable trace automata generalizes that of simulation for deterministic automata as shown below.

Automata are the objects of a category with morphisms as follows. A morphism from $\mathcal{A}^1 = (E^1, Q^1, T^1, \ast^1)$ to $\mathcal{A}^2 = (E^2, Q^2, T^2, \ast^2)$ is a triple of maps $(\eta, \sigma, \tau)$ such that $\tau : T^1 \to T^2$, $\sigma : Q^1 \to Q^2$, $\eta : E^1 \to E^2$ and these maps preserve the initial state,
domains, codomains and composition, i.e.

$$\sigma(\star) = \star_1;$$
$$\sigma \cdot \text{dom} = \text{dom} \cdot \tau;$$
$$\sigma \cdot \text{com} = \text{com} \cdot \tau;$$
$$\eta \cdot \text{ev} = \text{ev} \cdot \tau.$$

Two automata are therefore isomorphic when they are identical up to renamings of states and events. Note that any morphism of automata with deterministic codomain is determined by its components on states and events. Thus, morphisms between deterministic automata are commonly defined by the two latter components. According to this, if $$\mathcal{A}^1$$ and $$\mathcal{A}^2$$ are accessible and deterministic automata, a morphism $$(\eta, \sigma)$$ from $$\mathcal{A}^1$$ to $$\mathcal{A}^2$$ is a simulation, when for any state $$q$$ of $$\mathcal{A}^1$$ the restriction of $$\eta$$ to $$q^+$$ is a bijection onto $$(\sigma(q))^+$$.

**Definition 2.14** (Folding morphisms). Let $$\mathcal{A}^1 = (Q^1, E^1, T^1, \star^1, ||^1)$$ and $$\mathcal{A}^2 = (Q^2, E^2, T^2, \star^2, ||^2)$$ be trace automata. A folding morphism from $$\mathcal{A}^1$$ to $$\mathcal{A}^2$$ is a pair of maps $$(\sigma : Q^1 \rightarrow Q^2, \eta : E^1 \rightarrow E^2)$$, such that:

1. $$\sigma(\star^1) = \star^2$$,
2. if $$(q, e, q') \in T^1$$ then $$(\sigma(q), \eta(e), \sigma(q')) \in T^2$$,
3. for any state $$q$$ the restriction of $$\eta$$ on $$q^+$$ is a bijection onto $$\sigma(q)^+$$ that preserves and reflects the independency predicate, i.e. $$e \parallel f$$ if and only if $$\eta(e) \parallel \eta(f)$$.

When there is such a folding morphism from $$\mathcal{A}^1$$ to $$\mathcal{A}^2$$, $$\mathcal{A}^1$$ covers or is a covering of $$\mathcal{A}^2$$.

**Proposition 2.15** (Badouel and Darondeau [2] and Bracho and Droste [6]). Given two trace automata, if one covers the other one then they generate isomorphic domains.

The class of stable trace automata with folding morphisms is a category.

**Proposition 2.16.** Let $$\mathcal{A}^1$$ and $$\mathcal{A}^2$$ be stable trace automata such that $$\mathcal{A}^1$$ covers $$\mathcal{A}^2$$. If $$\mathcal{A}^2$$ is full then also is $$\mathcal{A}^1$$.

**Proof.** Let $$\varphi = (\sigma, \eta)$$ be a folding morphism from $$\mathcal{A}^1 = (Q^1, E^1, T^1, \star^1, ||^1)$$ onto $$\mathcal{A}^2$$ which is full and has independency predicate $$||^2$$. If $$q$$ is a state of $$\mathcal{A}^1$$, $$a$$ and $$b$$ are events in $$\mathcal{A}^1$$ such that $$a$$ is enabled at $$q$$ and $$b$$ is enabled at $$q$$ but not at $$q$$. Then $$\eta(a)$$ is enabled at $$\sigma(q)$$ and $$\eta(b)$$ is enabled at $$\sigma(q) \cdot \eta(a)$$. Let us show by contradiction that $$a$$ and $$b$$ cannot be independent with respect to $$||^1$$. If $$a||^1 b$$ then $$\varphi$$ being some folding morphism, $$\eta(a)||^2 \eta(b)$$. Since $$\mathcal{A}^2$$ is full, $$\eta(b)$$ is enabled at $$\sigma(q)$$. Therefore, there is some $$c$$ enabled at $$q$$ with $$\eta(c) = \eta(b)$$. Since $$\eta(a)||^2 \eta(c)$$, $$c||^1 a$$ and $$c$$ is also enabled at $$q$$. Since the restriction of $$\eta$$ to the set of events enabled at $$q$$ is injective and $$\eta(b) = \eta(c)$$, $$b = c$$. Therefore, $$b$$ is enabled at $$q$$ which is a contradiction. $\square$

If $$\mathcal{A}$$ is a stable trace automaton, $$\mathcal{U}(\mathcal{A})$$ denotes the unfolding of $$\mathcal{A}$$ which is the automaton $$\mathcal{A}(D(\mathcal{A}))$$. A stable trace automaton $$\mathcal{A}$$ is unfolded when it is isomorphic to
\(U(\mathcal{A})\). The following results (Propositions 2.17, 2.18 and Corollary 2.19) – due to [2] (for trace automata) and to [6] (for concurrent ones), show that the unfolding process for automata is a closure operation. Actually \(U(\mathcal{A})\), the unfolding of \(\mathcal{A}\), consists in the "greater" object covering \(\mathcal{A}\). Again, left-cancelation rule on traces is crucial in the proofs of these results.

**Proposition 2.17.** For any stable trace automaton \(\mathcal{A} = (Q, E, T, \star, \|)\) there is a folding morphism \(\rho_\mathcal{A} = (\sigma, \tau) : U(\mathcal{A}) \to \mathcal{A}\). It is given by \(\sigma(\perp) = \star\), \(\sigma(x) = \text{cod}(x)\) for any finite trace \(x\), and \(\tau((x, [x, y] \succ y)) = e\) where \(e\) is the event of the transition \(t\) with \(y = x [t]\).

**Proposition 2.18.** Unfolded stable trace automata are the objects of a coreflective subcategory of the category of stable trace automata and folding morphisms. \(U\) previously defined is the object function of a right adjoint to the inclusion functor, the co-unit is \(\rho_\mathcal{A}\) (with values \(\rho_\mathcal{A}\) in \(\mathcal{A}\)). Moreover, any folding morphism whose domain is unfolded is an isomorphism.

**Corollary 2.19.** If \(\mathcal{A}\) is a stable trace automaton then \(U(\mathcal{A})\) is the unique (up to isomorphism) stable trace automaton covering \(\mathcal{A}\).

The following result will be used later.

**Proposition 2.20** (Bracho and Droste [6]). The category of stable trace automata and folding morphisms has fibered products.

Let us give the construction of the fibered product. Let \(\mathcal{A}' = (Q', E', T', \star', \|')\), \(\mathcal{A}'' = (Q'', E'', T'', \star'', \|'')\) and, \(\mathcal{A}''' = \text{stable trace automata and } \varphi_1 = (\sigma_1, \eta_1) : \mathcal{A}' \to \mathcal{A}'''\) and \(\varphi_2 = (\sigma_2, \eta_2) : \mathcal{A}'' \to \mathcal{A}'''\) be folding morphisms. The fibered product of \(\varphi_1\) and \(\varphi_2\) consists of the pair of morphisms \(\tilde{\varphi}_1 = (\tilde{\sigma}_1, \tilde{\eta}_1) : \mathcal{A}' \times (\varphi_1, \varphi_2) : \mathcal{A}'' \to \mathcal{A}'' \times (\varphi_1, \varphi_2) : \mathcal{A}'' \to \mathcal{A}'''\), \(\tilde{\varphi}_2 = (\tilde{\sigma}_2, \tilde{\eta}_2) : \mathcal{A}' \times (\varphi_1, \varphi_2) : \mathcal{A}'' \to \mathcal{A}''\), defined as follows. Consider the automaton \(\mathcal{A}''' = (Q, E, T, \star)\) where

- \(Q\) is the set of \((q_1', q_2')\) in \(Q' \times Q''\) with \(\sigma_1(q_1') = \sigma_2(q_2')\),
- \(\star\) is \((\star_1, \star_2)\),
- \(E\) is the set of \((e_1, e_2)\) in \(E' \times E''\) with \(\eta_1(e_1') = \eta_2(e_2')\),
- \(T\) is the set of triples of the form \(((q_1', q_2'), (e_1', e_2'), (r_1, r_2'))\) with \((q_1', q_2') \in Q, (e_1', e_2') \in E, (q_1', e_1', r_1') \in T'\) and, \((q_2', e_2', r_2') \in T''\).

States of \(\mathcal{A}' \times (\varphi_1, \varphi_2) : \mathcal{A}''\) are the accessible ones of \(\mathcal{A}'\), its transitions are those of \(\mathcal{A}'\) with accessible codomains, its events are those of the previous transitions. The independence predicate of \(\mathcal{A}' \times (\varphi_1, \varphi_2) : \mathcal{A}''\) is as follows. \(\|\) is defined in \{\((a_1, a_2)\) if and only if \((a_1, a_2)\) and \((b_1, b_2)\) are enabled at some accessible state \((q_1', q_2')\) of \(\mathcal{A}'\), if it is the case then \((a_1, a_2) \|| (b_1, b_2)\) if and only if \(\eta_1(a_1) \|\eta_1(b_1)\). \(\tilde{\varphi}_1\), \(\tilde{\varphi}_2\) are projections: \(\tilde{\sigma}_1(q_1', q_2') = q_1', \tilde{\sigma}_2(q_1', q_2') = q_2', \tilde{\eta}_1(e_1', e_2') = e_1', \tilde{\eta}_2(e_1', e_2') = e_2'.\)
2.6. Event structures

The representation theorem of coherent dI-domains by conflict prime event structures is recalled.

**Definition 2.21 (Conflict prime event structures).** A conflict prime event structure is a triple \( \mathcal{E} = (E, \prec, \#) \) where
- \( E \) is an at most denumerable set of events partially ordered by \( \prec \);
- for any \( e \) in \( E \), \( \downarrow e \) is finite;
- the immediate conflict relation, \( \# \), is a binary anti-reflexive symmetric relation on events such that
  \[
  \forall e, f \in E, \quad e \# f \Rightarrow e \not\preceq f
  \]

and
  \[
  \forall e, f, g \in E, \quad e \# f < g \Rightarrow -(e \# g).
  \]

Consider a structure \( \mathcal{E} \) as above. A set of events is conflict-free when for any pair \( e, f \) of its elements, \( \neg e \# f \). A configuration of \( \mathcal{E} \) is lower set w.r.t. \( \prec \), which is conflict-free. \( \text{Conf}(\mathcal{E}) \) denotes the set of configurations of \( \mathcal{E} \). \( D(\mathcal{E}) \), the domain of configurations of \( \mathcal{E} \) is \( \text{Conf}(\mathcal{E}) \) with inclusion ordering.

Consider some conflict prime event structure \( \mathcal{E} \). For any of its event \( e \), we let \( C'(e) = \downarrow e \), and \( C(e) = C'(e) \setminus \{e\} \). \( C(e) \) and \( C'(e) \) are configurations of \( \mathcal{E} \) and \( C'(e) \) is the smallest one containing \( e \).

**Theorem 2.22** (Winskel [19]). If \( \mathcal{E} \) is a conflict prime event structure, \( D(\mathcal{E}) \) is a coherent dI-domain whose compacts are the finite configurations. With any coherent dI-domain \( D \) is associated the event structure \( \mathcal{E}(D) = (E, \prec, \#) \) where \( s \) denoting the map defined in Proposition 2.11:
- events of \( E \) are the projectivity classes of prime intervals (or "events") of \( D \); and for any events \( e \) and \( f \),
- \( e \# f \) if and only if there are compacts \( x, y, z \) such that \( [x, y] \in e, [x, z] \in f \), and \( \neg y \uparrow z \);
- \( e \prec f \) if and only if for any compacts \( x, y \), \( [x, y] \in f \Rightarrow e \in s(x) \).

Then \( D \) is isomorphic via \( s \) to the set of configurations of \( \mathcal{E}(D) \) ordered by inclusion. Compacts of \( D \) correspond (via \( s \)) to finite configurations \( \mathcal{E}(D) \).

Let \( D \) be some coherent dI-domain. If \( e \) is an event of \( D \), \( x(e) \) and \( x'(e) \) will denote respectively the compact elements of \( D \) satisfying \( s(x(e)) = C(e) \) and \( s(x'(e)) = C'(e) \).

**Lemma 2.23.** Let \( e \) and \( f \) be events of a coherent dI-domain; then

(i) If \( e \not\prec f \) then there are compacts \( x, x', x'' \) with \( x \not\prec x' \not\prec x'' \), \( [x, x'] \in e \) and \( [x', x''] \in f \),
(ii) If there are compacts \( x, x', x'' \) with \( x \prec x' \prec x'' \) then \([x,x'] \prec [x',x''] \prec \) if and only if there is no compact \( x''' \) with \( x''' \neq x' \) and \( x \prec x''' \prec x'' \).

**Proof.** (1) Let \( e \) and \( f \) be events of a coherent dl-domain with \( e \prec f \). Then \( e \) is maximal in \( C(f) \) and \( C(f) \setminus \{e\} \) is a configuration. Let \( x \) be the compact with \( s(x) = C(f) \setminus \{e\} \). Then \( x \prec x(f) \prec x'(f) \), \([x,x'] \in e \) and \([x',x''] \in f \).

(2) Let \( x, x', x'' \) be compacts of a coherent dl-domain such that \( x \prec x' \prec x'' \). Let \( e = [x,x'] \prec \) and \( f = [x',x''] \prec \). If there is a compact \( x''' \) with \( x''' \neq x' \) and \( x \prec x''' \prec x'' \), then \([x,x'''] \in f \) and \([x'''',x''] \in e \), therefore \( C(f) \subseteq s(x) \), and \( e \notin C(f) \).

Suppose there is not such an \( x''' \). Since \( C(f) \cup \{e\} \subseteq s(x'') \), \( C(f) \notin s(x) \). Since \( C(f) \subseteq s(x') \) and \( s(x') \setminus s(x) = \{e\} \), \( e \in C(f) \) and \( C(f) \setminus \{e\} = C(f) \cap s(x) \). Therefore, \( C(f) \setminus \{e\} \) is a configuration and \( e \) is maximal in \( C(f) \). \( \square \)

If \( \mathcal{D} \) is a conflict prime event structure, the following predicates on pairs of events are defined:

- \( e \triangleleft f \Leftrightarrow e \neq f \land \exists X, (X,X \cup \{e\},X \cup \{f\} \in Conf(\mathcal{D})) \land (X \cap \{e,f\} = \emptyset) \);
- \( e \triangleleft f \Leftrightarrow e \neq f \land \exists X, (X,X \cup \{e\},X \cup \{f\},X \cup \{e,f\} \in Conf(\mathcal{D})) \land (X \cap \{e,f\} = \emptyset) \);
- \( e \nparallel f \Leftrightarrow \exists X \in Conf(\mathcal{D}), \{e,f\} \subseteq X \).

Events \( e \) and \( f \) are adjacent, (respectively concurrent), when \( e \triangleleft f \), (respectively \( e \nparallel f \)).

It is straightforward to check that

**Remark 2.24.** If \( \mathcal{D} \) is a conflict prime event structure then for any events \( e \) and \( f \),

\[ e \nparallel f \text{ if and only if } (e \triangleleft f \text{ or } e \nparallel f \text{ or } e \triangleleft f) \].

### 2.7. Labellings of domains

The purpose here is to characterize the various labellings of events of domains unfoldings of stable trace automata.

**Definition 2.25 (Event, labellings).** A labelling of a coherent dl-domain is a map from its set of events to some set of labels or alphabet. A labelling is finite when its alphabet is finite.

Let \( \mathcal{A} \) be some stable trace automaton. According to the Remark 2.8, an easy induction shows that for any compacts \( x, x', y, y' \), if \( x \prec x', y \prec y' \) and \([x,x'] \prec [y,y'] \) then \( x' = x[t] \) and \( y' = y'[t'] \) for some transitions \( t \) and \( t' \) of \( \mathcal{A} \) with \( ev(t) = ev(t') \). Therefore \( D(\mathcal{A}) \) is labelled by \( \lambda_{\mathcal{A}} \) which sends any event \( e \) onto \( ev(t) \) where \( x' = x[t] \) for some compacts \( x \) and \( x' \) such that \([x,x'] \in e \).

**Definition 2.26 (Deterministic labelling).** A labelling \( \lambda \) of a coherent dl-domain is deterministic when it satisfies

\[ (Edet): e \triangleleft f \Rightarrow \lambda(e) \neq \lambda(f). \]
Definition 2.27 (Trace labelling). A labelling $\lambda$ of a coherent dI-domain is *trace* when it is deterministic and its alphabet may be provided with some partial binary predicate $\parallel$ on pairs of (distinct!) events called *independency* such that $- a \parallel b$ (respectively $a \parallel b$) denoting the fact that $\parallel$ is defined at $\{a, b\}$ and holds (respectively does not hold):

\[(Etr1): e \circ f \Rightarrow \lambda(e) \parallel \lambda(f), \quad \text{and} \]

\[(Etr2): e \# f \Rightarrow \lambda(e) \nparallel \lambda(f). \]

Definition 2.28 (Full trace labelling). A trace labelling $\lambda$ of a coherent dI-domain is *full* when it satisfies

\[(Eas): \text{if } e \prec f \text{ then } \neg(\lambda(e) \parallel \lambda(f)) \text{ (i.e. } \parallel \text{ does not hold or is not defined at } \{e,f\}).\]

Suppose that $\lambda$ is the labelling of a coherent dI-domain and that $\lambda$ ranks in an alphabet which may be provided with some independency predicate $\parallel$ making $\lambda$ a trace labelling. Then there is a “smallest” predicate on the alphabet making $\lambda$ a trace labelling. $\parallel_\lambda$ denotes this one which is (⋆):

- $a \parallel_\lambda b$ if and only if there are $e$ and $f$ such that $\lambda(e) = a$, $\lambda(f) = b$ and $e \circ f$;
- $a \nparallel_\lambda b$ if and only if there are $e$ and $f$ such that $\lambda(e) = a$, $\lambda(f) = b$ and $e \# f$.

Note also that if $\lambda$ is a trace labelling of a coherent dI-domain then it is full with respect to some independency predicate, and then it is full for $\parallel_\lambda$. Further on, the independency predicates on alphabets of trace labellings will always be taken to be the smallest ones when they are not explicit.

Proposition 2.29. (1) If $\mathcal{A}$ is a stable trace automaton then $\lambda_\mathcal{A}$ is a trace labelling. Precisely, if $\parallel$ is the independency predicate of $\mathcal{A}$, $\parallel_\mathcal{A} = \parallel$.

(2) if $\mathcal{A}$ is moreover full then $\lambda_\mathcal{A}$ is full.

Proof. (1) Let $\mathcal{A}$ be a stable trace automaton with independency predicate $\parallel$. First we show that $\lambda_\mathcal{A}$ is deterministic. Let $e$ and $f$ be adjacent events in $D(\mathcal{A})$. There are compacts $x, x', x''$ with $x \prec x', x \prec x''$, $x' \neq x''$, $[x,x'] \in e$ and, $[x,x''] \in f$. According to Remark 2.8, there are transitions $t'$ and $t''$ with domain $\cod(x)$ with $x' = x[t']$ and $x'' = x[t'']$. Necessarily $t' \neq t''$. Then since $\mathcal{A}$ is deterministic, $ev(t') \neq ev(t'')$ i.e. $\lambda_\mathcal{A}(e) \neq \lambda_\mathcal{A}(f)$. Now we show that for any adjacent events $e$ and $f$ in $D(\mathcal{A})$, $e \circ f \iff \lambda_\mathcal{A}(e) \parallel \lambda_\mathcal{A}(f)$. Therefore $\lambda_\mathcal{A}$ is trace. Let $e$ and $f$ be adjacent. According to Remark 2.8(1), there is a compact $x$ in $D(\mathcal{A})$, two distinct transitions $t_1, t_2$ with $[x,x[t_1]] \in e$ and $[x,x[t_2]] \in f$. Then $\lambda_\mathcal{A}(e) = ev(t_1)$ and $\lambda_\mathcal{A}(f) = ev(t_2)$. According to Remark 2.8(2), $e \circ f$ if and only if there are transitions $t'_1$ and $t'_2$ with $t_1t'_1 \sim t_2t'_2$. Therefore $e \circ f$ if and only if $ev(t_1) \circ ev(t_2)$. According to the accessibility of $\mathcal{A}$, $a \parallel_\mathcal{A} b \iff a \parallel b$.

(2) Let $\mathcal{A}$ be a full trace automaton with independency predicate $\parallel$ and $e$ and $f$ be two events of $D(\mathcal{A})$ with $e \prec f$. Let us show that $\neg(\lambda_\mathcal{A}(e) \parallel \lambda_\mathcal{A}(f))$. $e$ is maximal in $C_f$. Let $x, x', x''$ be the compacts $D(\mathcal{A})$ such that $s(x) = C_f \setminus \{e\}$, $s(x') = C_f$ and $s(x'') = C_f \cup \{f\}$. Then $x \prec x' \prec x''$ and $[x,x'] \prec e$, $[x',x''] \prec f$. Therefore,
there are transitions \( t \) and \( t' \) of \( \mathcal{A} \) with \( x' = x[t] \) and \( x'' = x'[t'] \). Then \( \lambda_{\mathcal{A}}(e) \parallel \lambda_{\mathcal{A}}(f) \) if and only if \( \text{ev}(t) \parallel \text{ev}(t') \). Suppose \( \text{ev}(t) \parallel \text{ev}(t') \) and let us show this leads to some contradiction. \( \mathcal{A} \) being full, there are transitions \( u \) and \( u' \) with \( \text{ev}(u) = \text{ev}(t') \), \( \text{ev}(u') = \text{ev}(t) \) and \( uu' \sim t' \), \( [x,x[u]] \in f \) and \( [x[u],x[u'u']] \in e \). Then \( e \notin C_f \), which is a contradiction.

Let \( D \) be a coherent dl-domain. A labelling \( \lambda \) of \( D \) with alphabet \( A \) defines the partition, \( \mathcal{P}^\lambda \) of \( \text{Proj}(D) \), which is the set of non-empty subsets of the form \( \lambda^{-1}(a) \) for some \( a \in A \). For brevity, \( \lambda_0 \) denotes the class \( \lambda^{-1}(a) \). If \( \lambda_1 \) and \( \lambda_2 \) are two labellings of \( D \), \( \lambda_2 \) is finer than \( \lambda_1 \) — which is denoted \( \lambda_1 \subseteq \lambda_2 \), when any class of \( \mathcal{P}^\lambda_2 \) is included in some class of \( \mathcal{P}^\lambda_1 \). The previous relation \( \subseteq \) is a preorder on the class of labellings of \( D \).

