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Aspects of M-5 brane world volume dynamics

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Abstract

This Letter studies various aspects of the world volume dynamics of the M-theory five-brane, including: non-BPS solutions and solution generating symmetries; the scattering properties of world volume solutions; and the equivalence with probe brane dynamics.

1. Introduction

The M-theory five-brane still remains one of the most mysterious objects in M-theory. Its role in M-theory is similar to that of a D-brane in string theory in that it is a submanifold on which open membranes may end just as a D-brane is a submanifold on which open strings may end [1]. The world volume dynamics of a D-brane, for slowly varying fields, are governed by a Dirac Born–Infeld theory. A similar non-linear theory also exists for the five-brane world volume fields [2]. Importantly, for D-branes there is also a description derived from open strings of the low energy dynamics (that is energies below the string scale) of N coincident branes using $U(N)$ Yang–Mills theory. This description of D-branes has provided us with a rich insight in gauge theories. Unfortunately, there is no such equivalent description for coincident M-theory branes. There are signs that a description of the coincident five-brane theory will not be so simple. It is known from scattering calculations and black hole thermodynamics that the number of low energy degrees of freedom of N coincident five-branes grows as N^3 [3,4]. There has been little insight so far as to reason for this growth in the number of degrees of freedom. For D-branes, where is growth is N^2, the answer is due to the open strings stretching between different D-branes becoming light as the D-branes become coincident which from the Yang–Mills point of view is simply the W-bosons becoming massless so the gauge symmetry becomes enhanced to a $U(N)$ theory.

There is some hope that perhaps with a better understanding of the open membranes that stretch between different five branes one might hope to understand the theory of coincident five-branes using a similar construction. In this Letter we will be less ambitious and attempt to study some basic aspects of how a open membranes end on five-branes using the world volume and brane probe approaches. This approach was initiated for the BIon [5] by [6] and then numerous other works [7–10] investigated further

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scattering properties. This Letter will concentrate on aspects relevant to the M-theory five brane.

In particular, the first section of the Letter will deal with symmetries of the five brane equations of motion that we will then use to generate a family of static non-BPS solutions. The second section of the Letter will deal with scattering properties of world volume solutions and the equivalence with brane probes. This will allow us to speculate on the number of degrees of freedom carried by N self-dual strings in a theory with N' coincident five branes.

We should note that similar issues have been investigated recently in a series of papers by [11] using a linearised version of the field equations for the self-dual string coupled to a two form and background scalar.

2. Non-BPS solutions and solution generating symmetries

The bosonic fields on the five brane include 5 scalar fields, ϕ^I, $I = 1, \dots, 5$ and a two form potential b whose field strength $H = db$ obeys a non-linear self-duality constraint. This self-duality constraint together with the Bianchi identities imply the full equations of motion. There are various formulations for the five brane that impose this non-linear constraint [12,13]. The form that we will find useful here is the $SO(5)$ invariant formulation where one may write the self-duality constraint using the following "Hamiltonian density" [14]; (in fact strictly speaking what follows is is the Legendre transform of the Hamiltonian density also derived in [14] and is expressed in terms of *electric* fields only[1])

$$\mathcal{H} = \sqrt{\det(g_{ij} + iE_{ij})} - 1. \tag{1}$$

Note the determinant is even in E_{ij} so there is no imaginary part in the determinant. The metric g_{ij} is the induced metric given by the pull back of the background metric to the five-brane world volume.

$$g_{ij} = \partial_i X^I \partial_j X^J G_{IJ}. \tag{2}$$

In so-called static or Monge gauge where one identifies the world volume coordinates with the spacetime coordinates the induced metric becomes:

$$g_{ij} = G_{ij} + \partial_i \phi^A \partial_j \phi^B G_{AB} \tag{3}$$

with the fields ϕ^A are the scalar fields on the five brane.

The so-called *electric* and *magnetic* fields of the three form field strength $H = db$ are defined respectively by:

$$E_{ij} = H_{0ij}, \qquad B^{ij} = -\frac{1}{6}\epsilon^{ijklm} H_{klm}. \tag{4}$$

The non-linear self-duality constraint may now be expressed as a relation between the magnetic fields and the electric field given by:

$$B_{ij} = \frac{\partial \mathcal{H}}{\partial E_{ij}} = \frac{(1 + \frac{1}{2}\operatorname{Tr} E^2)E_{ij} + (E^3)_{ij}}{\sqrt{1 + \frac{1}{2}\operatorname{Tr} E^2 + U_i^2}}, \tag{5}$$

where

$$U_i = \frac{1}{8}\epsilon_{ijrst} E^{jr} E^{st} \tag{6}$$

and we have implicitly included the scalar fields by using the induced metric in the contractions of the indices in the above equation.

One must also impose the Bianchi identities which in this formulation are equivalent to:

$$\partial_i B_{ij} = 0, \qquad \partial_t B_{ij} + \frac{1}{2}\epsilon_{ijkrs}\partial_k E_{rs} = 0. \tag{7}$$

These Bianchi identities combined with the self-duality constraint (5) imply the equations of motion for the self-dual field. This formulation of the five brane is discussed in [14,15] and also connected to the various other equivalent formulations.

The equations of motion of the scalar fields are given by [2,13]:

$$G^{\mu\nu}\nabla_\mu^{(g)}\partial_\nu \phi^i = 0, \tag{8}$$

$$G_{\mu\nu} = \frac{1+K}{2K}\left(g_{\mu\nu} + \frac{\ell_p^6}{4} H_{\mu\nu}^2\right), \tag{9}$$

$$K = \sqrt{1 + \frac{\ell_p^6}{24} H^2}, \tag{10}$$

where ∇ denotes a covariant derivative with respect to the induced metric $g_{\mu\nu}$. $G_{\mu\nu}$ is conformally related to the so-called open membrane metric described in [17].

[1] We have made a trivial field redefinition $4H \to H$ as compared with the formulation given in [14].

We are interested in solving these equations to find the M-theory version of the non-supersymmetric BIon. We will make a string-like ansatz for the fields motivated by the following physical arguments. On a D-brane where one has a one form potential there are point like solutions. A five brane has a two form and so it is natural to look for string like solutions. From the string/M-theory point of view this is just the usual addition of a new dimension as one goes to strong coupling in string theory and strings become membranes.

We will look for solutions with a single excited scalar field ϕ and a non-trivial two form potential $b_{\mu\nu}$ in a flat background with metric $G_{IJ} = \eta_{IJ}$ as follows:

$$\phi^1 = \phi(r), \qquad b_{01} = \chi(r),$$
$$b_{\phi\psi} = \frac{q}{r^2}(\pm 1 - \cos\theta). \tag{11}$$

We have decomposed the six-dimensional world volume in a "$2+4$" split, that is t and x_1 will be the coordinates of the string world sheet and the remaining four coordinates (r, θ, ϕ, ψ) are the spherical coordinates of the space transverse to the string. This ansatz also of course makes a gauge choice for the form of the two form potential $b_{\mu\nu}$. In this gauge, the field $\chi(r)$ can be thought of as an electric potential since the *electric* components of the field strength will be given by derivatives of χ. It is necessary to also include *magnetic* components of $b_{\mu\nu}$ so that the self-duality constraint will be obeyed. In fact, the ansatz made here includes a Dirac "monopole" type potential for $b_{\phi\psi}$ to ensure the string will be magnetically charged. With this ansatz, the Hamiltonian density greatly simplifies to:

$$\mathcal{H} = \sqrt{1 + \left(\partial_r\phi(r)\right)^2 - \left(\partial_r\chi(r)\right)^2} - 1. \tag{12}$$

The first thing to notice is that the Hamiltonian density in this form has a global $SO(1,1)$ symmetry rotating the ϕ and χ fields. This symmetry was pointed out for BIons in [5,16] and is similar to the Harrison transformations that relate a charged black hole to a neutral black hole via a boost like transformation. In the case described here the consequences of this symmetry are, given a solution with $(0, \chi)$ one can generate another solution with (ϕ', χ') given by:

$$\phi' = \frac{v}{\sqrt{1-v^2}}\chi, \qquad \chi' = \frac{1}{\sqrt{1-v^2}}\chi. \tag{13}$$

One should also remark that although we took a spherical ansatz (relevant to the case at hand) in fact this symmetry persists for any static configuration in static gauge. (This may be demonstrated using standard determinant manipulations.)

In [15] a solution of the five brane field equations was found with $\phi = 0$. This solution is given by:

$$E_r = \partial_r\chi(r) = \frac{q}{\sqrt{q^2 + r^6}}. \tag{14}$$

Note that q is the same parameter that appears in the magnetic potential and is of course related the charge of string as can be seen by looking at the asymptotics of $E(r)$,

$$E(r) \to \frac{q}{r^3}, \quad r \to \infty. \tag{15}$$

Integrating this electric field strength over the S^3 at infinity will give an electric charge q. Note that the maximum field strength is as $r \to 0$, $E(r) \to 1$. This is the presence of a critical field strength that is typical of Born–Infeld theories [18] (and indeed was part of the original motivation).