**Lemma 2.30.** Let \( D \) be a coherent dl-domain and \( \lambda, \lambda' \) be a trace labellings of \( D \) with \( \lambda' \) finer than \( \lambda \). Then
- if \( \lambda \) is trace then \( \lambda' \) is trace;
- if \( \lambda \) is full then \( \lambda' \) is full.

**Proof.** For short, let \( \parallel \) and \( \parallel' \) denote, respectively, the binary relations on events, \( \parallel \lambda \) and \( \parallel' \lambda \) as in \((*)\) above and, \( A \) and \( A' \) denote the respective alphabets of \( \lambda \) and \( \lambda' \).

Since \( \lambda \leq \lambda' \): (1) For any events \( e \) and \( f \), \( \lambda(e) \neq \lambda(f) \Rightarrow \lambda'(e) \neq \lambda'(f) \).

Let us check that \( \parallel' \) corresponds to some predicate on pairs of events i.e. that the relations \( \parallel' \) and \( \parallel' \) are indeed antireflexive. If \( a = \lambda'(e), b = \lambda'(f) \) and \( e \circ f \vee e \# f \) then \( \lambda(e) \neq \lambda(f) \) since \( \lambda \) is deterministic and \( a \neq b \) according to (1).

Let us show: (2) Given events \( e \) and \( f \), \( \lambda'(e) \parallel' \lambda'(f) \Rightarrow \lambda(e) \parallel \lambda(f) \). Let \( e \) and \( f \) be events of \( D \) with \( \lambda'(e) \parallel' \lambda'(f) \). By definition of \( \parallel' \), there are some events \( e' \) and \( f' \), with \( \lambda'(e') = \lambda'(e), \lambda'(f') = \lambda'(f) \) and, \( e' \circ f'. \lambda \) satisfying \((Etr1), \lambda(e') \parallel \lambda(f') \). Since \( \lambda(e') = \lambda(e) \) and \( \lambda(f') = \lambda(f) \), \( \lambda(e) \parallel \lambda(f) \).

Now given events \( e \) and \( f \) with \( e \# f \), \( \lambda(e) \parallel \lambda(f) \), and according to (2), \( \lambda'(e) \parallel' \lambda'(f) \), i.e. \( \lambda' \) satisfies Axiom \((Etr2)\).

Suppose that \( \lambda \) satisfies moreover \((Eas)\). If \( e \) and \( f \) are events and \( e \prec f \), then \( \lambda(e) \parallel \lambda(f) \)) and according to (2), \( \lambda'(e) \parallel' \lambda'(f) \)). I.e. \( \lambda' \) satisfies \((Eas) \). \( \square \)

### 2.8. Lemmas

Miscellaneous results are grouped here. They will be used in Section 3.

Let \( D \) denote a coherent dl-domain. In order to avoid some ambiguities \( \succcurlyeq_D \) shall stand for the projectivity relation on events of \( D \). If \( x \) is a compact of \( D \), the partial order \( \uparrow x \) is also a coherent dl-domain. For short, \( \succcurlyeq_x \) denotes \( \succcurlyeq_{\uparrow x} \). Prime intervals of \( \uparrow x \) are prime intervals of \( D \) of the form \([y,y']\) where \( y' \) is compact in \( D \) and \( x \leq y \). Thus the restriction of \( \succcurlyeq_D \) on the set of the latter intervals contains \( \succcurlyeq_x \). Therefore any event \([y,y']\) \( \succcurlyeq_x \) of the domain \( \uparrow x \) corresponds to some event \([y,y']\) \( \succcurlyeq_D \). This correspondence is functional. It is also injective as shown below.
Lemma 2.31. Let $D$ be a coherent dI-domain and $x$ be a compact in $D$. $\uparrow x$ is a dI-domain and $\uparrow x$ and $\uparrow x$ coincide on the set of prime intervals of $\uparrow x$.

Proof. If $D$ is a coherent dI-domain, it is straightforward to check that $\uparrow x$ also is. Let $x$, $y$, $y'$, $z$, $z'$ be compacts of $D$ such that $x \leq y$, $x \leq z$, $y \prec y'$, $z \prec z'$, and $[y, y'] \prec [z, z']$. Then $x \leq y \wedge z$. Using the representation of elements of $D$ by configurations, one obtains first that $s(y') \setminus s(y)$ is a singleton and $s(y') \setminus s(y) = s(z') \setminus s(z)$. Therefore $(s(y') \cap s(z')) \setminus (s(y) \cap s(z)) = s(y') \setminus s(y)$ and according to Proposition 2.11, $y \wedge z \prec y' \wedge z'$, $[y \wedge z, y' \wedge z'] \prec [y, y']$, $[y \wedge z, y' \wedge z'] \prec [z, z']$ and $(y \wedge z) \vee (y' \wedge z') = (y \wedge z) \vee (y' \wedge z') = y \wedge (z \vee z')$. Consider a covering chain $y_1, \ldots, y_n$ from $y \wedge z$ to $y$. Then for any integer $i$ with $1 \leq i \leq n$, according to (C1), $y_i \prec y_i \vee (y' \wedge z')$ and $[y_i, y_i \vee (y' \wedge z')] \prec [y, y']$. Then for any $i$ such that $1 \leq i < n$, $[y_i, y_i \vee (y' \wedge z')] \prec [y_{i+1}, y_{i+1} \vee (y' \wedge z')]$. Finally $[y \wedge z, y' \wedge z'] \prec [y, y']$. One proves analogously $[y \wedge z, x' \wedge y'] \prec [z, z']$ and therefore $[y, y'] \prec [z, z']$. □

Therefore, given some coherent dI-domain $D$ and some compact $x$ of $D$, events of the domain $\uparrow x$ are identified with those of $D$ above $x$, i.e. projectivity classes of prime intervals $[y, y']$ where $x \leq y \prec y'$.

Lemma 2.32. Let $D$ be a coherent dI-domain, $y$ be a compact of $D$ and $e$ and $f$ be events of $\uparrow x$. Then

(i) $x(e) \vee x \prec x'(e) \vee x$, $[x(e) \vee x \prec x'(e) \vee x] \in e$ and $x(e) \vee x$ is the smallest compact $z$ of $\uparrow x$ such that there is $z'$ with $z \prec z'$ and $[z, z'] \in e$.

(ii) If $e \prec f$ then there are compacts $y$, $y'$, $y''$, with $x \leq y$, $y \prec y' \prec y''$, $[y, y'] \in e$, and $[y', y''] \in f$.

(iii) If $e$ and $f$ are adjacent then there are compacts $y$, $y'$, $y''$ with $x \leq y$, $y \prec y' \prec y''$, $y \prec y''$, $y' \neq y''$, $[y, y'] \in e$ and $[y, y''] \in f$ and $e \circ f$ if and only if $y'$ and $y''$ have a common upper bound.

Proof. (i) $e$ being an event of $\uparrow x$, there are compacts $y$ and $y'$ with $x \leq y \prec y'$ and $[y, y'] \in e$. Then $y'$ is an upper bound of $x$ and $x'(e)$. Therefore $e \notin s(x)$, $x \vee x(e) \prec x \vee x'(e)$, $[x \vee x(e), x \vee x'(e)] \in e$ and $x \vee x(e) \leq y$.

(ii) $e$ being maximal in $C(f)$, there is a compact $z$ with $s(z) = C(f) \setminus \{e\}$. Let $y = x \vee z$, $y' = x \vee x(f)$ and, $y'' = x \vee x'(f)$. Since $e \notin s(x)$ and $f \notin s(x)$, $s(y'') \setminus s(y') = \{f\}$, $s(y') \setminus s(y) = \{e\}$ and then $y \prec y' \prec y''$, $[y, y'] \in e$ and $[y', y''] \in f$.

(iii) $e$ and $f$ being events of $\uparrow x$, $x'(e) \uparrow x$ and $x'(f) \uparrow x$. If $e$ and $f$ are adjacent then $x'(e) \uparrow x(f)$ and $x(e) \uparrow x'(f)$. The domain being coherent, the sets $\{x(x(e), x'(f))\}$ and $\{x(x'(f), x(f))\}$ have both upper bounds. Let $y = x \vee x(e) \vee x(f)$, $y' = x \vee x'(e) \vee x(f)$ and $y'' = x \vee x'(e) \vee x'(f)$. Since $e \notin C(f)$ and $f \notin C(e)$, $y \prec y'$, $y' \prec y''$, $[y, y'] \in e$, $[y, y''] \in f$. If $e \circ f$ then $x'(e) \uparrow x'(f)$. The domain being coherent, its set of compacts satisfy Axiom (V). Therefore, $e \circ f$ if and only if $y', y''$ have a common upper bound. □
If $\sigma$ is an isomorphism from $D_1$ onto $D_2$, $\sigma$ provides a bijection, still denoted $\sigma$, from the set of events of $D_1$ to the set of events of $D_2$, defined by $\sigma(e) = [\sigma(x), \sigma(x')] \succ$ for any prime interval $[x, x']$ of $e$.

**Lemma 2.33.** Let $D, D'$ be two coherent $dt$-domains, $x$ and $y$ be compacts respectively of $D$ and of $D'$ and $\sigma$ be an isomorphism from $\uparrow x$ onto $\uparrow y$. If $e$ and $f$ are events of $\uparrow x$ then

(i) $\sigma(x(e) \lor x) = x(\sigma(e)) \lor y$ and $\sigma(x'(e) \lor x) = x'(\sigma(e)) \lor y$;

(ii) $e \prec f \Rightarrow \sigma(e) \prec \sigma(f)$;

(iii) $e \circ f = \sigma(e) \circ \sigma(f)$;

(iv) $e \# f \Rightarrow \sigma(e) \# \sigma(f)$.

**Proof.** (i) According to Lemma 2.32(i), $x \lor x(e)$ is the smallest compact $z$ lower or equal to $x$ such that there is $z'$ with $x \prec z' \prec z''$ and $[z, z'] \in e$, and $x \lor x'(e)$ is the unique $z'$ with $x \lor x'(e) \prec z'$ and $[x \lor x'(e), z'] \in e$. $f = \sigma(e) = [\sigma(x(e) \lor x), \sigma(x'(e) \lor x)] \succ$. $\sigma$ being an isomorphism, $\sigma(x(e) \lor x)$ is the smallest $z$ of $\uparrow y$ such that there is $z'$ with $\sigma(z') \in f$. According to Lemma 2.32(i), $\sigma(x(e) \lor x) = x(f) \lor y$ and $\sigma(x'(e) \lor x) = x'(f) \lor y$.

(ii) According to Lemma 2.31(ii) consider compacts $z, z', z''$ with $x \leq z, z \prec z' \prec z''$, $[z, z'] \in e$ and, $[z', z''] \in f$. Since $e \prec f$, according to Lemma 2.23(ii), there is no compact $z'''$ with $z''' \neq z'$ and $z \prec z''' \prec z''$. $\sigma$ being an isomorphism, $y \leq \sigma(z)$, $\sigma(z) \prec \sigma(z') \prec \sigma(z'''),$ $[\sigma(z), \sigma(z')] \in \sigma(e)$, $[\sigma(z'), \sigma(z'')] \in \sigma(f)$ and there is no compact $z''''$ with $z'''' \neq \sigma(z')$ and $\sigma(z) \prec z'''' \prec \sigma(z''')$. Then according to Lemma 2.23(ii), $\sigma(e) \prec \sigma(f)$.

(iii) and (iv) If $e$ and $f$ are adjacent, according to Lemma 2.32(iii), there are compacts $z, z'$ and $z''$ with $x \leq z, z \prec z' \prec z''$, $[z, z'] \in e$, $[z', z''] \in f$, then $e \circ f$ if and only if $z' \uparrow z''$. $\sigma$ being an isomorphism, $y \leq \sigma(z)$, $\sigma(z) \prec \sigma(z')$, $\sigma(z) \prec \sigma(z'')$, $\sigma(z') \neq \sigma(z'')$, $\sigma(e) = [\sigma(z), \sigma(z')] \succ$, $\sigma(f) = [\sigma(z), \sigma(z'')] \succ$ and $\sigma(z') \uparrow \sigma(z'')$ if and only if $z' \uparrow z''$. According to Lemma 2.32(iii), this means $\sigma(e) \circ \sigma(f)$ if and only if $e \circ f$. □

**Proposition 2.34.** If $D_1$ and $D_2$ are two coherent $dt$-domains with respective deterministic labellings $\lambda_1$ and $\lambda_2$ with values in a same alphabet then there is at most one isomorphism $\sigma$ from $D_1$ onto $D_2$ preserving labellings, i.e. such that $\lambda_2(\sigma(e)) = \lambda_1(e)$ for any event $e$ of $D_1$.

**Proof.** Let $D_1$ and $D_2$ be coherent $dt$-domains with deterministic labellings $\lambda_1$ and $\lambda_2$ and, $\sigma$ and $\tau$ two isomorphisms from $D_1$ onto $D_2$ preserving the labellings. Let us show that for any compact $x$ of $D_1$, $\sigma(x) = \tau(x)$. By contradiction. Let $x$ be a compact of $D_1$, such that $\sigma(x) \neq \tau(x)$. Consider a covering chain $x_1, \ldots, x_n$ from $\bot$ to $x$. There is a smaller index $i$ such that $\sigma(x_i) \neq \tau(x_i)$ and this index is not 1. Then $\sigma(x_{i-1}) = \tau(x_{i-1})$, and since $\sigma$ and $\tau$ preserve labellings, $\lambda_2([\sigma(x_{i-1}), \sigma(x_i)] \succ) = \lambda_2([\tau(x_{i-1}), \tau(x_i)] \succ)$. According to (Edet2), $\sigma(x_i) = \tau(x_i)$, which is a contradiction. Now if $x \in D_1$, $x = \bigvee \{ y \in \ldots \}$
Comp(D_1) \mid y \leq x \}, \text{ and}

\sigma(x) = \sqrt{\{\sigma(y) \mid y \in \text{Comp}(D_1) \land y \leq x\} \land y \in \text{Comp}(D_1) \land y \leq x\} = \tau(x). \quad \square

3. Semantics of finite automata

This section focuses on unfoldings of finite stable trace automata. Characterization of these unfoldings are given and some useful lemmas are presented in Section 3.1. The proof of the main result of this paper is in Section 3.2.

3.1. Regularity

Domains unfoldings of finite stable trace automata are characterized: they are the coherent dI-domains admitting some particular finite regular labellings. This result is mainly an adaption of results in [6] (Corollary 3.22) to the stable trace automata setting. The notions of regularity for labelled domains are studied.

A type is generally a class of isomorphic objects. If \( P \) is some partial order, a residual of \( P \) is some partial order of form \( r \times \), for some \( x \in P \). We let \( P \) be regular when the set of types of its residuals is finite.

Lemma 3.1. If \( \mathcal{A} \) is a stable trace automaton then for any compact \( x \) in \( D(\mathcal{A}) \), the map \( \sigma_x \) from \( \uparrow x \) onto the set of traces with domain cod(x) sending \( y \) onto \( \sigma_x(y) = z \) where \( y = xz \) is an order isomorphism.

Proof. Let \( x \) be a compact in \( D(\mathcal{A}) \). By definition, \( y \geq x \) if and only if \( y = xz \), for some trace \( z \) with domain \( \text{cod}(x) \). According to the left-cancellation rule, for a given \( y \) lower than \( x \), such a \( z \) is unique. Therefore \( \sigma_x \) is well-defined and is a bijection from \( \uparrow x \) onto the set of traces with domain \( \text{cod}(x) \). Let us show that \( \sigma_x \) is an order isomorphism. Let \( y \leq x \leq y' \). Then \( x\sigma(y)z = x\sigma(y') \) for some trace \( z \). According to left cancellation, \( \sigma(y) \leq x \sigma(y') \). \( \square \)

For short, a coherent dI-domain will be called regular when its ordered sets of compacts is. Residuals of a domain \( D \) will be the partial orders \( \uparrow x \), \( x \) ranging in \( \text{Comp}(D) \).

If \( (D, \lambda) \) denotes a coherent dI-domain with labelling \( \lambda \), then for any compact \( x \) of \( D \), the domain \( \uparrow x \) is also labelled by the map sending any event \([y, y'] \rangle \lambda \), onto \( \lambda([y, y']) \rangle \lambda \). Further on, \( \lambda \) still denotes the latter map. If \( D \) is a CR-domain labelled by \( \lambda \), an equivalence on residuals labelled by \( \lambda \) – called for short \( \lambda \)-residuals, is defined in the following way: \( (\uparrow x, \lambda) \) and \( (\uparrow y, \lambda) \) are equivalent when there is a domain
isomorphism \( \sigma \) from \( \uparrow x \) onto \( \uparrow y \) preserving \( \lambda \), i.e. such that \( \lambda(\sigma(e)) = \lambda(e) \) for any event \( e \) in \( \uparrow x \).

**Proposition 3.2.** If \( \lambda \) is a trace labelling of a coherent \( \mathcal{DL} \)-domain \( D \) then \( D \) is isomorphic to \( D(\mathcal{A}^m(D, \lambda)) \) where \( \mathcal{A}^m(D, \lambda) \) is the following stable trace automaton, \( \tau_1, \ldots, \tau_i, \ldots \) being an enumeration of the types of \( \lambda \)-residuals,

- states are the \( \tau_i \);
- events are the labels of \( \lambda \);
- \( (\tau_i, a, \tau_j) \) is a transition if and only if there are compacts \( x \) and \( x' \) such that \( (\uparrow x, \lambda) \) has type \( \tau_i \), \( (\uparrow x', \lambda) \) has type \( \tau_j \), \( x \prec x' \) and, \( \lambda([x, x']_{\prec}) = a \);
- the independency predicate \( \parallel \) is \( \parallel_\lambda \).

If moreover \( \lambda \) is full then \( \mathcal{A}^m(D, \lambda) \) is full.

**Proof.** Consider a coherent \( \mathcal{DL} \)-domain \( D \) with trace labelling \( \lambda \). It is straightforward (and tedious!) to check that \( \mathcal{A}^m(D, \lambda) \) is a well-defined stable trace automaton with independency predicate \( \parallel_\lambda \), and that moreover \( \mathcal{A}^m(D, \lambda) \) is full when \( \lambda \) is full. In order to prove that \( D \) is isomorphic to \( D(\mathcal{A}^m(D)) \), it is enough to find some folding morphism from \( \mathcal{A}(D) \) onto \( \mathcal{A}^m(D, \lambda) \). Such one is \( (\sigma, \eta) \) where:

- for any compact \( x \) of \( D \), \( \sigma(x) \) is the type of \( (\uparrow x, \lambda) \);
- \( \eta = \lambda \). \( \square \)

Let us note the minimality of the previously defined automaton \( \mathcal{A}^m(D, \lambda) \).

**Property 3.3.** If \( \mathcal{A} \) is some stable trace automaton then \( \mathcal{A} \) covers \( \mathcal{A}^m(D(\mathcal{A}), \lambda_{\mathcal{A}}) \)

**Proof.** A folding morphism from \( \mathcal{A} \) to \( \mathcal{A}^m(D(\mathcal{A}), \lambda) \) is \( (\sigma, 1) \) where for any state \( q \), \( \sigma(q) \) is the type of \( \lambda \)-residuals \( (\uparrow x, \lambda) \) for some arbitrary compact \( x \) in \( D(\mathcal{A}) \) with \( \text{cod}(x) = q \). \( \square \)

**Definition 3.4** (Regular labelling). A labelling \( \lambda \) of a regular coherent \( \mathcal{DL} \)-domain \( D \) is regular when it leads to a finite number of types of \( \lambda \)-residuals for \( D \).

**Proposition 3.5.** If \( \mathcal{A} \) is a finite stable trace automaton then \( D(\mathcal{A}) \) is regular and \( \lambda_{\mathcal{A}} \) is a finite regular labelling of \( D(\mathcal{A}) \).

**Proof.** Obviously, \( \lambda_{\mathcal{A}} \) is finite. According to Proposition 2.29, \( \lambda_{\mathcal{A}} \) is trace. According to Lemma 3.1, for any compacts \( x \) and \( y \) in \( D(\mathcal{A}) \) with \( \text{cod}(x) = \text{cod}(y) \), \( \sigma_y^{-1} \cdot \sigma_x \) is an isomorphism between residuals of \( \text{Comp}(D(\mathcal{A})) \), sending \( \uparrow x \) onto \( \uparrow y \). It preserves \( \lambda_{\mathcal{A}} \). \( \mathcal{A} \) having a finite number of states, \( D(\mathcal{A}) \) is regular and \( \lambda_{\mathcal{A}} \) is regular. \( \square \)

Conversely, if \( \lambda \) is some finite regular trace labelling of a regular coherent \( \mathcal{DL} \)-domain \( D \) then \( D \) is isomorphic to \( D(\mathcal{A}^m(D, \lambda)) \) and \( \mathcal{A}^m(D, \lambda) \) is finite. According to this and to Propositions 3.2, 2.29 and 3.5,
Proposition 3.6. (1) A coherent dI-domain is generated by a finite stable trace automaton if and only if it is regular and admits a finite regular trace labelling. (2) A coherent dI-domain is generated by a finite full trace automaton if and only if it is regular and admits a finite regular full trace labelling.

Lemma 3.7. If $D$ is a coherent dI-domain and $\lambda^1$ and $\lambda^2$ are two deterministic labellings with $\lambda^1$ finer than $\lambda^2$ then $\mathcal{A}^m(D, \lambda^1)$ covers $\mathcal{A}^m(D, \lambda^2)$.

Proof. If $\lambda^2 = f \circ \lambda^1$ for some map $f$, a folding morphism $\varphi = (\sigma, \eta)$ from $\mathcal{A}^m(D, \lambda^1)$ onto $\mathcal{A}^m(D, \lambda^2)$ is as follows. $\sigma$ sends the type of $(\uparrow x, \lambda^1)$ to those of $(\uparrow x, \lambda^2)$ and $\eta$ sends any label $a$ to $f(a)$. $\square$

Definition 3.8 (Labeling juxtaposition). Given a coherent dI-domain $D$, the juxtaposition of a family of labellings $(\lambda_i)_{i \in I}$ of $D$ is denoted $((\lambda_i)_{i \in I})$ and is the labelling of $D$ sending any event $e$ on the family of labels $(\lambda_i(e))_{i \in I}$.

Remark 3.9. The juxtaposition of two regular labellings is generally not regular.