Substituting this field strength into the self-duality condition (5) implies the magnetic field must be:

$$B_{1r} = \frac{q}{r^3} \tag{16}$$

which is consistent with the ansatz (11) given above; and implies that the solution has equal electric and magnetic charge, q. This charge q will be quantised using a generalisation of the usual Dirac argument for monopole charge quantisation.

We can now use the transformation (13) to generate a continuous family of solutions. This is illustrated in Fig. 1. The scalar field strength is drawn indicating how the brane becomes more and more deformed as v is increased and the electric field is "boosted" into the scalar field. These solutions will of course all be "spacelike" from the $SO(1,1)$ point of view. One can also find "timelike" solutions that are disconnected from the these solutions where the scalar field dominates. These are the five brane versions of the catenoid given in [16]. The "lightlike" solutions, i.e., solutions that are fixed points under this $SO(1,1)$ symmetry are in fact the self-dual string solutions found by Lambert and West, [19]. This solution (with charge N) is given

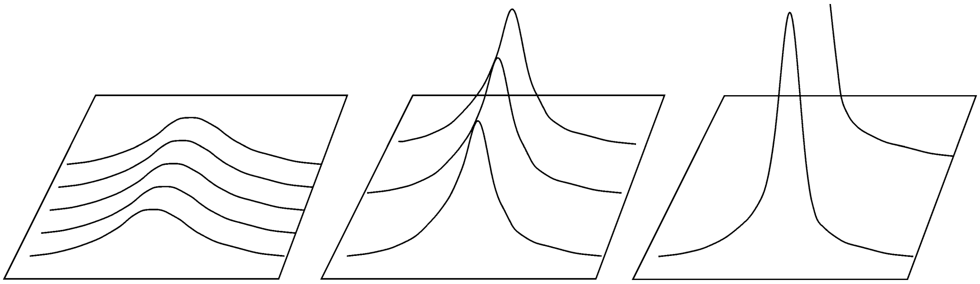

Fig. 1. Scalar field strength from small to large v.

by:

$$\phi = f = \frac{N}{r^2}, \qquad H_{01p} = \partial_p f,$$
$$H_{mnp} = \epsilon_{mnpq}\partial_q f. \tag{17}$$

This string solution may be connected to the space-like solutions by considering the divergent limit where the boost parameter, v is sent to one causing the scalar field strength to diverge. This self-dual string solution is one half BPS state from the world volume theory point of view and corresponds to the emergence of the membrane from the five brane and as such has been the subject of much study.

Note, the other solutions (that transform under this $SO(1,1)$ symmetry) are non-BPS. It is interesting to remark that it is the BPS solutions that are fixed points—one may wonder whether this is a generic property of hidden solution generating symmetries in M-theory [20].

It is not clear how to interpret these non-BPS solutions and they will certainly receive quantum corrections since no supersymmetry is preserved. However, these solutions may be tuned (for v near 1) to be very close to the BPS bound and as such it may prove interesting to study in this near BPS limit.

3. World volume scattering and probe brane scattering

In this section we will examine two related situations from which we will be able to study the scattering properties of membranes and five branes. The relationship between brane probes of supergravity solutions and world volume solitons was pointed out in [21].

The first is the situation we have already considered above, a single M-five brane with N membranes ending on it. This may be described by a charge N, BPS self-dual string solution living on the world volume of a single five brane.

The second is where we have N' five branes and a single membrane ending on it. This may be described by a membrane probing a supergravity solution with N' units of five brane flux. (The solution will be given below.)

These two scattering problems turn out to be mathematically equivalent once a suitable identification of parameters has been made. (This was pointed out for the case of BPS BIons on D-branes in [7].)

Let us first write down the equation of motion an s-wave scalar fluctuation of energy ω that is transverse to both the M-five brane and the self-dual string. This is obtained by expanding the scalar field equations (8) to linear order around the charge N self-dual string solution (17). This was first described in [22],

$$\left(\rho^{-3}\frac{d}{d\rho}\rho^3\frac{d}{d\rho} + 1 + \frac{R^6\omega^6}{\rho^6}\right)\phi(\rho) = 0, \tag{18}$$

where $\rho = r\omega$ the dimensionless distance transverse to the string and

$$R^3 = Nl_p^3. \tag{19}$$

To calculate the absorption cross section of the string one calculates solutions for the interior and exterior regions (exactly what we mean by interior and exterior is defined below) and then match the solutions. One may then take the ratio of the incoming fluxes in the interior region to the exterior region to obtain a (dimensionless) absorption cross section.

The asymptotic solution (for the outer region where $\rho \gg R\omega$) are given by simple Bessel functions.

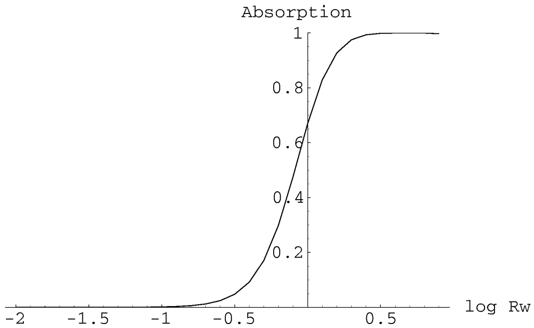

Fig. 2. Absorption as a function of $\log R\omega$.

One may then change to variables that are suited to the inner region of the solution (essentially one changes coordinates to those adapted to the membrane emerging from the five brane).

$$z = \frac{(R\omega)^3}{2\rho^2}. \tag{20}$$

In these variables the equation of motion becomes:

$$\left(\frac{d^2}{dz^2} + \frac{(\frac{1}{2}R\omega)^3}{z^3} + 1\right)\phi(z). = 0 \tag{21}$$

Solutions in the interior region (where $z \gg R\omega$) corresponding to the membrane fluctuations maybe written in terms of simple trigometric functions.

As shown in [22], the solutions in these two asymptotic regions can be matched provided $R\omega \ll 1$ to obtain a low energy scattering (dimensionless) cross section given by:

$$\sigma \sim (R\omega)^3 \sim N. \tag{22}$$

The implication of this calculation is that the number of degrees of freedom of N coincident self-dual strings is proportional to N. (Recall how for a D-brane the cross section scattering of bulk supergravity fields is proportional to N^2 which of course gives the correct number of degrees of freedom for a $U(N)$ gauge theory [3]. It is slightly puzzling when one compares this to the absorption cross section of a membrane in eleven-dimensional supergravity where the absorption cross section of Q membranes is proportional to $Q^{3/2}$.

To calculate the absorption cross section for generic $R\omega$ one may also solve these equations numerically. This was done using a simple Mathematica script. For small $R\omega$ the numerical solutions matched the analytic solutions of the asymptotic regions as they must. As $R\omega$ approaches one the absorption greatly increases till for high energies the waves are completely absorbed. This is shown in Fig. 2, where we have plotted the dimensionless absorption cross section against $\log(R\omega)$. As $R\omega$ nears 1 the membrane becomes fully absorbing, numerically confirming the analysis given in [6] for BIons.

We will now examine the alternative situation of one membrane ending on N' five branes by placing a probe membrane in a supergravity background describing N' coincident five branes. This was first discussed in [21] a similar scattering calculation was done in the string/D-brane context by [8,10].

The five brane background is given by:

$$ds^2 = H^{-\frac{1}{3}}\left(-dx_0^2 + \cdots + dx_5^2\right) + H^{\frac{2}{3}}\left(dx_6^2 + \cdots + dx_{10}^2\right), \tag{23}$$

$$F_{mnpq} = \epsilon_{mnpqr}\partial_r H, \tag{24}$$

where

$$H = 1 + \frac{\tilde{R}^3}{r^3}, \qquad \tilde{R}^3 = N' l_p^3,$$
$$r^2 = x_6^2 + \cdots + x_{10}^2. \tag{25}$$

The membrane probe action is

$$S = \int d^3\sigma \sqrt{-\det(G_{mn})} + \int \left(f^*C\right)_{mnp}\epsilon^{mnp}, \tag{26}$$

where f^*C denotes the pull back of the 3-form potential C and G_{mn} is the induced metric given by the pull back from the spacetime metric.

We will consider a membrane stretched radially outward from the five brane stack. This can be described after picking suitable coordinates by:

$$\sigma^0 = x^0, \qquad \sigma^1 = x^1, \qquad \sigma^2 = x^{10}. \tag{27}$$

It is trivial to check that this configuration solves the equations of motion given by varying the above action.