Actually, any deterministic labelling of the complete binary tree ranging in $\{a, b\}$ is regular (there is only one type of residuals). One may find a juxtaposition of two such labellings which is not regular – see Fig. 3 for two such labellings and Fig. 4 for their juxtaposition.

\[ \lambda_1 : \]
\[ \lambda_2 : \]

Fig. 3. $\lambda_1$, $\lambda_2$. 
It is straightforward to check that

Properties 3.10. Let $\lambda_1$ and $\lambda_2$ be labellings of a coherent $dI$-domain such that $\lambda_1 \leq \lambda_2$ then

1. if $\lambda_1$ is deterministic then also is $\lambda_2$;
2. if $\lambda_2$ is regular then also is $\lambda_1$.

Lemma 3.11. Given a regular coherent $dI$-domain with deterministic labelling $\lambda$ and a finite family $(\beta_i)_{i \in \{1, \ldots, n\}}$ of regular labellings all finer that $\lambda$ then $(\beta_1, \ldots, \beta_n)$ is regular.

Proof. Let $\beta = (\beta_1, \ldots, \beta_n)$. $\beta$, like any $\beta_i$ for some $i \in \{0, \ldots, n\}$, is deterministic since it is finer than $\lambda$. Note therefore that if $x$ and $y$ are compacts then, according to Fact 2.34, there is at least one isomorphism from $\uparrow x$ onto $\uparrow y$ preserving $\beta$ (respectively $\beta_i$), and if such an isomorphism exists, this one is also the unique isomorphism from $\uparrow x$ onto $\uparrow y$ preserving $\lambda$.

Now let us prove the lemma by contradiction. Suppose $(\beta_1, \ldots, \beta_n)$ is not regular. $\lambda$ being regular, there is therefore an infinite set $S$ of compacts such that labelled residuals $(\uparrow x, \lambda)$, $x$ ranging over $S$, have the same type though for any distinct $x$ and $y$ in $S$ labelled residuals $(\uparrow x, (\beta_1, \ldots, \beta_n))$ and $(\uparrow y, (\beta_1, \ldots, \beta_n))$ have distinct types. For any $x, y \in S$, let $\sigma_{x,y}$ denote the unique isomorphism preserving $\lambda$ of $\uparrow x$ onto $\uparrow y$. 
Let $x, y$ be distinct in $S$. Since there is no isomorphism preserving $\beta$ from $\uparrow x$ onto $\uparrow y$, there is an event $e$ of $\uparrow x$ with $\langle \beta_1, \ldots, \beta_n \rangle(\sigma_{x,y}(e)) \neq \langle \beta_1, \ldots, \beta_n \rangle(e)$. Then there is some $i \in \{1, \ldots, n\}$ with $\beta_i(\sigma_{x,y}(e)) \neq \beta_i(e)$ and, according to the initial remark, $(\uparrow x, \beta_i)$ and $(\uparrow y, \beta_i)$ have different types. Applying Ramsey's theorem (infinitary version, see Theorem A.5 in appendix), one deduces the existence of some $i \in \{0, \ldots, n\}$ and of some infinite subset $S'$ of $S$ such that if $x$ and $y$ are distinct in $S'$, then $(\uparrow x, \beta_i)$ and $(\uparrow y, \beta_i)$ have distinct types. Therefore $\beta_i$ is not regular, which is a contradiction. \[\]

3.2. Stable trace vs. full trace automata

This section is devoted to the proof of the following theorem.

**Theorem 3.12.** Any finite stable trace automaton is covered by some finite full trace automaton.

This is indeed a consequence of the following result.

**Theorem 3.13.** For any finite regular trace labelling of a regular coherent dI-domain, there is a finer labelling which is full, finite and regular.

**Proof of Theorem 3.12.** Let $\mathcal{A}$ be a finite stable trace automaton. Then $\mathcal{A}$ folds onto $\mathcal{A}^m(D(\mathcal{A}, \lambda_{\mathcal{A}}))$ — Property 3.3, let $\phi$ be the folding morphism from $\mathcal{A}$ onto $\mathcal{A}^m(D(\mathcal{A}, \lambda_{\mathcal{A}}))$. $\lambda_{\mathcal{A}}$ being finite trace and regular, there is some finite full trace regular labelling $\lambda'$ finer than $\lambda_{\mathcal{A}}$. According to Lemma 3.7, $\mathcal{A}^m(D(\mathcal{A}, \lambda'))$ folds onto $\mathcal{A}^m(D(\mathcal{A}, \lambda_{\mathcal{A}}))$, let $\psi$ denote the corresponding folding morphism. Consider the fibered product of $\phi$ and $\psi$, the stable trace automaton $\mathcal{A} \times_{\phi, \psi} \mathcal{A}^m(D(\mathcal{A}, \lambda'))$ is finite and folds onto $\mathcal{A}$. It also folds onto $\mathcal{A}^m(D(\mathcal{A}, \lambda'))$ which is full. According to Proposition 2.16, $\mathcal{A} \times_{\phi, \psi} \mathcal{A}^m(D(\mathcal{A}, \lambda'))$ is full. \[\]

Let us prove Theorem 3.13. A little material is required. Given a coherent dI-domain, binary relations $R^1$, $R^2$, $\bowtie 1$, $\bowtie 2$ and $\bowtie$ are defined on the set on the set of events by

- $eR^1e'$ if and only if there exists $e''$ with $e \prec e'' \circ e'$;
- $eR^2e'$ if and only if there exists $e''$ with $e \circ e'' \prec e'$;
- $\bowtie 1$ is the symmetric closure of $R^1$;
- $\bowtie 2$ is the symmetric closure of $R^2$;
- $\bowtie = \bowtie 1 \cup \bowtie 2$.

Given a trace labelling $\lambda$ of a coherent dI-domain, for any labels $a$ and $b$ with $a \parallel b$ \(\lambda(a,b)\) stands for the labelling with values in \{0,1,2\} sending any event $e$ to:

- 0 if $\lambda(e) \neq a$,
- 1 if $\lambda(e) = a$ and if there exists some $f$ with $\lambda(f) = b$ and $f \prec e$,
- 2 if $\lambda(e) = a$ and there is no $f$ with $\lambda(f) = b$ and $f \prec e$.

Finally let $\lambda^*$ be the juxtaposition of $\lambda$ and the family $(\lambda(a,b))_{(a,b) \in \mathcal{A}^2 \cap \parallel}$. Note

**Remark 3.14.** If $\lambda$ is finite then also is $\lambda^*$. 

Recall

**Terminology 3.15.** A coloring of a simple graph with no loops is a map from the set of its vertices such that two adjacent vertices have different images. If $G$ is a simple graph with no loops, $\chi(G)$, the chromatic number of $G$ is the minimal cardinal $\alpha$ such that there is some coloring of $G$ with rank of cardinality $\alpha$.

The sketch of the proof of Theorem 3.13 is given below. The result follows from the three next lemmas.

**Lemma 3.16.** Given a coherent di-domain with some trace labelling $\lambda$, which is a coloring for $\bowtie^1$, $\lambda^*$ is full trace.

**Lemma 3.17.** Given a regular coherent di-domain with a regular finite trace labelling $\lambda$, there is some regular labelling $\lambda^1$, finer than $\lambda$, which is a finite coloring for the graph $\bowtie^1$.

and

**Lemma 3.18.** Given a regular coherent di-domain with a finite regular trace labelling $\lambda$, $\lambda^*$ is regular.

Indeed, let $\lambda$ be a trace labelling of a coherent di-domain $D$. If $\lambda^1$ is a labelling finer than $\lambda$ and also a coloring for the graph $\bowtie^1$ then $\lambda^1$ is trace according to Lemma 2.30 and $(\lambda^1)^*$ is a full trace labelling according to Lemma 3.16. If moreover $D$ is regular and $\lambda$ is finite and regular, such a labelling $\lambda^1$ may be chosen finite and regular according to Lemma 3.17 and $(\lambda^1)^*$ is a finite regular full trace labelling of $D$ according to Remark 3.14 and Lemma 3.18. Moreover $(\lambda^1)^*$ is finer than $\lambda^1$ and then finer than $\lambda$.

Let us prove Lemma 3.16.

**Proof of Lemma 3.16.** According to Lemma 2.30, because $\lambda^*$ is finer than $\lambda$, $\lambda^*$ is trace. It remains to show that it satisfies Axiom (Eas). Let $A$ and $A^*$ denote, respectively, the alphabets of $\lambda$ and $\lambda^*$ and let $||$ and $||^*$ stand, respectively, for $||\lambda$ and $||\lambda^*$. By definition of $\lambda^*$, for any $a,b \in A$ with $a||b$ and $b' \in A^*$, if $\lambda^* b' \subseteq \lambda b$ then either $\lambda^* b' \subseteq (\lambda_a)^- \cup (\lambda^* b' \cap \lambda_a)^-$ or $\lambda^* b' \cap \lambda_a^- = \emptyset$ where $\lambda_a^-$ is the set of events $f$ for which there is some $e \in \lambda_a$ with $e \rightarrow f$. Suppose now that $\lambda^*$ does not satisfy (Eas), let us show this leads to a contradiction. There are some events $e,f$, and labels $a', b' \in A^*$ and $a,b \in A$, with $e \rightarrow f$, $e \in \lambda_a \subseteq \lambda_a$, $f \in \lambda^* b' \subseteq \lambda b$ and $a'||b'$. By definition of $||$, there are events $e', f'$ such that $\lambda^*(e') = a'$, $\lambda^*(f') = b'$ and $e' \rightarrow f'$ and therefore $a||b$. $f' \notin \lambda_a^-$ because if there is $e''$ in $\lambda_a$ with $e'' \rightarrow f'$ then $e'' \bowtie^1 e'$, which is impossible because $\lambda$ is a coloring for $\bowtie^1$. Then $f' \notin \lambda^* b' \setminus \lambda_a^-$ and $\lambda^* b' \notin \lambda_a^-$. But $f \notin \lambda^* b' \cap \lambda_a^- = \emptyset$, which is a contradiction. $\square$

Note that with the assumptions of Lemma 3.16.
Corollary 3.19. \( \lambda^* \) is a coloring of \( \infty^2 \).

In order to prove Lemmas 3.17 and 3.18, we need:

Properties 3.20. If \( \lambda \) is a trace labelling of a coherent dl-domain and \( a \) is a label of \( \lambda \) then

1. if \( e \prec f \), \( f \prec g \) and, \( \lambda(e) = \lambda(f) \) then \( e = f \);
2. if \( e \) and \( f \) belong to \( \lambda_a \) then \( e \) and \( f \) are not in immediate conflict and if \( \{e, f\} \) is consistent then \( e \) and \( f \) are comparable;
3. if \( e, f \in \lambda_a \) and \( eR^1f \) then \( e < f \);
4. if \( e_1, e_2, e_3 \in \lambda_a \), \( e_1 < e_2 < e_3 \) and \( e_1 \prec f \circ e_3 \) then \( f \circ e_2 \).

Proof. (1) If \( e \) and \( f \) are distinct, \( e \prec g \) and, \( f \prec g \) then \( f \) and \( e \) are not comparable. Since \( e \uparrow f \) is consistent, according to Remark 2.24, \( e \circ f \). Then, since \( \lambda \) satisfies \( (Etr1) \), \( \lambda(e) = \lambda(f) \) and then \( \lambda(e) \neq \lambda(f) \).

(2) Let \( e, f \in \lambda_a \). Since \( \lambda \) satisfies \( (Etr2) \), \( e \) and \( f \) are not in immediate conflict. Consider moreover \( e \uparrow f \). Since \( \lambda(e) = \lambda(f) \) and \( \lambda \) satisfies \( (Etr1) \), \( \{e, f\} \notin \circ \) and \( e \) and \( f \) are comparable according to Remark 2.24.

(3) Let \( e, f \in \lambda_a \) and \( g \) such that \( e \prec g \circ f \). Necessarily \( e \uparrow f \), otherwise because \( g > e \), \( \{g, f\} \) would be inconsistent. \( f \notin e \) otherwise by transitivity \( f \leq g \). According to (2), \( e < f \).

(4) Let \( e_1, e_2, e_3 \in \lambda_a \), \( e_1 < e_2 < e_3 \) and \( e_1 \prec f \circ e_3 \). \( e_1 \prec f \) and \( e_1 < e_2 \) implies \( e_2 \notin f \). Because \( e_3 \circ f \), \( f \notin e_2 \) (otherwise by transitivity \( f \leq e_3 \)). From \( e_3 \uparrow f \) and \( e_2 < e_3 \), one deduces \( e_2 \uparrow f \). According to Remark 2.24, \( e_2 \circ f \).

Furthermore, if \( \lambda \) is a trace labelling of a coherent dl-domain then for any label \( a \), \( R^1_{\lambda_a} \) and \( \infty^1_{\lambda_a} \) denote the respective restrictions of \( R^1 \) and \( \infty^1 \) on \( \lambda_a \).

Let us prove Lemma 3.17 which we recall:

Given a regular coherent dl-domain with a regular finite trace labelling \( \lambda \), there is a regular labelling \( \lambda^1 \), finer than \( \lambda \) which is a finite coloring for the graph \( \infty^1 \).

Proof. Consider a regular coherent dl-domain with a finite regular trace labelling \( \lambda \). First we show the existence of a finite coloring \( \beta_{\lambda,a} \) of the graph induced by \( \infty^1 \) on the class \( \lambda_a \) for each label \( a \). Letting \( \lambda^1 \) be the juxtaposition of \( \lambda \) and of the latter colorings, \( \lambda^1 \) is finite, finer than \( \lambda \) and it is also a coloring for \( \infty^1 \). Finally we show that \( \lambda^1 \) is regular.

Recall,

Terminology 3.21. A clique in a graph (simple, with no loops) is a set of pairwise adjacent vertices. For any graph \( G \), \( \omega(G) \), when it is defined, denotes the maximal cardinality of cliques of \( G \).

Lemma 3.22. If \( \lambda \) is a finite regular trace labelling of a regular coherent dl-domain then for any label \( a \), \( \omega(\infty^1_{\lambda,a}) \) is finite.
Proof. Let \( n \) be a non null integer such that there is a clique for \( \succ^1_{\lambda, a} \) with size \( n \) in \( \lambda_a \). We show then that there are at least \( n \) distinct types of residuals labelled by \( \lambda \) (see (10) above). \( \lambda \) being regular, \( \omega(\succ^1_{\lambda, a}) \) is therefore finite.

Consider such a clique. According to Property 3.20(3), it is a chain \( e_1 < \cdots < e_i < \cdots < e_n \) such that for any integers \( i, j \) with \( 1 \leq i < j \leq n \), \( e_i R^1 e_j \), i.e. there is some \( f_{i,j} \) with \( e_i \prec f_{i,j} \prec e_j \). According to Property 3.20(4), such an \( f_{i,j} \) satisfies also \( f_{i,j} \circ e_k \) for any integer \( k \) with \( i < k \leq j \). For any \( i \) with \( 1 \leq i < n \), let \( f_i = f_{i,n} \). Then for any \( i, j \) with \( 1 \leq i < j \leq n \), \( e_i \prec f_i \) and \( f_i \circ e_j \). The inclusion order on the configurations \( C(e_i) \) and \( C(f_i) \) is depicted in Fig. 5. Consider then the following sequences \((X_j)_{1 \leq j \leq n}\) and \((Y_{i,j})_{1 \leq i < j \leq n, i < n}\) of sets of events (see Fig. 6), defined inductively by \( X_n = C(e_n) \); if \( 1 \leq j < n \), \( X_j = C(e_j) \cup Z_j \), where \( Z_j \) is the set \( (C(f_1) \cup \cdots \cup C(f_{j-1})) \cup (C(f_j) \setminus \{e_j\}) \) \((Y_{i,j})_{1 \leq i < j \leq n, i < n}\) and, for any \( i, j \) such that \( 1 \leq i < j \leq n \), \( Y_{i,j} = X_j \cup (C(f_j) \setminus \{e_j\}) \) if \( j \neq n \), \( Y_{i,j} = X_j \cup C(f_i) \).
Let us show that for any integers \( i \) and \( j \) with \( 1 \leq i \leq j \leq n \) and \( i < n \):

1. \( X_j \subseteq X_{j+1} \);
2. \( e_j \notin X_j \) and \( e_j \notin Y_{i,j} \);
3. \( X_j \) is a configuration;
4. \( Y_{i,j} \) is a configuration.

**Proof.** (1) If \( j \) is an integer with \( 1 \leq j \leq n \) then \( C(e_j) \subseteq X_j \). Therefore, if \( j < n \), \( C(e_j) \subseteq C(e_{j+1}) \subseteq X_{j+1} \) and then \( X_j \subseteq X_{j+1} \).

(2) For any integers \( i, j \) with \( 1 \leq i < j < n \), \( e_j \notin C(f_i) \) (because \( e_j \circ f_i \) then \( e_j \notin X_j \), \( e_j \notin Y_{i,j} \) and if \( j \neq n \), \( e_j \notin Y_{i,j} \).

(3) By induction. \( X_n \) is a configuration. Note that for any \( j \) with \( 1 \leq j < n \), \( e_j \) is a maximal in \( C(f_j) \), therefore \( C(f_j) \setminus \{e_j\} \) is a configuration. Suppose that \( X_{j+1} \) is a configuration. Because \( X_j = C(e_j) \cup C(f_j) \cap X_{j+1} \cup \ldots \cup ((C(f_j) \setminus \{e_j\}) \cap X_{j+1}) \), \( X_j \) is a union of configurations. According to (1), \( X_j \subseteq C(e_n) \), \( X_j \) is then consistent and \( X_j \) is a configuration.

(4) For any \( i, j \) with \( 1 \leq i \leq j \leq n \) and \( i < n \), since \( X_j \subseteq C(e_n) \) (according to (1)) and \( f_i \circ e_n \), \( C(f_i) \cup C(e_n) \) is a configuration containing \( Y_{i,j} \). Therefore \( Y_{i,j} \) is consistent. Since according to (3), \( X_j \) is a configuration, \( Y_{i,j} \) is an union of configurations. Therefore \( Y_{i,j} \) is a configuration.

Sequences \( (X_j)_{1 \leq j \leq n} \) and \( (Y_{i,j})_{1 \leq i \leq j \leq n} \) satisfy the following properties:

5. \( X_j \cup \{e_j\} \) is a configuration and \( X_j \subset X_j \cup \{e_j\} \) if \( 1 \leq j \leq n \);
6. \( Y_{i,j} \subset Y_{i,j} \cup \{e_j\} \), if \( 1 \leq i \leq j \leq n \) and \( i < n \);
7. \( Y_{i,j} \subset Y_{i,j} \cup \{f_i\} \), if \( 1 \leq i \leq j < n \);
8. \( Y_{i,j} \subset Y_{i,j} \cup \{f_i\} \), if \( 1 \leq j < n \);
9. \( Y_{i,j} \subset Y_{i,j+1} \setminus X_{j+1} \), if \( 1 \leq i \leq j < n \).

(5) Let \( j \) be an integer with \( 1 \leq j \leq n \). According to (1), \( X_j \cup \{e_j\} \subseteq C'(e_n) \), and because \( C(e_j) \subseteq X_j \), \( X_j \cup \{e_j\} \) is a configuration. According to (2), \( e_j \notin X_j \).

(6) Let \( i \) and \( j \) be integers with \( 1 \leq i \leq j \leq n \) and \( i < n \). Then \( C(e_j) \subseteq X_j \subseteq Y_{i,j} \), \( Y_{i,j} \cup \{e_j\} \subseteq C'(e_n) \cup C(f_i) \), and \( C'(e_n) \cup C(f_i) \) is a configuration since \( f_i \circ e_n \), therefore \( Y_{i,j} \cup \{e_j\} \) is a configuration. According to (2), \( e_j \notin Y_{i,j} \).

(7) Let \( i \) and \( j \) be integers with \( 1 \leq i < j \leq n \). Since \( f_i \circ e_n \), \( C(f_i) \subseteq C'(e_n) \) and \( C'(f_i) \cup C'(e_n) \) are configurations and \( f_i \notin C(f_i) \cup C(e_n) \), \( C(f_i) \subseteq Y_{i,j} \) and according to (1), \( Y_{i,j} \subseteq C(f_i) \cup X_n = C(f_i) \cup C(e_n) \). \( Y_{i,j} \) is a configuration not containing \( f_i \). \( Y_{i,j} \cup \{f_i\} \subseteq C(f_i) \cup C'(e_n) \) and then \( Y_{i,j} \cup \{f_i\} \) is also a configuration.

(8) The result follows from the fact that if \( j \) is an integer with \( 1 \leq j \leq n \) then \( e_j \in C(f_j) \) and according to (2), \( e_j \notin Y_{i,j} \).

(9) Let \( i \) and \( j \) be integers with \( 1 \leq i < j \leq n \). Then \( C(f_i) \cap X_{j+1} \subseteq X_j \) and \( (C(f_j) \setminus \{e_j\}) \cap X_{j+1} \subseteq X_j \). Therefore \( Y_{i,j+1} \cap X_{j+1} = C(f_i) \cap X_{j+1} = C(f_i) \cap X_j = Y_{i,j} \cap X_j \) and \( Y_{j+1} \cap X_{j+1} = C(f_j) \cap X_{j+1} = (C(f_j) \setminus \{e_j\}) \cap X_j = Y_{j+1} \cap X_j \).

Let \( s \) denote the isomorphism associated with the domain in consideration (see Proposition 2.11). For any integers \( i \) and \( j \) with \( 1 \leq i \leq j \leq n \) and \( i < n \), \( X_i \) and \( Y_{i,j} \) are configurations so let \( x_i \) and \( y_{i,j} \) denote the compacts of the domain with \( s(x_i) = X_i \)
and \( s(y_{i,j}) = Y_{i,j} \). We are going to prove

(10) For any integers \( i \) and \( j \) with \( 1 \leq i < j \leq n \), the types of \((x_i, \lambda)\) and \((x_j, \lambda)\) are distinct.

For this purpose, let us show first

(11) If \( 1 \leq i < n \) then there is no event \( f \) of the form \([y_{i,i}, z] \rhd \) with \( \lambda(f) = \lambda(f_i) \).

Actually, since \( f_i \circ e_j \), according to \((Etr1)\) such an \( f \) would satisfy \( \lambda(f) \parallel \lambda(e_i) \) since \( f_i \circ e_j \), then, since according to \((6)\) \( e_i \) has the form \([y_{i,i}, z] \rhd \), one would have according to Axiom \((Etr2)\), \( f \circ e_i \) and according to \((Edet)\), \( f = f_i \), contradicting \((8)\). Nevertheless, \((7)\) shows that \( f_i \) has the form \([y_{i,i}, z] \rhd \) when \( i < j \).

Proof. (10) By contradiction. Consider two integers \( i, j \) with \( 1 \leq i < j \leq n \) and such there is an isomorphism \( \sigma_{i,j} \) preserving \( \lambda \) from \( x_i \) onto \( x_j \). One shows that necessarily such a \( \sigma_{i,j} \) sends \( y_{i,i} \) onto \( y_{i,j} \), which contradicts \((11)\). Actually, consider a covering chain \((s_p)_{1 \leq p \leq q} \) from \( x_i \) to \( y_{i,i} \). According to \((9)\), \((Y_{i,i} \setminus x_i) \times (X_j \setminus x_j) \subseteq \circ \) and \( Y_{i,i} \setminus x_j = Y_{i,i} \setminus X_i \). \((s_p \setminus x_j)_{1 \leq p \leq q} \) is therefore a covering chain from \( x_j \) to \( y_{i,j} \) and \([s_p \setminus x_j, s_{p+1} \setminus x_j] \rhd \). Since if \( 1 \leq p < q \), \( \lambda([s_p \setminus x_j, s_{p+1} \setminus x_j]) = \lambda([s_p \setminus x_j, s_{p+1} \setminus x_j]) \) and \( \lambda \) is deterministic \((Edet)\), an induction shows that if \( 1 \leq p < q \), \( \sigma_{i,j}(s_p) = s_p \setminus x_j \) and \([s_i, s_{i,j}](s_{p+1}) \rhd [s_p \setminus x_j, s_{p+1} \setminus x_j] \) and finally \( \sigma_{i,j}(s_q) = s_q \setminus x_j \), i.e. \( \sigma_{i,j}(y_{i,i}) = y_{i,i} \setminus x_j \).

Recall

Terminology 3.23. The coloring number of a simple graph with no loops is the smallest cardinal \( \alpha \) such that there is a well ordering of its set of vertices such that any vertex strictly dominates at most \( \alpha \) vertices. It follows that the chromatic number of a graph is lower or equal to its coloring number plus one.

Lemma 3.24. If \( \lambda \) is a regular trace labelling of a regular coherent dl-domain, then for any label \( a \) the graph \( \phi^a_{\lambda,a} \) has a finite coloring number. Precisely, this number is \( \omega(\phi^a_{\lambda,a}) - 1 \) and for any event \( e \) in \( \lambda_a \),

- there is at least \( \omega(\phi^a_{\lambda,a}) - 1 \) events \( f \) in \( \lambda_a \) with \( e \notin f \) and \( e \phi^a f \);
- \( (f \in \lambda_a \wedge f \not\geq e \wedge f \phi^a e) \Rightarrow f < e \).

Proof. If the event \( f \) lies in the set \( S = \{ f \in \lambda_a | e \not\geq f \wedge f \phi^a e \} \) then according to Property 3.20(3) \( f \phi^a e \) and \( f < e \). According to Property 3.20(2), \( S \) is a chain, denoted \((f_i)_{1 \leq i \leq n} \) where \( n \) is an integer and \( f_i < f_j \) for any integers \( i, j \) with \( 1 \leq i < j \leq n \). According to Property 3.20(4), \( f_i \phi^a f_j \) if \( 1 \leq i < j \leq n \). Therefore, \( S \cup \{ e \} \) is a clique for \( \phi^a \) and its cardinality is strictly lower than \( \omega(\phi^a_{\lambda,a}) \) which is finite according to Lemma 3.22. 

Consider a regular coherent dl-domain with a finite regular trace labelling \( \lambda \). If \( a \) is a label for \( \lambda \), a consequence of Lemma 3.24 is that \( \chi(\phi^a_{\lambda,a}) = \omega(\phi^a_{\lambda,a}) \). According again
to Lemma 3.24, one may define by induction on \( \prec \) a coloring \( \beta_{\lambda, a} \) of \( \varpi_{\lambda, a} \) ranging in \( \{0, \ldots, \omega(\varpi_{\lambda, a})\} \):

- If \( e \in \lambda_{a} \), \( \beta_{\lambda, a}(e) \) is the smallest \( l \) in \( \{1, \ldots, \omega(\varpi_{\lambda, a})\} \) such that there is no \( f \in \lambda_{a} \) with \( f \prec e \), \( f \triangleright l \) \( \omega \), and \( \beta_{\lambda, a}(f) = l \),
- If \( e \not\in \lambda_{a} \), then \( \beta_{\lambda, a}(e) = 0 \).

Let \( \lambda^{1} \) be the juxtaposition of \( \lambda \) and of the labellings \( \beta_{\lambda, a} \) ranging in the alphabet of \( \lambda \). \( \lambda^{1} \) is finite, finer than \( \lambda \) and it is a coloring for \( \varpi^{1} \). According to Lemma 3.11, \( \lambda^{1} \) is regular if for any label \( a \), the labelling \( \langle \lambda, \beta_{\lambda, a} \rangle \) is regular. In order to conclude the proof of Lemma 3.17, let us show

**Lemma 3.25.** If \( \lambda \) is a finite regular trace labelling of a regular coherent \( \mathcal{D} \)-domain then for any label \( a \), \( \langle \lambda, \beta_{\lambda, a} \rangle \) is regular.

**Proof.** By contradiction. Suppose there is some label \( a \) such that \( \langle \lambda, \beta_{\lambda, a} \rangle \) is not regular. We show then that the following situation holds. There is

- an infinite chain \( I \);
- a family of distinct compacts \( (x_{i})_{i \in I} \);
- a family of isomorphisms \( (\sigma_{i,j})_{(i,j) \in I^{2}} \) such that:
  - labelled residuals \( (\uparrow x_{i}, \lambda) \) have the same type and \( \sigma_{i,j} \) is the unique isomorphism from \( \uparrow x_{i} \) onto \( \uparrow x_{j} \) preserving \( \lambda \);
  - \( (e_{i,j}, i,j) \) is an event of \( \uparrow x_{i} \), \( \lambda(e_{i,j}) = a \) and, \( \sigma_{i,j}(e_{i,j}) = e_{j,i} \),
  - for any \( i, j \) in \( I \) with \( i < j \), \( \lambda(f_{i,j}) = a \), \( f_{i,j} < e_{i,j} \), \( f_{i,j} \varpi^{1} e_{i,j} \), and:
    - either (case 1): for any \( i, j \) in \( I \) with \( i < j \), \( \lambda(f_{i,j}) = a \), \( f_{i,j} \varpi^{1} e_{i,j} \), and \( (e_{j,i}, j,i) \) \( \not\in \mathcal{R}^{1} \);
    - or (case 2): for any \( i, j \) in \( I \) with \( i < j \), \( f_{i,j} \in s(x_{i}) \) and there is no \( f \) with \( \lambda(f) = a \), \( f \varpi^{1} e_{i,j} \) and \( \beta_{\lambda, a}(f) = \beta_{\lambda, a}(f_{i,j}) \).

It is shown later that the two cases mentioned above lead to contradictions (Lemma 3.26 for the case (1) and Lemma 3.27 for the case (2).

Because \( \lambda \) is regular and \( \langle \lambda, \beta_{\lambda, a} \rangle \) is not, there is some infinite subset \( S \) of compacts such that labelled residuals \( (\uparrow x, \lambda) \), \( x \) ranging over \( S \), have the same type though for any distinct \( x \) and \( y \) in \( S \), \( (\uparrow x, \langle \lambda, \beta_{\lambda, a} \rangle) \) and \( (\uparrow y, \langle \lambda, \beta_{\lambda, a} \rangle) \) have distinct types.

If \( x, y \in S \) let \( \sigma_{x,y} \) stand for the unique domain isomorphism preserving \( \lambda \) from \( \uparrow x \) onto \( \uparrow y \). If \( x \) and \( y \) are distinct such an isomorphism does not preserve \( \langle \lambda, \beta_{\lambda, a} \rangle \), i.e., there is an event \( e \) in \( \uparrow x \) with \( \beta_{\lambda, a}(\sigma_{x,y}(e)) \neq \beta_{\lambda, a}(e) \).

Then for any \( x, y \) in \( S \) there is some event, namely \( e_{x,y} \), which is minimal among events \( e \) in \( \uparrow x \) satisfying \( \beta_{\lambda, a}(\sigma_{x,y}(e)) \neq \beta_{\lambda, a}(e) \). Necessarily \( \lambda(e_{x,y}) = a \). Let \( e_{x,y} = \sigma_{x,y} \) \( (e_{x,y}) \). One notes then that \( e_{x,x} \) is also a minimal element of \( \uparrow y \) among the \( e \) in \( \uparrow y \) with \( \beta_{\lambda, a}(\sigma_{x,y}(e)) \neq \beta_{\lambda, a}(e) \).

If \( \lambda(f) = a \) and \( f \varpi^{1} e_{x,y} \), then \( f < e_{x,y} \) according to Property 3.20(3), then \( \{f\} \cup s(x) \) is consistent and either \( f \in s(x) \) or \( f \) is an event of \( \uparrow x \).
For any \( x, y \in S \), \( k \in \{ 1, \ldots, \omega_{\lambda, a} \} \) and \( l \in \{ 0, 1 \} \), one defines the following predicates:

- \( P(x, y, k, l) : \) "there is an event \( f \) with \( f \in \lambda_a \land f R^1 \varepsilon_{x, y} \land \beta_{\lambda, a}(f) = k \) and if \( l = 0 \) then \( f \in s(x) \) and if \( l = 1 \) then \( f \) is an event of \( \uparrow x \)"

- \( Q(x, y, k) : \) "there is an event \( f \) with \( f \in \lambda_a \land f R^1 \varepsilon_{x, y} \land \beta_{\lambda, a}(f) = k \)."

Then \( Q(x, y, k) = P(x, y, k, 0) \lor P(x, y, k, 1) \).

Let \( x, y \in S \). Since \( \beta_{\lambda, a}(e_{x, y}) \neq \beta_{\lambda, a}(e_{y, y}) \) there is some \( k \) in \( \{ 1, \ldots, \omega_{\lambda, a} \} \) such that, either \( Q(x, y, k) \), or \( Q(y, x, k) \) holds. Applying the corollary of Ramsey's theorem (Corollary A.6 in appendix), one shows the existence of an infinite chain \( I \), of a family of distinct compacts \( (x_i)_{i \in I} \) in \( S \), of \( k \in \{ 1, \ldots, \omega(\omega_{\lambda, a}) \} \) and of \( l \in \{ 0, 1 \} \) such that if \( i < j \) then \( P(x_i, x_j, k, l) \) holds and \( Q(x_j, x_i, k) \) does not.

For any distinct \( i, j \) in \( I \), let \( \sigma_{i, j} = \sigma_{x_i, x_j} \), \( e_{i, j} = e_{x_i, x_j} \) and let us reformulate the previous result. There is some infinite chain \( I \), a family of distinct compacts \( (x_i)_{i \in I} \) in \( S \), \( l \in \{ 0, 1 \} \) such that if \( i < j \), then there is some \( f_{i, j} \) with:

- \( f_{i, j} \in s(x_i) \) if \( l = 0 \) and \( f_{i, j} \) is an event of \( \uparrow x_i \) if \( l = 1 \),
- \( \lambda(f_{i, j}) = a \),
- \( f_{i, j} \notin e_{i, j} \),
- \( f_{i, j} R^1 e_{i, j} \) and there is no \( f \) with \( f \in \lambda_a \land f R^1 e_{i, j} \land \beta_{\lambda, a}(f_{i, j}) = \beta_{\lambda, a}(f) \).

In case \( l = 1 \) above, if \( i, j \in I \) and \( i < j \) then \( f_{i, j} \) is in \( \uparrow x_{i, j} \). Then due to the choice of the \( e_{x, y} \), \( \beta_{\lambda, a}(f_{i, j}) = \beta_{\lambda, a}(\sigma_{i, j}(f_{i, j})) \) and \( (\sigma_{i, j}(f_{i, j}), e_{j, j}) \notin R^1 \). \( \square \)

**Lemma 3.26.** Given a regular coherent \( \lambda \)-domain with a finite regular trace labelling \( \lambda \) and a label \( a \) of \( \lambda \), there is no infinite chain \( I \) with

- a family of distinct compacts \( (x_i)_{i \in I} \);  
- a family of isomorphisms \( (\sigma_{i, j})_{\{ (i, j) : i, j \in I, i \neq j \}} \);  
- families of events \( (e_{i, j})_{\{ (i, j) : i, j \in I, i \neq j \}} \) and \( (f_{i, j})_{\{ (i, j) : i, j \in I, i \neq j \}} \);  
- labelled residuals \( (\varepsilon_{x_i, \lambda}) \) have all the same type and \( \sigma_{i, j} \) is the unique isomorphism of \( \uparrow x_i \) onto \( \uparrow x_j \) preserving \( \lambda \);  
- \( e_{i, j} \) and \( f_{i, j} \) are events of \( \varepsilon_{x_i, \lambda} \), \( \lambda(e_{i, j}) = a \), \( \lambda(f_{i, j}) = a \), \( \sigma_{i, j}(e_{i, j}) = e_{j, j} \) and \( \sigma_{i, j}(f_{i, j}) = f_{j, j} \);  
- for any \( i, j \) in \( I \) with \( i < j \), \( f_{i, j} \neq e_{i, j} \), \( f_{i, j} R^1 e_{i, j} \), and \( (f_{i, j}, e_{i, j}) \notin R^1 \).

**Proof.** By contradiction. Suppose the existence of \( \lambda \), and \( I \) as described above. We show that this situation leads to a contradiction (see (7) below).

Let \( G_I \) denote the simple graph with no loops of the order on \( I \) (\( I \) is the set of its vertices and its edges corresponds to pairs \( (i, j) \) with \( i < j \)). Consider \( i, j \in I \) with \( i < j \), because \( f_{i, j} R^1 e_{i, j} \), there is some event namely \( g_{i, j} \) with \( f_{i, j} \prec g_{i, j} \circ e_{i, j} \). \( g_{i, j} \) does not belong to \( s(x_i) \) since \( f_{i, j} \prec g_{i, j} \). Let \( \delta_{i, j} = \{ h \in s(x_i) \mid \{ h, g_{i, j} \} \notin Con \} \). The \( \delta_{i, j} \) are upper closed sets, therefore the \( s(x_i) \setminus \delta_{i, j} \) are configurations. So let \( y_{i, j} \) denote the compact of the domain with \( s(y_{i, j}) = s(x_i) \setminus \delta_{i, j} \). Then \( y_{i, j} \preceq x_i \) and \( g_{i, j} \) is an event of \( \uparrow y_{i, j} \).

Because \( \lambda \) is regular, applying Ramsey's theorem, one proves the existence of an infinite subchain \( I' \) of \( I \) such that if \( i, j \in I' \) and \( i < j \), residuals \( \uparrow y_{i, j} \) have the
same type. Furthermore, $I$ denotes such an $I'$ and for any $i, j, k \in I$ with $i < j < k$, $\tau_{i, j, k}$ denotes the unique isomorphism of $\uparrow y_{i, j}$ onto $\uparrow y_{j, k}$.

Let us show

(1) If $i, j, k \in I$ and $i < j < k$ then $\tau(i, j, k)(f_{i, j}) \neq f_{j, i}$ or $\tau(i, j, k)(e_{i, j}) \neq e_{j, i}$.

Let $i, j, k \in I$ and $i < j < k$. Then $f_{i, j} = g_{i, j} \circ e_{i, j}$ and $e_{i, j}$, $f_{i, j}$, $g_{i, j}$ being events of $\uparrow y_{i, j}$, according to Lemma 2.33(ii) and (iii), $\tau_{i, j, k}(f_{i, j}) = \tau_{i, j, k}(g_{i, j}) \circ \tau_{i, j, k}(e_{i, j})$. By assumption, $(f_{j, i}, e_{j, i}) \notin R^l$, therefore $\tau_{i, j, k}(f_{i, j}) \neq f_{j, i}$ or $\tau(i, j, k)(e_{i, j}) \neq e_{j, i}$.

For any $i, j \in I$ with $i < j$, one defines moreover $\gamma_{i, j} = C'(e_{i, j}) \in \mathcal{S}(x_i)$ (refer to Fig. 7).

Let us show the following points. For any $i, j \in I$ with $i < j$:

(2) $\delta_{i, j} \cap C'(e_{i, j}) = \emptyset$;
(3) $\gamma_{i, j} = \gamma'(e_{i, j}) \in \mathcal{S}(y_{i, j})$;
(4) $\gamma_{i, j} \times \delta_{i, j} \subseteq \emptyset$;
(5) $\lambda(\gamma_{i, j}) \times \lambda(\delta_{i, j}) \subseteq \|\lambda\|$.

**Proof.** (2) Let $i, j \in I$ with $i < j$, and $h \in \delta_{i, j}$. $h \notin e_{i, j}$ since $\{g_{i, j}, h\} \notin \text{Con}$ and $\{g_{i, j}, e_{i, j}\} \in \text{Con}$.

(3) Simple consequence of (2).

(4) Let $i, j \in I$ with $i < j$. If $h \in \delta_{i, j}$ and $g \in \gamma_{i, j}$, then $g \notin h$ since $g \notin \mathcal{S}(x_i)$ and $h \notin g$ since according to (1), $h \notin C'(e_{i, j})$. $g$ and $h$ are compatible since they both belong to the configuration $s(x_i) \cup C'(e_{i, j})$. Finally according to Remark 2.24, $g \circ h$.

(5) Immediate consequence of (4).

Let us show

(6) If $i, j, k \in I$ and $i < j < k$, then there is some event $h_{i, j, k}$ in $C'(\tau_{i, j, k}(e_{i, j})) \setminus \mathcal{S}(y_{j, k})$ and some event $g_{i, j, k}$ in $\delta_{i, j, k}$ with $-\lambda(h_{i, j, k}) \| \lambda(g_{i, j, k})$.

Consider $i, j, k \in I$ with $i < j < k$. There is some covering chain $\{s_p\}_{1 \leq p \leq q}$ from $y_{i, j}$ to $y_{i, j} \lor x'(e_{i, j})$ via $y_{i, j} \lor x(f_{i, j})$ and $y_{i, j} \lor x(f_{i, j}')$ and with $[s_{q-1}, s_q] \in e_{i, j}$, i.e. there is some index $q'$ with $1 < q' \leq q$, $s_{q' - 1} = y_{i, j} \lor x(f_{i, j})$ and, $s_{q'} = y_{i, j} \lor x'(f_{i, j})$ and $s_{q - 1} = y_{i, j} \lor x(e_{i, j})$ (see Fig. 8 describing the domain $\uparrow y_{i, j}$). According to (3), the set of events $\{[s_p, s_{p+1}] \lor x_{i, j} \forall x_{i, j} \mid 1 \leq p \leq q\}$ is $\gamma_{i, j}$ and according to (4), $\{s_p \lor x_{i, j} \mid 1 \leq p \leq q\}$ is a covering chain from $x_i$ to $x_i \lor x'(e_{i, j})$ with $[s_p, s_{p+1}] \lor x_{i, j} \forall x_{i, j}$ if $1 \leq p \leq q$. Especially, $f_{i, j} = [s_{q' - 1} \lor x_{i, j}, s_{q'} \lor x_{i, j}] \lor x_{i, j}$, $s_{q' - 1} \lor x_{i, j} = x_i \lor x(f_{i, j})$, $s_{q'} \lor x_{i, j} = x_i \lor x'(f_{i, j})$, $e_{i, j} = [s_{q - 1} \lor x_{i, j}, s_{q} \lor x_{i, j}] \lor x_{i, j}$ and $s_{q - 1} \lor x_{i, j} = x_i \lor x(e_{i, j})$. 

**Fig. 7.** Set of events $\delta_{i, j}$ and $\gamma_{i, j}$. 

\[ e_{i, j} \]
\( \sigma_{i,j} \) being an isomorphism from \( \uparrow x_i \) onto \( \uparrow x_j \) with \( \sigma_{i,j}(f_{i,j}) = f_{i,j} \) and \( \sigma_{i,j}(e_{i,j}) = e_{i,j} \), according to Lemma 2.33(i), \( O_{i,j}(x_i V x(e_{i,j})) = x_j V x(e_{j,i}) \), \( \sigma_{i,j}(x_i V x(e_{i,j})) = x_j V x(e_{j,i}) \), \( \sigma_{i,j}(x_i V x'(e_{i,j})) = x_j V x'(e_{j,i}) \) and \( (\sigma_{i,j}(x_i V x'(e_{i,j})))_1 \leq p \leq q \) is a covering chain from \( x_j \) to \( x_j V x'(e_{j,i}) \) with \( \sigma_{i,j}(s_{q-1} V x_i) = \sigma_{i,j}(x_{s'} V x_i) \) \( \in f_{i,i} \) and \( [\sigma_{i,j}(s_{q-1} V x_i), \sigma_{i,j}(x_{s'} V x_i)] \in e_{i,i} \). One should refer to Fig. 9 describing domain \( \uparrow y_{j,k} \).

Suppose now that the assertion (6) does not hold. Then \( \lambda(g) g \neq \lambda(\tau_{i,j,k}(\{s_p, s_{p+1}\} \leq q)) \) for any \( g \in \delta_{i,j,k} \) and any \( p \), with \( 1 \leq p \leq q \). An induction on \( p \) shows that:

- \( \tau_{i,j,k}(s_{p}) \neq \tau_{i,j,k}(s_{p+1}) \) \( \xrightarrow{\tau_{i,j,k}(s_{p})} \) for any \( g \in \delta_{i,j,k} \) according to Axiom (Exr2); and

- \( \tau_{i,j,k}(s_{p}) \neq \tau_{i,j,k}(s_{p+1}) \) \( \xrightarrow{\tau_{i,j,k}(s_{p})} \) for any \( g \in \delta_{i,j,k} \) according to Axiom (Edet) since \( \lambda(\tau_{i,j,k}(s_{p}), \tau_{i,j,k}(s_{p+1})) \). Especially, \( \tau_{i,j,k}(f_{i,j}) = [\tau_{i,j,k}(s_{q'}-1), \tau_{i,j,k}(s_{q})] \) \( \xrightarrow{\tau_{i,j,k}(s_{q'})} \neq [\tau_{i,j,k}(s_{q'}), \tau_{i,j,k}(s_{q})] \) \( \xrightarrow{\tau_{i,j,k}(e_{i,j})} \) \( \xrightarrow{\tau_{i,j,k}(e_{i,j})} = [\tau_{i,j,k}(s_{q}), \tau_{i,j,k}(s_{q-1})] \) \( \xrightarrow{\tau_{i,j,k}(s_{q})} = e_{i,i} \), which contradicts (1) above.
Let $i, j, k \in I$ with $i < j < k$. According to (6), there are events $h_{i,j,k}$ in $C'(\tau_{i,j,k}(e_{i,j}))\setminus s(x_{j,k})$ and $g_{i,j,k} \in \delta_{j,k}$ with $-\langle \lambda(h_{i,j,k}) \rangle \mu\lambda(g_{i,j,k})$. Choosing such $g_{i,j,k}$ and $h_{i,j,k}$, let $c_{i,j,k}^1 = \lambda(g_{i,j,k})$ and $c_{i,j,k}^2 = \lambda(h_{i,j,k})$.