We wish to examine small fluctuations of this configuration. That is we will do a background field expansion around the above solution as follows,

$$X^\alpha(\sigma) = X^\alpha_{\rm cl} + x^\alpha(\sigma), \tag{28}$$

where $X^\alpha_{\rm cl}$ denote the classical configuration described by Eq. (27) above and $x^\alpha(\sigma)$ is the fluctuation field.

We then decompose the spacetime coordinates into those longitudinal to the five brane, transverse to the five brane and membrane and finally transverse to the membrane but longitudinal to the membrane as follows,

$$x^\alpha \to \left(x^a, x^\mu, x^{10}\right), \tag{29}$$

where $a = 0, \ldots, 5$, $\mu = 6, \ldots, 9$. We then derive the equations of motion for the s-wave fluctuations transverse to both the five brane and membrane by inserting the background field expansion (28) with background field (27) and coordinate decomposition (29) into the action (26). Then, examining the fluctuations transverse to both branes implies the equation of motion for $x^\mu = x^\mu(\sigma^2, t)$ is given by:

$$\partial_r^2 x^\mu - H(r)\partial_t^2 x^\mu = 0. \tag{30}$$

Solutions with energy ω, are given by $x^\mu = e^{-i\omega t} x^\mu(z)$ where we have introduced the dimensionless distance $z = r\omega$, with x^μ now obeying the following equation of motion:

$$\left(\frac{d^2}{dz^2} + 1 + \frac{(\tilde{R}\omega)^3}{z^3}\right) x^\mu(z) = 0. \tag{31}$$

Now we remark that this is the identical equation to that derived using the background field expansion of the five brane equations of motion around the self-dual string solution, i.e., the previous set up described earlier in the section. This should not come as a surprise since the scattering processes we are considering are clearly related. Using the previous results for solving this equation implies the dimensionless absorption cross section in this case is:

$$\sigma \sim \tilde{R}^3 \sim N', \tag{32}$$

where N' is the number of five branes. This means that in this process the absorption cross section and hence the number of degrees of freedom available for the probe membrane to scatter into is proportional to the number of five branes.

This leads us to conjecture that for a generic configuration of N' five branes and N membranes the cross section would be $N \times N'$. Note, this is different to the case described in [8] with D-branes and strings where the identification between parameters is more complicated. If one wishes to interpret the scattering as being related to the number of degrees of freedom accessible to the object then this would imply the number of low energy degrees of freedom of N self-dual strings contained in N' five branes goes as $N \times N'$.

The scattering of the non-BPS solutions found earlier maybe calculated using a simple generalisation of the above and shows that the scattering cross section decreases for the non-BPS solution as compared with the BPS solution in concordance with the results for BIons discussed in [23] and of course the N dependence remains the same.

4. Conclusions

This Letter has attempted to study various aspects of how membranes end on five-branes. These investigations were originally prompted by the lack of a current understanding of a "non-Abelian" five-brane theory and also by previous work concerning the near horizon of the self-dual string [22]. In [22] there was a conjectured low energy duality between a five-brane with $AdS_3 \times S^3$ geometry and the theory of N coincident self-dual strings. From the scattering calculations described above we have seen how the number of low energy degrees of freedom of the self-dual string depend on N (the self-dual string charge) and N' (the number of five branes); the answer may not be very surprising but given the previous surprises in M-theory (e.g., the $Q^{3/2}$ dependence of the membrane cross section) it is worth obtaining the answer. This result also seems to be consistent with the intuition gained by looking at N' five branes intersecting with N five branes on a string (even though that set up is quite different to the one described here one might imagine a relationship) for that situation, studied in [24] anomaly arguments also imply $N' \times N$ degrees of freedom for the string intersection.

It would also be useful to make a quantitative comparison with the model of self-dual strings described in [11] where Thompson scattering was calculated using an alternative description of the five-brane coupled to a self-dual string.

We have also constructed non-BPS solutions of the five-brane. It is not clear what role they play in M-theory or indeed what the non-BPS BIons play in string theory. Since these objects will have large quantum corrections one perhaps should not give them too much credence, however (using the $SO(1,1)$ symmetry), one may construct near BPS solutions where one can take the solution to be as near BPS as required. These near BPS solutions may well prove interesting to study just as near BPS solutions have proved more tractable in theories with gravity. One possible interpretation of these non-BPS solutions[2] is that they describe a situation with n membranes ending on one side of the five brane and m membranes ending on the other side. The scalar charge is then related to $m - n$ but the two form charge is related to $m + n$. The BPS condition which equates the scalar and two form charges then implies that $n = 0$ and one recovers the usual charge m self-dual string solution.

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