Let us show

(7) For any $i, j, k, l \in I$ with $i < j < k < l$, $c_{i,j,k}^1 \neq c_{j,k,l}^1$.

Let $i, j, k, l \in I$ with $i < j < k < l$. $c_{i,j,k}^2 = \lambda(h_{i,j,k}) = \lambda((\tau_{j,k,l})^{-1}(h_{j,k,l}))$ and $(\tau_{j,k,l})^{-1}(h_{j,k,l}) \in \gamma_{j,k,l}$. $c_{i,j,k}^1 = \lambda(g_{i,j,k})$ and $g_{i,j,k} \in \delta_{j,k}$. According to (4), $c_{i,j,k}^1 \| c_{i,j,k}^2$ and since by definition $-\langle c_{i,j,k}^1 \rangle \mu c_{j,k,l}^2$, $c_{i,j,k}^1 \neq c_{j,k,l}^1$.

(7) implies that $c^1$ is a coloring of the simple graph with no loops whose vertices are the triples $(i, j, k)$ where $(i, j, k) \in I$ and $i < j < k$, and whose edges correspond to pairs $\{(i, j, k), (j, k, l)\}$. The latter graph is indeed isomorphic to $\text{Arc}(\text{Arc}(\text{Arc}(G_I)))$ (refer to the definition of $\text{Arc}(\cdot)$, Definition A.1, in appendix). $G_I$ being an infinite clique, $\chi(G_I)$ is infinite the according to Proposition A.2, $\chi(\text{Arc}(\text{Arc}(\text{Arc}(G_I))))$ is infinite. This contradicts the finiteness of $\lambda$. \qed

Lemma 3.27. Given a regular coherent $dI$-domain with a regular finite trace labelling $\lambda$ and a label $a$ of $\lambda$, there is no infinite chain $I$, with:

- a family of distinct compacts $(x_i)_{i \in I}$;
- a family of isomorphisms $(\sigma_{i,j})_{(i,j) \in I^2 \land i \neq j}$;
- families of events $(e_{i,j})_{(i,j) \in I^2 \land i \neq j}$ and $(f_{i,j})_{(i,j) \in I^2 \land i < j}$;

such that:

- labelled residuals $(x_i, \lambda)$ have the same type and $\sigma_{i,j}$ is the unique isomorphism from $\uparrow x_i$ onto $\uparrow x_j$ preserving $\lambda$;
- $e_{i,j}$ is an event of $\uparrow x_i$, $\lambda(e_{i,j}) = a$, and $\sigma_{i,j}(e_{i,j}) = e_{j,i}$;
- for any $i, j$ in $I$ with $i < j$, $\lambda(f_{i,j}) = a$, $f_{i,j} < e_{i,j}$, $f_{i,j}R^1 e_{i,j}$, $f_{i,j} \in s(x_i)$ and there is no $f$ with $\lambda(f) = a$, $fR^1 e_{j,i}$ and $\beta_{\lambda,a}(f) = \beta_{\lambda,a}(f_{i,j})$.

Proof. By contradiction. Suppose the existence of $\lambda$, $a$ and $I$ as described above. We show that the following situation holds. There is

- a label $b$ with $b \parallel a$;
- an infinite chain $I$;
- a family of distinct compacts $(x_i)_{i \in I}$;
- a family of isomorphisms $(\sigma_{i,j})_{(i,j) \in I^2 \land i \neq j}$;
- families of events $(e_{i,j})_{(i,j) \in I^2 \land i \neq j}$, $(f_{i,j})_{i \in I}$, $(d_{i,j})_{(i,j) \in I^2 \land i < j}$;

such that:

- labelled residuals $(x_i, \lambda)$ have all the same type and $\sigma_{i,j}$ is the unique isomorphism from $\uparrow x_i$ onto $\uparrow x_j$ preserving $\lambda$;
- $e_{i,j}$ is an event of $\uparrow x_i$, $\lambda(e_{i,j}) = a$, and $\sigma_{i,j}(e_{i,j}) = e_{j,i}$;
- for any $i \in I$, $f_{i} \in s(x_i)$ and $\lambda(f_{i}) = a$,
- for $i, j \in I$ with $i < j$, $\lambda(d_{i,j}) = b$, $f_{i} \prec d_{i,j} \circ e_{i,j}$ and there is no event $d$ with $\lambda(d) = b$ and $f_{j} \prec d \circ e_{j,i}$, and

- either (case 2.1): for any $i, j \in I$ with $i < j$, $d_{i,j} \not\in s(x_i)$,
- or (case 2.2): for any $i, j \in I$ with $i < j$, $d_{i,j} \in s(x_i)$.
It is shown that each of the two previous case leads to a contradiction—Lemmas 3.28 and 3.30, proving therefore 3.27.

If \( x \) is compact, according to Property 3.20(2) the set \( s(x) \cap \lambda_a \) is a chain, and according to Property 3.20(4), the set \( P(x) \) made of the \( f \) in \( s(x) \cap \lambda_a \) such that there is \( e \) of \( \tau x \) with \( f \models e \) is a clique whose cardinality is at most \( \omega(\lambda_1,0)^{\alpha} - 1 \).

Therefore there is some chain \( I \) as above such that the set \( P(x_i) \) have the same cardinality, namely \( k \). Then for any \( i \in I \), \( P(x_i) \) is necessarily non-empty. Now, consider enumerations \( (f_{i,j})_{1 \leq j \leq k} \) of sets \( P(x_i) \). Since \( \beta_{k,a} \) is finite such a set \( I \) may be chosen such a way that for any \( i \) and \( i' \) in \( I \) and any \( j \in \{1, \ldots, k\} \), \( \beta_{k,a}(f_{i,j}) = \beta_{k,a}(f_{i',j}) \).

Applying Corollary A.6, one deduces the existence of a chain \( I \) as above and such that there is some \( l \in \{1, \ldots, k\} \) with for any \( i, j \in I \) with \( i < j \), \( f_{i,j} = f_{l,i} \). So let \( f_i \) denote \( f_{l,i} \) for any \( i \).

For any \( i, j \in I \) with \( i < j \), there is then some event \( d_{i,j} \) satisfying \( f_i \models d_{i,j} \circ e_{i,j} \), though there is no \( d \) satisfying \( f_j \models d \circ e_{j,i} \). Applying Ramsey’s theorem again, one deduces the existence of a label \( b \) with \( b \models a \), of a family of compacts \( (x_i)_{i \in I} \) as described above satisfying moreover \( \lambda(d_{i,j}) = b \) for any \( i, j \in I \) with \( i < j \) and,

- either (case 2.1): for any \( i, j \in I \) with \( i < j \), \( d_{i,j} \notin s(x_i) \),
- or (case 2.2): for any \( i, j \in I \) with \( i < j \), \( d_{i,j} \notin s(x_i) \).

**Lemma 3.28.** Given a regular coherent dI-domain with a finite regular trace labelling \( \lambda \) and labels \( a \) and \( b \) with \( a \models b \), there is no infinite chain \( I \), with

- a family of distinct compacts \( (x_i)_{i \in I} \);
- a family of isomorphisms \( (\sigma_{i,j})_{(i,j) \in I \setminus I \setminus j} \);
- family of events \( (e_{i,j})_{(i,j) \in I \setminus I \setminus j} \), \( (f_i)_{i \in I} \), \( (d_{i,j})_{(i,j) \in I \setminus I \setminus j} \), such that:
  - labelled residuals \( \langle \tau x, \lambda \rangle \) have the same type and \( \sigma_{i,j} \) is the unique isomorphism from \( \tau x_i \) onto \( \tau x_j \) preserving \( \lambda \);
  - \( e_{i,j} \) is an event of \( \tau x_i \), \( \lambda(e_{i,j}) - a \), and \( \sigma_{i,j}(e_{i,j}) = e_{j,i} \);
  - for any \( i \in I \), \( f_i \in s(x_i) \) and \( \lambda(f_i) = a \);
  - for any \( i, j \in I \) with \( i < j \), \( \lambda(d_{i,j}) = b \), \( f_i \models d_{i,j} \circ e_{i,j} \), and there is no event \( d \) with \( \lambda(d) = b \) and \( f_j \models d \circ e_{j,i} \), and for any \( i, j \in I \) with \( i < j \), \( d_{i,j} \notin s(x_i) \).

**Proof.** By contradiction. Suppose the existence of \( \lambda \) and \( I \) as described above. Let \( G_I \) denote the simple graph with no loops of the order of \( I \). For any \( i, j \) with \( i < j \), let \( \delta_{i,j} = \{g \in s(x_i) : \{g, d_{i,j}\} \notin \text{Con}\} \). The \( \delta_{i,j} \) are upper closed sets, the sets \( s(x_i) \setminus \delta_{i,j} \) are therefore configurations, so let \( y_{i,j} \) denote the compact of the domain with \( s(y_{i,j}) = s(x_i) \setminus \delta_{i,j} \). It satisfies \( y_{i,j} \leq x_i \). Since \( \lambda \) is regular, applying Ramsey’s theorem, one deduces the existence of an infinite subchain \( I' \) of \( I \) such that if \( i, j \in I' \) with \( i < j \), residuals \( (\top y_{i,j}, \lambda) \) have the same type. Furthermore, \( I \) denotes such an \( I' \) and for any \( i, j, k \in I \), with \( i < j < k \), \( y_{i,j,k} \) denotes the unique isomorphism from \( \top y_{i,j} \) onto \( \top y_{j,k} \). Applying again Ramsey’s theorem, one may consider that \( I \) satisfies moreover for any \( i, j, k \in I \)
with $i < j < k$:
- either (case 2.1.1): $\tau_{i,j,k}(e_{i,j}) = e_{j,i};$
- or (case 2.1.2): $\tau_{i,j,k}(e_{i,j}) \neq e_{j,i}.$

Furthermore, it is proved that each of the two cases above cannot occur.

First we suppose that case (2.1.2) holds and shows this leads to a contradiction (see (6) below).

For any $i, j \in I$ with $i < j$, let $\gamma_{i,j} = C'(e_{i,j}) \setminus s(x_i)$ (see Fig. 10). Let us show the following points. For any $i, j \in I$ with $i < j$:

1. $\delta_{i,j} \cap C'(e_{i,j}) = \emptyset$;
2. $\gamma_{i,j} = C'(e_{i,j}) \setminus s(y_{i,j});$
3. $\lambda(\gamma_{i,j}) \times \delta_{i,j} \subseteq \emptyset$;
4. $\lambda(\gamma_{i,j}) \times \lambda(\delta_{i,j}) \subseteq \emptyset$.

**Proof.** (1) Let $i, j \in I$ with $i < j$, and $h \in \delta_{i,j}$. $h \notin e_{i,j}$ since $\{d_{i,j}, h\} \notin Con$ and $\{d_{i,j}, e_{i,j}\} \in Con$.

(2) Simple consequence of (1).

(3) Let $i, j \in I$ with $i < j$. If $h \in \delta_{i,j}$ and $g \in \gamma_{i,j}$, $g \notin h$ since $g \notin s(x_i)$, and $h \notin g$ since according to (1), $h \notin C'(s_{i,j})$. $g \neq h$ since they belong both to the configuration $s(x_i) \cup C'(e_{i,j})$. Therefore, according to Remark 2.24 $g \circ h$.

(4) Immediate consequence of (3).

Let us show (5) If $i, j, k \in I$ and $i < j < k$, there is an event $h_{i,j,k}$ in $C'(\tau_{i,j,k}(e_{i,j})) \setminus s(y_{i,j,k})$ and an event $g_{i,j,k}$ in $\delta_{i,j,k}$ such that $\langle \lambda(h_{i,j,k}) \| i, \lambda(g_{i,j,k}) \rangle$.

Consider $i, j, k \in I$ with $i < j < k$. There is some covering chain $(s_p)_{1 \leq p \leq q}$ from $y_{i,j}$ to $y_{i,j} \lor x'(e_{i,j})$ with $[s_{q-1}, s_q] \in e_{i,j}$, i.e. $s_{q-1} = y_{i,j} \lor x(e_{i,j})$ (see Fig. 11 describing the domain $\uparrow y_{i,j}$). According to (2), the set of events $\{[s_p, s_{p+1}] \}_{1 \leq p \leq q}$ is $\gamma_{i,j}$ and according to (3), $(s_p \lor x_i)_{1 \leq p \leq q}$ is a covering chain from $x_i$ to $x_i \lor x'(e_{i,j})$ with $[s_p, s_{p+1}] \lor [s_p \lor x_i, s_{p+1} \lor x_i]$ if $1 \leq p < q$. Especially, $e_{i,j} = [s_{q-1} \lor x_i, s_q \lor x_i] \lor x'(e_{i,j})$ and $(s_{i,j}(s_{q-1} \lor x_i), s_{i,j}(s_q \lor x_i))$.

$\sigma_{i,j}$ being an isomorphism from $\uparrow x_i$ onto $\uparrow x_j$ with $\sigma_{i,j}(e_{i,j}) = e_{j,i}$, according to Lemma 2.33(i), $\sigma_{i,j}(x_i \lor x(e_{i,j})) = x_j \lor x(e_{j,i}), \sigma_{i,j}(x_i \lor x'(e_{i,j})) = x_j \lor x'(e_{j,i})$ and $(\sigma_{i,j}(s_{p} \lor x_j))_{1 \leq p \leq q}$ is a covering chain from $x_j$ to $x_j \lor x'(e_{j,i})$ with $[\sigma_{i,j}(s_{q-1} \lor x_i), \sigma_{i,j}(s_q \lor x_i)]$.
Suppose now that the assertion (5) does not hold. Then $\lambda(g) \parallel \lambda((\tau_{i,j,k}(s_p), s_{p+1}))$ for any $g \in \delta_{j,k}$ and any $p$ with $1 \leq p < q$. An induction shows that for any $p$ with $1 \leq p < q$:

- $[\tau_{i,j,k}(s_p), \tau_{i,j,k}(s_{p+1})] \gg \sigma_i$ for any $g \in \delta_{j,k}$ - according to Axiom (Etr2); and
- $[\tau_{i,j,k}(s_p), \tau_{i,j,k}(s_{p+1})] \gg [\sigma_j(s_p \lor t_i), \sigma_j(s_{p+1} \lor t_i)]$ - according to Axiom (Edet) since $\lambda([\tau_{i,j,k}(s_p), \tau_{i,j,k}(s_{p+1})]) \gg \lambda([\sigma_j(s_p \lor t_i), \sigma_j(s_{p+1} \lor t_i)]) \gg \sigma_i$.

Especially, $\tau_{i,j,k}(e_{i,j}) = [\tau_{i,j,k}(s_q-1), \tau_{i,j,k}(s_q)] \gg [\sigma_i(s_q-1), \sigma_i(s_q)] \gg e_{j,i}$, which is a contradiction.

Let $i, j, k$ in $I$ with $i < j < k$. According to (5), there are events $h_{i,j,k}$ in $C'(\tau_{i,j,k}(e_{i,j})) \setminus s(y_{j,k})$ and $g_{i,j,k}$ in $\delta_{j,k}$ such that $-(\lambda(h_{i,j,k}) || \lambda(g_{i,j,k}))$. Choosing such $g_{i,j,k}$ and $h_{i,j,k}$, let $c_{i,j,k} = \lambda(g_{i,j,k})$ and $c_{i,j,k}^2 = \lambda(h_{i,j,k})$.

---

**Fig. 11.** Domain $\uparrow y_{i,j}$.

**Fig. 12.** Domain $\uparrow y_{j,k}$. 

$e_{j,i}$. One should refer to Fig. 12 describing the domain $\uparrow y_{j,k}$. Since $\tau_{i,j,k}$ is an isomorphism from $\uparrow y_{i,j}$ onto $\uparrow y_{j,k}$, then $(\tau_{i,j,k}(s_p))_{1 \leq p \leq q}$ is a covering chain from $y_{i,j}$ to $y_{j,k} \land \tau_{i,j,k}(x'(e_{i,j}))$ and $\tau_{i,j,k}(y_{i,j} \land x(e_{i,j})) = y_{j,k} \land x(\tau_{i,j,k}(e_{i,j}))$. 

Let $i, j, k$ in $I$ with $i < j < k$. According to (5), there are events $h_{i,j,k}$ in $C'(\tau_{i,j,k}(e_{i,j})) \setminus s(y_{j,k})$ and $g_{i,j,k}$ in $\delta_{j,k}$ such that $-(\lambda(h_{i,j,k}) || \lambda(g_{i,j,k}))$. Choosing such $g_{i,j,k}$ and $h_{i,j,k}$, let $c_{i,j,k} = \lambda(g_{i,j,k})$ and $c_{i,j,k}^2 = \lambda(h_{i,j,k})$. 

---
Let us show

(6) For any \( i, j, k, l \in I \) with \( i < j < k < l \), \( c_{i,j,k}^1 \neq c_{j,k,l}^1 \).

Let \( i, j, k, l \in I \) with \( l < i < j < k \). \( c_{i,j,k,l}^2 = \lambda(h_{j,k,l}) = \lambda((\tau_{j,k,l})^{-1}(h_{j,k,l})) \) and \((\tau_{j,k,l})^{-1}(h_{j,k,l}) = (h_{j,k,l})_{(j,k,l)} \). According to (4), \( c_{i,j,k}^1 || c_{j,k,l}^2 \). and by definition \( \neg(c_{i,j,k,l}^1 \neq c_{j,k,l}^1) \), therefore \( c_{i,j,k}^1 \neq c_{j,k,l}^1 \).

(6) implies that \( c^1 \) is a coloring of the simple graph with no loops whose vertices are the triples \( (i, j, k) \) where \( (i, j, k) \in I \) and \( i < j < k \) and whose edges correspond to pairs \( \{(i, j, k), (j, k, l)\} \). The latter graph is indeed isomorphic to \( Arc(Arc(G_I)) \). \( G_I \) being an infinite clique, \( \chi(G_I) \) is infinite and according to Proposition A.2, \( \chi(Arc(Arc(G_I))) \) is infinite. This contradicts the finiteness of \( \lambda \).

Now, suppose that case (2.1.1) holds. We show that this leads also to a contradiction. Consider the graph \( Arc(G_I) \), its vertices are the couples \( (i, j) \) with \( i < j \) and \((u, v) \in Arc(G_I) \) if and only if \( u = (i, j) \) and \( v = (j, k) \). Let \( S \) denote the set of vertices of \( Arc(G_I) \). One defines family of compacts \( (\tilde{\sigma}_{u,v})_{(u,v) \in Arc(G_I)} \), family of isomorphisms \( (\tilde{\varepsilon}_{u,v})_{(u,v) \in Arc(G_I) \cup (Arc(G_I))^{-1}} \) and families of events \( (\tilde{\vartheta}_{u,v})_{(u,v) \in Arc(G_I) \cup (Arc(G_I))^{-1}}, (\tilde{f}_{u,v} \in S) \), \((\tilde{\tau}_{u,v})_{(u,v) \in Arc(G_I)} \) in this way:

\[
\text{if } u = (i, j),
\begin{align*}
- \tilde{x}_u &= y_{i,j}, \\
- \tilde{f}_u &= f_i,
\end{align*}
\]

\[
\text{if } u = (i, j) \text{ and } v = (j, k),
\begin{align*}
- \tilde{\sigma}_{u,v} &= \tau_{i,j,k}, \\
- \tilde{\varepsilon}_{u,v} &= \varepsilon_{i,j}, \\
- \tilde{\theta}_{u,v} &= \theta_{i,j}, \\
- d_{u,v} &= d_{i,j}.
\end{align*}
\]

One checks the following points.

If \((u, v) \in Arc(G_I) \cup (Arc(G_I))^{-1} \), then

(1) \( \tilde{\sigma}_{u,v} \) is the unique isomorphism preserving \( \lambda \) from \( \uparrow \tilde{x}_u \) onto \( \uparrow \tilde{x}_v \) and

(2) \( \tilde{\varepsilon}_{u,v} \) is an event of \( \uparrow \tilde{x}_v \).

(3) For any vertex \( u \) of \( Arc(G_I), f_u \in s(\tilde{x}_u) \) and \( \lambda(f_u) = \alpha \).

(4) If \((u, v) \in Arc(G_I) \), \( \lambda(d_{u,v}) = b \), \( f_u \prec d_{u,v} \circ \tilde{e}_{u,v} \), and there is no event \( d \) with \( \lambda(d) = b \) and \( f_v \prec d \circ \tilde{e}_{v,u} \).

(5) For any \((u, v) \in Arc(G_I), d_{u,v} \) is an event of \( \uparrow \tilde{x}_u \).

Only points (3) and (5) require explanations.

(3) If \( u = (i, j), f_u = f_i \) and \( f_i \notin \delta_{i,j} \) because \( f_i \prec d_{i,j} \). Then \( f_i \in s(y_{i,j}) = s(\tilde{x}_u) \).

(5) If \( u = (i, j), d_u = d_{i,j}, \tilde{x}_u = y_{i,j} \) and \( d_{i,j} \) is an event of \( \uparrow y_{i,j} \) since \( \{d_{i,j}, s(y_{i,j})\} \) \( \in Con \) and \( d_{i,j} \notin s(x_i) \supseteq s(y_{i,j}) \).

Since \( G_I \) is an infinite clique \( \chi(G_I) \) is infinite and according to Proposition A.2 (in Appendix A), \( \chi(Arc(G_I)) \) is infinite. The contradiction which we look for is then given by Lemma 3.29. □
Lemma 3.29. If \( \lambda \) is a regular trace finite labelling of a regular dI-domain and \( a \) and \( b \) are such that \( a \parallel b \) then there is no graph \( G \) simple and with no loops with infinite chromatic number and with, \( S \) denoting the set of its vertices:
- a family of distinct compacts \( (x_i)_{i \in S} \);
- a family of isomorphisms \( (\sigma_{i,j})_{(i,j) \in G \cup G^{-1}} \);
- families of events \( (e_{i,j})_{(i,j) \in G \cup G^{-1}}, (f_i)_{i \in S}, (d_{i,j})_{(i,j) \in G} \) such that:
- residuals \((\uparrow x_i, \lambda)\) have all the same type and \( \sigma_{i,j} \) is the unique isomorphism from \( \uparrow x_i \) onto \( \uparrow x_j \) preserving \( \lambda \);
- \( e_{i,j} \) is an event of \( \uparrow x_i \), \( \lambda(e_{i,j}) = a \), and \( \sigma_{i,j}(e_{i,j}) = e_{j,i} \);
- for any \( i \) in \( S \), \( f_i \in s(x_i) \) and \( \lambda(f_i) = a \);
- for any \( (i,j) \in G \), \( \lambda(d_{i,j}) = b \), \( f_i \prec d_{i,j} \circ e_{i,j} \), and there is no event \( d \) with \( \lambda(d) = b \) and \( f_j \prec d \circ e_{j,i} \), and for any \( (i,j) \in G \), \( d_{i,j} \) is an event of \( \uparrow x_i \).

Proof. By contradiction. Suppose the existence of a labelling \( \lambda \) and of a graph \( G \) as described above. We show that this situation leads to some contradiction (see (8) below). For any vertex \( i \) in \( G \), \( s(x_i) \setminus \uparrow f_i \) is a configuration, so let \( y_i \) denote the compact with \( s(y_i) = s(x_i) \setminus \uparrow f_i \). Consider the partition of the set of vertices of \( G \) whose classes are the subsets of vertices \( i \) with labelled residuals \((\uparrow y_i, \lambda)\) of a given type. \( \lambda \) being regular, this partition is finite and necessarily one among the induced subgraph on the previous classes has an infinite chromatic number. Furthermore, let \( G \) denote such a graph. Then for any pair \( \{i,j\} \) of vertices of \( G \), \((\uparrow y_i, \lambda)\) and \((\uparrow y_j, \lambda)\) have the same type, so let \( \tau_{i,j} \) stand for the isomorphism preserving \( \lambda \) from \( \uparrow y_i \) onto \( \uparrow y_j \). Let also \( \tilde{S} \) be the set of non terminal vertices of \( G \). According to Remark A.4, \( \tilde{G} \), the induced subgraph of \( G \) on \( \tilde{S} \) has an infinite chromatic number.

If \( i \) a vertex of \( G \), because \( f_i \in s(x_i) \), \( s(x_i) \setminus \uparrow f_i \supseteq C(f_i) \). Then there is some compact namely \( y'_i \) with \( y_i \prec y'_i \) and \((y_i, y'_i) \supseteq f_i = f_i \), it satisfies \( y'_i \prec x_i \). If \( (i,j) \in G \), since \( \lambda(\tau_{i,j}(f_j)) = \lambda(f_j) = \lambda(f_j) \) and according to (Edet), \( \tau_{i,j}(f_j) = f_j \) and \( \tau_{i,j}(y'_j) = y'_j \). Given \( (i,j) \in G \), \( d_{i,j} \) is an event of \( \uparrow x_i \) and so let \( d_{i,i} = \sigma_{i,j}(d_{i,j}) \), which is an event of \( \uparrow x_j \).

Let us show a few points.

1) If \( (i,j) \in G \), \( \tau_{i,j}(d_{i,j}) \neq d_{i,j} \).

\( \sigma_{i,j} \) is an isomorphism from \( \uparrow x_i \) onto \( \uparrow x_j \), \( \sigma_{i,j}(d_{i,j}) = d_{j,i} \) and \( \sigma_{i,j}(e_{i,j}) = e_{j,i} \) then according to Lemma 2.33(iii), \( d_{j,i} \circ e_{j,i} \).

Since \( \tau_{i,j} \) is defined on \( \uparrow y_i \), \( d_{i,j} \) is an event of \( \uparrow y_i \), \( f_j \prec d_{i,j} \) and \( f_j = \tau_{i,j}(f_j) \), according to Lemma 2.33(ii) \( f_j \prec \tau_{i,j}(d_{i,j}) \). Since \( d_{j,i} \circ e_{j,i} \) and \( \lambda(d_{j,i}) = b \), according to the assumptions \( d_{j,i} \neq \tau_{i,j}(d_{i,j}) \).

2) If \( (i,j) \in G \), \( s(x_i) \cap \uparrow f_i \times (C(d_{i,j}) \setminus s(x_i)) \subseteq \emptyset \).

Let \( (i,j) \in G \), \( g \in s(x_i) \setminus \uparrow f_i \) and \( h \in C(d_{i,j}) \setminus s(x_i) \). Then \( \{g,h\} \subseteq s(x_i) \cup C(d_{i,j}) \supseteq s(x_i) \vee d(x_{i,j}) \) therefore \( g \uparrow h \). \( g \nless h \) since \( h \notin s(x_i) \), \( h \nless g \) since \( g \geq f_i \) and \( f_i \) is maximal in \( C(d_{i,j}) \).

Therefore \( g \circ h \).

3) If \( (i,j) \in G \), \( g \geq f_i \) and \( g \in s(x_i) \) then \( g \circ d_{i,j} \).

Suppose \( (i,j) \in G \), \( g \geq f_i \) and \( g \in s(x_i) \). Then \( g \uparrow d_{i,j} \) since \( d_{i,j} \) is an event of \( \uparrow x_i \), \( d_{i,j} \notin g \) since \( d_{i,j} \notin s(x_i) \) and \( g \notin d_{i,j} \) since \( f_i \prec d_{i,j} \).
For any \((i,j) \in G\), let:

\[
\gamma^1_{j,i} = (C(d_{j,i}) \setminus s(x_j)) \setminus f_j,
\]

\[
\gamma^2_{j,i} = (C(d_{j,i}) \setminus s(x_j)) \cap \uparrow f_j \quad \text{and}
\]

\[
\gamma^3_{j,i} = C(d_{j,i}) \cap s(x_j) \cap (\uparrow f_j \setminus \{f_j\}).
\]

(see Fig. 13).

Let \(v_{j,i} = y_j' \lor (x_j \land x(d_{j,i}))\). Then \(v_{j,i} = x_j \lor (y_j' \land x(d_{j,i})) \leq x_j\) and

\[
s(v_{j,i}) = s(y_j') \lor (x_j \land x(d_{j,i}))
\]

\[
= s(y_j') \cup (s(x_j \land x(d_{j,i})) \setminus s(y_j'))
\]

\[
= s(y_j') \cup \gamma^3_{j,i}.
\]

Note also that \(\gamma^1_{j,i} \cup \gamma^2_{j,i} = C(d_{j,i}) \setminus s(x_j)\).

Let us show the following points. If \((i, j) \in G\) then

(4) If \(g \in \gamma^1_{j,i}\), \(h < g\) and \(h \notin \gamma^1_{j,i}\) then \(h \in s(y_j)\);

(5) \(\gamma^1_{j,i} \times \{f_j\} \cup \gamma^3_{j,i} \subseteq \infty\);

(6) \((s(x_j) \setminus s(v_{j,i})) \times (\gamma^1_{j,i} \cup \gamma^2_{j,i}) \subseteq \infty\).

**Proof.** (4) Let \(g \in \gamma^1_{j,i}\) and \(h < g\) and \(h \notin \gamma^1_{j,i}\) then \(h \in C(d_{j,i}) \setminus f_j\). If \(h \notin \gamma^1_{j,i}\) then \(h \in (C(d_{j,i}) \setminus f_j) \setminus s(x_j)\). Therefore \(g \cdot h\).

(5) Let \(g \in \gamma^1_{j,i}\) and \(h \in \gamma^3_{j,i} \cup \{f_j\}\). According to (4), \(h \notin f_j\) and \(g \notin h\). \(g, h \in s(x_j \lor x(d_{j,i}))\), then \(g \cdot h\). Therefore \(g \cdot h\).

(6) Since \(x_j \lor x(d_{j,i})\), \((s(x_j) \setminus C(d_{j,i})) \times (C(d_{j,i}) \setminus s(x_j)) \subseteq \infty\). \(v_{j,i} \geq x_j \land x(d_{j,i})\), then \(s(x_j) \setminus s(v_{j,i}) \subseteq s(x_j) \setminus C(d_{j,i})\) and \((s(x_j) \setminus s(v_{j,i})) \times (\gamma^1_{j,i} \cup \gamma^2_{j,i}) \subseteq \infty\).

Let us show

(7) If \((i, j) \in \overline{G}\) then there are some events \(g_{j,i}\) in \(\gamma^3_{j,i}\) and \(h_{j,i}\) in \(\gamma^2_{j,i}\) with \(\neg(g_{j,i} \parallel h_{j,i})\).
Let \((i,j) \in G\). According to (4) and (5) \(s(y'_j) \cup \gamma_{j,i}^1\) is a configuration. So let \(w_{j,i}\) denote the compact with \(s(w_{j,i}) = s(y'_j) \cup \gamma_{j,i}^1\). It satisfies \(w_{j,i} \triangleq y'_j \lor x(d_{j,i})\) (see Fig. 14). Therefore there is some covering chain \((s_p)_{1 \leq p \leq q}\) from \(y'_j\) to \(w_{j,i}\), and for such a chain \([s_p, s_{p+1}] \gg \gamma_{j,i}^1\) if \(1 \leq p < q\). According to (5), \((s_p \lor v_{j,i})_{1 \leq p \leq q}\) is a covering chain from \(v_{j,i}\) to \(v_{j,i} \lor w_{j,i}\) and \([s_p \lor v_{j,i}, s_{p+1} \lor v_{j,i}] \gg [s_p, s_{p+1}]\) if \(1 \leq p < q\).

Since \(v_{j,i} \lor w_{j,i} \triangleq v_{j,i} \lor x(d_{j,i}) = y'_j \lor x(d_{j,i})\), this chain may be continued into a covering chain \((t_p)_{1 \leq p \leq q'}\) from \(v_{j,i}\) to \(v_{j,i} \lor x(d_{j,i})\), i.e. \(q' > q\) and \(t_p = s_p \lor v_{j,i}\) if \(1 \leq p < q\). Then \([t_p, t_{p+1}] \gg \gamma_{j,i}^2\) if \(q \leq p < q'\). Now since \(x_j \land x(d_{j,i}) \leq v_{j,i} \land x_j\), \(C(d_{j,i}) \setminus s(x_j) = C(d_{j,i}) \setminus s(v_{j,i})\) and according to (6), \((t_p \lor x_j)_{1 \leq p \leq q'}\) is a covering chain from \(x_j\) to \(x_j \lor x(d_{j,i})\) and \([t_p \lor x_j, t_{p+1} \lor x_j] \gg [t_p, t_{p+1}]\) if \(1 \leq p < q'\).

Because \(\sigma_{j,i}\) is an isomorphism from \(\lfloor x_j \rfloor \lor x_j\) onto \(\lfloor x_{j,i} \rfloor \lor x_{j,i}\) and \(\sigma_{j,i}(d_{j,i}) = x_{j,i} \lor x(d_{j,i})\) is a covering chain from \(x_{j,i}\) to \(\sigma_{j,i}(x_{j,i} \lor x(d_{j,i}))\) according to Lemma 2.3(i) (see Fig. 15). The set of events of the form \([\sigma_{j,i}(t_p \lor x_j)\), \((\sigma_{j,i}(l_{p+1} \lor x_j))\) \(\gg \gamma_{j,i}^1\) if \(1 \leq p < q'\) is then \(C(d_{j,i}) \setminus s(x_j)\) which is also \(C(d_{j,i}) \setminus s(y'_j)\) according to (3). Therefore there is some covering chain \((u_p)_{1 \leq p \leq q'}\) from \(y'_j\) to \(y'_j \lor x(d_{j,i})\) with \(u_p \lor x_j = \sigma_{j,i}(t_p \lor x_j)\) if \(1 \leq p \leq q'\), and \([u_p \lor x_j, u_{p+1} \lor x_j] \gg [u_p, u_{p+1}]\) if \(1 \leq p < q'\).

Since \(\tau_{i,j}\) is an isomorphism from \(\lfloor y_j \rfloor \lor y_j\) onto \(\lfloor y_{j,i} \rfloor \lor y_{j,i}\) and \(\tau_{i,j}(y'_j) = y'_j\), \((\tau_{i,j}(u_p))\) is a covering chain from \(y'_j\) to \(y'_j \lor x(\tau_{i,j}(d_{j,i}))\). Since \(\lambda([\tau_{i,j}(u_p), \tau_{i,j}(u_{p+1})] \gg \gamma_{j,i}^1\) \(\gg \gamma_{j,i}^1\) if \(1 \leq p < q'\) and \(\lambda([s_p, s_{p+1}] \gg \gamma_{j,i}^1\) \(\gg \gamma_{j,i}^1\) if \(1 \leq p < q\), an induction shows, according to Axiom (Edet), that \(\tau_{i,j}(u_p) = s_p\) if \(1 \leq p \leq q\).
Let us show (7) by contradiction. Suppose $\lambda(\gamma_{ji}^2) \times \dot{\lambda}(\gamma_{ji}^3) \subseteq \|A\|$. Since if $q \leq p < q'$, 
$\lambda([t_{ji}(u_p), t_{ji}(u_{p+1})]) \gg \lambda([t_p, t_{p+1}]) \gg g$ for any $g \in \gamma_{ji}^3$ — according to Axiom (Etr2); and 
$- [\tau_{ji}(u_p), \tau_{ji}(u_{p+1})] \gg \tau_{ji}(d_{ji})$ for any $g \in \gamma_{ji}^3$ — according to Axiom (Etr2).

Since $j \in S$, there is some $k$ with $(j, k) \in G$. For any $g$ in $\gamma_{ji}^3$, since $g \gg f_j$ and $g \in s(x_j)$ according to (3), $g \circ d_{jk}$ and $\dot{\lambda}(g) \parallel b$. According to (Etr2), $\tau_{ji}(d_{ji}) \circ g$ if $g \in \gamma_{ji}^3$ and finally $\tau_{ji}(d_{ji}) = d_{ji}$. This contradicts (1).

Let $(i, j) \in G$. According to (7), there are events $g_{ji}$ and $h_{ji}$ with $g_{ji} \in \gamma_{ji}^3$, $h_{ji} \in \gamma_{ji}^2$ and $-(\dot{\lambda}(g_{ji}) || \dot{\lambda}(h_{ji}))$. Choosing such $g_{ji}$ and $h_{ji}$, let $c^1(i, j) = \dot{\lambda}(g_{ji})$ and $c^2(i, j) = \dot{\lambda}(h_{ji})$.

Let us show:

(8) If $(i, j), (j, k) \in G$, $c^1(i, j) \neq c^1(j, k)$.

Let $(i, j), (j, k) \in G$. $\sigma_{kj}(h_{kj}) \in C(d_{kj}) \backslash s(x_j)$, $g_{ji} \in \gamma_{ji}^3 \subseteq s(x_j) \cap \uparrow f_j$ and according to 2) $\sigma_{kj}(h_{kj}) \circ g_{ji,i}$, then $c_{ji}^1 || c^2_{j,k}$. By definition, $-(c_{ji}^1 || c^2_{j,k})$, therefore $c_{ji}^1 \neq c^2_{j,k}$.

(8) shows that $c^1$ is a coloring of $Arc(\tilde{G})$. $\chi(G)$ being infinite, $\chi(\tilde{G})$ is infinite according to Remark A.4 and $\chi(Arc(\tilde{G}))$ is infinite according to Proposition A.2. This contradicts the finiteness of $\lambda$. \[\square\]

Lemma 3.30. If $\lambda$ is a finite regular trace labelling of a regular $df$-domain and $a$ and $b$ are labels with $a \parallel b$ then there is no infinite chain $I$ with
- a family of distinct compacts $(x_i)_{i \in I}$;
- a family of isomorphisms $(\sigma_{ij})_{(i,j) \in I \cap i > j}$;
- families of events $(e_{ij})_{(i,j) \in I \cap i < j}$ and $(d_{ij})_{(i,j) \in I \cap i < j}$ such that:
- residuals $\uparrow x_i, \lambda$ have all the same type and $\sigma_{ij}$ is the unique isomorphism from $\uparrow x_i$ onto $\uparrow x_j$ preserving $\lambda$;
- $e_{ij}$ is an event of $\uparrow x_i$, $\lambda(e_{ij}) = a$, and $\sigma_{ij}(e_{ij}) = e_{ji}$;
- for any $i$ in $I$, $f_i \in s(x_i)$ and $\lambda(f) = a$;
- $\rightarrow$
for any \( i < j \), \( \lambda(d_{i,j}) = b \), \( f_i < d_{i,j} \) and \( e_{i,j} \) and there is no event \( d \) with \( \lambda(d) = b \) and \( f_j < d \) and \( e_{j,i} \) and for any \( i, j \in I \) with \( i < j \), \( d_{i,j} \subseteq s(x_i) \).

**Proof.** By contradiction. Suppose the existence of \( \lambda \) and \( I \) as described above. We show that this situation leads to a contradiction (see (9) below).

Let us show

1. If \( i, j, k \in I \), \( i < j \) and \( j < k \), then \( d_{i,j} = d_{i,k} \).

Let \( i, j, k \in I \) with \( i < j \) and \( i < k \). \( f_i < d_{i,j} \) and \( f_i < d_{i,k} \), then \( d_{i,j} \neq d_{i,k} \). \((d_{i,j}, d_{i,k}) \notin \circ \) since \( \lambda(d_{i,j}) = b = \lambda(d_{i,k}) \) and \( d_{i,j} \uparrow d_{i,k} \) since \( d_{i,j}, d_{i,k} \in s(x_i) \). Necessarily \( d_{i,j} = d_{i,k} \).

Let \( \bar{I} \) be the subchain of \( I \) made of the non-maximal elements. According to (1), for any \( i \in \bar{I} \), we let \( d_i = d_{i,j} \) where \( j \in I \) and \( j > i \). If \( i \in \bar{I} \) then \( s(x_i) \setminus \uparrow d_i \) is a configuration, so let \( y_i \) be the compact with \( s(y_i) = s(x_i) \setminus \uparrow d_i \). Consider the partition of \( \bar{I} \) whose classes are the subsets of elements \( i \) with \((\uparrow y_i, \lambda)\) of a given type. \( \lambda \) being regular this partition is finite and at least one of its classes is infinite. Furthermore \( \bar{I} \) will denote such a class. Then for any \( i \in \bar{I} \) there is \( j \in I \) with \( j > i \). For any \( i, j \in \bar{I} \), \((\uparrow y_i, \lambda)\) and \((\uparrow y_j, \lambda)\) have the same type, so let \( \tau_{i,j} \) stand for the isomorphism preserving \( \lambda \) from \( \uparrow y_i \) onto \( \uparrow y_j \).

If \( i \in \bar{I} \) then since \( d_i \triangleleft s(x_i) \), \( s(x_i) \setminus \uparrow d_i \supseteq C(d_i) \), there is a compact \( y'_i \) with \( y_i < y'_i \) and \([y_i, y'_i] \notin d_i \), it satisfies \( y'_i \triangleleft x_i \). If \( i, j \in \bar{I} \) and \( i < j \) then since \( \lambda(\tau_{i,j}(d_i)) = \lambda(d_i) = \lambda(d_j) \), according to Axiom \((\text{End})\), \( \tau_{i,j}(d_i) = d_j \) and \( \tau_{i,j}(y'_i) = y'_j \).

Let us show

2. If \( i, j \in \bar{I} \) and \( i < j \) then \( \tau_{i,j}(e_{i,j}) \neq e_{j,i} \).

Since \( \tau_{i,j} \) is an isomorphism from \( \uparrow y_i \) onto \( \uparrow y_j \), and \( d_i, e_{i,j} \) are events of \( \uparrow y_i \) and \( d_i \circ e_{i,j} \) according to Lemma 2.33(iii) \((\tau_{i,j}(d_i)) \circ \tau_{i,j}(e_{i,j}) \). But \( \tau_{i,j}(d_i) = d_j \) and since \( f_j < d_j \) and \( \lambda(d_j) = b \), according to the assumptions, \( (d_j, e_{j,i}) \notin \circ \). Therefore \( e_{j,i} \neq \tau_{i,j}(e_{i,j}) \).

If \( i, j \in I \) and \( i < j \) then

3. \( d_j < e_{j,i} \) if \( j \in \bar{I} \) and

and

4. \((\uparrow d_i \setminus s(x_i)) \times (C'(e_{i,j}) \setminus s(x_i)) \subseteq \circ \).

**Proof.** (3) Let \( i, j \in \bar{I} \) with \( i < j \). \( \{d_j, e_{j,i}\} \subseteq s(x_i) \cup C'(e_{j,i}) \) then \( d_j \uparrow e_{j,i} \). Since \( e_{j,i} \notin s(x_j) \) \( d_j \neq e_{j,i} \). Thus necessarily \( d_j < e_{j,i} \), since \( (d_j, e_{j,i}) \notin \circ \).

(4) Let \( i, j \in I \) with \( i < j \), \( g \in \{d_j \cap s(x_i)\} \) and \( h \in C'(e_{i,j}) \setminus s(x_i) \). Then \( \{g, h\} \subseteq s(x_i) \cup C'(e_{i,j}) \) therefore \( g \uparrow h \). \( g \neq h \) since \( g \geq d_i \circ e_{i,j} \) and \( h \neq g \) since \( h \notin s(x_i) \). Necessarily \( g \circ h \).

If \( i, j \in \bar{I} \) and \( i < j \), let

\[ \gamma_{i,j}^1 = (C(e_{i,j}) \setminus s(x_j)) \setminus \uparrow d_j, \]
\[ \gamma_{i,j}^2 = (C(e_{i,j}) \setminus s(x_j)) \cap \uparrow d_j, \]
\[ \gamma_{i,j}^3 = C(e_{i,j}) \cap s(x_j) \cap (\uparrow d_j \setminus \{d_j\}). \]

(see Fig. 16).
Let also $v_{j,i} = y_j' \vee (x_j \land x(e_{j,i}))$. Then $v_{j,i} = x_j \land (y_j' \vee x(e_{j,i})) \leq x_j$ and

$$s(v_{j,i}) = s(y_j' \vee (x_j \land x(e_{j,i}))),$$

$$= s(y_j') \cup (s(x_j \land x(e_{j,i})) \setminus s(y_j')),$$

$$= s(y_j') \cup y_j^{3}.$$

Note that $y_j^{1} \cup y_j^{2} = C(e_{j,i}) \setminus s(x_j)$.

Let us show the following points. If $i, j \in \bar{T}$ and $i < j$ then

(5) If $g \in y_j^{1}$, $h \notin y_j^{1}$ and $h < g$ then $h \in s(y_j)$;

(6) $y_j^{1} \times (\{d_j\} \cup y_j^{3}) \subseteq \varnothing$;

(7) $s(x_j) \setminus s(v_{j,i})) \times (y_j^{1} \cup y_j^{2}) \subseteq \varnothing$.

Proof. (5) Let $g \in y_j^{1}$ and $h < g$. $h \in C(e_{j,i})$ and $h \notin d_j$. $h \in C(e_{j,i}) \setminus d_j \cap s(x_j)$ i.e. $h \in s(y_j)$.

(6) Let $g \in y_j^{1}$ and $h \in \{d_j\} \cup y_j^{3}$. According to (5) $h \notin g$. Since $g \notin s(x_j)$, $g \notin h$.

Since $g, h \in s(x_j) \cup C(e_{j,i})$, $g \uparrow h$. Therefore $g \circ h$.

(7) Since $x_j \uparrow x(e_{j,i})$, $s(x_j) \setminus C(e_{j,i}) \times (C(e_{j,i}) \setminus s(x_j)) \subseteq \varnothing$. $v_{j,i} \geq x_j \land x(e_{j,i})$ then $s(x_j) \setminus s(v_{j,i}) \subseteq s(x_j) \setminus C(e_{j,i})$ and $(s(x_j) \setminus s(v_{j,i})) \times (y_j^{1} \cup y_j^{2}) \subseteq \varnothing$.

Let us show

(8) If $(i, j) \in \bar{T}$ then there are an $g_{j,i}$ in $y_j^{3}$ and an event $h_{j,i}$ of $y_j^{1}$ with $-(g_{j,i} \uparrow h_{j,i})$.

Consider $i, j \in \bar{T}$ with $i < j$. According to (5) and (6), $s(y_j') \cup y_j^{1}$ is a configuration. So let $w_{j,i}$ be the compact with $s(w_{j,i}) = s(y_j') \cup y_j^{1}$. It satisfies $w_{j,i} \leq y_j' \vee x(e_{j,i})$ (see Fig. 17). Therefore there is some covering chain $(s_{p})_{1 \leq p < q}$ from $y_j'$ to $w_{j,i}$, and for such a chain $[s_{p}, s_{p+1}]_{1 \leq p < q}$ if $1 \leq p < q$. According to (6), $(s_{p} \cup v_{j,i})_{1 \leq p < q}$ is a covering chain from $v_{j,i}$ to $v_{j,i} \lor w_{j,i}$ and $[s_{p} \lor v_{j,i}, s_{p+1} \lor v_{j,i}] = [s_{p}, s_{p+1}]_{1 \leq p < q}$. Since $v_{j,i} \lor w_{j,i} \leq v_{j,i} \lor x(e_{j,i}) = y_j' \lor x(e_{j,i})$, the latter chain may be continued into a covering chain $(t_{p})_{1 \leq p < q}$ from $v_{j,i}$ to $y_j' \lor x(e_{j,i})$, i.e. $q' \geq q$ and $t_{p} = s_{p} \lor v_{j,i}$ if $1 \leq p < q$. Then $[t_{p}, t_{p+1}]_{1 \leq p < q}$ if $q \leq p < q'$. Now since $x_j \land x(e_{j,i}) \leq v_{j,i} \leq x_j, C(e_{j,i}) \setminus
Fig. 17. Domain $\uparrow y_j$.

$s(x_j) = C(e_{j,i}) \setminus s(v_{j,i})$ and according to (7), $(t_p \lor x_j)_{1 \leq p \leq q'}$ is a covering chain from $x_j$ to $x_j \lor x(e_{j,i})$ and $[t_p \lor x_j, t_{p+1} \lor x_j] \prec [t_p, t_{p+1}]$ if $1 \leq p < q'$.

Because $\sigma_{j,i}$ is an isomorphism from $\uparrow x_j$ onto $\uparrow x_i$ and $\sigma_{j,i}(e_{j,i}) = e_{i,j}, (\sigma_{j,i}(t_p \lor x_j))_{1 \leq p \leq q'}$ is a covering chain from $x_i$ onto $\sigma_{j,i}(x_j \lor x(e_{j,i}))$ which is $x_i \lor x(e_{i,j})$ according to Lemma 2.33(i) (see Fig. 18). The set of events of the form $[(\sigma_{j,i}(t_p \lor x_j)), (\sigma_{j,i}(t_{p+1} \lor x_j))] \prec$ where $1 \leq p < q'$ is therefore $C(e_{i,j}) \setminus s(x_j)$ which is also $C(e_{i,j}) \setminus s(y'_j)$ according to (4). Then there is a covering chain $(u_p)_{1 \leq p \leq q'}$ from $y'_j$ to $y'_j \lor x(e_{j,i})$ with $u_p \lor x_i = \sigma_{j,i}(t_p \lor x_j)$ if $1 \leq p < q'$, and $[u_p \lor x_i, u_{p+1} \lor x_i] \prec [\sigma_{j,i}(t_p \lor x_j), \sigma_{j,i}(t_{p+1} \lor x_j)]$ if $1 \leq p < q$.

Because $\tau_{i,j}$ is an isomorphism from $\uparrow y_i$ onto $\uparrow y_j$ and $\tau_{i,j}(y'_j) = y'_j, (\tau_{i,j}(u_p))$ is a covering chain from $y'_j$ to $y'_j \lor x(\tau_{i,j}(e_{i,j}))$. Since $\lambda([(\tau_{i,j}(u_p), \tau_{i,j}(u_{p+1})], \prec, \lambda([s_{p}, s_{p+1}]) \prec)$ pour $1 \leq p < q$, an induction shows, according to Axiom (Ede1), that $\tau_{i,j}(u_p) = s_p$ if $1 \leq p \leq q$.

Let us prove (8) by contradiction. Suppose $\lambda(\gamma_{j,i}^2) \times \lambda(\gamma_{j,i}^3) \subseteq ||a||$. Since if $q \leq p < q'$ then $\lambda([(\tau_{i,j}(u_p), \tau_{i,j}(u_{p+1})]) \prec) = \lambda([t_p, t_{p+1})] \prec)$, an induction shows that if $q \leq p < q'$:

- $[\tau_{i,j}(u_p), \tau_{i,j}(u_{p+1})] \prec \circ g$ for any $g \in \gamma_{j,i}^3$ (according to Axiom (Etr2)) and,
- $[\tau_{i,j}(u_p), \tau_{i,j}(u_{p+1})] \prec [t_p, t_{p+1})$.

Since $j \in J$, there is some $k$ in $I$ with $j < k$. For any $g$ in $\gamma_{j,i}^3$, $g > f_j$ and $g \in s(x_j)$ and according to (4), $g \circ e_{j,k}$ therefore $\lambda(g)||a||a$. According to Axiom (Etr2), $\tau_{i,j}(e_{j,i}) \circ g$ if $g \in \gamma_{j,i}^3$, and finally $\tau_{i,j}(e_{j,i}) = e_{j,i}$ which contradicts (2) above.

Let $i, j \in I$ with $i < j$. According to (8), there are events $g_{j,i}$ and $h_{j,i}$ with $g_{j,i} \in \gamma_{j,i}^3, h_{j,i} \in \gamma_{j,i}^3$ and $-(\lambda(g_{j,i}||a||a(\lambda(h_{j,i})))$. Choosing such $g_{j,i}$ and $h_{j,i}$, let $c^1(i,j) = \lambda(g_{j,i})$ and $c^2(i,j) = \lambda(h_{j,i})$. 

Let us show (9) If $i, j, k \in \mathcal{T}$ and $i < j < k$ then $c^1(i, j) \neq c^1(j, k)$.

Let $i, j, k \in \mathcal{T}$ with $i < j < k$. $\sigma_{k,j}(h_{k,j}) \in C(e_{j,k}) \setminus s(x_j)$, $g_{j,i} \in \gamma_{j,i}^3$. According to (4) $\sigma_{k,j}(h_{k,j}) \circ g_{j,i}$ then $c_{i,j}^1 c_{j,k}^2$. By definition $-(c_{j,k}^1 || c_{j,k}^2)$, therefore $c_{i,j}^1 \neq c_{j,k}^1$.

(9) shows that $c^1$ is a coloring of $Arc(G_f)$ where $G_f$ is the graph of the order on $\mathcal{T}$. $G_f$ being an infinite clique, $\chi(G_f)$ is infinite and according to Proposition A.2, $\chi(Arc(G_f))$ is infinite. This contradicts the finiteness of $\lambda$. □

We prove now Lemma 3.18 which we recall:

Given a regular coherent dI-domain with a finite regular trace labelling $\lambda$, $\lambda^*$ is regular.

Proof of Lemma 3.18. According to Property 3.10(2) and Lemma 3.11, $\lambda^*$ is regular if and only if for any labels $a, b$ with $a \parallel b$, the labelling $\langle \lambda, \lambda_a, \lambda_b \rangle$ is regular. So let us prove

Lemma 3.31. Given a regular coherent dI-domain with a finite regular trace labelling $\lambda$, for any labels $a, b$ with $a \parallel b$, $\langle \lambda, \lambda_a, \lambda_b \rangle$ is regular.

Proof. By contradiction. Suppose that there are some labels $a$ and $b$ with $a \parallel b$ and $\langle \lambda, \lambda_a, \lambda_b \rangle$ is not regular. Then we show that the following situation holds. There is

- an infinite chain $I$;
- a family of distinct compacts $(x_i)_{i \in I}$;
- a family of isomorphisms $(\sigma_{i,j})_{(i,j) \in I \times I, i \neq j}$;
- families of events $(e_{i,j})_{(i,j) \in I \times I, i \neq j}$ and $(f_j)_{j \in I}$; such that:
- labelled residuals $(\uparrow x_i, \lambda)$ have all the same type and $\sigma_{i,j}$ is the unique isomorphism from $\uparrow x_i$ onto $\uparrow x_j$ preserving $\lambda$;
- $e_{i,j}$ is an event of $\uparrow x_i$, $\lambda(e_{i,j}) = a$, and $\sigma_{i,j}(e_{i,j}) = e_{j,i}$;
– for any \( i \) in \( I \), \( f_i \in s(x_i) \), \( \lambda(f_i) = b \);
– for any \( i, j \in I \) with \( i < j \), \( f_i \not\prec e_{i,j} \) and there is no event \( f \) with \( \lambda(f) = b \) and \( f \not\prec e_{j,i} \), and
  * either (case 1): for any \( i, j \in I \) with \( i < j \), \( f_j \not\in C(e_{i,j}) \),
  * or (case 2): for any \( i, j \in I \) with \( i < j \), \( f_j \in C(e_{j,i}) \).

Since \( \lambda \) is regular and \( \langle \lambda, \lambda_{a,b} \rangle \) is not, there is an infinite set \( S \) of compacts such that labelled residuals \( (\uparrow x, \lambda, x) \) ranging over \( S \) have the same type though for any distinct \( x \) and \( y \) in \( S \), \( (\uparrow x, \langle \lambda, \lambda_{a,b} \rangle) \) and \( (\uparrow y, \langle \lambda, \lambda_{a,b} \rangle) \) have distinct types. For distinct \( x \) and \( y \) in \( S \), let \( \sigma_{x,y} \) be the unique isomorphism from \( \uparrow x \) onto \( \uparrow y \) preserving \( \lambda \). Such and isomorphism does not preserve \( \langle \lambda, \lambda_{a,b} \rangle \), i.e. if \( x \) and \( y \) are distinct, there is an event \( e \) of \( \uparrow x \) with \( \lambda_{a,b}(e) \neq \lambda_{a,b}(\sigma_{x,y}(e)) \). For such an \( e \), \( \lambda(e) = a \) and,
– either there is some \( f \in \lambda_b \) with \( f \not\prec e \) and there is no \( f \in \lambda_b \) with \( f \not\prec \sigma_{x,y}(e) \),
– or there is no \( f \in \lambda_b \) with \( f \not\prec e \) and there is \( f \in \lambda_b \) with \( f \not\prec \sigma_{x,y}(e) \).

Applying Ramsey’s theorem, one deduces the existence of an infinite chain \( I \) and of a family \( \{x_i\}_{i \in I} \) of pairwise distinct compacts of \( S \) such that if \( i, j \in I \) and \( i < j \) there are events \( e_{i,j} \) and \( f_{i,j} \) with \( e_{i,j} \in \uparrow x_i \) \( \lambda(e_{i,j}) = a \), \( \lambda(f_{i,j}) = b \), \( f_{i,j} \not\prec e_{i,j} \) and such that there is no \( f \) with \( \lambda(f) = b \) and \( f \not\prec \sigma_{x,y}(e) \). For shortness, if \( i < j \), let \( \sigma_{i,j} \) and \( e_{i,j} \) denote, respectively, \( \sigma_{x,y}(e_{i,j}) \) and \( e_{i,j} \).

If \( i < j \), \( f_{i,j} \in s(x_i) \) (otherwise \( f_{i,j} \) would be an event of \( \uparrow y \) \( \sigma_{y,x}(f_{i,j}) \not\prec \sigma_{y,x}(e_{i,j}) = e_{j,i} \)). Consider now \( i, j, k \in I \), with \( i < j, i < k \) and \( f_{i,j} \neq f_{i,k} \). Then, \( f_{i,j} \uparrow f_{i,k} \), because \( f_{i,j} \) and \( f_{i,k} \) belong both to \( s(x_i) \). Suppose \( f_{i,j} < f_{i,k} \). Then, \( f_{i,k} \not\prec e_{i,j} \) since \( f_{i,j} < e_{i,j}, e_{i,j} \not\prec s(x_i) \) implies \( e_{i,j} \not\prec f_{i,k} \), and \( f_{i,k} \not\prec e_{i,j} \) since \( f_{i,k} \in s(x_i) \) and \( e_{i,j} \) is an event of \( \uparrow x_i \). Therefore \( f_{i,k} \not\prec e_{i,j} \) and \( f_{i,j} \not\prec f_{i,k} \).

According to Lemma 3.22 \( \omega(\infty^1_b) \) is finite. Therefore for any \( i \in I \), the set \( F_i = \{f_{i,j} \mid j \in I, j > i \} \) which is a clique for \( \infty^1_b \), has cardinality lower or equal to \( \omega(\infty^1_b) \). Let \( g_1, \ldots, g_l, \ldots \) be respective enumerations of the \( F_i \)'s, \( i \) ranging over \( I \). One defines a partition \( (Q_l)_{l < \omega} \) on pairs in \( I \) in the following way: \( \{i, j\} \in Q_l \) if and only if \( i < j \Rightarrow f_{i,j} = g_l \). Applying again Ramsey’s theorem one deduces that there are some particular \( l \) and an infinite subset \( I' \) of \( I \) with \( i, j \in I' \Rightarrow \{i, j\} \in Q_l \). Furthermore, \( I \) will denote such an \( I' \). Then \( I \) satisfies moreover: for any \( i, j, k \in I \) with \( i < j \) and \( i < k \), \( f_{i,j} = f_{i,k} \). If \( i \) is not the greatest element of \( I \) then let \( f_i \) denote the unique \( f \in \lambda_b \) such that for any \( j > i \), \( f \not\prec e_{i,j} \). Considering the subchain of \( I \) made of non-maximal elements (in order that \( f_i \) is defined for any \( i \) of this subchain) and applying again Ramsey’s theorem, one shows there is some chain \( I \) as described above satisfying moreover
– either (case 1): for any \( i, j \in I \) with \( i < j \), \( f_j \not\in C(e_{i,j}) \),
– or (case 2): for any \( i, j \in I \) with \( i < j \), \( f_j \in C(e_{i,j}) \).

We prove in Lemmas 3.32 and 3.33 that these two cases above cannot occur. □

**Lemma 3.32.** If \( \lambda \) is a finite regular trace labelling of a regular \( dl \)-domain and \( a \) and \( b \) are labels with \( a \parallel b \) then there is no infinite chain \( I \) with:
– a family of distinct compacts \( (x_i)_{i \in I} \);
– a family of isomorphisms \( (\sigma_{i,j})_{\{i,j\} \in I^2, i \neq j} \).
families of events \((e_{i,j})_{\{i,j\} \in I: i \neq j}, (f_i)_{i \in I}\);

such that:

- labelled residuals \((\uparrow x_i, \lambda)\) have all the same type and \(\sigma_{i,j}\) is the unique isomorphism from \(\uparrow x_i\) onto \(\uparrow x_j\) preserving \(\lambda\);
- \(e_{i,j}\) is an event of \(\uparrow x_i\), \(\lambda(e_{i,j}) = a\), and \(\sigma_{i,j}(e_{i,j}) = e_{j,i}\);
- for any \(i \in I\), \(f_i \in s(x_i), \lambda(f_i) = b\);
- for any \(i,j \in I\) with \(i < j\), \(f_i = e_{i,j}\) and there is no event \(f\) with \(\lambda(f) = b\) and \(f \neq e_{j,i}\), and for any \(i,j \in I\) with \(i < j\), \(f_j \notin C(e_{i,j})\).

**Proof.** By contradiction. Suppose the existence of \(\lambda, a, b\) and \(I\) as described above.

We show this situation leads to a contradiction (see (5) below). For any \(j \in I\), let \(Y_j = s(x_j) \cap (\bigcup_{i < j} C(e_{j,i})) \cup \bigcup_{k > j} C(e_{j,k}) \setminus\{f_j\}\). Let us show,

For any \(j \in I\), \(Y_j \cup \{f_j\}\) are configurations and \(f_j \notin Y_j\).

For any \(k\) with \(j < k\), \(f_j\) is maximal in \(C(e_{j,k})\), therefore \(C(e_{j,k}) \setminus\{f_j\}\) is a configuration and \(Y_j\) is an union of configurations. By assumption if \(i < j\) in \(I\) then \(f_j \notin C(e_{i,j})\) and therefore \(f_j \notin Y_j\). Since \(f_j \in C(e_{j,k})\) if \(k > j\), \(Y_j \cup \{f_j\} = s(x_j) \cap \bigcup_{i \neq j} C(e_{i,i})\) and \(Y_j \cup \{f_j\}\) is an union of configurations.

Since \(Y_j \cup \{f_j\} \subseteq s(x_j)\), \(Y_j\) and \(Y_j \cup \{f_j\}\) are consistent and then they are configurations.

If \(j \in I\), let \(y_j\) and \(y'_j\) denote the compacts with \(s(y_j) = Y_j\) and \(s(y'_j) = Y_j \cup \{f_j\}\). If \(i < j\), let \(\gamma_{i,j} = C(e_{i,j}) \setminus s(x_i)\) and \(\gamma_{j,i} = C(e_{j,i}) \setminus s(x_j)\). Let us show,

If \(i < j\), then (1) \(\gamma_{i,j} = C(e_{i,j}) \setminus s(x'_j)\),

(2) \(f_i \cup \gamma_{i,j}\),

(3) \(\gamma_{j,i} = C(e_{j,i}) \setminus Y_j\).

(1) is easy to check.

**Proof.** (2) Let \(h \in \gamma_{i,j}, f_i\) is maximal in \(C(e_{i,j})\) then \(f_i \neq h, h \notin s(x_i)\) then \(h \notin f_i\). \(h \cup f_i\) since \(\{h, f_i\} \subseteq s(x_i \cup x(e_{i,j}))\). Necessarily \(h \cup f_i\).

(3) It results from the fact that \(f_j \notin C(e_{i,j})\).

Let \(i, j \in I\) and \(i < j\). According to (1) and (2), \(Y_i \cup \gamma_{i,j}\) is a configuration. So let \(w_{i,j}\) denote the compact with \(s(w_{i,j}) = s(y_j) \cup \gamma_{i,j}\). Note that according to 1), \(s(y'_j) \cup x(e_{i,j}) = s(y'_j) \cup \gamma_{i,j}\) and since \(f_i \notin \gamma_{i,j}\), \(w_{i,j} \neq y'_j \cup x(e_{i,j})\) and \([w_{i,j} \neq y'_j \cup x(e_{i,j})]_{\gamma_{i,j}} = f_i\) (the domain \(\uparrow y_j\) is depicted in Fig. 19). According to (3), \(s(y_j \cup x(e_{j,i})) \setminus s(y_j) = \gamma_{j,i}\) (the domain \(\uparrow y_j\) is depicted in Fig. 20).

We show

(4) If \(i, j \in I\) and \(i < j\), \(\uparrow (w_{i,j}, \lambda)\) and \(\uparrow (y_j \cup x(e_{j,i}), \lambda)\) have distinct types.

Note there is no event \(e\) with \(\lambda(e) = a\) and of the form \([w_{i,j}, z]_{\gamma_{i,j}}\): such an \(e\) would satisfy \(e \cup f_i\) according to \((Etra2)\) and \(e\) would be \(e_{i,j}\) according to \((Edet)\), but this is impossible since \(f_i \neq e_{i,j}\). Note also that \(\lambda(e_{j,i}) = a\) and \(e_{j,i} = [y_j \cup x(e_{j,i})] \setminus y_j \cup x'(e_{j,i})\) \(\gamma_{i,j}\).

We show

(5) If \(i, j \in I\) and \(i < j\), \((y_i, \lambda)\) and \((y_j, \lambda)\) have distinct types.

The latter point contradicts the regularity of \(\lambda\).
Proof. (5) Consider a covering chain \((s_p)_{1 \leq p \leq q}\) from \(y_i\) to \(w_{i,j}\). According to (1) and (2) \((s_p \lor x_i)_{1 \leq p \leq q}\) is a covering chain from \(x_i\) to \(x_i \lor x(e_{i,j})\) and \([s_p, s_{p+1}] \rightarrow [s_p \lor x_i, s_{p+1} \lor x_i]\) for any \(p\) with \(1 \leq p \leq q\).

Since \(\sigma_{i,j}\) is an isomorphism from \(\uparrow x_i\) onto \(\uparrow x_j\) sending \(e_{i,j}\) onto \(e_{j,i}\), \(\sigma_{i,j}(x_i \lor x(e_{i,j})) = x_j \lor x(e_{j,i})\) and \((\sigma_{i,j}(s_p \lor x_i))_{1 \leq p \leq q}\) is a covering chain from \(x_j\) to \(x_j \lor x(e_{j,i})\).

According to (3), \(s(y_j \lor x(e_{j,i})) \setminus s(y_i) = \gamma_{j,i}\). Therefore there is a covering chain \((u_p)_{1 \leq p \leq q}\) from \(y_j\) to \(y_j \lor x(e_{j,i})\) with \(u_p \lor x_j = \sigma_{i,j}(s_p \lor x_i)\) for any \(p\) with \(1 \leq p \leq q\).

Let us prove (5) by contradiction. Suppose there is some isomorphism \(\tau\) preserving \(\lambda\) from \(\uparrow y_i\) onto \(\uparrow y_j\). An induction shows, according to (Edet) that for any \(p\) with \(1 \leq p \leq q\), \(\tau(s_p) = u_p\) and finally \(\tau(w_{i,j}) = y_j \lor x(e_{j,i})\). This contradicts (4). \(\square\)

Lemma 3.33. If \(\lambda\) is a finite regular trace labelling of a regular \(df\)-domain and \(a\) and \(b\) are labels with \(a \parallel b\) then there is no infinite chain \(I\) with
- a family of distinct compacts \((x_i)_{i \in I}\);
- a family of isomorphisms \((\sigma_{i,j})_{\{i,j\} \in I^2 : i \neq j}\).
families of events \((e_{i,j})_{(i,j) \in I^2 \setminus \{i \neq j\}}, (f_i)_{i \in I}\);

such that:
- labelled residuals \(\uparrow x_i, \lambda\) have all the same type and \(\sigma_{i,j}\) is the unique isomorphism from \(\uparrow x_i\) onto \(\uparrow x_j\) preserving \(\lambda\); 
- \(e_{i,j}\) is an event of \(\uparrow x_i\), \(\lambda(e_{i,j}) = a\), and \(\sigma_{i,j}(e_{i,j}) = e_{j,i}\); 
- for any \(i \in I\), \(f_i \in s(x_i)\), \(\lambda(f_i) = b\); 
- for any \(i,j \in I\) with \(i < j\), \(f_i \prec e_{i,j}\) and there is no event \(f\) with \(\lambda(f) = b\) and \(f \prec e_{j,i}\), and for any \(i,j \in I\) with \(i < j\), \(f_j \in C(e_{j,i})\).

Proof. By contradiction: Suppose the existence of \(\lambda, a, b\) and \(I\) as described above.

We show this situation leads to a contradiction (see (9) below).

If \(j \in I\), let \(Y_j = s(x_j) \cap \bigcup_{i \neq j} (C(e_{i,j}) \setminus f_j)\), then 
(1) \(Y_j\) and \(Y_j \cup f_j\) are configurations and \(f_j \notin Y_j\).

If \(j \in I\), let \(y_j\) and \(y'_j\) be the compacts with \(y_j = Y_j\) and \(s(y'_j) = Y_j \cup \{f_j\}\), then \(y_j \prec y'_j\) and \([y_j, y'_j] \in f_j\).

Consider the partition of \(I\) whose classes are the subsets of \(i\) with \((\uparrow x_i, \lambda)\) of a given type. \(\lambda\) being regular this partition is finite and at least one of its class is infinite. Further on \(I\) will denote such a class. Then for any \(i, j \in I\), \((\uparrow x_i, \lambda)\) and \((\uparrow x_j, \lambda)\) have the same type, so let \(\tau_{i,j}\) denote the isomorphism preserving \(\lambda\) from \(\uparrow x_i\) onto \(\uparrow x_j\). If \(i, j \in I\) and \(i < j\) then since \(\lambda(\tau_{i,j}(f_i)) = \lambda(f_i) = \lambda(f_j)\), according to Axiom (Edet), \(\tau_{i,j}(f_i) = f_j\) and \(\tau_{i,j}(y'_i) = y'_j\).

Let us show
(2) If \(i, j \in I\) and \(i < j\) then \(\tau_{i,j}(e_{i,j}) \neq e_{j,i}\).

Since \(\tau_{i,j}\) is an isomorphism from \(\uparrow y_i\) onto \(\uparrow y_j\) and \(f_i, e_{i,j}\) are events of \(\uparrow y_i\) and \(f_i \prec e_{i,j}\), according to Lemma 2.33(ii) \(\tau_{i,j}(f_i) \prec \tau_{i,j}(e_{i,j})\). But \(\tau_{i,j}(f_i) = f_j\) and according to the assumptions, \(f_j\) is not covered by \(e_{j,i}\). Necessarily \(\tau_{i,j}(e_{i,j}) \neq e_{j,i}\).

If \(i, j \in I\) and \(i < j\) then
(3) \((\uparrow f_i \cap s(x_i)) \times (C(e_{i,j}) \setminus s(x_i)) \subseteq \emptyset\); 
(4) if \(g > f_i\) and \(g \in s(x_i)\) then \(g \cap e_{i,j}\).

Proof. (3) Let \(i, j \in I\) with \(i < j\), \(g \in \uparrow f_i \cap s(x_i)\) and \(h \in C(e_{i,j}) \setminus s(x_i)\), then \(\{g, h\} \subseteq s(x_i) \cup C(e_{i,j})\) and \(g \uparrow h\). Since \(g \geq f_i\) and \(f_i \prec e_{i,j}\), either \(g = f_i\) or \(g \notin e_{i,j}\) therefore \(g \not\in h\). \(h \not\in g\) since \(h \not\in s(x_i)\). Necessarily \(h \cap g\).

(4) Let \(i, j \in I\) with \(i < j\), \(g \in s(x_i)\) and \(g > f_i\). Since \(\{g, e_{i,j}\} \subseteq s(x_i) \cap \lambda'(e_{i,j})\), \(g \uparrow e_{i,j}\).

Since \(g > f_i\) and \(f_i \prec e_{i,j}\), \(g \not\in e_{i,j}\) and since \(e_{i,j} \notin s(x_i)\), \(e_{i,j} \not\in g\).

If \(i, j \in I\) and \(i < j\), let
\[
\gamma_{j,i}^1 = (C(e_{j,i}) \setminus s(x_j)) \setminus f_j,
\gamma_{j,i}^2 = (C(e_{j,i}) \setminus s(x_j)) \cap f_j,
\gamma_{j,i}^3 = C(e_{j,i}) \cap s(x_j) \cap (\uparrow f_j \setminus \{f_j\}).
\]

(see Fig. 21).
Let also $v_{j,i} = y_j' \lor (x_j \land x(e_{j,i}))$. Then $v_{j,i} = x_j \land (y_j' \lor x(e_{j,i})) \leq x_j$ and

$$s(v_{j,i}) = s(y_j') \lor (x_j \land x(e_{j,i})),$$

$$= s(y_j') \lor (s(x_j \land x(e_{j,i}))) \setminus s(y_j'),$$

$$= s(y_j') \cup \gamma_{j,i}^3.$$

Note also $\gamma_{j,i}^1 \cup \gamma_{j,i}^2 = C(e_{j,i}) \setminus s(x_j)$. Let us show the following points. If $i,j \in I$ and $i < j$ then:

1. If $g \in \gamma_{j,i}^1$, $h \not\in \gamma_{j,i}^1$, $h < g$ then $h \in s(y_j)$;
2. $\gamma_{j,i}^1 \times \{ f_j \} \cup \gamma_{j,i}^3 \subseteq \gamma_{j,i}$;
3. $(s(x_j) \setminus s(v_{j,i})) \times (\gamma_{j,i}^1 \cup \gamma_{j,i}^2) \subseteq \gamma_{j,i}$.

**Proof.** (5) Let $g \in \gamma_{j,i}^1$ and $h < g$. $h \not\in C(e_{j,i})$ and $h \not\in f_j$, then $h \in C(e_{j,i}) \setminus f_j$. If $h \not\in \gamma_{j,i}^1$ then $h \in (C(e_{j,i}) \setminus f_j) \cap x(e_i)$. i.e. $h \in s(y_j)$.

(6) Let $g \in \gamma_{j,i}^1$ and $h \in \{ f_j \} \cup \gamma_{j,i}^3$. According to (5), $h \notin g$. Since $g \not\subseteq s(x_j)$, $g \subseteq h$. Since $g \in s(x_j) \cup C(e_{j,i})$, $g \nsubseteq h$. Therefore $g \nsubseteq h$.

(7) Since $x_j \cap x(e_{j,i})$, $(s(x_j) \setminus C(e_{j,i})) \times (C(e_{j,i}) \setminus s(x_j)) \subseteq \gamma_{j,i}$, $v_{j,i} \geq x_j \land x(e_{j,i})$ therefore $(s(x_j) \setminus s(v_{j,i})) \times (\gamma_{j,i}^1 \cup \gamma_{j,i}^2) \subseteq \gamma_{j,i}$.

Let us show

(8) If $i,j \in I$, $i < j$ and $j$ is not maximal in $I$, there are some events $g_{j,i}$ in $\gamma_{j,i}^1$ and $h_{j,i}$ in $\gamma_{j,i}^2$ with $\neg(g_{j,i} \cup h_{j,i})$.

Let $i,j \in I$ with $i < j$ and $j$ non maximal in $I$. According to (5) and (6) $s(y_j') \cup \gamma_{j,i}^1$ is a configuration. Let $w_{j,i}$ be the compact with $s(w_{j,i}) = s(y_j') \cup \gamma_{j,i}^1$. It satisfies $w_{j,i} \leq y_j' \lor x(e_{j,i})$ (see Fig. 22). Therefore, there is a covering chain $(s_p)_{1 \leq p \leq q}$ from $y_j'$ to $w_{j,i}$, and for such a chain $[s_p, s_{p+1}]_{\geq \leq} \in \gamma_{j,i}^1$ if $1 \leq p < q$. According to (6), $(s_p \cup v_{j,i})_{1 \leq p \leq q}$ is a covering chain from $v_{j,i}$ to $v_{j,i} \lor w_{j,i}$ and $[s_p \lor v_{j,i}, s_{p+1} \lor v_{j,i}]_{\geq \leq} [s_p, s_{p+1}]$ if $1 \leq p < q$. Since $v_{j,i} \lor w_{j,i} \leq v_{j,i} \lor x(e_{j,i}) = y_j' \lor x(e_{j,i})$, this chain may be continued.
into a covering chain \((t_p)_1 \leq p \leq q'\) from \(v_{j, i}\) to \(y'_j \lor x(e_{j, i})\), i.e. \(q' \geq q\) and \(t_p = s_p \lor v_{j, i}\) if \(1 \leq p \leq q\). Then \([t_p, t_{p+1}] > \gamma^2_{j, i}\) if \(q \leq p < q'\). Now since \(x_j \land x(e_{j, i}) \equiv v_{j, i} \leq x_j\), \(s(x(d_{j, i})) \land s(x(e_{j, i})) \land s(v_{j, i})\) and according to (7), \((t_p \lor x_j)_1 \leq p \leq q'\) is a covering chain from \(x_j\) to \(x_j \lor x(e_{j, i})\) and \([t_p \lor x_j, t_{p+1} \lor x_j] > [t_p, t_{p+1}]\) if \(1 \leq p < q'\).

Because \(\sigma_{j, i}\) is an isomorphism from \(\uparrow x_j\) onto \(\uparrow x_i\) and \(\sigma_{j, i}(e_{j, i}) = e_{i, j}\), \((\sigma_{j, i}(t_p \lor x_j))_1 \leq p \leq q'\) is a covering chain from \(x_i\) to \(\sigma_{j, i}(x_j \lor x(e_{j, i}))\) which is \(x_i \lor x(e_{i, j})\) according to Lemma 2.33(i). (see Fig. 23). Therefore, the set of events of the form \([(\sigma_{j, i}(t_p \lor x_j)), (\sigma_{j, i}(t_{p+1} \lor x_j))] >\) with \(1 \leq p \leq q'\) is \(C_{e_{i, j}} \land s(x_i)\) which is also \(C_{e_{i, j}} \land s(y_i)\) according to (4). Then there is a covering chain \((u_p)_1 \leq p \leq q'\) from \(y'_i\) to \(y'_i \lor x(e_{i, j})\) with \(u_p \lor x_i = \sigma_{j, i}(t_p \lor x_j)\) if \(1 \leq p \leq q'\), and \([u_p \lor x_i, u_{p+1} \lor x_i] = [\sigma_{j, i}(t_p \lor x_j), \sigma_{j, i}(t_{p+1} \lor x_j)]\) if \(1 \leq p < q\).

Because \(\tau_{i, j}\) is an isomorphism from \(\uparrow y_j\) onto \(\uparrow y_i\) and \(\tau_{i, j}(y'_j) = y'_i\), \((\tau_{i, j}(u_p))\) is a covering chain from \(y'_j\) onto \(y'_j \lor x(\tau_{i, j}(e_{i, j}))\). Since \(\lambda([\tau_{i, j}(u_p), \tau_{i, j}(u_{p+1}]) > \lambda([s_p, s_{p+1}]) >\) if \(1 \leq p < q\), an induction shows, according to (Edet) that \(\tau_{i, j}(u_p) = s_p\) if \(1 \leq p \leq q\).

Now let us prove (8) by contradiction. Suppose \(\lambda(\gamma^2_{j, i}) \times \lambda(\gamma^3_{j, i}) \not\equiv \|b\). Since \(\sigma_{j, i}\) and \(\tau_{j, i}\) preserve \(\lambda\), if \(q \leq p < q'\), \(\lambda([\tau_{i, j}(u_p), \tau_{i, j}(u_{p+1})]) >\) \(\lambda([t_p, t_{p+1}] >\) ). An induction shows that if \(q \leq p \leq q'\):

- \([\tau_{i, j}(u_p), \tau_{i, j}(u_{p+1})] >\) \(\circ g\) for any \(g \in \gamma^3_{j, i}\) (according to Axiom (Etr2)) and,
- \([\tau_{i, j}(u_p), \tau_{i, j}(u_{p+1})] >\) \([t_p, t_{p+1}] >\).

\(j\) not being maximal in \(L\), let \(k\) in \(L\) with \(j < k\). Then for any \(g\) in \(\gamma^3_{j, i}\), \(g > f_j\) and \(g \in s(x_j)\) and according to (4), \(g \circ e_{j, k}\). Therefore \(\lambda(g)\|b\) for any \(g\) de \(\gamma^3_{j, i}\), then

Fig. 22. Domain \(\uparrow y_j\).
according to (Etr2), \( \tau_{i,j}(e_{i,j}) \circ g \) if \( g \in \gamma_{j,i}^3 \), and finally according to (Edet), \( \tau_{i,j}(e_{i,j}) = e_{j,i} \) which contradicts (2).

Furthermore let \( \bar{I} \) denote the subchain of \( I \) made of the non maximal elements. Let \( i, j \in \bar{I} \). According to (8), there are events \( g_{j,i} \) and \( h_{j,i} \) with \( g_{j,i} \in \gamma_{j,i}^3 \), \( h_{j,i} \in \gamma_{j,i}^2 \) and \( -\langle \lambda(g_{j,i}) \rangle \lambda(h_{j,i}) \). Choosing such \( g_{j,i} \) and \( h_{j,i} \), let \( c'(i,j) = \lambda(g_{j,i}) \) and \( c'(i,j) = \lambda(h_{j,i}) \).

Then:

(9) If \( i, j, k \in \bar{I} \) and \( i < j < k \) then \( c'(i,j) \neq c'(j,k) \).

Let \( i, j, k \in I \) with \( i < j < k \). \( \sigma_{k,j}(h_{k,j}) = C(e_{j,k}) \Delta(x_{j}) \), \( g_{j,i} \in \gamma_{j,i}^3 \) and according to (4) \( \sigma_{k,j}(h_{k,j}) \circ g_{j,i} \) therefore \( c_{i,j}^1 || c_{k,k}^2 \). By definition \( -(c_{i,k}^1 || c_{i,k}^2) \), therefore \( c_{i,j}^1 \neq c_{i,k}^1 \).

(9) shows that \( c^1 \) is a coloring of \( \text{Arc}(G_{\bar{I}}) \) where \( G_{\bar{I}} \) is graph of the order \( \bar{I} \). \( G_{\bar{I}} \) being an infinite clique, \( \chi(G_{\bar{I}}) \) is infinite and according to Proposition A.2, \( \chi(\text{Arc}(G_{\bar{I}})) \) infinite. This contradicts the finiteness of \( \lambda \). □

A few remarks are in order. Though unfoldings of finite stable trace automata are exactly unfoldings of full trace automata, no similar property holds for automata on finite alphabets. Actually, the class of unfoldings of stable trace automata on finite alphabets contains strictly those of unfoldings of full trace automata on finite alphabets. Fig. 24 shows a (non-regular) coherent dl-domain admitting some finite trace labelling but no finite full trace labelling. A finite trace labelling of the domain may be \( \lambda \) with values in \( \{a, b\} \) with \( a \parallel b \) and for any integer \( i \), \( \lambda(e_i) = a \) and \( \lambda(f_i) = b \). The relation \( \Join = \ast \circ \) has an infinite clique: for any integers \( i, j \) with \( i < j \), events \( e_i \), \( e_j \), \( f_i \) satisfy \( e_i \Join \neq f_i \) and \( f_i \circ e_j \) and therefore \( e_i \Join e_j \). This shows that there is no finite full trace labelling of the domain.

The proof above does not rely on any algorithm of “translation” of stable trace automata into some full trace one. It seems difficult to relate the size of a stable trace automaton to the size of its full trace coverings.
4. Conclusions

The result presented here seems to confirm that the recognizability of coherent dI-domains by trace automata is a robust notion. Since stability is a necessary and sufficient condition for concurrent automata to unfold into distributive domains, a distributive domain is the unfolding of some finite concurrent automaton if and only if it is the unfolding of some finite stable one. Questions about conflict remain unsolved. It is not known whether conflict event domains unfoldings of finite concurrent automata are also unfoldings of finite trace automata. A simpler question is whether coherent dI-domains unfoldings of finite stable concurrent automata are also unfoldings of finite trace ones. Problems lie also in finding pure order theoretic characterization of recognizable domains – i.e. such characterizations should not refer to any labellings of events. Nevertheless, these questions seem difficult which may be due to the fact that they are too specific. In this sense, the unfolding process for automata should perhaps be redefined and studied in a more general setting. For example, the unfolding process may be simply extended to some kinds of graphs [16]. Therefore, it is certainly worth studying the domains unfoldings of finite such graphs and compare them with the unfoldings of automata. Anyway, a general recognizability notion associated with some unfolding process has still to be found.
Appendix A

A.1. Graphs

For any vertex $v$ of a graph, $v^+$ (respectively $v^-$) denote the set of edges with domain (respectively with codomain) $v$. A simple graph is considered as a set of couples of vertices.

Definition A.1. If $G$ is a simple graph with no loops, $\text{Arc}(G)$ is the simple graph whose vertices are the edges of $G$ and whose edges are the couples $(u, u')$ with $u \in v^-$ and $u' \in v^+$ for vertex $v$ of $G$. Note that $\text{Arc}(G)$ has no loop.

Proposition A.2. $G$ has a finite coloring if and only if $\text{Arc}(G)$ has a finite coloring.

Proof. Let $\lambda$ be a coloring of $G$. It provides the following map $\tilde{\lambda}$. For any vertex $(v^1, v^2)$ of $\text{Arc}(G)$, $\tilde{\lambda}((v^1, v^2)) = \lambda(v^1)$. Let us check that $\tilde{\lambda}$ is a coloring of $\text{Arc}(G)$. Suppose that $(u^1, u^2)$ is some edge of $\text{Arc}(G)$, then $u^1 = (v^1, v^2)$ and $u^2 = (v^2, v^3)$ for some vertices of $G$ $v^1$, $v^2$, $v^3$. Then $\tilde{\lambda}(u^1) \neq \tilde{\lambda}(u^2)$ since, by definition $\tilde{\lambda}(u^1) = \lambda(v^1)$ and $\tilde{\lambda}(u^2) = \lambda(v^2)$ and, $\lambda$ being a coloring, $\lambda(v^1) \neq \lambda(v^2)$. Note that if $\lambda$ is finite then also is $\tilde{\lambda}$. Conversely, if $\tilde{\lambda}$ is a coloring of $\text{Arc}(G)$, one defines the map $\lambda$ on the set of vertices of $G$ by $\lambda(v) = \{\lambda((v, v'))| (v, v') \in G\}$. Let us check now that $\lambda$ is a coloring of $G$. Suppose $(v, v') = u \in G$. Then $\lambda(u) \in \lambda(v)$ by definition, and $\lambda(u) \notin \lambda(v')$ since $\lambda$ is a coloring of $\text{Arc}(G)$, finally $\lambda(v) \neq \lambda(v')$. Note that if $\lambda$ is finite, then also is $\tilde{\lambda}$. □

Corollary A.3 (Berge [5]). The simple graph with no loops whose vertices are couples of natural integers of the form $(i, j)$ with $i < j$ and whose edges correspond to pairs $((i, j), (j, k))$ has infinite chromatic number (though no clique of size three).

Proof. This graph is (isomorphic to) $\text{Arc}(K)$ where $K$ is the strict order on natural integers, and $K$, which is a complete graph on some infinite set of vertices, has no finite coloring. □

A set of vertices of a simple graph with no loops is stable when any two distinct vertices in it are not adjacent. If $S$ is stable in a simple graph with no loops $G$, then the chromatic number of $G$ equals the chromatic number of the induced subgraph on the set of vertices not in $S$, plus one. A vertex of a graph is terminal when it has no successors. The set of terminal vertices of a simple graph is stable, therefore

Remark A.4. The chromatic number of a simple graph with no loops equals to the chromatic number of the induced subgraph on the set of non-terminal vertices plus one.

A.2. Ramsey's theorem

A labelling is a map with values in a set of "labels" or "colors".
Theorem A.5 (Ramsey-infinitary version). For any couple of integers \((r, m)\) with \(m > 0\), for any infinite set \(E\) and for any labelling with \(m\) colors of the subsets of \(E\) with \(r\) elements, there is a non finite subset of \(E\) whose subsets with \(r\) elements have the same color.

A consequence of Ramsey’s theorem is

Corollary A.6. Let \(X\) be an infinite set. If \((R_i)_{1 \leq i \leq n}\) is a finite family of binary relations on \(X\) such that for any distinct \(x\) and \(y\) in \(X\) there is some \(l\) with \(xR_iy\) or \(yR_ix\), then there is some \(l \in \{1, \ldots, n\}\), some infinite chain \(I\) and a sequence \((x_i)_{i \in I}\) of distinct elements such that

- either \(i < j \Rightarrow x_iR_ix_j\),
- or \(i < j \Rightarrow x_jR_ix_i\).

Proof. Consider some (total) order on \(X\) and apply Ramsey’s theorem (Theorem A.5) to some labelling of pairs of distinct elements of \(X\) with values in \(\{1, \ldots, n\} \times \{1, 2\}\) made as follows. For any pair \(\{x, y\}\), let \(l(\{x, y\})\) be some \(l\) such that \(xR_iy\) or \(yR_ix\).

Then for some pair \(\{x, y\}\) with \(x < y\), let \(\lambda(\{x, y\}) = (l(\{x, y\}), 1)\) if \(xR_i(l(\{x, y\}))y\), and let \(\lambda(\{x, y\}) = (l(\{x, y\}), 2)\), otherwise. \(\Box\)

References


